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FINITE ISOSPIN GROUPS AND THEIR EXPERIMENTAL CONSEQUENCES I*

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Abstract

Under the assumption that the observed isospin symmetry is the manifestation of the group structures of hadrons and their interactions, it is attempted to determine the order of the symmetry group, if finite, and to clarify the physical meanings of each group element. Our scheme is based on the observations that (1) the classifications of particles according to the irreducible representations of both the finite and the continuous groups are possible under certain restrictions, (2) the transformation laws of the particles under the continuous rotations in the isospin space cannot be established directly by experiments.

In particular, we will consider the polyhedral kaleidoscope groups. The consistent formulation by finite groups need a selection rule to exclude the unobserved exotic states, which turns out to be the requirement of the charge conservation. Several experiments are suggested to test our assumptions in the strong-interaction processes.

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I. INTRODUCTION

The approximate isospin symmetry is one of the most important concepts of the particle physics. This symmetry or the charge independence, which originated in the study of the nuclear forces, is usually formulated as a continuous, rotational symmetry in a hypothetical isospin space. The particles are classified into the irreducible representations of the group SU(2) in analogy with the ordinary angular momentum. Then one asks the consequences of the assumption that the scattering processes due to the strong interactions conserve the total isospin and that the scattering amplitudes depend, as for the isospin quantum numbers, only on the total isospin. This procedure is applied for most of the actual analyses. This symmetry has been successfully extended to the SU(3) symmetry by including the strangeness. The predictions and the subsequent discovery of Ω^-, in particular, seem to indicate that the group principle is really working in nature. (We remember that the J/ψ was not predicted although the SU(4) symmetry had been known.) The experiments have revealed, however, a remarkable property of hadrons that their isospins seem to have upper limits, $I = 1$ for the mesons, and $I = 3/2$ for the baryons.[1]

Now it is well known that the finite groups have only the finite-dimensional irreducible representations. So the question naturally arises: Is the isospin group finite or infinite?

The continuous group is a special case of the latter. This question should be answered before forming the Clebsch-Gordan series for the products of two irreducible representations. A similar problem has also occurred for the SU(3) symmetry, in which the mesons are classified into $\underset{\sim}{1}$ and $\underset{\sim}{8}$, while the baryons are classified into $\underset{\sim}{1}$, $\underset{\sim}{8}$, and $\underset{\sim}{10}$. The reason why higher dimensional multiplets do not appear in nature has never been explained in a satisfactory way. An interesting fact in this case is that these representations are constructed in the way[2]: $\underset{\sim}{3} \times \underset{\sim}{3}^* = \underset{\sim}{1} + \underset{\sim}{8}$, and $\underset{\sim}{3} \times \underset{\sim}{3} \times \underset{\sim}{3} = \underset{\sim}{1} + \underset{\sim}{8} + \underset{\sim}{8} + \underset{\sim}{10}$. However, the group SU(3) itself does not contain any inherent rule to exclude the higher dimensional representations. One of the motivations of our work presented here may be considered as an attempt to find such a framework. In order to answer the question raised above in connection with the isopsin symmetry, it will be necessary to examine the way in which the group SU(2) has been used. For the ordinary spin, the relative angles between the polarization vectors are measurable in principle to any degree of accuracy. This is the key point in establishing that the electrons behave as spinors in the ordinary space. Similarly, in order to establish the transformation laws under the continuous group, it is necessary to find some ways to observe the pions, the nucleons, and other particles at every angle θ_i ($i = 1,2$, and 3) with respect to some fixed coordinate system in the isospin space, if such ever exists.

The customary reason to believe in the SU(2) symmetry comes from the entirely different, indirect observations. The charge independence of the systems with relatively small isospins can be conveniently described by adopting this symmetry.[3] Thus the far weaker symmetry than the SU(2) symmetry may be sufficient to classify the particles. In this work we will try to formulate the isospin symmetry by using the finite subgroups of SU(2). If this method can explain all the evidence of the charge independence, then we will lose the argument for the isospin symmetry under the continuous group. On the other hand, if such an attempt turns out to be impossible, we must perhaps go to the stronger symmetry. Thus we find the formal similarity with the questions asked many years ago by Case et al[4], and Fairbairn et al.[5]

In the formulation we shall encounter with the basic problem: What is the physical meaning of each group element? This problem is not peculiar to the formulation by finite groups and arises because we may think the observed isospin symmetry to be just like the bilateral- and the rotational- symmetries of the various objects and the dynamical laws in the real world.

One suggestion to this problem comes from the original formulation by Heisenberg.[6] Three Pauli matrices were introduced there to a hypothetical space to describe the different states of the nucleons and the transitions between them. According to our interpretation, three matrices and the group generated by them are related to the dicyclic group of order 8. The group elements, or more rigorously three Pauli matrices, have the physical interpretations in this example. The finite group appeared here is also generated by three quaternions i, j, and k ($i^2 = j^2 = k^2 = ijk = -1$) and is denoted by $\langle 2, 2, 2\rangle$.

The finite groups considered in our work are the slight generalizations of it. They are the binary tetrahedral, the binary octahedral, and the binary icosahedral groups. These groups are often denoted by $\langle 3, 3, 2\rangle$, $\langle 4, 3, 2\rangle$, and $\langle 5, 3, 2\rangle$, respectively.

Now it is not difficult to make the unitary representations of these groups. We can identify one generator with the rotation around the Z-axis and diagonalize it. Then what are the possible correspondences between the basis of the irreducible representations and the electric charges?

In Sec. II we will consider one natural choice of the correspondence. The other choices will be mentioned.

We then assign the mesons and the baryons to the irreducible representations of the binary tetrahedral group and the binary octohedral group, respectively. This is based on the possible dimensions of the irreducible representations of these groups.

The observed isospins and the several arguments suggest that the hadrons actually belong to the representations of these groups. However, it should be stressed that we have no conclusive evidence for it at present. In the course of the analyses, we shall find that the decomposition of the product of the two irreducible representations into the Clebsch-Gordan series contains in general the components that are not the eigenstates of the electric charge. If such components are realized as particles, then they will lead to the violation of the charge-conservation law through the scattering process. A possible interpretation is suggested. In particular, we shall assume that only the states with the definite electric-charges can be realized as particles. This assumption still allows the appearance of the incomplete isospin multiplets such as the doublets with the charges +1 and -1. These multiplets, if realized as particles, will lead to the violation of the charge independence, yet will conserve the electric charges through the scattering processes. We will suggest the experiments to test such a possibility. If the incomplete multiplets are suppressed or forbidden as a whole, then our result is essentially the same, with respect to the classifications of the particles, as the conventional result with all the exotic-contributions omitted. In this case, the test of our hypotheses will need a far more advanced framework and will be postponed. Our work presented here should be considered to be of preliminary nature in this sense.

In Sec. III, the scattering processes are considered and several experiments are suggested as possible tests of our hypotheses.

Sec. IV. contains the concluding remarks.

II. CLASSIFICATION OF PARTICLES BY FINITE GROUPS

The conventional isospin group SU(2) or O(3) can classify the particles with any values of the isospin ($I = 0, \frac{1}{2}, 1, \ldots$). The observed mesons and the baryons, however, seem to have only the limited values of the isospin. The satisfactory explanation of this remarkable fact is hitherto unknown. We are therefore tempted to classify the particles by taking this restriction into account. In order to formulate it mathematically, the use of the finite groups seems to be the most attractive way for this purpose. The many-body systems such as the heavy nuclei and the neutron stars will be assumed to belong to the reducible representations of these symmetry groups.

Now let us begin our discussions with the finite subgroups of O(3). The possible finite subgroups are the cyclical, the dihedral, the tetrahedral, the octahedral, and the icosahedral groups.

We know that the quantum-mechanical states are represented by rays, rather than by vectors. So let us consider the ray representations of these groups. They are the same as the ordinary representations of the corresponding finite subgroups of SU(2).[7] The character tables for them are given in Tables I to III.[8]

The familiar form for the generator of the cyclic group is given by

$$A = \exp\,(2\pi\, i/n) = \cos\frac{2\pi}{n} + i\,\sin\frac{2\pi}{n}\,, \tag{1}$$

where n is a positive integer. The group elements are $1, A, \ldots, A^{n-1}$. In a similar way, the finite subgroups of SU(2) are generated by three quaternions:

$$A = \exp\,(P\pi/p),\ B = \exp\,(Q\pi/q),\ C = \exp\,(R\pi/r) \tag{2}$$

They are the cyclic, the dicyclic, the binary tetrahedral, the binary octahedral, and the binary icosahedral groups. These groups are exactly related to the subgroups of O(3) as in the same way $S\bar{U}(2)$ is related to O(3). In (2), P, Q, and R are pure unit quaternions and p, q, and r are positive integers. Geometrically P expresses a point on the unit sphere in the three dimensional space which is spanned by three unit-quaternions i, j, and k. Thus P, Q, and R can express a spherical triangle with angles π/p at P, π/q at Q, and π/r at R. All possible reflections on the sphere of this triangle, which is often called a "fundamental region", generate the desired finite group.

This notion is known to be quite general.[9] In another way, these groups are completely specified by the defining relations:

$$A^p = B^q = C^r = ABC = Z,\ Z^2 = 1 \quad . \tag{3}$$

The resultant group is denoted by $\langle p, q, r\rangle$.

Let us turn to the representations of the finite groups. It is easy to read off the character tables the possible dimensions of the irreducible representations. They are 1, 2, 3 for $\langle 3, 3, 2\rangle$, 1, 2, 3, 4 for $\langle 4, 3, 2\rangle$ and 1, 2, 3, 4, 5, 6 for $\langle 5, 3, 2\rangle$. The group mentioned in Sec. I, $\langle 2, 2, 2\rangle$, has only the one- and the two-dimensional irreducible representations.

We may identify one generator of the finite groups with the rotation around the Z-axis of the three-dimensional Euclidean space and diagonalize it. Such a generator is conveniently expressed by using a discrete angle $\theta = 2\pi/n$ (n = 2, 3, 4, and 5) and the usual infinitesimal generator I_z. Then it is $\exp(i\,\theta\, I_z)$.

Next, in order to apply to the physical problems, we need to assume some correspondence between the basis of the irreducible representations and the electric charges. The most natural way is clearly to retain the Gell-Mann-Nishijima relation in the integrated form. We may require the equation:

$$\exp\ (i\theta I_Z) = \exp\ \{i\theta\ (Q - Y/2)\} \tag{4}$$

to hold for all possible discrete values of θ corresponding to a given finite group. In this equation, Q is the electric charge and Y is the hypercharge of a particle. Another possibility is realized if the charge states are permuted among themselves in an arbitrary way. We note that the quantum numbers Q, Y, and others, if needed, specify the eigenvalues of the matrices $\exp\ (i\ \theta\ I_Z)$, but that they are not the group elements.

It is clear that the finite groups considered as the subgroups of SU(2) contain the finite number of the discrete rotation angles.[10] However, it may be too early to conclude that such angles have the direct physical meanings unless the metric is introduced into the underlying space in a physically meaningful way. The generators A, B, and C in (2) are the fundamental ingredients of the finite groups. Therefore in any physical applications, their meanings should be clarified.

We simply note that the generating relations (3) can be realized by isodoublet fermion-fields in the following way:

$$\begin{aligned} A\ \psi &= A_{op}^{-1}\ \psi\ A_{op}\ , \\ A_{op} &= \exp\ [\int d^3\vec{x}\ \psi^+(x)\ P\ (\pi/p)\ \psi(x)] \end{aligned} \tag{5}$$

and similar relations for B and C. In (5), P should be identified with $i\ \sigma_1$, which is a two-dimensional realization of the pure unit quaternion P in terms of the Pauli matrix, and p=2.

III. SCATTERING AMPLITUDES

The isospin coordinate was introduced to describe the protons and the neutrons as the different states of the same particles. The group SU(2) is usually employed for the classification of the particles. But the hypothesis of the charge independence is a more complicated matter.

The experimental analyses have shown that the isospin symmetry of the scattering amplitudes can be understood if we assume that the same symmetry holds for the vertices of the diagram corresponding to the scattering and that the propagators (which are either resonances or Reggeons) constitute the complete isospin multiplets.[11]

This situation seems to be very general. In this scheme, the scattering amplitudes can be constructed diagramatically by combining the vertices and the propagators with no loop. One can even imagine that the vertices are actually 3 - vertices (Fig. 1).

We know that the apparent absence of the exotic states of the mesons and baryons has been confirmed by analyses based on such diagrams. The basic observation is that, to a good approximation, such 3-vertices are actually allowed only when three lines correspond to the non-exotic particles.

So let us try to formulate the above rule as a basic law of the scattering processes. We may require that the allowed vertices in the above sense should also occur in the Clebsch-Gordan decomposition of the product of two irreducible representations corresponding to a and b in Fig. 1. In the $\pi\pi$ scattering, the usual decomposition contains I = 0, 1, and 2 states. Then, is it possible to suppress the unobserved direct-channel resonances with I = 2 by a composition rule for the initial two-pion state?

In order to answer this question, we now propose the following set of assumptions:

(a) There is no exotic stable or resonance state,

(b) The mesons (the baryons) belong to the irreducible representations of the binary tetrahedral (binary octahedral) group,

(c) The assignment of the electric charges to each member of the multiplet is done in the conventional way,

(d) Only the eigenstates of the electric charges are realized as particles.

Before assigning the particles to the basis of each representation, Q and Y must be known beforehand for each particle. This is clear for any experimental situation. The converse problem, i.e., to define Q or Y from (4), does not arise.

Let us turn to Clebsch-Gordan series for the two-pion system. It will contain the doubly charged components, e.g., $\pi^+\pi^+$ and $\pi^-\pi^-$. If these components are realized as particles, it will contradict with our assumption (a). Evidently this process can lead to the states with any values of the electric charges for sufficiently many pions if they are allowed. So, some way to exclude such exotic states is essential for the success of our procedure. It may be accomplished by a selection rule. We will find such a rule in the following.

Next we assign the mesons to Γ_0, $\Gamma_{\frac{1}{2}}$, and Γ_1 of <3, 3, 2>, and baryons to Γ_0, $\Gamma_{\frac{1}{2}}$, Γ_1, and $\Gamma_{3/2}$ of <4, 3, 2>. These representations are the same as the corresponding representations D_0, $D_{\frac{1}{2}}$, D_1 and $D_{3/2}$ of the group SU(2). Of these representations, Γ_0, $\Gamma_{\frac{1}{2}}$, and Γ_1 belong to both <3, 3, 2> and <4, 3, 2>.

Our assumption (b) implies that the system with the baryon number 0 is to be decomposed into the irreducible representations of <3, 3, 2>, while the system with the baryon number ±1 is always an irreducible representation of <4, 3, 2>. In particular, the baryon-antibaryon systems should be first decomposed into the irreducible representations of <4, 3, 2>, and the latter representations should be further decomposed, if reducible under <3, 3, 2>, into the irreducible representations of <3, 3, 2>. For the πN state, the decomposition is the same as the conventional case. This is seen from

$$\Gamma_{\frac{1}{2}} \times \Gamma_1 = \Gamma_{\frac{1}{2}} + \Gamma_{3/2} \tag{6}$$

In (6), N belongs to $\Gamma_{\frac{1}{2}}$ of <4, 3, 2>, while π belongs to Γ_1 of <3, 3, 2>. Thus $\Gamma_{\frac{1}{2}}$ and Γ_1 are apparently two irreducible representations of the different groups. Now we know that

$$SU(2) \supset \langle 4,\ 3,\ 2\rangle \supset \langle 3,\ 3,\ 2\rangle\ . \tag{7}$$

So we may use the decomposition-rule of SU(2), followed by subduction (i.e., restriction to the subgroup), to reach (6). The total baryon number determines whether the representations should be subduced to those of <4, 3, 2> or <3, 3, 2>. The baryon states $\pi\Sigma$, $\bar{K}\Delta$, and $\pi\Delta$ have the unconventional components in the decompositions.

For $\pi\,\Sigma$ state we obtain

$$\Gamma_1 \times \Gamma_1 = \Gamma_0 + \Gamma_1 + \Gamma_1^{*} + \Gamma_x\ ,$$

$$|\pi\rangle \times |\Sigma\rangle = \frac{1}{\sqrt{3}}\,(\pi^-\Sigma^+ - \pi^o\Sigma^o + \pi^+\Sigma^-),$$

$$\left\{\begin{array}{l} \frac{1}{\sqrt{2}}\,(\pi^+\Sigma^o - \pi^o\Sigma^+) \\[2ex] \frac{1}{\sqrt{2}}\,(\pi^+\Sigma^- - \pi^-\Sigma^+) \\[2ex] -\frac{1}{\sqrt{2}}\,(\pi^-\Sigma^o - \pi^o\Sigma^-)\quad , \end{array}\right. \tag{8}$$

$$+\begin{cases} \frac{1}{\sqrt{2}}\,(\pi^-\Sigma^o + \pi^o\Sigma^-) \\ \frac{1}{\sqrt{2}}\,(\pi^+\Sigma^+ - \pi^-\Sigma^-) \\ \frac{1}{\sqrt{2}}\,(\pi^+\Sigma^o + \pi^o\Sigma^+) \end{cases},$$

$$+\begin{cases} \frac{1}{\sqrt{2}\,i}\,(\pi^+\Sigma^+ + \pi^-\Sigma^-) \\ \frac{1}{\sqrt{6}}\,(\pi^-\Sigma^+ + 2\pi^o\Sigma^o + \pi^+\Sigma^-) \end{cases}.$$

The assignments of $\pi\,\Sigma$ state to $\Gamma_1 \times \Gamma_1^*$ and $\Gamma_1^* \times \Gamma_1^*$ lead to the same decompositions.

The $\overline{K}\Delta$ state is decomposed as

$$\Gamma_{\frac{1}{2}} \times \Gamma_{3/2} = \Gamma_1 + \Gamma_1^* + \Gamma_x ,$$

$$|\overline{K}> \times |\Delta> = \begin{cases} \frac{1}{2}\,(\Delta^+ \overline{K}^o - \sqrt{3}\,\Delta^{++} K^-) \\ \frac{1}{\sqrt{2}}\,(\Delta^o \overline{K}^o - \Delta^+ K^-) \\ \frac{1}{2}\,(\sqrt{3}\,\Delta^- \overline{K}^o - \Delta^o K^-) \end{cases}, \qquad (9)$$

$$+\begin{cases} \frac{1}{2}\,(\Delta^-\bar{K}^o + \sqrt{3}\;\Delta^oK^-) \\[2ex] \frac{1}{\sqrt{2}}\,(\Delta^{++}K^o - \Delta^-K^-) \\[2ex] -\frac{1}{2}\,(\sqrt{3}\;\Delta^+\bar{K}^o + \Delta^{++}K^-)\quad, \end{cases}$$

$$-\begin{cases} \frac{1}{\sqrt{2}\,i}\,(\Delta^{++}\bar{K}^o + \Delta^-K^-) \\[2ex] \frac{1}{\sqrt{2}}\,(\Delta^o\bar{K}^o + \Delta^+K^-)\;. \end{cases}$$

The assignment of the $\bar{K}\Delta$ state to $\Gamma_{\frac{1}{2}}^* \times \Gamma_{3/2}$ leads to the same decomposition. The $\pi\Delta$ state is decomposed to $\Gamma_{3/2} \times \Gamma_1 = \Gamma_{3/2} \times \Gamma_1^* = \Gamma_{3/2} + \Gamma_{\frac{1}{2}} + \Gamma_{3/2} + \Gamma_{\frac{1}{2}}^*$, $|\pi> \times |\Delta>=$

$$\begin{cases} \sqrt{\frac{3}{5}}\;\Delta^{++}\pi^o - \sqrt{\frac{2}{5}}\;\Delta^+\pi^+ \\[2ex] \sqrt{\frac{2}{5}}\;\Delta^{++}\pi^- - \frac{1}{\sqrt{15}}\;\Delta^+\pi^o - \sqrt{\frac{8}{15}}\;\Delta^o\pi^+ \\[2ex] \sqrt{\frac{8}{15}}\;\Delta^+\pi^- - \frac{1}{\sqrt{15}}\;\Delta^o\pi^o - \sqrt{\frac{2}{5}}\;\Delta^-\pi^+ \\[2ex] \sqrt{\frac{2}{5}}\;\Delta^o\pi^- - \sqrt{\frac{3}{5}}\;\Delta^-\pi^o\quad, \end{cases}$$

$$+\begin{cases} \frac{1}{\sqrt{2}}\,\Delta^{++}\pi^{-} - \frac{1}{\sqrt{3}}\,\Delta^{+}\pi^{o} + \frac{1}{\sqrt{6}}\,\Delta^{o}\pi^{+} \\ \frac{1}{\sqrt{6}}\,\Delta^{+}\pi^{-} - \frac{1}{\sqrt{3}}\,\Delta^{o}\pi^{o} + \frac{1}{\sqrt{2}}\,\Delta^{-}\pi^{+}\ , \end{cases} \tag{10}$$

$$+\begin{cases} \frac{1}{\sqrt{10}}\,\Delta^{+}\pi^{+} + \frac{1}{\sqrt{15}}\,\Delta^{++}\pi^{o} + \sqrt{\frac{5}{6}}\,\Delta^{-}\pi^{-} \\ -\sqrt{\frac{3}{10}}\,\Delta^{o}\pi^{+} + \sqrt{\frac{3}{5}}\,\Delta^{+}\pi^{o} + \frac{1}{\sqrt{10}}\,\Delta^{++}\pi^{-} \\ \frac{1}{\sqrt{10}}\,\Delta^{-}\pi^{+} + \sqrt{\frac{3}{5}}\,\Delta^{o}\pi^{o} + \sqrt{\frac{3}{10}}\,\Delta^{+}\pi^{-} \\ -\sqrt{\frac{5}{6}}\,\Delta^{++}\pi^{+} - \frac{1}{\sqrt{15}}\,\Delta^{-}\pi^{o} - \frac{1}{\sqrt{10}}\,\Delta^{o}\pi^{-} \end{cases}$$

$$+\begin{cases} \frac{1}{\sqrt{6}}\,\Delta^{++}\pi^{+} - \frac{1}{\sqrt{3}}\,\Delta^{-}\pi^{o} - \frac{1}{\sqrt{2}}\,\Delta^{o}\pi^{-} \\ -\frac{1}{\sqrt{2}}\,\Delta^{+}\pi^{+} - \frac{1}{\sqrt{3}}\,\Delta^{++}\pi^{o} + \frac{1}{\sqrt{6}}\,\Delta^{-}\pi^{-} \end{cases}$$

In (7), Γ_0 and Γ_1 are the same as the conventional I = 0 and I = 1 states. Γ_1^* contains a component which does not correspond to a definite charge. If such a component is realized as a particle, it has no definite charge and the conservation law of the charges will be violated through the process: $\pi^+ + \Sigma^+ \rightarrow (1/\sqrt{2})\ (\pi^+\Sigma^+ - \pi^-\Sigma^-) \rightarrow \pi^- + \Sigma^-$. This is the motivation for our assumption (d). It is interesting to note that all the exotic components appear in combination with ones of different charges. (This is also the case for the mesons).

So, if we can select only the states with definite charges from the representations, the exotic states are completely excluded. Then the resulting incomplete isospin-multiplet may be realized as a particle multiplet. This is one possibility. The basis of the incomplete multiplet no longer constitute the irreducible representations of the group. If the incomplete multiplet is forbidden as a whole, then it is equivalent to omit all the exotic multiplets in the conventional way ($I = 2$ for the $\pi\pi$ state). This is another possibility. One should ask whether such an incomplete multiplet, if realized as a particle multiplet, violate the well-established principles of physics. As noted before, the charge independence will be violated in the scattering processes if such particles are exchanged. We found no further difficulty. For the $\pi\,\Sigma$ and the $\bar{K}\Delta$ scatterings, the gapped-charge states will be observed if Γ_1^* is dominant. These states simulate the $I = 1$ states but are different from the latter in that the neutral counterparts are absent in the particle spectrum.[12]

This is not a so strange possibility. We know one such example in the case of the ordinary spin, i.e., the polarization states of the real photon. One difficulty in identifying such resonances in the existing data lies in that the usual analyses are always done by assuming the conventional isospin symmetry.[13] A more definite conclusion may be obtained from the ratio:

$$\sigma(\Delta^o K^- \to \Delta^o K^-) : \sigma(\Delta^o K^- \to \Delta^- \bar{K}^o) : \sigma(K^- \Delta^{++} \to \Delta^+ \bar{K}^o) : \sigma(K^- \Delta^{++} \to K^- \Delta^{++})$$

$$= 1:\ 3:\ 3:\ 9 \qquad \text{for } \Gamma_1,$$

$$= 9:\ 3:\ 3:\ 1 \qquad \text{for } \Gamma_1^*. \tag{11}$$

If the Δ^o -exchange contribution is sufficiently well separated from the N-exchange in the K^-p backward scattering, then this ratio will give us an interesting test of our assumptions.[14] We have yet no conclusive evidence on these points.

It is interesting to see what should happen if the assumption (c) is relaxed. Clearly, a quite general way of the assignment of the charges is obtained by a replacement: $\vec{\pi} \to M\vec{\pi}$, for pions for example, by using a unitary 3 x 3 matrix M. The permutations between the definitely-charged components will be the special case of it. However, we have no experimental evidence for such a generalization at present.

Finally we study the decompositions for the two-meson states.

The $\pi\pi$ state is decomposed

$$\Gamma_1 \times \Gamma_1 = \Gamma_0 + \Gamma_1 + \Gamma_1 + \Gamma_0' + \Gamma_o'' ,$$

$$= \frac{1}{\sqrt{3}} (\pi^-\pi^+ - \pi^o\pi^o + \pi^+\pi^-) ,$$

$$+ \left\{ \begin{array}{l} \frac{1}{\sqrt{2}} (\pi^+\pi^o - \pi^o\pi^+) \\ \\ \frac{1}{\sqrt{2}} (\pi^+\pi^- - \pi^-\pi^+) \\ \\ -\frac{1}{\sqrt{2}} (\pi^-\pi^o - \pi^o\pi^-) , \end{array} \right. \qquad (12)$$

$$
+\begin{cases} \frac{1}{\sqrt{2}}\,(\pi^-\pi^o + \pi^o\pi^-) \\ \frac{1}{\sqrt{2}}\,(\pi^+\pi^+ - \pi^-\pi^-) \\ -\frac{1}{\sqrt{2}}\,(\pi^+\pi^o + \pi^o\pi^+) \end{cases} ,
$$

$$
+ \left[\frac{1}{\sqrt{6}}\,(\pi^-\pi^+ - 2\pi^o\pi^o + \pi^+\pi^-) - \frac{1}{\sqrt{2}}\,(\pi^+\pi^+ + \pi^-\pi^-) \right] C_1 ,
$$

$$
+ \left[\frac{1}{\sqrt{6}}\,(\pi^-\pi^+ - 2\pi^o\pi^o + \pi^+\pi^-) + \frac{1}{\sqrt{2}}\,(\pi^+\pi^+ + \pi^-\pi^-) \right] C_2 ,
$$

where C_1 and C_2 are normalization constants.[15]

Another interesting example is πK state. It is given by

$$
\Gamma_1 \times \Gamma_{\frac{1}{2}} = \Gamma_{\frac{1}{2}} + \Gamma_{\frac{1}{2}}' + \Gamma_{\frac{1}{2}}'' ,
$$

$$
|\pi\rangle \times |K\rangle = \begin{cases} \frac{1}{\sqrt{3}}\,\pi^oK^+ - \sqrt{\frac{2}{3}}\,\pi^+K^o \\ \sqrt{\frac{2}{3}}\,K^+\pi^- - \frac{1}{\sqrt{3}}\,\pi^oK^o \end{cases} ,
$$

$$
+\begin{cases} -\frac{1}{\sqrt{3}}\,\pi^-K^+ - \sqrt{\frac{2}{3}}\,\pi^oK^o \\ \sqrt{\frac{2}{3}}\,\pi^+K^+ + \frac{1}{\sqrt{3}}\,\pi^-K^o \end{cases} , \tag{13}
$$

$$\rightarrow \begin{cases} \dfrac{1}{\sqrt{3}}\,\pi^{+}K^{+} - \sqrt{\dfrac{2}{3}}\,\pi^{-}K^{o} \\ -\sqrt{\dfrac{2}{3}}\,\pi^{o}K^{+} - \dfrac{1}{\sqrt{3}}\,\pi^{+}K^{o} \end{cases} .$$

In (12), $\Gamma_{\frac{1}{2}}$, $\Gamma_{\frac{1}{2}}{'}$, and $\Gamma_{\frac{1}{2}}{''}$ are equivalent ray-representations. If the incomplete multiplets can be neglected, we get only a conventional $I = \frac{1}{2}$ state ($\Gamma_{\frac{1}{2}}$). This decomposition is particularly interesting when we note that in the customary theory the effective Hamiltonian for the nonleptonic weak interactions is assumed to have the same isospin-structure as the $K\pi$ system. More specifically, $H_{wk} = \text{const.} \times \int d^3\vec{x}\ J_\mu(x)(\Delta S=0, \Delta I=1) \times J_\mu(x)(\Delta S=\frac{1}{2}, \Delta I=0)$. Thus, under the assumption that the incomplete multiplets ($\Gamma_{\frac{1}{2}}{'}$ and $\Gamma_{\frac{1}{2}}{''}$) do not occur, we are naturally led to the $\Delta I = \frac{1}{2}$ rule. Experimentally $\Delta I = 3/2$ part certainly exists and this fact may indicate either that the incomplete multiplets has the contributions or that the assumed form of the Hamiltonian is not appropriate.[16] The most attractive way will be to relate the current operators with the generators of the finite groups in the way of (5) and to write the Hamiltonian in terms of them in the usual way. However, we must wait the much more detailed, quantitative analyses on this subject before reaching the definite conclusions.

IV. CONCLUDING REMARKS

If we accept the group concept seriously in any physical applications, we should find some physical method to determine the precise structure of the group. The symmetry under the continuous groups can be established and meaningful only when some experimental procedure is actually given to prove the symmetry at every value of the continuous parameters.

The precise group structure responsible for the isospin symmetry is not yet known in this sense, even if the group concept is actually relevant in this symmetry. We note, however, that this is essential in forming the Clebsch-Gordan series.

In this work we made a preliminary attempt to clarify the isospin symmetry. We classified the particles into the irreducible ray representations of the finite subgroups of O(3). If the exotic mesons and baryons are established by experiments, then we may still classify them by the icosahedral group as far as the isospins are sufficiently small.

In atomic physics, the discrete energy levels of the hydrogen atoms were explained by the standing-wave condition for the de Broglie wave. Our assignment of the mesons and baryons into the ray representations of the finite subgroups of O(3) is therefore in close similarity to it. It is quite a remarkable fact that these finite groups are generated by reflections. They are analogous to the familiar C, P, T transformations.

Finally we stress again that we have not yet any conclusive evidence for our scheme. If the incomplete multiplets are not confirmed, then a far more elaborate framework is certainly necessary to test our assumptions. The detailed knowledge from the electromagnetic- and the weak-interaction processes will be indispensable in that case.

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13. We thank F. J. Gilman for the helpful conversations on this point.
14. We thank D.W.G.S. Leith and H. Miettinen for the helpful conversations on this point.

15. The choice of the basis for $\Gamma_{\frac{1}{2}}'$ and $\Gamma_{\frac{1}{2}}''$ was done by following (7).

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Table captions

I. Character table of the binary tetrahedral group <3, 3, 2>, of order 24. $\omega = \exp(2\pi i/3)$. Γ_0, $\Gamma_{1/2}$, and Γ_1 are the representations D_0, $D_{1/2}$, and D_1 of the group SU(2).

II. Character table of the binary octahedral group <4, 3, 2> of order 48. Γ_0, $\Gamma_{1/2}$, Γ_1, and $\Gamma_{3/2}$ are the representations D_0, $D_{1/2}$, D_1, and $D_{3/2}$ of the group SU(2).

III. Character table of the binary icosahedral group <5, 3, 2> of the order 120. $\alpha = (1 + \sqrt{5})/2$; $\beta = (1 - \sqrt{5})/2$. Γ_0, $\Gamma_{1/2}$, Γ_1, $\Gamma_{3/2}$, Γ_2, and $\Gamma_{5/2}$ are the representations D_0, $D_{1/2}$, D_1, $D_{3/2}$, D_2 and $D_{5/2}$ of the group SU(2).

Table I

Class / Rep.	E	R	$6C_2$	$4C_3$	$4C_3^{1}$	$4C_3^{a}$	$4C_3^{a'}$
Γ_0	1	1	1	1	1	1	1
Γ_0'	1	1	1	ω	ω	ω^2	ω^2
Γ_0''	1	1	1	ω^2	ω^2	ω	ω
Γ_1	3	3	-1	0	0	0	0
$\Gamma_{\frac{1}{2}}$	2	-2	0	1	-1	1	-1
$\Gamma_{\frac{1}{2}}'$	2	-2	0	ω	$-\omega$	ω^2	$-\omega^2$
$\Gamma_{\frac{1}{2}}''$	2	-2	0	ω^2	$-\omega^2$	ω	$-\omega$

Table II

Rep. \ Class	E	R	$6C_2$	$6C_4'$	$6C_4$	$12C_2^a$	$8C_3$	$8C_3'$
Γ_0	1	1	1	1	1	1	1	1
Γ_0^*	1	1	1	-1	-1	-1	1	1
Γ_x	2	2	2	0	0	0	-1	-1
Γ_1	3	3	-1	1	1	-1	0	0
Γ_1^*	3	3	-1	-1	-1	1	0	0
$\Gamma_{1/2}$	2	-2	0	$\sqrt{2}$	$-\sqrt{2}$	0	1	-1
$\Gamma_{1/2}^*$	2	-2	0	$-\sqrt{2}$	$\sqrt{2}$	0	1	-1
$\Gamma_{3/2}$	4	-4	0	0	0	0	-1	1

Table III

Rep. \ Class	E	R	$12C_5$	$12C_5{}'$	$12C_5{}^a$	$12C_5{}^{a'}$	$20C_3$	$20C_3{}'$	$30C_2$
Γ_0	1	1	1	1	1	1	1	1	1
Γ_1	3	3	α	α	β	β	0	0	-1
$\Gamma_1{}^*$	3	3	β	β	α	α	0	0	-1
Γ_y	4	4	-1	-1	-1	-1	1	1	0
Γ_2	5	5	0	0	0	0	-1	-1	1
$\Gamma_{1/2}$	2	-2	α	$-\alpha$	$-\beta$	β	1	-1	0
$\Gamma_{1/2}{}^*$	2	-2	β	$-\beta$	$-\alpha$	α	1	-1	0
$\Gamma_{3/2}$	4	-4	1	-1	-1	1	-1	1	0
$\Gamma_{5/2}$	6	-6	-1	1	1	-1	0	0	0

Figure caption

1. The basic diagram for the scattering process. Three lines a, b, and c represent the non-exotic particles.

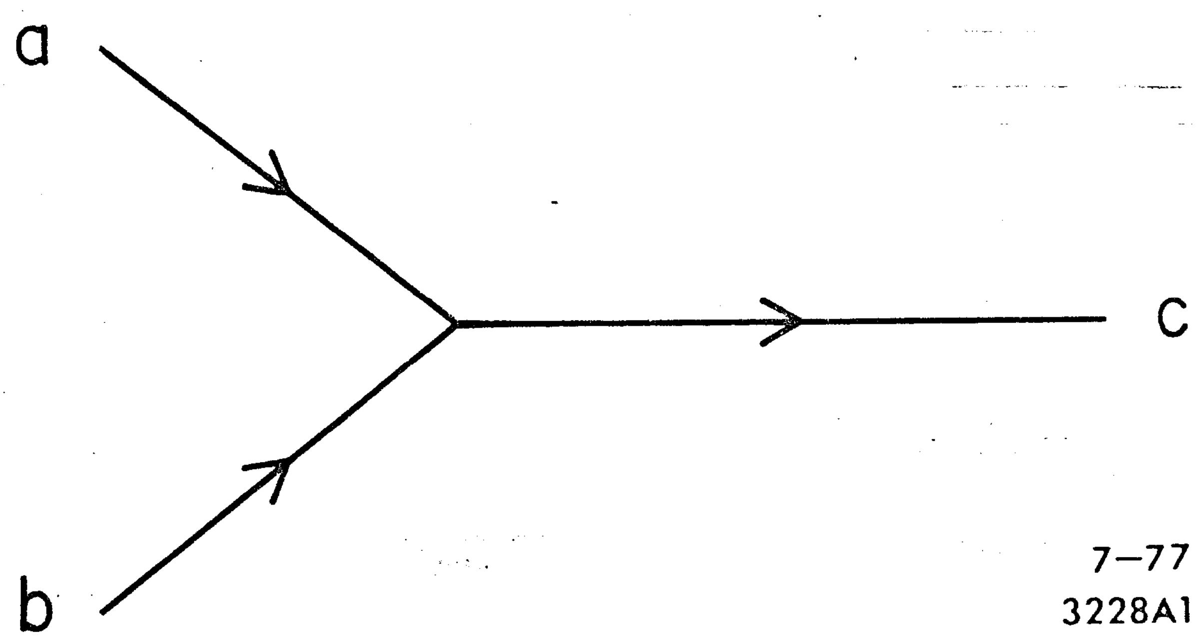

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Fig. 1