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THE NAMBU MECHANICS OF NON-ABELIAN DYONS*

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ABSTRACT

Non-Abelian top-like realizations of Nambu's mechanics are found in nonrelativistic bound systems of a point singular SU(2) monopole and a test particle bearing G-spin embedded in a general compact gauge group G. The exactly soluble cases of G = U(1), SU(2) and SU(3) dyons are studied in detail. The naturalness of the new formalism for these Keplerian systems is exhibited.

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I. INTRODUCTION

In the current climate of superunified and quark confining theories, there is a renewed feeling that some generalization of our idea of space-time[1] and/or extension of quantum mechanics[2] may be needed to reach a conceptually unified scheme for all interactions. If one chooses to proceed from Dirac's Hamiltonian method[3] as a first step to a quantum theory, then extensions of classical mechanics should be made. It is in this spirit that not long ago, with Liouville theorem as his guiding principle, Nambu[4] wrote down a new analytic mechanics and discussed its quantization. This mechanics is remarkable in several respects. First, it treats all conserved quantities of a mechanical system on the same footing. This is clearly a most attractive feature from a quantum perspective. Second it allows for a phase space of <u>odd</u> as well as even dimensionality, a property shared by another new mechanics, pseudomechanics, whose phase space is partly spanned by anticommuting Grassmann variables.[5] The latter permit a consistent inclusion of spin and other internal degrees of freedom into classical mechanics. It has been shown that at the classical level, this new mechanics is connected to a singular Hamiltonian mechanics.[6] As regards quantization, Nambu showed that the usual Heisenberg equations are recovered in several examples. This lends further support to the idea of the uniqueness of quantum mechanics. Notably, he found that both Jordan's non-associative algebra, be they special or exceptional, can be readily incorporated in his new mechanics. Thus it was later demonstrated that most of all the irreducible Fock representations of Green's trilinear algebra are consistent with Nambu's quantization.[7] This invites the conjecture that quarks may well find their natural dynamical description within the new mechanics. Our work advances a small step in fulfilling such an expectation: We show that beyond the asymmetric top Nambu mechanics find its simplest physical realizations in non-Abelian dyons. The latter have provided many models for strong interaction dynamics.[8]

Before stating our results, it is helpful to recall the essential elements of the new mechanics.[4]

In a general Nambu mechanics, the phase space is spanned by a n-tuple of dynamical variables x_i, $i=1,2,\ldots n$. There are (n-1) Hamiltonians $H_1,\ldots,H_{n-1}$ which are constants of motion and which define the dynamics via the generalized Hamiltonian equations

$$\dot{x}_i = \frac{\partial(x_i, H_1, H_2, \ldots, H_{n-1})}{\partial(x_1, x_2, \ldots, x_n)}$$

$$= \epsilon_{ijk\ldots\ell}\, \partial_j H_1 \ldots \partial_\ell H_{n-1} \quad , \tag{1.1}$$

where $\epsilon_{ijk\ldots\ell}$ is the Levi-Civita symbol and $\partial_j \equiv \partial/\partial x_j$. The $F_i = F_i(x_1,\ldots,x_n)$ is an arbitrary differentiable function of the variables x_i. The totally antisymmetric n-linear bracket

$$\left\{F_1,\ldots,F_n\right\} \equiv \epsilon_{ij\ldots\ell}\, \partial_i F_1\, \partial_j F_2 \ldots \partial_\ell F_n \tag{1.2}$$

takes the place of the usual Poisson bracket. Like the latter, it stands as the fundamental algebraic object of the theory. Instead of the canonical Poisson brackets $\left\{q_i, p_j\right\} = \delta_{ij}$, one has the Nambu bracket

$$\left\{x_1, x_2, \ldots, x_n\right\} = 1 \tag{1.3}$$

and in place of $\left\{q_j, q_i\right\} = \left\{p_i, p_j\right\} = 0$, one has for instance

$$\left\{x_1, x_1, x_3, \ldots, x_n\right\} = 0 \quad , \tag{1.4}$$

etc. The corresponding Nambu bracket preserving canonical transformations are defined as

$$x \rightarrow x'$$

such that

$$\left\{x_1', \ldots, x_n'\right\} = \left\{x_1, \ldots, x_n\right\} = 1 \quad . \tag{1.5}$$

From Eq. (1.1) it ensues that the velocity field $\dot{\vec{x}}$ is divergenceless or equivalently Liouville theorem is fulfilled in the phase space. This fact opens the door to the corresponding new statistical mechanics yet to be investigated. As the simplest generalization of the usual canonical doublets (p_n, q_n), Nambu considers the interesting situation of N canonical triplets (P_n, Q_n, R_n) $n=1, 2, \ldots, N$. Then one has from Eq. (1.2) the trilinear bracket

$$[A, B, C] = \sum_{n=1}^{N} \frac{\partial(A, B, C)}{\partial(P_n, Q_n, R_n)} \tag{1.6}$$

and

$$\dot{F} = \left[F, H_1, H_2\right] \quad , \tag{1.7}$$

where $F = F(P, Q, R)$, $H_1(P, Q, R)$ and $H_2(P, Q, R)$ are two Hamiltonian functions. To make a case for the physical relevance of his new formalism, Nambu pointed out a specific realization for the n=3, one triplet case; namely the asymmetric top. Here the triplet $\vec{x}$ is naturally identified with the angular momentum $\vec{L}$ in the body fixed frame. There are two guaranteed conserved quantities: $H_1 = 1/2 \sum L_i^2/I_i$ is the total kinetic energy $(L_i = I_i \omega_i)$ and $H_2 = 1/2 \vec{L}^2$ the Casimir invariant. The Nambu equations Eq. (1.7) are given by

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_3 - I_2)\, \omega_2\omega_3 \quad , \\ I_2\dot{\omega}_2 &= (I_1 - I_3)\, \omega_1\omega_3 \quad , \\ I_3\dot{\omega}_3 &= (I_2 - I_1)\, \omega_1\omega_2 \quad , \end{aligned} \tag{1.8}$$

which are just the Euler force-free rigid body equations. We observe that they correspond to a special case of the Lie equation[9] when the algebra is SO(3, R) of the rotation group.

This example is important on two scores. From the structural viewpoint it illustrates one essential advantage of the new mechanics which reveal prima facie the key structures of constrained systems, i.e., their Lie algebra at the level of the equations of motion. This situation contrasts with Dirac's mechanics where the constraints appear as subsidiary conditions. We recall that the essential structure shared by classical and quantum mechanics lies in their partaking in the same Lie algebra. While the usual bilinear bracket structure is connected to some Lie or Jordan algebra, the n-linear bracket can in principle accommodate other algebra as well. It is this algebraic naturalness and flexibility of Nambu mechanics which make it most attractive in one's search for generalized dynamics encompassing both classical and quantum mechanics.

From the physics viewpoint, top-like systems are plentiful in nature. It suffices to mention the Bohr-Mottelson nuclear model and the dynamics of colored, electric strings in the non-Abelian lattice gauge theory.[10] Since its inception, Nambu's mechanics has been the object of a few investigations all focused solely on its formal aspects.[6,11,12] What is clearly more desirable to the model building physicist are further realizations of the new mechanics, possibly ones which bear directly on our current ideas about the basic constituents of matter. In this work we depart minimally from Nambu's simple example of the top and yet we do exhibit realizations which obey the mentioned criterion of selection.

Basically our work stems from Fierz's old observation[13] that a dyon, a system of a charged particle interacting with a magnetic monopole, has the algebraic structure of a symmetric rigid top. Hence we infer that natural realizations of Nambu mechanics are to be found in these Keplerian systems. We have particularly in mind their non-Abelian generalizations which have been the objects of several recent investigations.

In a short note, one of us (M.H.)[14] studied in the context of Nambu mechanics the motion of a test particle bearing isotopic spin $\vec{T}$ in the field of a 't Hooft-Polyakov[15] monopole. Such a classical dyon is conceived as a non-relativistic, point-singular limit of the field theory model of a SU(2) monopole interacting with a Higgs field carrying a representation T^a of SU(2). While entirely bosonic in composition such a complex yet admits integral or half-integral total angular momentum depending on the tensor or spinor nature of T^a.[16] In the latter case we witness a realistic example of the dynamical fermionization phenomenon which has its counterpart in the duality between Sine-Gordon soliton and the massive fermion of the Thirring model.[17] The spin and statistics connection for such systems has been elucidated by A. S. Goldhaber.[18] We have a system with superselection rule sectors set by Dirac or Schwinger's quantization condition. The superselection rule manifests itself in a striking group theoretical manner which makes the dyons distinctly different from the usual top. While in the latter only the Lie algebra is of importance, with the dyon, the self-adjointness of the Hamiltonian and rotational invariance, are realized only if the algebra of the system can be integrated to yield the group. This globality constraint is nothing but the monopole quantization condition.[19]

In this work, we extend the work of Ref. 14 to the general cases where the composite SU(2) monopole $\oplus$ test particle, a SU(2) dyon, is embedded in a compact gauge group G.

We do not lose any nontrivial topological feature by restricting ourselves to the limiting case of dyons involving a point-singular monopole and a test particle with internal quantum numbers. We will proceed rather systematically from the standard problem of the U(1) dyon to that of the SU(2) dyon which receives here an improved treatment over Ref. 14. Then we go on to the natural extension of the SU(2) problem, namely when SU(2) is embedded in SU(3) in two possible ways, respectively the SO(3) and the U(2) embeddings. All these exactly soluble systems are cast into Nambu mechanics. The physical motivation for studying this SU(3) extension is enticing. In the case of the SU(3)/U(2) dyon the resulting monopole quantization restricts the electric charge of the test particle to be quark-like. So the resulting Nambu system describes a very massive composite quark-monopole complex endowed with dynamical fermionic spin. Such a structure might be of eventual interest. In particular for such a SU(3) dyon we obtain the corresponding SU(3) top equations of motion in the form conjectured by Nambu. Our analysis readily generalizes to the instance of a G dyon.

Our paper is organized as follows: In Section II, we introduce the U(1) dyon in its new garment as a Nambu mechanics. In Section III we first recall the essential results of the embedding SO(3) point singular monopole in a general compact Lie group. Then we set up the dynamical equations in the usual Hamiltonian mechanics. We consider the dynamics of the SU(2) and SU(3) dyons as Nambu mechanics of non-Abelian tops. Our results are in such a form that generalization to a G dyon is manifest. Finally in Section IV we discuss the possible advantages of the new formalism and its pending problems.

II. THE U(1) DYON

In many ways, magnetic monopoles may play a role in particle physics. On one hand, they are soliton solutions to gauge theories of weak and electromagnetic interactions. They can also be identified with the endpoints of vortex lines which then serve as field theoretical building blocks for dual string theories.[20] As non-Abelian analogs of Cooper pairs in a color supervacuum, they may play a controlling role in the confining plasma phase of the quark-gluon dynamics.[21] On the other hand, even if these physical pictures fail to materialize the possibility still exists that the relevance of their underlying algebraic structures will survive under a different guise. So it is hoped that when recast into a new mathematical framework, the explicit examples given here may yet lead to possible generalizations of monopole systems. These observations are justification enough for the reformulation of monopole dynamics into Nambu mechanics.

In this work we deal exclusively with dyons by which we mean composites of an electrically charged particle moving in the field of a fixed monopole. As the kernel of a Nambu mechanics of dyons is present in the Abelian case, it will be our first concern to study the U(1) dyon.

The dynamics of a nonrelativistic dyon has received ample treatment in the literature; we only gather the ingredients necessary for our work.[22]

Since a true dyon does not have bound states, to secure binding we shall assume an added nonelectromagnetic potential V(r) taken spherically symmetric and free of singularities for radii $r > 0$.

The Hamiltonian of the system is

$$H = \frac{1}{2m}(\vec{p} - e\vec{A})^2 + V(r) \tag{2.1}$$

with

$$\vec{A} = g\,\frac{(\vec{r}\cdot\vec{n})(\vec{r}\times\vec{n})}{r\{r^2 - (\vec{r}\cdot\vec{n})^2\}} \tag{2.2}$$

Notably $\vec{\nabla}\times\vec{A}$ differs from the field $\vec{B} = g\vec{r}/r^3$ of a fixed monopole by the Dirac string singularity starting at the pole and ending at infinity along the unit direction $\vec{n}$. For our problem the group kinematical structure of the system is of central importance. First, we identify the constants of motion, foremost among them, the angular momenta. Due to the contribution $-\mu\hat{r}$ ($\mu = eg$) of the gauge field, the conserved angular momentum is

$$\vec{J} = \vec{r} \times (\vec{p} - e\vec{A}) - \mu\hat{r} \tag{2.3}$$

already considered by Poincaré.[23]

The constraint $\vec{J}\cdot\hat{r} = -\mu$ implies the motion of the test charge is confined to a cone. If we are to use the $\vec{J}$ as a new dynamical variable, we are naturally led to introduce an object complementary to $\vec{J}$, the radial dilatation generator

$$D = \vec{r}\cdot(\vec{p} - e\vec{A}) \quad . \tag{2.4}$$

This is so since $\vec{J}$ does not carry any information about the radial component of $\vec{p}$. Then one can check that the Poisson brackets of the system can be cast into a closed Lie algebra.[22]

$$\begin{aligned} &\{x_i, x_j\} = 0 \quad , \\ &\{J_i, x_j\} = \epsilon_{ijk} x_k \quad , \\ &\{J_i, J_j\} = \epsilon_{ijk} J_k \quad , \\ &\{D, x_i\} = -x_i \quad , \\ &\{J_i, D\} = 0 \quad . \end{aligned} \tag{2.5}$$

So the corresponding group is the 3-dimensional symmetric group S^3 made up of the Euclidean group $E_3 = R(3) \times T_3$ of rotations and translations plus the dilatations. The irreducible representation of S^3 are labelled by the "helicity" invariants

$$\vec{J}\cdot\hat{r} = 0 \,, \quad \pm 1/2, \quad \pm 1, \; \ldots \quad . \tag{2.6}$$

And since $\vec{J}\cdot\hat{r} = -eg$, they provide an algebraic derivation of Dirac's quantization condition. The latter is seen as a global superselection rule required for rotational invariance and self-adjointness of the Hamiltonian.[19] It implies the fictitiousness of the Dirac string. Instead of the momentum variables $\vec{p}$, we can use $\vec{J}$ and D. Then

$$H = \frac{1}{2mr^2}(D^2 + J^2 - \mu^2) + V(r) \quad . \tag{2.7}$$

Since $\{\vec{r}^2, J\} = 0$, we see that for <u>fixed</u> r^2, the system is isomorphic to a symmetric top, wherefore the natural motivation to reformulate it as a Nambu mechanics.

From the above ingredients, we deduce the following equations of motion

$$\dot{\vec{J}} = 0 \quad , \tag{2.8}$$

$$\dot{r} = \frac{D}{mr} \quad , \tag{2.9}$$

$$\dot{D} = 2H - 2V - r\frac{dV}{dr} \quad . \tag{2.10}$$

The geometry of the motion leads us to define two angles θ and ϕ such that

$$\cos\theta = \frac{\vec{J}\cdot\vec{r}}{Jr} \tag{2.11}$$

and

$$r\sin\theta\,\dot{\phi} = \dot{\vec{r}}\cdot\frac{\vec{J}\times\vec{r}}{|\vec{J}\times\vec{r}|} \quad . \tag{2.12}$$

Due to the constraint $\cos\theta = -\mu/J$, θ is not an independent variable. From (2.12) we have

$$\dot{\phi} = \frac{J}{mr^2} \tag{2.13}$$

So the equations of motion for $\vec{r}$ and $\vec{p}$ are equivalent to the set (2.8) - (2.10) and (2.13). We note that besides the triplet $\vec{J}$, the variables r, D and ϕ form another triplet. So these kinematics of the dyon system naturally invite a reformulation into a Nambu mechanics. Yet the variables r, D, ϕ are still unsuitable for that purpose. From Eqs. (1.1) and (1.2), any differentiable function $F(x_i)$

serves just as well as a new coordinate. It follows that if an $F(x_i)$ can be found such that $\dot{F}(x_i) = 0$, we then know its orbit to be restricted to a submanifold of the phase space, one defined by $F(x_i) = \text{const}$. One such constant is clearly the Hamiltonian H, (2.7), which defines the energy shell. Besides $\vec{J}$ and H, the remaining fifth constant of motion of our problem can be chosen to be the angular variable

$$\Phi = \phi - J \int^{r} \frac{dr'}{r'\, f(r')} \tag{2.14}$$

given as an Abelian integral where f(r) is given by

$$\{f(r)\}^2 = D^2 = 2mr^2\{H - V(r)\} - J^2 + \mu^2 \tag{2.15}$$

Usually the five constants of motion are taken as $\vec{J}$, H and $|\vec{L}| = m\,|\vec{r} \times \dot{\vec{r}}|$. However Φ is a better choice than $|\vec{L}|$ since it is conserved in the SU(N) $(N \geqq 2)$ dyon problem while $|\vec{L}|$ is not. This point will be made clear in Section III.

The equations of motion are now

$$\dot{\vec{J}} = \dot{H} = \dot{\Phi} = 0 \tag{2.16}$$

and the corresponding five constants of motion determine a one dimensional submanifold, the orbit of the six dimensional phase space. For our sixth Nambu-coordinate completing the set $\vec{J}$, H, Φ, we can define

$$S = m \int^{r} \frac{r'\, dr'}{f(r')} \tag{2.17}$$

such that $\dot{S}=1$.

Exploiting the two triplet structure of the U(1) dyon kinematics, contact with Nambu mechanics can be established in the following manner. We denote the two triplets as $(J_1, J_2, J_3) = (P_1, Q_1, R_1)$ and $(S, \Phi, H) = (P_2, Q_2, R_2)$. Then the equation for an arbitrary $F = F(Q, P, R)$ is given by

$$\dot{F} = \left[F, H_1, H_2\right] \tag{2.18}$$

with $H_1 = Q_2 = \Phi$ and $H_2 = R_2 = H$ and

$$[A, B, C] = \sum_{n=1}^{2} \frac{\partial(A, B, C)}{\partial(P_n, Q_n, R_n)} \tag{2.19}$$

Then we get

$$\left[P_n, Q_n, R_n\right] = 1, \quad \text{etc.} \tag{2.20}$$

The above equations (2.18) - (2.20) realize Nambu mechanics for the U(1) dyon system.

In subsequent sections, it will turn out that for systems with correlation between spatial and internal degrees of freedom, there exists a more convenient time variable τ defined by

$$\tau = \frac{\phi(t)}{J} \quad . \tag{2.21}$$

Then instead of the t-derivative $\dot{F} = dF/dt$, we make use of the τ derivative

$$\overset{*}{F} \equiv \frac{dF}{d\tau} = \frac{J}{\dot{\phi}} \dot{F} = mr^2 \dot{F} \quad . \tag{2.22}$$

It is of course easy to rewrite (2.18) - (2.20) into Nambu mechanics with the proper time variable τ

$$\overset{*}{F} = \left[F, H_1, H_2\right] \quad . \tag{2.23}$$

The only modification necessary is that P_2 should be identified not with S but with

$$U = \int^{r} \frac{dr'}{r' f(r')} \tag{2.24}$$

which obeys

$$\overset{*}{U} = 1 \quad . \tag{2.25}$$

Next, we proceed to the instance of non-Abelian dyons.

III. NON-ABELIAN DYONS

A. Generalities

In this section we reformulate in the context of Nambu mechanics a general class of non-Abelian top-like systems. They are given by the non-relativistic dynamics of a classical point particle with mass m and G-spin T_a in the field of a fixed SU(2) monopole embedded in a general compact semi-simple gauge group G. T_a are the generators of some representation of G. For this task we need to know all spherically symmetric 3-dimensional point monopole solutions for the simply connected universal covering $\tilde{G}$ of G. This purely group theoretical problem has been analyzed by many people and has been in particular solved completely in a very simple and compact manner by Wilkinson and Goldhaber.[24] Here we only gather the results relevant to our work. Generally one seeks static solutions in the Coulomb gauge $A_0=0$ of the Higgs-Kibble field equations

$$\epsilon_{ijk} D_j B_k = ie\left[\Phi, D_i\Phi\right] \quad ,$$
$$D_i D_i \Phi = \frac{\partial V}{\partial \Phi} \quad , \qquad (3.1)$$
$$B_i = \frac{1}{2}\epsilon_{ijk} G_{jk} \quad ,$$

where $G_{\mu\nu} = \partial_{[\mu} A_{\nu]} - ie\left[A_\mu, A_\nu\right]$ and $D_\mu \Phi = \partial_\mu \Phi - ie\left[A_\mu, \Phi\right]$. A_μ and Φ are matrix fields in some faithful representation of G. $V(\Phi)$ is the standard G invariant quartic polynomial. Specifically the point-monopole solutions are obtained in the London limit[25] when $\delta V/\delta\Phi = \vec{D}\Phi = 0$ everywhere except at the singularity positioned at the origin. The simple recipe[24] for constructing these solutions comes from using a singular Abelian gauge

$$\vec{A}_D = \frac{\hat{r} \times \vec{n}}{r(1 - \vec{r} \cdot \vec{n})} \qquad (3.2)$$

with the Dirac string singularity pointing from the pole along $\vec{n}$ to infinity. Thus a solution to Eq. (3.1) is

$$\Phi = \Phi_0 \quad ,$$
$$e\vec{A} = Q\vec{A}_D \quad , \tag{3.3}$$

Φ_0 and Q are constant matrices such that $(\partial V/\partial\Phi)_{\Phi_0} = 0$ and $\left[Q, \Phi_0\right] = 0$. Then the generalized Dirac quantization condition is given by

$$e^{i4\pi Q} = \mathbb{1} \quad . \tag{3.4}$$

Provided the eigenvalues of Q are all integers or half-integers, the Dirac string is unobservable and with $\widetilde{G}$ being simply connected, it can be removed by way of a singular gauge transformation to get to a no-string gauge.

The spherical symmetry of the problem is underscored by the conserved angular momenta

$$\vec{J} = \vec{r} \times \vec{\pi} - ie\, r^2 \vec{B} \tag{3.5}$$

with $\vec{\pi} = \vec{p} - e\vec{A}$ and the associated gauge covariant properties

$$\left[J_i, J_j\right] = i\,\epsilon_{ijk} J_k \quad ,$$
$$\left[J_i, \pi_j\right] = i\,\epsilon_{ijk} \pi_k \quad , \tag{3.6}$$
$$\left[J_i, \Phi\right] = 0 \quad .$$

The main result of Ref. 24 is to provide the necessary and sufficient conditions for transferring the solution (3.3) to a regular gauge A'_j with manifest spherical symmetry, i.e.,

$$\left[L_i + T_i, A'_j\right] = i\,\epsilon_{ijk} A'_k$$
$$\left[L_i + T_i, \Phi'\right] = 0 \tag{3.7}$$

when $\vec{L} = \vec{r} \times \vec{p}$ and T_i generate some SU(2) subgroup of G. We shall use such radial gauge in the following examples of the SU(2) and SU(3) dyons. The above summary constitutes the minimum knowledge we need. For further details on actual determination of point monopoles for a general group G, we refer the reader to the literature and shall only quote the results needed below.

B. SU(2) Dyon

The equations of motion of a test particle with mass m and isospin T_i ($i=1,2,3$) interacting with a SU(2) magnetic monopole are[14, 16]:

$$\dot{x}_i = \frac{1}{m}\left(p_i - eA_i^a T_a\right) \quad , \tag{3.8}$$

$$\dot{p}_i = \frac{1}{m}\left(p_j - eA_j^a T_a\right) \frac{\partial A_j^b}{\partial x_i} eT_b - \frac{\partial V}{\partial x_i} \tag{3.9}$$

with

$$\dot{T}_a = -\epsilon_{abc} \frac{1}{m}\left(p_i - eA_i^d T_d\right) eA_i^b T_c \quad . \tag{3.10}$$

The notation is self-explanatory.

These equations can be derived from

$$\dot{F}(x,p,T) = \langle F, H \rangle \tag{3.11}$$

with

$$H = \frac{1}{2m}\left(\vec{p} - e\vec{A}^a T_a\right)^2 + V(r) \tag{3.12}$$

where a generalized Poisson bracket is defined as

$$\langle A, B \rangle = \left\{A, B\right\} + (A, B) \quad . \tag{3.13}$$

Here

$$\left\{A, B\right\} = \frac{\partial A}{\partial x_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial x_i} \tag{3.14}$$

is the standard bracket and

$$(A, B) \equiv \epsilon_{abc} \frac{\partial A}{\partial T_a}\frac{\partial B}{\partial T_b} T_c \tag{3.15}$$

is characteristic of classical true spin systems.[26] The latter involve only the variables T_a and no others. This is certainly not the case for a rigid body or our SU(2) dyon since the T_i do not form a complete set of coordinates, neither in the kinematic nor dynamic sense.

In the radial gauge and the London limit where the Higgs field is 'rigid', i.e., everywhere constant, we have

$$A_i^a(x) = -\frac{1}{e}\epsilon_{iak}\frac{x_k}{r^2} \tag{3.16}$$

for all $\vec{r} \in R^3-(0)$. This potential corresponds to the only spherically symmetric point monopole for SU(2).[24] The conserved angular momentum is[16]

$$\vec{J} = \vec{r} \times \vec{p} + \vec{T} \tag{3.17}$$

and since

$$\vec{J}\cdot\hat{r} = \vec{T}\cdot\hat{r} \tag{3.18}$$

the total angular momentum takes on integral or half-integral values depending on the tensor or spinor character of the representation of $\vec{T}$.

We define the angles θ and ϕ just as in the previous section. θ and ϕ fix the direction of $\vec{r}$ relative to $\vec{J}$. It is convenient to introduce three orthonormal vectors $\vec{m}_1$, $\vec{m}_2$ and $\vec{m}_3$ such that

$$\vec{m}_1 = \frac{(\vec{J}\times\hat{r})\times\hat{r}}{|(\vec{J}\times\hat{r})\times\hat{r}|} , \tag{3.19}$$

$$\vec{m}_2 = \frac{\vec{J}\times\hat{r}}{|\vec{J}\times\hat{r}|} \tag{3.20}$$

and

$$\vec{m}_3 = \hat{r} . \tag{3.21}$$

They satisfy

$$\vec{m}_i\cdot\vec{m}_j = \delta_{ij} \tag{3.22}$$

and

$$\vec{m}_1 \times \vec{m}_2 = \vec{m}_3 \ , \quad \text{etc.} \tag{3.23}$$

Next we define the variables w_1, w_2 and w_3 by

$$\vec{T} = w_1\vec{m}_1 + w_2\vec{m}_2 + w_3\vec{m}_3 \quad . \tag{3.24}$$

It should be noted that w_3 is given by

$$w_3 = J \cos \theta \quad . \tag{3.25}$$

After some manipulations we find that $\vec{p}$ can be expressed as

$$\vec{p} = -\frac{w_2}{r}\vec{m}_1 + \frac{w_1 + J\sin\theta}{r}\vec{m}_2 + \frac{D}{r}\vec{m}_3 \tag{3.26}$$

where D is given by

$$D = \vec{r}\cdot\vec{p} \ . \tag{3.27}$$

Using (3.12), (3.16), (3.24), and (3.26) we have

$$\begin{aligned} H &= \frac{1}{2m}\left(\vec{p} + \frac{\vec{T}\times\vec{r}}{r^2}\right)^2 + V(r) \\ &= \frac{1}{2mr^2}\left(D^2 + J^2 - w_3^2\right) + V(r) \quad . \end{aligned} \tag{3.28}$$

It is easy to check that the set of nine variables J_1, J_2, J_3, r, ϕ, D, w_1, w_2, and w_3 constitutes independent variables defining the SU(2) dyon system.

The nine equations of motion for $\vec{r}$, $\vec{p}$ and $\vec{T}$ are then equivalent to[14]

$$\begin{aligned} &\dot{J}_1 = \dot{J}_2 = \dot{J}_3 = 0 \quad , \\ &\dot{r} = \frac{D}{mr} \ , \quad \dot{\phi} = \frac{J}{mr^2} \ , \quad \dot{D} = 2H - 2V - r\frac{dV}{dr} \quad , \\ &\dot{w}_1 = \frac{w_2 w_3}{mr^2} \ , \quad \dot{w}_2 = -\frac{w_3 w_1}{mr^2} \ , \quad \text{and} \quad \dot{w}_3 = 0 \quad . \end{aligned} \tag{3.29}$$

In Eq. (3.29) we recognize the U(1) problem in the subset $\{\vec{J}, r, \phi, D\}$. The remaining set of $\{w_1, w_2, w_3\}$ relates specifically to the SU(2) problem. We note that while $L^2 = (\vec{r} \times \vec{p})^2$ is conserved in the U(1) problem, it is no longer so here. We find $L^2 = T^2 + 2w_1 \, J \sin\theta$. T^2, J and $\sin\theta$ are conserved but w_1 is not, hence L^2 is also not conserved.

Just as in Section II we are led to define new variables

$$\Phi = \phi - J \int^r \frac{dr'}{r' f(r')} \tag{3.30}$$

and

$$S = m \int^r \frac{r' \, dr'}{f(r')} \tag{3.31}$$

where

$$\{f(r)\}^2 = D^2 = 2mr^2 (H - V(r)) - J^2 + w_3^2 \quad . \tag{3.32}$$

We further define u and σ by

$$u = \sqrt{w_1^2 + w_2^2} \tag{3.33}$$

and

$$\sigma = \tan^{-1} \frac{w_1}{w_2} + \frac{\phi}{J} w_3 \tag{3.34}$$

Then Eqs. (3.29) are equivalent to

$$\dot{\vec{J}} = \dot{H} = \dot{\Phi} = \dot{\theta} = \dot{u} = \dot{\sigma} = 0 \tag{3.35}$$

and

$$\dot{S} = 1 \quad .$$

This form of the equations allows an immediate transcription into Nambu mechanics. Here we have a three triplet structure of dynamical variables. The simplest identification is $(P_1, Q_1, R_1) = (J_1, J_2, J_3)$, $(P_2, Q_2, R_2) = (S, \Phi, H)$ and $(P_3, Q_3, R_3) = (u, \sigma, \theta)$ and $H_1 = \Phi$ and $H_2 = H$. In general we can identify the eight variables Q_2, Q_3, P_1, P_2, P_3, R_1, R_2 and R_3 with any independent eight functions of the Φ, u, σ, H, θ, $\vec{J}$. By setting $Q_1 = S$, $H_1 = P_1$ and $H_2 = R_1$ any arbitrary differentiable function f(P, Q, R) will obey Eq. (1.7).

An alternative and more elegant choice of variables is provided by the set $\Omega = \{\vec{J}, U, \Phi, H, \vec{w}\}$ where $\vec{w} = (w_1, w_2, w_3)$ and

$$U = \int^r \frac{dr'}{r' f(r')} \quad . \tag{3.36}$$

The motivation is apparent. The first six variables constitute the U(1) dyon system while $\vec{w}$ describes the internal rotation of the isospin. The nine equations of motion in (3.29) can be reexpressed as

$$\begin{aligned} &\overset{*}{J}_1 = \overset{*}{J}_2 = \overset{*}{J}_3 = 0 \quad , \\ &\overset{*}{\Phi} = \overset{*}{H} = 0 \ , \qquad \overset{*}{U} = 1 \ , \\ &\overset{*}{w}_1 = w_2 w_3 \ , \quad \overset{*}{w}_2 = -w_3 w_1 \quad \text{and} \quad \overset{*}{w}_3 = 0 \end{aligned} \tag{3.37}$$

where *-symbol denotes the τ-derivative introduced at the end of Section II.

It should be noted that $\vec{w}$ really satisfies the "SU(2)-top" equation of motion:

$$\begin{aligned} I_1 \overset{*}{w}_1 &= (I_3 - I_2)\, w_2 w_3 \ , \\ I_2 \overset{*}{w}_2 &= (I_1 - I_3)\, w_3 w_1 \ , \\ I_3 \overset{*}{w}_3 &= (I_2 - I_1)\, w_1 w_2 \end{aligned} \tag{3.38}$$

with

$$I_3 = 2I_1 = 2I_2 \quad . \tag{3.39}$$

It is now easy to cast (3.37) into the form of Nambu mechanics. By making the identifications

$$
\begin{aligned}
(P_1, Q_1, R_1) &= (J_1, J_2, J_3) \quad , \\
(P_2, Q_2, R_2) &= (U, \Phi, H) \quad , \\
(P_1, Q_3, R_3) &= (w_1, w_2, w_3) \quad , \\
H_1 &= \frac{1}{2}\left(w_1^2 + w_2^2 + w_3^2\right) + \Phi
\end{aligned}
\tag{3.40}
$$

and

$$H_2 = \frac{1}{2} w_3^2 + H \quad ,$$

we get to

$$\overset{*}{F} = \sum_{n=1}^{3} \frac{\partial(F, H_1, H_2)}{\partial(P_n, Q_n, R_n)} \quad . \tag{3.41}$$

In Eq. (3.38) we have observed that $\vec{w}$ behaves just like the angular momentum vector of a force-free rigid body in the body fixed frame. The coordinate system defined by $\vec{m}_1$, $\vec{m}_2$ and $\vec{m}_3$ corresponds to the body fixed frame of a rigid body. The interconnection between internal and spatial motions of the test particle rests in the space-dependent definition of $\vec{w}$ and in the redefinition of the time variable.

C. The SU(3) Dyon

We now proceed to the SU(3) dyon, which turns out to have some interesting new structure.

By a SU(3) dyon we understand the composite of a SU(2) monopole embedded in the group SU(3) and a test particle carrying a SU(3) unitary spin λ_a. From the preliminaries in Section III.A, we know that all the SU(3) monopole solutions must be invariant under simultaneous spatial and isospatial rotations generated by $\vec{J}+\vec{T}$. There are two distinct embeddings of SU(2) into SU(3), [24] the "U spin" $U(2) \sim SU(2) \otimes U(1)$ case with $\vec{T}$ identified with $(1/2\,\lambda_1,\ 1/2\,\lambda_2,\ 1/2\,\lambda_3)$ and the "nuclear physics" SO(3) case with $\vec{T} = (\lambda_7, -\lambda_5, \lambda_2)$.

The corresponding nonrelativistic Hamiltonian for the SU(3) dyon is then given as

$$H = \frac{1}{2m}\left(\vec{p} - e\vec{A}^a \frac{\lambda_a}{2}\right)^2 + V(r) \tag{3.42}$$

The c-number $1/2\,\lambda_a$ $(a = 1, 2, \ldots, 8)$ is the unitary spin carried by the test particle. V(r) is a spherically symmetric binding potential.

The equations of motion are derived from

$$\dot{F} = \langle F, H\rangle \tag{3.43}$$

where the generalized Poisson bracket is given by

$$\langle A, B\rangle = \sum_{j=1}^{3} \frac{\partial(A,B)}{\partial(x_j, p_j)} + \sum_{a,b,c=1}^{8} 2f_{abc} \frac{\partial A}{\partial\lambda_a} \frac{\partial B}{\partial\lambda_b} \lambda_c \ . \tag{3.44}$$

with f_{abc} being the totally antisymmetric structure constants of SU(3). From Eqs. (3.42) and (3.43) we obtain

$$\begin{aligned} \dot{x}_i &= \frac{1}{m}\left(p_i - eA_i^a \frac{\lambda_a}{2}\right) , \\ \dot{p}_i &= \frac{e}{m}\left(p_j - eA_j^a \frac{\lambda_a}{2}\right) \frac{\partial A_j^b}{\partial x_i} \frac{\lambda_b}{2} - \frac{\partial V(r)}{\partial x_i} \\ \frac{1}{2}\dot{\lambda}_a &= -\frac{e}{m}\left(p_j - eA_j^d \frac{\lambda_d}{2}\right) f_{abc} A_j^b \frac{\lambda_c}{2} \end{aligned} \tag{3.45}$$

which gives

$$m\ddot{x}_i = e\,\dot{x}_j\, G_{ij}^a \frac{\lambda_a}{2} - \frac{\partial V}{\partial x_i} \tag{3.46}$$

where

$$G^a_{ij} = \partial_i A^a_j - \partial_j A^a_i + ef_{abc} A^b_i A^c_j \quad . \tag{3.47}$$

Following the detailed analysis of Corrigan et al.[27] the two distinct ansatzes for the asymptotic forms of the gauge potentials are

(a)

$$eA^a_i \frac{\lambda_a}{2} = i\left[\psi_1, \partial_i \psi_1\right] \quad ,$$

$$\psi_1 = \sum_{i=1}^{3} \hat{r}_i \frac{\lambda_i}{2} \tag{3.48}$$

$$T_a = \frac{\lambda_a}{2} \,, \qquad a = 1, 2, 3 \quad ,$$

and

(b)

$$eA^a_i \frac{\lambda_a}{2} = i\left[\phi_1, \partial_i \phi_1\right] \quad ,$$

$$(\phi_1)_{\alpha\beta} = \hat{r}_\alpha \hat{r}_\beta - \frac{1}{3}\delta_{\alpha\beta} \quad , \tag{3.49}$$

$$\vec{T} = (\lambda_7, -\lambda_5, \lambda_2) \quad .$$

For the point singular limit of interest here, the above forms are valid over the whole space. We have for both cases (a) and (b)

$$eA^a_i \frac{\lambda_a}{2} = -\frac{\epsilon_{ia\ell} T_a x_\ell}{r^2} \quad . \tag{3.50}$$

The conserved angular momentum is $\vec{J} = \vec{r} \times \vec{p} + \vec{T}$ which obeys the O(3) algebra $<J_i, J_j> = \epsilon_{ijk} J_k$ and $\vec{T}$ identified with the λ_a's according to the respective embeddings.

It follows from Eqs. (3.42) and (3.50) that

$$H = \frac{1}{2m}\left(\vec{p} + \frac{\vec{T} \times \vec{r}}{r^2}\right)^2 + V(r) \quad . \tag{3.51}$$

which is identical to the SU(2) problem. The structure of the Poisson brackets for $\vec{r}$, $\vec{p}$ and $\vec{T}$ is also exactly that of the SU(2) dyon problem. Thus there is no difference between the SU(3)/U(2) and SU(3)/SO(3) dyons as far as the quantities $\vec{r}$, $\vec{p}$ and $\vec{T}$ are concerned. As for the solutions for the remaining SU(3) generators, we select to analyze in detail the U(2) embedding case. Our motivation is as follows. The SU(3)/SO(3) monopole is a stable pure monopole and the quantization condition assigns to the test particle multiplet integral charges. So this system differs minimally from the SU(2) problem. On the other hand, the SU(3)/U(2) dyon is endowed with a more interesting new structure. Indeed its monopole is colored as it carries both electromagnetic and isomagnetic charges with the consequence that the quantization condition is no longer Dirac's. The charges of the test particle are then quark-like.[27] So this SU(3) dyon is an object bearing both quark-like quantum numbers and dynamical half-integral spin.

Denoting $T_a = \lambda_a/2$, $a=4,5,\ldots,8$, the equation (3.45) becomes

$$\dot{T}_a = \frac{e}{m}\left(p_j + \frac{\epsilon_{jk\ell} T_k x_\ell}{r^2}\right) f_{abc} \frac{\epsilon_{jbm} x_m}{er^2} T_c \tag{3.52}$$

or

$$\dot{T}_a = f_{abc} K_b T_c = \langle T_a, K \rangle \tag{3.53}$$

where

$$K_b = \begin{cases} \frac{1}{mr^2}\left\{J_b - (\vec{J}\cdot\hat{r})\,\hat{r}_b\right\} , & b=1,2,3 \\ 0 , & b=4,5,6,7,8 \end{cases} \tag{3.54}$$

and

$$K = \frac{1}{2} K_b J_b = \frac{1}{2mr^2} (\vec{J} \times \hat{r})^2 \tag{3.55}$$

The second equality in Eq. (3.53) follows from the fact that

$$H = K + \frac{(\vec{r}\cdot\vec{p})^2}{2mr^2} + V(r) \quad . \tag{3.56}$$

To make an appropriate choice of isospin variables, we go to the "body fixed frame." As $\vec{J}$ is conserved, we can, without loss of generality, fix $\vec{J}$ to be parallel to the third axis. In that system, $\vec{J}$ and $\hat{r}$ are

$$\vec{J} = \begin{pmatrix} 0 \\ 0 \\ J \end{pmatrix} \tag{3.57}$$

and

$$\hat{r} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \tag{3.58}$$

Then $\vec{m}_1$, $\vec{m}_2$ and $\vec{m}_3$ are

$$\vec{m}_1 = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \quad \vec{m}_2 = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{m}_3 = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\cos\phi \\ \cos\theta \end{pmatrix} \quad . \tag{3.59}$$

$\vec{T}$ and $\vec{w}$ are related by

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \tag{3.60}$$

where w_1, w_2 and w_3 are given by (3.24). w_1, w_2 and w_3 are the isospin variables in the body fixed frame.

We define the variables $w_4, w_5, \ldots, w_8$ which takes the place of $T_4, T_5, \ldots, T_8$ as follows:

$$\begin{pmatrix} T_7 - iT_6 \\ T_5 - iT_4 \end{pmatrix} = e^{-\frac{i\phi}{2}\sigma_3} e^{\frac{i\theta}{2}\sigma_3} W \quad ,$$

$$W = \begin{pmatrix} w_7 - iw_6 \\ w_5 - iw_4 \end{pmatrix} \tag{3.61}$$

and

$$T_8 = w_8 \quad , \tag{3.62}$$

σ_i's being Pauli matrices. The definitions (3.61) and (3.62) should be regarded as the isospinor and isoscalar variants of (3.60), respectively. The decomposition of $T_1, T_2, \ldots, T_8$ into a triplet $\vec{w}$, a complex doublet W and a singlet w_8 corresponds to the SU(2) decomposition of an SU(3) octet:

$$8 = 3 + 2 + 2 + 1 \quad .$$

Now it turns out that W and w_8 satisfy

$$\overset{*}{W} = \frac{i}{2} w_3 \sigma_3 W \tag{3.63}$$

and

$$\overset{*}{w}_8 = 0 \quad . \tag{3.64}$$

The triplet $\vec{w}$ of course satisfies

$$\overset{*}{w}_1 = w_2 w_3 \,, \qquad \overset{*}{w}_2 = -w_3 w_1 \quad \text{and} \quad \overset{*}{w}_3 = 0 \quad . \tag{3.65}$$

It is straightforward to recognize that (3.63), (3.64) and (3.65) can be combined in a simple form

$$\overset{*}{w}_a = f_{abc} \frac{\partial G_1}{\partial w_b} \frac{\partial G_2}{\partial w_c} \,, \qquad a = 1, 2, \ldots, 8 \tag{3.66}$$

where

$$G_1 = \frac{1}{2} \sum_{a=1}^{8} w_a^2 \tag{3.67}$$

and

$$G_2 = \frac{1}{2} w_3^2 \quad . \tag{3.68}$$

Equation (3.66) could indeed be called the "SU(3) top" equation.

By making the identifications

$$(u_1, u_2, u_3) = (J_1, J_2, J_3) \quad , \tag{3.69}$$

$$(v_1, v_2, v_3) = (S, \Phi, H) \quad , \tag{3.70}$$

$$H_1 = G_1 + \Phi \tag{3.71}$$

and

$$H_2 = G_2 + H \quad , \tag{3.72}$$

we are then led to the generalized Nambu mechanics

$$\overset{*}{F} = \sum_{x=u,v,w} f_{abc} \frac{\partial F}{\partial x_a} \frac{\partial H_1}{\partial x_b} \frac{\partial H_2}{\partial x_c} \tag{3.73}$$

where F is an arbitrary differentiable dynamical quantity. In Eq. (3.73) $u_4, u_5, \ldots, u_8$ and $v_4, v_5, \ldots, v_8$ are regarded as the frozen degrees of freedom and reflect the three dimensionality of physical space. We should also mention that an equation of the form of (3.73) was suggested to us by Nambu as the natural generalization of his original scheme.[28] It is very pleasing to see such a realization among non-Abelian dyon systems as we recall Dirac's invitation to exploit the kinship between mathematical and physical structures, a discourse which prefaced his celebrated introduction of monopoles into theoretical physics.[29] The form (3.73) of the SU(3) dyon dynamics then invites the abstract algebraic generalization in which none of the u's, v's, and w's are frozen out. What would be a physical realization of such an extension?

IV. FINAL REMARKS

As evidenced by studies of dynamical groups and supersymmetries, the underlying Lie algebra plays an essential role both in the formulation and the solutions of dynamical problems. It is a striking feature of Nambu mechanics that the Lie algebraic structure appear at the very level of the equations of motion. Another notable feature is the equal status of the space-time and internal symmetry dynamical variables. This has all the hallmarks of a unified theoretical framework. Indeed while the generalized Poisson bracket structure for the dyons looks rather unnatural both in the Dirac formalism or in pseudo-mechanics, no such an objection can be raised against the Nambu brackets. Of course to achieve this uniform symmetry with respect to all the variables in the new mechanics, a price must be paid. In general, our classical intuition is lost which may not be a handicap if one is seeking to generalize quantum mechanics not via any correspondence principle but, say, through some algebraic structures and their deformations.[30] In the examples considered in our work, the choice of dynamical variables and conserved quantities is relatively easy since we understand the physics behind our systems and can solve for the mathematical structure. This removes some of the arbitrariness of the choice of dynamical quantities inherent to Nambu mechanics. Though we only consider the instance of the U(1), SU(2) and SU(3) dyons, our treatment makes obvious the structure of Nambu mechanics of the SU(N) and the G dyons. Namely if $G \supset SU(2)$ with g_{abc} $(a, b, c = 1, \ldots, n)$ are the structure constants, then defining the w_a in the body fixed frame, we have

$$G_1 = \frac{1}{2} \sum_{a=1}^{n} w_a^2 , \quad G_2 = \frac{1}{2} w_3^2 \tag{4.1}$$

and

$$H_1 = G_1 + \Phi \quad , \quad H_2 = G_2 + H \tag{4.2}$$

then the Nambu equations of motion are

$$\overset{*}{F} = \sum_{x=u,v,w} g_{abc} \frac{\partial F}{\partial x_a} \frac{\partial H_1}{\partial x_b} \frac{\partial H_2}{\partial x_c} \quad . \tag{4.3}$$

In the introduction we give the main motivations for reformulation of the dynamics of dyons in the new mechanics. There the gauge field degrees of freedom are frozen down to a static potential. To go to a Nambu field theory is no easy task. As a first step beyond the Keplerian systems studied here, it would be natural to go over to the system of a Nambu string with monopoles at their ends.[31, 20] Here the gauge field is dynamical but still frozen into the line geometry of a string.

The hope in any reformulation of old problems in a new language lies in new possibilities which might show up in the new context. In the dyon systems studied here the correlation between internal symmetry and space-time degrees of freedom shows a richness and beauty of algebraic structures. Seen as Nambu mechanics, do they suggest completely novel algebraic dynamical structures? The generalized top-like structures we have obtained here are certainly suggestive. Indeed Nambu mechanics still has to come of age; its relevance awaits further and more radical illustrations than those provided here. Moreover the quantization of dyons in the new formalism should be studied. Already at the classical level of the dyons we see inviting connections of the new mechanics to dynamical groups,[32] to pseudomechanics,[5] and in a larger context to the rich symplectic structure[33] of the usual canonical formalism. We also note the analogy between dyon systems with Yang-Mills[34] and gravitational instantons;[35] would a Nambu statistical mechanics of non-Abelian tops be helpful in our understanding of the physical vacua?[21] Our investigations are continuing in these directions.[36]

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