# A NOVEL SOLUTION <br> OF THE YANG－MILLS FIELD EQUATIONS＊ 

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#### Abstract

We derive a new explicit solution of the Euclidean Yang－Mills field equations．Our solution becomes an instanton when some param－ eters vanish．Otherwise，it is singular on infinitely many shells that cluster at a point．We suggest an interpretation，and comment on the generality of this type of singular behavior．


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[^0]In the Euclidean formulation of non－Abelian gauge theories，the equations of（anti－）self－duality of the field－strength tensor，${ }^{1} \mathrm{~F}= \pm \widetilde{F}$ ，together with the requirement of finite action，in principle possess considerable physical signif－ icance（Ref．1）。By now，the mathematical theory of finite solutions of these equations has become rather sophisticated（Refs。2，3，4）。

We have found a class of singular explicit solutions to these equations． They exhibit some curious properties．In this note we derive our solutions， point out these properties，and offer some comments about implications．

For definiteness，we consider the equation $\mathrm{F}=-\widetilde{\mathrm{F}}$ 。 The tensor $\mathrm{F}_{\mu \nu}$ is de－ fined in terms of the gauge fields and the coupling constant by $\mathrm{F}_{\mu \nu} \equiv \partial_{\mu} \mathrm{A}_{\nu}$ $-\partial_{\nu} A_{\mu}-\operatorname{ig}\left[A_{\mu}, A_{\nu}\right]$ 。Our starting point is the following ansatz

$$
\begin{equation*}
\mathrm{A}_{\mu}=-\frac{\sigma^{\mathrm{a}}}{2} \eta_{\mu \nu}^{\mathrm{a}} \psi_{\nu} \tag{1}
\end{equation*}
$$

Our gauge group is $\operatorname{SU}(2)$ 。The summed index a runs from 1 to 3．The $\sigma^{\text {a }}$ are the Pauli matrices．The symbol $\eta_{\mu \nu}^{\mathrm{a}}$ is defined in Ref．5．

This ansatz has been written down before（Ref．3），but in previous analyses （Refs．2，3，6）it was assumed that $\psi_{\nu}$ was a gradient．We can simplify the field equations without imposing this constraint．Using Eq。（1）and the formulae in the appendix of Ref．5，we find the following expression for the sum of the field－ strength tensor and its dual：
$\mathrm{F}_{\mu \nu}+\widetilde{\mathrm{F}}_{\mu \nu}=\frac{\sigma^{\mathrm{a}}}{2}\left[-\eta_{\nu \lambda}^{\mathrm{a}}\left(\partial_{\mu} \psi_{\lambda}-\partial_{\lambda} \psi_{\mu}\right)+\eta_{\mu \lambda}^{\mathrm{a}}\left(\partial_{\nu} \psi_{\lambda}-\partial_{\lambda} \psi_{\nu}\right)+\eta_{\mu \nu}^{\mathrm{a}}\left(\partial_{\gamma} \psi_{\gamma}+\mathrm{g} \psi_{\gamma} \psi_{\gamma}\right)\right]$.
Now let us（i）set this equal to zero，（ii）multiply by $\eta_{\rho \beta}^{\mathrm{b}}{ }^{\mathrm{b}} \mathrm{b}$（summed over b）and take the group trace，and（iii）set $\mu=\beta$ and sum over $\mu_{\circ}$ ．The result is
$0=2\left(\partial_{\nu} \psi_{\rho}-\partial_{\rho} \psi_{\nu}\right)+\epsilon_{\nu \rho \mu \lambda}\left(\partial_{\mu} \psi_{\lambda}-\partial_{\lambda} \psi_{\mu}\right)+3 \delta_{\nu_{\rho}}\left(\partial_{\gamma} \psi_{\gamma}+g \psi_{\gamma} \psi_{\gamma}\right)$.

The first two terms in（3）are odd in $\nu$ and $\rho$ ，whereas the last term is even。 Thus we may replace（3）by the following two equations：

$$
\begin{gather*}
\partial_{\gamma} \psi_{\gamma}+g \psi_{\gamma} \psi_{\gamma}=0,  \tag{4}\\
\partial_{\nu} \psi_{\rho}-\partial_{\rho} \psi_{\nu}=-\frac{1}{2} \epsilon_{\nu \rho \mu \lambda}\left(\partial_{\mu} \psi_{\lambda}-\partial_{\lambda} \psi_{\mu}\right) 。 \tag{5}
\end{gather*}
$$

We see that，if $\mathrm{F}_{\mu \nu}$ is anti－self－dual，so is the curl of $\psi_{\mu}$ 。
Without loss of generality，we write（Ref．7）

$$
\begin{equation*}
\mathrm{g}_{\mu} \equiv \partial_{\mu} \psi+\partial_{\lambda} \mathrm{V}_{\lambda \mu}, \tag{6}
\end{equation*}
$$

where the tensor $\mathrm{V}_{\lambda \mu}$ is self－dual。 Using（6），and the self－duality of V ，Eq。（5） becomes，after some algebra，

$$
\begin{equation*}
\Delta V_{\mu \nu}=\left(\partial_{\lambda} \partial_{\mu}-\partial_{\mu} \partial_{\lambda}\right) V_{\lambda \nu}-\left(\partial_{\lambda} \partial_{\nu}-\partial_{\nu} \partial_{\lambda}\right) V_{\lambda \mu} . \tag{7}
\end{equation*}
$$

When $V_{\mu \nu}$ is singularity－free，this reduces to $\Delta V_{\mu \nu}=0$ 。 Finally，defining $\log \phi \equiv \psi$ ，Eq。（4）becomes

$$
\begin{equation*}
\phi^{-1} \sum_{\gamma=1}^{4}\left[\partial_{\gamma}+\left(\partial_{\lambda} V_{\lambda \gamma}\right)\right]^{2} \phi=0 \tag{8}
\end{equation*}
$$

Eq．（8）is essentially linear in $\phi$ 。Eqs。（7）and（8）are the fundamental equa－ tions of our analysis．

Our explicit solution is obtained by choosing

$$
\begin{equation*}
v_{\mu \nu}=\frac{\mathrm{K}_{\mu \nu}}{\mathrm{x}^{2}} \tag{9}
\end{equation*}
$$

with $K_{\mu \nu}$ constant and self－dual，and insisting that $\phi$ also depend only on $x^{2}$.
Given（9），the most general real such solution of（8）is

$$
\begin{equation*}
\phi=\sin \alpha\left(\frac{1}{\mathrm{x}^{2}}+\frac{1}{\mathrm{~L}^{2}}\right) . \tag{10}
\end{equation*}
$$

L is an arbitrary（real or imaginary）scale，and $\alpha$ is defined by

$$
\begin{equation*}
\mathrm{K}_{\lambda \gamma} \mathrm{K}_{\gamma \nu}=-\alpha^{2} \delta_{\lambda \nu} \tag{11}
\end{equation*}
$$

Thus our solution for $A_{\mu}$ is

$$
\begin{equation*}
\rightarrow \mathrm{A}_{\mu}=\frac{1}{\mathrm{~g}} \sigma^{\mathrm{a}} \eta_{\mu \nu}^{\mathrm{a}}\left[\mathrm{x}_{\lambda} \mathrm{K}_{\lambda \nu}+\alpha \mathrm{x}_{\nu} \cot \left(\frac{1}{\mathrm{x}^{2}}+\frac{1}{\mathrm{~L}^{2}}\right) \alpha\right]\left(\mathrm{x}^{4}\right)^{-1} \tag{12}
\end{equation*}
$$

A corresponding solution to $F=+\widetilde{F}$ is obtained from the ansatz（1）by replacing $\eta_{\mu \nu}^{\mathrm{a}}$ with＇t Hooft＇s $\widetilde{\eta}_{\mu \nu}^{\mathrm{a}}$ ．

We discovered this solution during a study of small perturbations about the multiinstanton configurations described in Ref．2。 In the limit $\mathrm{K}_{\mu \nu} \rightarrow 0, \mathrm{~L}$ fixed and real，（12）approaches＇t Hooft＇s form of the one－anti－instanton solu－ tion．In this limit，the gauge fields are singular at $x=0$ ，but all gauge－in－ variant quantities are singularity－free．When expanded to first order in $\mathrm{K}_{\mu \nu}$ ， our solution coincides with one of the approximate solutions of $F=-\widetilde{F}$ con－ structed by Jackiw and Rebbi（Ref．3）．（In fact，their Eq。（2．21）is the same as our Eq．（8），expanded to first order in V，when differences in notation are eliminated。）According to their analysis，this is，to first order in $\mathrm{K}_{\mu \nu}$ ，gauge－ equivalent to the solution obtained by letting $K_{\mu \nu}$ approach zero。

However，when all orders in $\mathrm{K}_{\mu \nu}$ are included，our solution has singulari－ ties that cannot be anticipated in a perturbative analysis．Specifically，$A_{\mu}$ be－ comes infinite whenever

$$
\begin{equation*}
\alpha\left(\frac{1}{\mathrm{x}^{2}}+\frac{1}{\mathrm{~L}^{2}}\right)=\mathrm{n} \pi \tag{13}
\end{equation*}
$$

with n an integer．These singularities are not artifacts of a poor choice of gauge．Near $x^{2}=R_{n}^{2} \equiv$ solution of（13），the gauge－invariant action density sat－ isfies

$$
\begin{equation*}
\operatorname{Tr} \mathrm{F}_{\mu \nu} \mathrm{F}_{\mu \nu} \approx\left(\frac{96}{\mathrm{~g}^{2}}\right) \mathrm{R}_{\mathrm{n}}^{4}\left[\mathrm{R}_{\mathrm{n}}^{2}-\mathrm{x}^{2}\right]^{-4} \tag{14}
\end{equation*}
$$

Eq．（13）indicates that the field is singular on infinitely many shells that cluster at the origin．It＇s difficult to say what external source is implicitly
responsible for this．To do that，we would have to subtract away the source that the gauge field generates for itself．That involves evaluating ill－defined integrals of infinite quantities．A simpler alternative is to look for points at which either Eq．（7）or Eq．（8）is not satisfied．

First consider（7），integrated over the interior of a sphere of radius $R$ ， centered at the origin．By Gauss＇s theorem，we can change it into a surface integral．The result is

$$
\begin{equation*}
\mathrm{R}^{2} \oint \mathrm{~d} \Omega \mathrm{x}_{\lambda} \partial_{\lambda} \mathrm{V}_{\mu \nu}=\mathrm{R}^{2} \oint \mathrm{~d} \Omega\left[\mathrm{x}_{\lambda} \partial_{\mu} \mathrm{V}_{\lambda \nu}-\mathrm{x}_{\lambda} \partial_{\nu} \mathrm{V}_{\lambda \mu}-\mathrm{x}_{\mu} \partial_{\lambda} \mathrm{V}_{\lambda \nu}+\mathrm{x}_{\nu} \partial_{\lambda} \mathrm{V}_{\lambda \mu}\right] \tag{15}
\end{equation*}
$$

Using（9），the left－hand side of（15）is $-4 \pi^{2} \mathrm{~K}_{\mu \nu}$ 。 The right－hand side is zero。 Thus，our choice（9）satisfies（7）everywhere except at the origin．

Now consider（8），using（9）and（10）．Away from the origin，the numerator is a combination of very smooth functions；thus it unqualifiedly vanishes every－ where in that region．Therefore，despite the denominator，which is zero when－ ever（13）holds，we are inclined to say that Eq。（8）is satisfied everywhere ex－ cept possibly at the origin，where $\phi$ ceases to be defined．

Thus we tentatively suggest that the singularity at the origin represents the presence of an external source，while the singularities on shells clustering about the origin represent the response of the field。 ${ }^{2}$

By performing a coordinate inversion about any point between two of these shells，we obtain a new solution with an intriguing aspect．It too has a point about which infinitely many nested singular shells cluster，but it also has a dis－ joint system of finitely many singular shells nested about a region containing no isolated point singularity．It would be interesting to imagine how such a point source could induce the formation both of singular shells that contain it and of singular＂bubbles＂that exclude it．

When $V_{\mu \nu}$ is not in the form shown in（9），we cannot explicitly solve Eq． （8）．Hevertheless，we can draw some general conclusions．Eq。（8）resembles a Schroedinger equation，with $\left(\partial_{\lambda} \mathrm{V}_{\lambda \gamma}\right)^{2}$ roughly analogous to a potential。 If $\mathrm{V}_{\mu \nu}$ solves（7），any of its point singularities must be at least as strong（in their power behavior）as the singularity in（9）；therefore in the neighborhood of such a singularity we expect to see at least as many zeros in $\phi$ as we see in（10）。 Thus clustering singularities in the gauge fields should occur wherever $V_{\mu \nu}$ has a point singularity，because $\phi$ appears in the gauge fields through $\psi=\log \phi$ 。 The phenomenon is not a special property of our explicit solution．

We hope it＇s possible to learn something about particle physics by observ－ ing the variety of singularities that arise naturally，as these appear to do，in classical Yang－Mills theories．

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NOTE：We draw the reader＇s attention to a recent paper by Bars（Ref．8），in which the same subject is discussed with a somewhat different emphasis．We learned of this work shortly after completing our own．

## Footnotes

1．The usual Yang－Mills equations follow easily from the equations of（anti－） self－duality，with the help of the Bianchi identity， $\mathrm{D}_{\mu} \widetilde{\mathrm{F}}_{\mu \nu}=0$ 。
2．In light of the first footnote，this interpretation assumes that some kind of sense can be made of the Bianchi identity on the singular shells．This is not obvious．

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