# (2+1)-DIMENSIONAL ABELIAN LATTICE GAUGE THEORY* 

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#### Abstract

A modified form of the Wilson action is studied for the Abelian field in $2+1$ dimensions. The system is shown to be identical to the infinite spin component generalization of the Ising model. In the weak coupling limit, the system factorizes into the ordinary Coulomb interaction, and a new set of interactions which are a direct reflection of the compactness of the Abelian lattice field. This new interaction is divergent, for $\mathrm{d} \leq 4$, if expanded about the naive vacuum. An exact calculation shows the existence of at least two phases. A mean field calculation shows the system to be in the same phase for all coupling $g^{2}$, except for $g^{2}=0-$ where there is a phase transition.


(Submitted to Phys。Revo)

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## Section I

The lattice gauge theory proposed by Wilson ${ }^{1,2}$ is motivated by the requirement of exact local gauge invariance for the gauge field theory with a lattice cut-off. This requirement leads to a redefinition of the conventional continuum action, ${ }^{1,2}$ such that the lattice action is made a function of the finite group elements of the gauge group.

We discuss the Wilson action for the Abelian field, and then propose a modification of it for the $2+1$ dimensional case. We will always consider a finite d-dimensional Euclidean lattice which is periodic, i. $e_{0}$, a lattice of $N^{d}$ lattice points.

Let $B_{n \mu}$ be the local spacetime Abelian degree of freedom at the lattice site $\mathrm{n}_{0} \quad\left(\mathrm{~B}_{\mathrm{n} \mu}=\mathrm{a}^{(\mathrm{d}-2) / 2} \mathrm{gA}_{\mu}(\mathrm{x}=\mathrm{na})\right.$, where a is the lattice spacing, g the bare coupling constant, and $A_{\mu}(x)$ the continuum field. $)^{1,2}$ Let

$$
\begin{equation*}
\mathrm{f}_{\mathrm{n} \mu \nu}=\mathrm{B}_{\mathrm{n} \mu}+\mathrm{B}_{\mathrm{n}+\hat{\mu}, \nu}-\mathrm{B}_{\mathrm{n}}+\hat{\nu}, \mu-\mathrm{B}_{\mathrm{n} \nu} \tag{1.1}
\end{equation*}
$$

Then the conventional continuum action is defined by

$$
\begin{equation*}
\mathrm{e}^{\mathrm{A}} \text { continuum }=\underset{\mathrm{n} \mu \nu}{\prod^{\prime}} \mathrm{e}^{-\frac{1}{4 \mathrm{~g}^{2}} \mathrm{f}_{\mathrm{n} \mu \nu}^{2}} \tag{1,2}
\end{equation*}
$$

(where the prime denotes $\mu \neq \nu$ ).
The Wilson action is defined by ${ }^{1,2}$

$$
\begin{align*}
\mathrm{e}^{\mathrm{A}} \mathrm{continuum} & \rightarrow \underset{\mathrm{n} \mu \nu}{I I^{\prime}} \mathrm{e}^{\frac{1}{2 \mathrm{~g}^{2}}\left(\mathrm{e}^{\mathrm{if}} \mathrm{n} \mathrm{\mu} \mathrm{\nu}-1\right)}  \tag{1.3}\\
& ={ }_{\mathrm{n} \mu \nu}^{\prime} \mathrm{e}^{\frac{1}{4 \mathrm{~g}^{2}}\left(\cos \mathrm{f}_{\mathrm{n} \mu \nu}-1\right)} \tag{1.4}
\end{align*}
$$

Note that since the action is now only a function of $e^{i B_{n} \mu}$, the Feynman path
integral is defined on the compact space of $-\pi \leq \mathrm{B}_{\mathrm{n} \mu} \leq \pi$ for all $\mathrm{n}, \mu$ and hence leading to a theory which needs no gauge-fixing.

We define the modified Wilson action in $\mathrm{d}=2+1$ by the following prescription

$$
\begin{align*}
& \mathrm{e}^{\mathrm{A} \text { continuum }}=\prod_{\mathrm{nij}}^{\prime} \mathrm{e}^{-\frac{1}{4 \mathrm{~g}^{2}} \mathrm{f}_{\mathrm{nij}}^{2}} \\
&=\prod_{\mathrm{ni}} \mathrm{e}^{-\frac{1}{2 \mathrm{~g}^{2}}\left(\frac{1}{2} \epsilon_{\mathrm{ijk}} \mathrm{f}_{\mathrm{njk}}\right)^{2}} \\
&\left.\mathrm{e}^{\mathrm{A}_{\text {continuum }}} \rightarrow \mathrm{e}^{\mathrm{A}}=\prod_{\mathrm{ni}} \sum_{\ell \mathrm{ni}}^{+\infty} \sum_{-\infty}^{-\frac{1}{2 g^{2}}\left(\frac{1}{2} \epsilon_{i j k} f_{n j k}-2 \pi \ell\right.}{ }_{\mathrm{ni}}\right)^{2} \tag{1.5}
\end{align*}
$$

Using the identity ${ }^{3}$

$$
\begin{equation*}
\sum_{n=-\infty}^{+\infty} \delta(x-n)=\sum_{\ell=-\infty}^{+\infty} \mathrm{e}^{2 \pi i l x} \tag{1.6}
\end{equation*}
$$

we also have (dropping an overall constant)

$$
\begin{align*}
& e^{A}=\sum_{\{\ell\}} e^{-\frac{g^{2}}{2} \sum_{n i} \ell_{n i}^{2}+i \sum_{n} \epsilon_{i j k} f_{n j k}^{\ell} \ell_{n i} / 2}  \tag{1.7}\\
& {\left[\sum_{\{\ell\}}=\Pi \quad \begin{array}{ll}
\Pi & \sum_{n i}^{+\infty} \\
\ell_{n i}=-\infty
\end{array}\right]}
\end{align*}
$$

Similar systems have been studied in statistical mechanics. ${ }^{4}$ Note that $e^{A}$ is a function of $e^{\text {if }}$ nij and hence is periodic in the $B_{n i}$ variables with period $2 \pi$ 。 In fact, it was primarily to have this periodicity in the $f_{n i j}$ that the construction in
(1.5) was carried out. In the strong coupling limit (g $\gg 1$ ) we have

$$
\begin{align*}
\mathrm{e}^{\mathrm{A}} & \simeq \prod_{\mathrm{ni}}\left\{1+\mathrm{e}^{-\mathrm{g}^{2} / 2}\left(\mathrm{e}^{\left.\left.\mathrm{i} \epsilon_{i j k} \mathrm{f}_{n j k}^{/ 2}+\mathrm{e}^{-\mathrm{i} \epsilon_{i j k} \mathrm{f}_{n j k} / 2}\right)+0\left(\mathrm{e}^{-2 \mathrm{~g}^{2}}\right)\right\}}\right.\right. \\
& \simeq \exp \left\{\mathrm{e}^{-\mathrm{g}^{2} / 2} \sum_{\mathrm{nij}} \mathrm{e}^{\mathrm{if}}\right\} \tag{1.8}
\end{align*}
$$

We see that $(1,8)$ is the strong coupling Wilson action, but with a coupling constant renormalization. See Appendix A for further discussion on this action.

## Section II

Since the action is a function only of $e^{i B_{n i}}$, we cover all possible unique values of the field $\mathrm{B}_{\mathrm{ni}}$ by letting $-\pi \leq \mathrm{B}_{\mathrm{ni}} \leq \pi$. Hence the Feynman path integral is defined by

$$
\begin{equation*}
\mathrm{z}=\prod_{\mathrm{ni}} \int_{-\pi}^{+\pi} \mathrm{dB} \mathrm{~B}_{\mathrm{ni}} \mathrm{e}^{\mathrm{A}} \tag{2.1}
\end{equation*}
$$

The gauge transformation on the lattice is defined by

$$
\begin{equation*}
e^{i B_{n i}} \rightarrow e^{i \phi_{n}} e^{i B_{n i}} e^{-i \phi_{n+\hat{i}}} \tag{2,2}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathrm{B}_{\mathrm{ni}} \rightarrow \mathrm{~B}_{\mathrm{ni}}-\delta_{\mathrm{i}} \phi_{\mathrm{n}+\hat{\mathrm{i}}}+2 \pi \mathrm{~h}_{\mathrm{ni}} \tag{2.3}
\end{equation*}
$$

where $h_{n i}=$ integers, $\phi_{\mathrm{n}}$ is a (compact) scalar quantum field, and

$$
\delta_{\mathrm{i}} \mathbf{f}_{\mathrm{n}} \equiv \mathbf{f}_{\mathrm{n}}-\mathbf{f}_{\mathrm{n}-\hat{\mathbf{r}}}
$$

is the finite lattice derivative。

The most general integer-valued external current which is gaugeinvariantly coupled to the Abelian field is

$$
\begin{equation*}
e^{-i \epsilon_{i j k} \sum_{n} j_{n i} \delta_{j} B_{n}+\hat{j}, k}=e^{i \epsilon_{i j k} \sum_{n} \delta_{i} j_{n j} B_{n k}} \tag{2,4}
\end{equation*}
$$

where $\left\{j_{n i}\right\}$ is an arbitrary collection of integers.
We evaluate the generating functional

$$
\begin{aligned}
& Z[j]=\prod_{n i} \int_{-\pi}^{+\pi} d B_{n i} e^{i \epsilon_{i j k}} \sum_{n} \delta_{i} j_{n j} B_{n k} e^{A}
\end{aligned}
$$

$$
\begin{align*}
& =\sum_{\{\ell\}} \cdot e^{-\frac{g^{2}}{2} \sum_{n i}\left(\ell_{n i}-j_{n i}\right)^{2}} \prod_{n i} \delta_{i j k} \delta_{j} \ell_{n k, 0} \tag{2,5}
\end{align*}
$$

Note that the Kronecker $\delta$-functions in (2.5) imply that

$$
\begin{equation*}
\epsilon_{\mathrm{ijk}} \delta_{\mathrm{j}} \ell_{\mathrm{nk}}=0 \quad \not \quad \mathrm{n}, \mathrm{i} \tag{2,6}
\end{equation*}
$$

We show in Appendix A that Eq. (2.6) is uniquely solved by

$$
\begin{equation*}
\ell_{\mathrm{ni}} \equiv \delta_{\mathrm{i}} \ell_{\mathrm{n}} \tag{2.7}
\end{equation*}
$$

$\left(\ell_{n}\right.$ is an integer-valued scalar fieldo) Also, the functional sum $\prod_{n i} \sum_{\ell_{n i}}^{+\infty}=-\infty$ is restricted by (2.6) to become $\prod_{n} \sum_{\mathrm{n}}^{+\infty}=-\infty$ giving the final answer

$$
\begin{equation*}
\mathrm{Z}[\mathrm{j}]=\prod_{\mathrm{n}} \sum_{\ell_{\mathrm{n}}=-\infty}^{+\infty} e^{-\frac{g^{2}}{2} \sum_{\mathrm{ni}}\left(\delta_{i} \ell_{n}-j_{n i}\right)^{2}} \tag{2.8}
\end{equation*}
$$

We see that $(2.8)$ is simply the generalized Ising－model with the infinite spin field $\ell_{n}$ at the lattice site $n$ ．The theory is well－defined，on a finite lattice $(\mathrm{N}<\infty)$ ，for all $\mathrm{d} \geq 1$ 。 The limit of $\mathrm{N} \rightarrow \infty$ seems to exist only for $\mathrm{d}>2$［see Section III］。 Z［j］reduces to the ordinary Ising model in the presence of a mag－ netic field if $\ell_{n}= \pm 1$ ．We call $(2,8)$ the strong coupling representation of $Z[j]$ 。

Equation（2，8）is suitable for the strong coupling sector，but unsuited for weak coupling．We rewrite $(2.8)$ using the identity of $(1.6)$ ．To do so，we de－ fine the following（for an $\mathrm{N}^{\mathrm{d}}$ periodic lattice）：

$$
\begin{aligned}
& \ell_{n}=\frac{1}{N^{d}}\left\{\prod_{i=1}^{d} \sum_{k_{i}=0}^{2 \pi(N-1) / N} e^{i k_{i} n_{i}}\right\} \ell_{k} \equiv \sum_{k} e^{i k n_{k}} \ell_{k} \\
& \delta_{k, q} \equiv N^{d} \prod_{i=1}^{d} \delta_{k_{i}, q_{i}}, \quad r_{q i}=1-e^{-i q_{i}} \\
& d_{q}=\sum_{i}\left|r_{q i}\right|^{2}, \quad D_{n}=\sum_{q} e^{i q n} / d_{q}
\end{aligned}
$$

Then, from ( 2.8 ) and ( 1.6 ), we have

$$
\begin{align*}
& Z[j]=\sum_{\{\ell\} n} \int_{-\infty}^{+\infty} d x_{n} e^{2 \pi i \sum_{n} x_{n} \ell} e^{-\frac{g^{2}}{2} \sum_{n i}\left(\delta_{i} x_{n}-j_{n i}\right)^{2}} \\
& \left.=e^{-\frac{g^{2}}{2} \sum_{n i} j_{n i}^{2}} \sum_{\{\ell\} n} \Pi \int_{-\infty}^{+\infty} d x_{n} e^{\sum_{n} x_{n}\left(2 \pi i \ell n-g^{2} \delta_{i} j_{n}+\hat{i}, i\right.}\right)^{-\frac{g^{2}}{2} \sum_{n i}\left(\delta_{i} x_{n}\right)^{2}} \\
& =e^{F} \tilde{Z}[j] \tag{2,9}
\end{align*}
$$

where

$$
\begin{equation*}
F_{c}=-\frac{g^{2}}{2} \sum_{n i} j_{n i}^{2}+\frac{g^{2}}{2} \sum_{n m} \delta_{i} j_{n}+\hat{i}_{, i} D_{n-m} \delta_{k} j_{m+\hat{k}, k} \tag{2,10}
\end{equation*}
$$

For $\mathrm{d}=3$, we can rewrite $(2,10)$ as

$$
F_{c}=-\frac{g^{2}}{2} \quad \sum_{n n^{\prime}} \epsilon_{i j k} \epsilon_{i a b} \delta_{i} j_{n k} D_{n-n^{\prime}} \delta_{a} j_{n^{\prime} a}
$$

Also

$$
\begin{equation*}
\widetilde{\mathrm{z}}[\mathrm{j}]=\prod_{\mathrm{n}} \sum_{\ell_{n}=-\infty}^{+\infty} e^{-\frac{2 \pi^{2}}{\mathrm{~g}^{2}} \sum_{\mathrm{nn}^{\prime}} \ell_{\mathrm{n}} D_{\mathrm{n}-\mathrm{n}^{\prime} \ell_{n^{\prime}}} e^{i} \sum_{\mathrm{n}} \ell_{\mathrm{n}} 0_{\mathrm{n}}} \tag{2.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\theta_{\mathrm{m}}=2 \pi \sum_{\mathrm{ni}} \mathrm{D}_{\mathrm{m}-\mathrm{n}}\left(\mathrm{j}_{\mathrm{n}}+\hat{\mathrm{i}}, \mathrm{i}-\mathrm{j}_{\mathrm{ni}}\right) \tag{2.12}
\end{equation*}
$$

The results obtained are exact, and note that no gauge-fixing was involved in obtaining the result. We call $(2,9)$ the weak coupling representation for $Z[j] . F_{c}$ is simply the ordinary Coulomb interaction in $2+1$ dimensions. To show this, we compute the closed gauge loop by choosing the appropriate $\left\{\mathrm{j}_{\mathrm{ni}}\right\}$ and compute its Coulomb part; i.e.,

$$
\left\langle e^{i \frac{\Phi}{\Gamma} B_{m} \lambda}\right\rangle_{c} \equiv e^{F_{c}[j]}
$$

where the contour $\Gamma$ is a planar square of side $L$ (see Fig。2)。 The sum along the closed contour $\Gamma$ we denote by $\frac{\underset{E}{\Gamma}}{\Gamma}$. Then one can easily show that, for large $L$,

$$
\begin{equation*}
\left\langle e^{i \sum_{\Gamma} B_{m \lambda}}\right\rangle_{c}=e^{-\frac{\mathrm{g}^{2}}{\pi} L \ln L}+\text { lower order } \tag{2,13}
\end{equation*}
$$

(See Appendix B for derivation.)
Let $E(L)$ be the energy of the state. Then ${ }^{1,2}$

$$
\left\langle e^{i \frac{\sum}{\Gamma} B_{m \lambda}}\right\rangle_{c}=e^{-L E_{c}(L)}
$$

giving the Coulomb part of the energy to be

$$
\begin{equation*}
E_{c}(L)=\frac{1}{\pi} g^{2} \ln L+0(1) \tag{2.14}
\end{equation*}
$$

We see that $\mathrm{E}_{\mathrm{c}}(\mathrm{L})$ is simply the expected logarithmically growing Coulomb potential for $d=2+1$ 。

The partition function $\mathbb{Z}[j]$ is the sector of the Abelian field which is a direct consequence of the compactness (periodicity) of the field variables $e^{i B_{n i}}$ (as construction in (1.5) shows). Note $\widetilde{\mathrm{Z}}[\mathrm{j}]$ is non-analytic about $\mathrm{g}^{2}=0$ due to the discreteness of the degrees of freedom $\left\{\ell_{n}\right\}$. The structure of $A$ is reminiscent of the actions studied by Polyakov ${ }^{5,6}$ and Savit. ${ }^{7}$ Similar actions have also been studied by Banks, Meyerson, and Kogut. ${ }^{8}$

We discuss gauge-invariance for the system in the $\left\{\ell_{n}\right\}$ basis. The gaugeinvariant coupling of $\left\{j_{n i}\right\}$ to the Abelian field implies invariance under the following transformation:

$$
\begin{equation*}
\mathrm{j}_{\mathrm{ni}} \rightarrow \mathrm{j}_{\mathrm{ni}}-\delta_{\mathrm{i}} \mathrm{~h}_{\mathrm{n}} \tag{2.15}
\end{equation*}
$$

where $\left\{h_{n}\right\}$ is an arbitrary collection of integers. We call (2.15) the gaugetransformation on the external current $\mathrm{j}_{\mathrm{ni}}{ }^{\circ}$ Under (2.15), we have

$$
\begin{equation*}
Z[j]=Z[j-\delta h] \tag{2.16}
\end{equation*}
$$

This invariance is realized quite differently for the strong and weak coupling representations of $Z[j]$. For the strong coupling representation, we have from (2.8)

$$
\begin{align*}
z[j-\delta h] & =\sum_{\{\ell\}} e^{-\frac{\mathrm{g}^{2}}{2}} \sum_{n i}\left(\delta_{i} \ell_{n}-j_{n i}+\delta_{i} h_{n}\right)^{2} \\
& =\sum_{\{\ell\}} e^{-\frac{g^{2}}{2}} \sum_{n i}\left(\delta_{i} \ell_{n}-j_{n i}\right)^{2} \tag{2.17}
\end{align*}
$$

where we displaced $\ell_{n}$ to $\ell_{n}-h_{n}$ to obtain (2.17). Any cutoff on $\ell_{n}$, say $-H \leq \ell_{n} \leq H$ ( $H=$ integer), will destroy invariance in (2.16). (Note if $e^{i B_{n i}}$ is made discrete, then $\left\{\ell_{n}\right\}$ becomes bounded.) Any strong coupling expansion of (2.8) about the $\ell_{n}=0$ vacuum respects gauge-invariance only to the order of the expansion, and not exactly. To define an exactly gauge-invariant strong i $_{n i}$ coupling expansion, it seems easier to go back to the $e^{1 \mathrm{~B}_{\mathrm{ni}}}$ variables.

Gauge-invariance of $\mathrm{Z}[\mathrm{j}]$ for the weak coupling representation is realized as follows: From (2.10) and (2.12)

$$
\begin{gather*}
F_{c}=-\frac{g^{2}}{2} \sum_{k} j_{-k i}-j_{k i}+\frac{g^{2}}{2} \sum_{k} j_{-k i} \frac{r_{k i} r_{k i}^{*}}{d_{k}} j_{k j}  \tag{2.18}\\
\theta_{n}=-2 \pi \sum_{k} e^{i k n} \frac{r_{k i}^{*} j_{k i}}{d_{k}} \tag{2.19}
\end{gather*}
$$

From (2.15)

$$
\begin{equation*}
\mathrm{j}_{\mathrm{ki}}^{\prime} \rightarrow \mathrm{j}_{\mathrm{ki}}^{\prime}+\mathrm{r}_{\mathrm{ki}} \mathrm{~h}_{\mathrm{k}} \tag{2.20}
\end{equation*}
$$

Therefore, under (2.20),

$$
\begin{align*}
F_{c}[j-\delta h]= & -\frac{g^{2}}{2} \sum_{k}\left(j_{-k i}+r_{k i}^{*} h_{-k}\right)\left(j_{k i}+r_{k i} h_{k}\right)+\frac{g^{2}}{2} \sum_{k}\left(j_{-k i}+r_{k i}^{*} h_{-k}\right) \frac{r_{k i} r_{k j}^{*}}{d_{k}}\left(j_{i j} r_{k j} h_{k}\right) \\
= & F_{c}[j]-\frac{g^{2}}{2} \sum_{k} d_{k} h_{-k} h_{k}-g^{2} \sum_{k} j_{-k i} r_{k i} h_{k} \\
& +\frac{g^{2}}{2} \sum_{k} \frac{d_{k}^{2}}{d_{k}} h_{-k} h_{k}+g^{2} \sum_{k} j_{-k i} r_{k i} \frac{d_{k}}{d_{k}} h_{k} \\
= & F_{c}[j] \tag{2.21}
\end{align*}
$$

and

$$
\begin{equation*}
\theta_{n}[j-\delta h]=-2 \pi \sum_{k} e^{i k n} \frac{r_{k i}^{*}\left(j_{k i}+r_{k i} h_{k}\right)}{d_{k}}=\theta_{n}[j]-2 \pi h_{n} \tag{2,22}
\end{equation*}
$$

Therefore, from (2.11),

$$
\begin{align*}
Z[j-\delta h] & =e^{F_{c}[j]} \sum_{\{l\}} e^{-\frac{2 \pi^{2}}{2}} \sum_{n n^{\prime}} \widetilde{l}_{n} D_{n-n^{\prime}} \tilde{l}_{n^{\prime}}\left[i \sum_{n} e_{n} \tilde{l}_{n} \theta_{n}[j]\right]\left[-2 \pi i \sum_{n} \tilde{l}_{n} h_{n}\right]  \tag{2.23}\\
& =Z[j] \tag{2.24}
\end{align*}
$$

Note that $(2,20)$ is gauge-invariant, order by order, in $\tilde{l}_{n}$; that is, any cutoff in $\tilde{\ell}_{n}$ preserves exact gauge-invariance. We can consequently define an exactly gauge-invariant weak-coupling expansion by imposing a cutoff on $\tilde{\ell}_{\mathrm{n}}$.

## Section III

We examine the weak coupling representation. We set $j_{n i}=0$, as this does not change the bulk properties (phase diagram) of the system. We make a low temperature expansion ${ }^{9}$ about the gauge-invariant naive vacuum given by $\tilde{\ell}_{\mathrm{n}}=0$, and obtain (for d dimensions)

$$
\begin{align*}
& Z=\prod_{n} \sum_{\ell_{n}=-\infty}^{+\infty} e^{-\frac{2 \pi^{2}}{g^{2}}} \sum_{n^{\prime}} \ell_{n} D_{n-n^{\prime} \ell_{n^{\prime}}} \\
& =1+2 N^{d} e^{-\frac{2 \pi^{2}}{g^{2}} D_{0}}+\frac{1}{2} e^{\frac{4 \pi^{2}}{2} \mathrm{D}_{0}} \cdot 4 \sum_{\mathrm{n} \neq \mathrm{n}^{\prime}} \cosh \mathrm{D}_{\mathrm{n}-\mathrm{n}^{\prime}} \\
& +o\left(e^{-6 \pi^{2} D_{0} / g_{0}^{2}}\right)^{\prime}  \tag{3.2}\\
& \therefore \frac{1}{N^{\mathrm{d}}} \ln Z \simeq 2 \mathrm{e}^{-\frac{2 \pi^{2}}{\mathrm{~g}^{2}} \mathrm{D}_{0}}+2 \mathrm{e}^{-\frac{4 \pi^{2}}{\mathrm{~g}^{2}} \mathrm{D}_{0}} \sum_{\mathrm{n} \neq 0}\left(\cosh D_{\mathrm{n}}-1\right) \\
& +o\left(e^{-6 \pi^{2} D_{0} / g^{2}}\right) \tag{3.3}
\end{align*}
$$

Note for $|n| \gg 1$, we have $(d>2)$

$$
\begin{equation*}
\mathrm{D}_{\mathrm{n}} \sim \frac{1}{\mathrm{n}^{\mathrm{d}-2}} \tag{3,4}
\end{equation*}
$$

Therefore, for some $R \gg 1$,
$-\sum_{n \neq 0}\left(\cosh D_{n}-1\right) \geq \sum_{|n|>R}\left(\cosh D_{n}-1\right)$

$$
\simeq \sum_{|n|>R} D_{n}^{2} \sim\left\{\begin{array}{lr}
N^{4-d} & 2<d<4  \tag{3.5}\\
\ln N & d=4 \\
\text { constant } & d>4
\end{array}\right.
$$

Hence, we see that $\ln Z / N^{d}$ is divergent (as $N \rightarrow \infty$ ) for $d \leq 4$. To understand the reason for this divergence ( $i_{0} e_{0}$, whether it arises because this theory undergoes a phase transition for $\mathrm{g}^{2}<\mathrm{g}_{\mathrm{c}}^{2}$, or whether this divergence arises due to an incorrect choice of the naive vacuum), we have to study the phase diagram for the system.

We are interested in the limit of $\mathrm{g}^{2}-0$. First, note that for $\mathrm{g}^{2}=0$, the gauge field becomes a massless free scalar field. To see this, let $\phi_{\mathrm{n}}=\mathrm{g} \ell_{\mathrm{n}}$. Then, from (2.8) (with $\mathrm{j}_{\mathrm{ni}}=0$ ), we have

$$
\begin{aligned}
\mathrm{Z} & =\lim _{\mathrm{g}^{2} \rightarrow 0}\left(\frac{1}{\mathrm{~g}}\right)^{\mathrm{N}^{\mathrm{d}}} \prod_{\mathrm{n}} \mathrm{~g} \sum_{\ell_{\mathrm{n}}} \mathrm{e}^{-\frac{1}{2} \sum_{\mathrm{ni}}\left[\delta_{\mathrm{i}}\left(\mathrm{~g} \ell_{\mathrm{n}}\right)\right]^{2}} \\
& =(\text { constant }) \prod_{\mathrm{n}} \int_{-\infty}^{+\infty} \mathrm{d} \phi_{\mathrm{n}} \mathrm{e}^{-\frac{1}{2} \sum_{\mathrm{ni}}\left(\delta_{\mathrm{i}} \phi_{\mathrm{n}}\right)^{2}}
\end{aligned}
$$

since $\phi_{n}$ becomes a continuous variable ranging from $-\infty$ to $+\infty$ in the limit of $\mathrm{g}^{2}=0$. Note that the massless free field is a lattice system at the critical point, and has infinite correlation length. The low temperature expansion of (3.3) can be generated by making all the discrete variables $\{\ln \}$ continuous in (2,8), except for a finite number of $\ell n$ 's, which are kept discrete.

The $g^{2}=\infty$ phase is well-defined. From (3.1), defining $\phi_{n}=\ell_{n} / g$, we have

$$
\mathrm{Z}\left(\mathrm{~g}^{2}=\infty\right)=(\text { const }) \prod_{\mathrm{n}} \int_{-\infty}^{+\infty} \mathrm{d} \phi_{\mathrm{n}} \mathrm{e}^{-2 \pi^{2} \sum_{\mathrm{n}, \mathrm{n}^{\prime}} \phi_{\mathrm{n}} \mathrm{D}_{\mathrm{n}-\mathrm{n}^{\prime}} \phi_{\mathrm{n}^{\prime}}}
$$

The propagator for the above system is non-zero only between nearest neighbors, giving a correlation length of unity. The correlation length for the system at $\mathrm{g}^{2}=0$ is infinity. Hence, the existence of (at least) two distinct phases is an exact result. We investigate if the phase transition from finite to infinite correlation length takes place at $\mathrm{g}^{2}=0$, or at some finite $\mathrm{g}_{\mathrm{c}}^{2}>0$. If the phase transition takes place at $\mathrm{g}^{2}=0$, then $\ln Z / \mathrm{N}^{\mathrm{d}}$ will have a finite limit for all $\mathrm{g}^{2}>0$ and $\mathrm{N}^{\mathrm{d}} \rightarrow \infty$ 。

The crudest (and easiest) first approximation for the phase diagram is given by mean field theory. ${ }^{1,9}$ In this approximation, one replaces the entire field theory by a problem involving only one degree of freedom.

Recall, from (2.8)

$$
\begin{equation*}
\mathrm{A}=-\frac{\mathrm{g}^{2}}{2} \sum_{\mathrm{ni}}\left(\ell_{\mathrm{n}}-\ell_{\mathrm{n}-\hat{\mathrm{i}}}\right)^{2} \tag{3.6}
\end{equation*}
$$

Consider a particular lattice site $n$. Replace the interaction of $\ell_{n}$ with its 2 d nearest neighbor's, by the effective interaction of $\ell_{n}$ with a constant background field of strength $M$. Then, for this one site

$$
\begin{equation*}
A \rightarrow-g^{2} d(\ell-M)^{2} \tag{3.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{Z}_{\mathrm{MF}}=\sum_{\ell=-\infty}^{+\infty} \mathrm{e}^{\mathrm{A}}=\sum_{\ell=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{g}^{2} \mathrm{~d}(\ell-\mathrm{M})^{2}} \tag{3,8}
\end{equation*}
$$

Let

$$
\begin{equation*}
f(\mathrm{M})=\frac{1}{\mathrm{Z}_{\mathrm{MF}}} \sum_{\ell=-\infty}^{+\infty} \ell \mathrm{e}^{-\mathrm{g}^{2} \mathrm{~d}(\ell-\mathrm{M})^{2}} \tag{3.9}
\end{equation*}
$$

Since M is the background field which is self-consistently created by all the $\left\{\ell_{\mathrm{n}}\right\}$, we have the following mean field equation for $M$

$$
\begin{equation*}
\mathrm{M}=\mathrm{f}(\mathrm{M}) \tag{3.10}
\end{equation*}
$$

which fixes $M$ as a function of $g^{2} d$; i.e., the solution of (3.10) gives $M=M\left(g^{2} d\right)$. Note $f(M+n)=f(M)+n(n=$ integer $)$, so we only consider $M \in(0,1)$. The solution of (3.10) for $\mathrm{g}^{2} \neq 0$ is rather trivial, since, from (3.9),

$$
\mathrm{f}\left(\frac{1}{2}\right)=1-\mathrm{f}\left(\frac{1}{2}\right)
$$

or

$$
\begin{equation*}
\mathrm{f}\left(\frac{1}{2}\right)=\frac{1}{2} \tag{3.11}
\end{equation*}
$$

Hence, we see from (3.10) and (3.11)

$$
\begin{equation*}
\mathrm{M}=\frac{1}{2} \tag{3.12}
\end{equation*}
$$

for all $\mathrm{g}^{2} \neq 0$.
To investigate the $\mathrm{g}^{2}=0$ behavior, note for $\mathrm{g}^{2} \simeq 0$

$$
\begin{equation*}
f(M)=M+\frac{2 \pi}{g^{2} d} e^{-\frac{\pi^{2}}{g^{2} d}} \sin 2 \pi M+O\left(e^{-\frac{4 \pi^{2}}{g^{2} d}}\right) \tag{3.13}
\end{equation*}
$$

Therefore, for $\mathrm{g}^{2}=0$, we see that $F(M)=M$; this gives all $M \epsilon(0,1)$ as solutions of the mean field equation ( 3.10 ). Hence, there is a singularity (instability) of the system at $\mathrm{g}^{2}=0$. The singularity of the system at $\mathrm{g}^{2}=0$ indicates the phase transition to the free massless scalar field. A numerical calculation
shows that（ 3.13 ）is good for $\mathrm{g}^{2} \mathrm{~d}<0$ 。4．Hence，the system is exponentially close to the $\mathrm{g}^{2}=0$ massless free field phase for $0<\mathrm{g}^{2} \mathrm{~d}<0.4$ ．The crossover from strong to weak coupling representation is for $0.4<\mathrm{g}^{2} \mathrm{~d}<0.6$ 。 We sche－ matically plot $\xi=$ correlation length as a function of g in Fig。3．

The mean field calculation in general is not reliable for quantitative in－ formation，although it is usually good enough to describe，qualitatively，the phase diagram of the system．We hence make the tentative conclusion that the divergence we found in perturbation about the $\mathrm{g}^{2}=0$ naive vacuum is due to the phase transition of the system at $\mathbf{g}^{2}=0$ ，and that defining a perturbation about the correct vacuum will give a convergent expansion of $\frac{1}{N^{d}} \ln Z$ for all $g^{2}>0$ 。 A more detailed study is required to confirm these expectations．

I am indebted to Y．J．Ng，F．Martin，S．Doniach，M．Weinstein， M．Nauenberg，M．Wortis，and M．Peshkin for helpful discussions．

## Appendix A

Here we shall prove that

$$
\begin{equation*}
\mathrm{z}[\mathrm{j}]=\prod_{\mathrm{n} \neq \mathrm{N}} \sum_{\mathrm{n}} \sum_{-\infty}^{+\infty} \mathrm{e}^{-\frac{\mathrm{g}^{2}}{2}} \sum_{\mathrm{n} \neq \mathrm{N}} \sum_{\mathrm{i}}\left(\delta_{\mathrm{i}} \ell_{\mathrm{n}}-\mathrm{j}_{\mathrm{ni}}\right)^{2} \tag{A.1}
\end{equation*}
$$

In the first place, for a precise definition of the modified action, note that for a periodic lattice

$$
\begin{equation*}
\sum_{\mathrm{n}} \mathrm{f}_{\mathrm{nij}}=0 \tag{A.2}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathrm{f}_{\mathrm{Nij}}=-\sum_{\mathrm{n} \neq \mathrm{N}} \mathrm{f}_{\mathrm{nij}} \tag{A.3}
\end{equation*}
$$

Hence, eliminating $f_{\text {nij }}$ from the action gives

$$
\begin{equation*}
e^{A}=\left\{{\underset{n}{7} \neq N i}^{\prod_{\ell}} \sum_{n i}^{+\infty}\right\}^{+\infty} e^{-\frac{1}{2 g^{2}}\left(\frac{1}{2} \epsilon_{i j k} f_{n j k}-2 \pi \ell \ell_{n i}\right)^{2}} \tag{A.4}
\end{equation*}
$$

and finally for $Z[j]$ from (2.5)

$$
\begin{equation*}
\mathrm{Z}[\mathrm{j}]=\sum_{\{\ell\}}^{\prime} \mathrm{e}^{-\frac{\mathrm{g}^{2}}{2} \sum_{\mathrm{ni}}^{\prime}\left(\ell \ell_{\mathrm{ni}}-\mathrm{j}_{\mathrm{ni}}\right)^{2}} \prod_{\mathrm{ni}}^{\prime} \delta_{\epsilon_{\mathrm{ijk}}} \delta_{\mathrm{j}}^{\ell} \mathrm{nk}, 0 \tag{A.5}
\end{equation*}
$$

where the primes mean that $\mathrm{n} \neq \mathrm{N}$.
We make a change of variables so as to saturate the constraint imposed by Kronecker $\delta$-functions. We choose the axial gauge ${ }^{1,10}$; for $\mathrm{d}=2+1$, factorize the lattice into domains as illustrated in Fig. 1. Make the following transformations (see Fig. 1):

$$
n \in D^{(i)}\left\{\begin{array}{l}
\ell_{n j}=\tilde{l}_{n j}+\delta_{j} \ell_{n}(j \neq i)  \tag{A.6}\\
\ell_{n i}=\delta_{i} \ell_{n} ; \tilde{l}_{n i} \equiv 0
\end{array}\right.
$$

In the new variables, the Kronecker $\delta$-functions reduce to the restriction that all $\left\{\tilde{\ell}_{n_{j}}\right\}$ be zero; hence, dropping the superfluous Kronecker $\delta$ 's, we have

$$
\begin{align*}
\mathrm{z} \mathrm{j} & =\Pi_{\mathrm{n} \alpha}^{\prime} \sum_{\ell_{\mathrm{n}}}^{+\infty} \sum_{-\infty} \mathrm{e}^{-\frac{\mathrm{g}^{2}}{2} \sum_{\mathrm{l} i}^{\prime}\left(\delta_{\mathrm{i}} \ell_{\mathrm{n}}-\tilde{l}_{\mathrm{n}_{\mathrm{i}}}-\mathrm{j}_{\mathrm{n}_{\mathrm{i}}}\right)^{2}} \prod_{\mathrm{n} \alpha} \delta_{\tilde{l}_{\mathrm{n} \alpha}, 0}  \tag{A.7}\\
& =\Gamma_{\mathrm{n}}^{\prime} \sum_{\ell_{\mathrm{n}}=-\infty}^{+\infty} \mathrm{e}^{-\frac{\mathrm{g}^{2}}{2} \sum_{\mathrm{n} 1}^{\prime}\left(\delta_{\mathrm{i}} \ell_{\mathrm{n}}-\mathrm{j}_{\mathrm{ni}}\right)^{2}} \tag{A.8}
\end{align*}
$$

We will usually ignore (in this paper) the special behavior of the lattice site $n=N$.

## Appendix B

Fo obtain the Coulomb force, we have to compute

$$
\left\langle\mathrm{e}^{\mathrm{i} \frac{\Phi}{\Gamma} \mathrm{~B}_{\mathrm{m} \mathrm{\lambda}}}\right\rangle
$$

using only the Coulomb part $\mathrm{F}_{\mathrm{c}}[\mathrm{j}]$. $\Gamma$ is a planar contour restricted to the 12 plane; let $e^{i B_{n i}}=\xrightarrow[n]{\longrightarrow}{ }_{n+\hat{i}}$ (see Fig。2).

In $d=2+1$, the planar loop can be represented (up to a gauge transformation) by the following choice of $\left\{\mathrm{j}_{\mathrm{ni}}\right\}$

$$
\begin{equation*}
j_{n i}=\delta_{i 3} \sum_{\ell_{1}, l_{2}=0}^{L-1} \delta_{n, \ell_{1}} \hat{1}+\ell_{2} \hat{2}=\delta_{i 3} j_{n} \tag{B.1}
\end{equation*}
$$

Note $\left\{\mathrm{j}_{\mathrm{n}}\right\}$ is nonzero on the planar grid of points enclosed by the contour $\Gamma$ 。
That is,

$$
\begin{equation*}
\left\langle e^{i \sum_{\Gamma} B_{m \lambda}}\right\rangle=\left\langle e^{i \sum_{n} \epsilon_{j k} \delta_{i} j_{n j} B_{n k}}\right\rangle \tag{B.2}
\end{equation*}
$$

Then, for the Coulomb part

$$
\begin{equation*}
\left\langle e^{i \sum_{\Gamma} B_{m \lambda}}\right\rangle=e^{F_{c}[j]} \tag{B.3}
\end{equation*}
$$

Using $\left\{\mathrm{j}_{\mathrm{n} i}\right\}$ given by (B. 1), and using (2.10) gives

$$
\begin{equation*}
F_{c}[j]=-\frac{\mathrm{g}^{2}}{2} \int_{-\pi}^{+\pi} \frac{\mathrm{d}^{3}{ }_{q}}{(2 \pi)^{3}}\left\{\frac{\left|r_{q 1}\right|^{2}+\left|r_{q 2}\right|^{2}}{d_{q}}\right\}\left|j_{q}\right|^{2} \tag{B.4}
\end{equation*}
$$

where

$$
\begin{align*}
j_{q} & =\sum_{n} e^{-i q n} j_{n} \\
& =\frac{1-e^{-i q_{1} L}}{1-e^{-i q_{1}}} \frac{1-e^{-i q_{2} L}}{1-e^{-i q_{2}}} \tag{B.5}
\end{align*}
$$

We compute $F_{c}$ for the case $L \gg 1$ 。 Keeping only the leading order effect, one ctan show (for $\mathrm{d}=2+1$ )

$$
\begin{equation*}
F_{c}[j]=-\frac{\mathrm{g}^{2}}{\pi} \mathrm{~L} \ln \mathrm{~L}+0(\mathrm{~L}) \tag{B.6}
\end{equation*}
$$

I thank F. Martin for helpful discussions on this section.

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## FIGURE CAPTIONS

1. Partition of the lattice points into disjoint sets denoted by the domains $D^{(i)}, i=1,2,3$. Lattice site $N=(N, N, N)$ is a domain by itself.
2. Contour product of the gauge field around the contour, which is denoted by $\Gamma$ 。
3. Schematic plot of the mean field result for the correlation length $\xi$ as a function of g 。


Fig. 1


Fig. 2


Fig. 3


[^0]:    *Work supported by the Energy Research and Development Administration.

