SLAC-PUB-1983
July 1977
(T)

A NEW CLASS OF SUPERALGEBRAS AND LOCAL GAUGE GROUPS IN SUPERSPACE*

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ABSTRACT

It is shown that there is a new class of superalgebras associated with a given Lie algebra or a superalgebra. The structure constants of the new algebras either vanish or else are directly related to those of the original algebra. The new algebraic structures provide a link between the local gauge groups constructed over superspace and those over ordinary space-time.

(Submitted to J. Math. Phys.)

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* Supported in part by the Energy Research and Development Administration.

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I. Introduction

The purpose of this paper is to construct a new class of superalgebras associated with a given Lie algebra or with another known superalgebra of simple or semi-simple type. The latter algebras have been extensively studied in the literature.[1,2,3] The proof of the existence of the new algebras is given by constructing explicit representations for them. This is carried out in Section II. Although the new algebraic structures are independent of any specific application, the interest in them is not purely from the mathematical point of view. There have been a number of suggestions[4-10] about the construction of locally supersymmetric gauge theories both in ordinary space-time and in superspace. One would therefore want to have an understanding of the meaning of local gauge groups in superspace. It is shown in Section III that a local gauge group element in superspace can be written in terms of a set of elements of a different local gauge group in space-time, which is based on one of the new algebras. We use the notation and conventions of references 5, 7, and 10.

II. The New Superalgebras

Let $\vec{G}$ be a Lie algebra with basis elements $\left\{X_A\right\}$ satisfying the commutation relations

$$\left[X_A, X_B\right] = f^C_{AB} X_C \tag{2.1}$$

where f^C_{AB} are the structure constants of the associated Lie group. Let $\theta^{\alpha_1}, \theta^{\alpha_2}, \ldots, \theta^{\alpha_m}$, be elements of a Grassmann algebra:

$$\left\{\theta^{\alpha_i}, \theta^{\alpha_j}\right\} = 0 , \quad i,j = 1, \ldots, N . \tag{2.2}$$

Construct the elements

$$X_A \; ; \; X_A^i = \theta^{\alpha_i} X_A \; ; \; X_A^{ij} = \theta^{\alpha_i}\theta^{\alpha_j} X_A \; , \quad i \neq j$$
$$X_A^{ij\ldots km} = \theta^{\alpha_i}\theta^{\alpha_j}\ldots\theta^{\alpha_k}\theta^{\alpha_m} X_A ; \; i \neq j \neq \ldots \neq k \neq m \; . \tag{2.3}$$

By construction all elements $X_A^{ij\cdots}$ involving products of more than N θ^{α_k}'s vanish identically. Recall[5] the definition of a superbracket

$$\left[X_A, X_B\right\} \equiv X_A X_B - (-)^{\sigma_A \sigma_B} X_B X_A \tag{2.4}$$

where $\sigma_A = 0$ if X_A is an even element of the algebra, and $\sigma_A = 1$ if X_A is an odd element. Then, by straightforward computation one can show that

$$\left[X_A^{\cdots ij\cdots}, X_B^{\cdots k\ell\cdots}\right\} = \begin{cases} 0 & \text{if there are more } ijk\ell \text{ indices than N} \\ f_{AB}^{C} X_C^{\cdots ij\ldots k\ell\cdots} & \text{otherwise} \; . \end{cases} \tag{2.5}$$

The order of the indices $\ldots ij\ldots k\ell\ldots$ are the same on both sides of the equality sign.

Similarly, if the Lie algebra (2.1) is replaced by a superalgebra

$$\left[X_A, X_B\right\} = f_{AB}^{C} X_C \tag{2.6}$$

a representation of the elements of the new superalgebra will be of the form given by (2.3). But this time

$$\left[X_A^{\cdots ij\cdots}, X_B^{\cdots k\ell\cdots}\right\} = \begin{cases} 0 & \text{if there are more } ijk\ell \text{ indices than N} \\ (-)^{\sigma_A(\ldots+\sigma_k+\sigma_\ell+\ldots)} f_{AB}^{C} X_C^{\cdots ij\ldots k\ell\cdots} & \text{otherwise} \; . \end{cases} \tag{2.7}$$

Notice that in this case even for non-vanishing brackets the sign of some structure coefficients will change when X_A is an odd element of the algebra.

As an example, suppose N = 4. Then the basis elements of the new algebra are the set

$$\left\{ X_A,\ X^i_A,\ X^{ij}_A,\ X^{ijk}_A,\ X^{ijk\ell}_A \right\} . \tag{2.8}$$

The non-vanishing superbrackets are listed below:

$$\begin{aligned}
\left[X_A, X_B\right\} &= f^C_{AB} X_C \quad ; \left[X_A, X^j_B\right\} = (-)^{\sigma_A \sigma_j} f^C_{AB} X^j_C \\
\left[X_A, X^{jk}_B\right\} &= f^C_{AB} X^{jk}_C \quad ; \left[X_A, X^{jk\ell}_B\right\} = (-)^{\sigma_A \sigma_j} f^C_{AB} X^{jk\ell}_C \\
\left[X_A, X^{jklm}_B\right\} &= f^C_{AB} X^{jk\ell m}_C \quad ; \left[X^i_A, X^j_B\right\} = (-)^{\sigma_A \sigma_j} f^C_{AB} X^{ij}_C \\
\left[X^i_A, X^{jk}_B\right\} &= f^C_{AB} X^{ijk}_C \quad ; \left[X^i_A, X^{jk\ell}_B\right\} = (-)^{\sigma_A \sigma_j} f^C_{AB} X^{jk\ell}_C \\
\left[X^{ij}_A, X^{k\ell}_B\right\} &= f^C_{AB} X^{ijk\ell} .
\end{aligned} \tag{2.9}$$

A typical vanishing bracket is

$$\left[X^{ij}_A, X^{klm}_B\right\} = 0 .$$

Consider some of the subalgebras of the superalgebra (2.8). Clearly, the elements $\left\{X_A\right\}$ of the original superalgebra from a subalgebra. The remaining elements, i.e. when excluding $\left\{X_A\right\}$, also form a subalgebra, in fact an invariant subalgebra. The latter subalgebra contains a hierarchy of invariant subalgebras obtained by first excluding $\left\{X^i_A\right\}$, then $\left\{X^i_A, X^{ij}_A\right\}$, etc. These properties are, of course, not limited to this illustrative example but are shared by the general algebras (2.5) and (2.7).

III. Local Group Elements in Superspace

In the neighborhood of the identity any element g of a symmetry group G can be written in the form

$$g = e^{\epsilon^A X_A},$$

where ϵ^A are the group parameters. A local group element over ordinary space-time can be constructed by requiring that

$$\epsilon^A \rightarrow \epsilon^A(x) .$$

Thus, one obtains a group with a continuous infinity of parameters, one for each $\left\{x^\mu\right\}$. At every value of $\left\{x^\mu\right\}$, the algebra of the group remains the same:

$$\left[X_A, X_B\right\} = f^C_{AB} X_C , \quad \text{for every} \left\{x^\mu\right\} . \tag{3.1}$$

Now suppose one wants to construct a local group element over superspace. Then one must require

$$\epsilon^A \rightarrow \epsilon^A(x, \theta)$$

where θ is an element of Grassmann algebra (2.2). For definiteness suppose the set $\left\{\theta^{\alpha_i}\right\}$ consists of four elements. Then, suppressing the x-dependence, the exponent $\epsilon^A(\theta)X_A$ can be expanded in powers of θ:

$$\epsilon^A(\theta)X_A = \epsilon^A X_A + \epsilon^A_\alpha X^\alpha_A + \epsilon^A_{\alpha\beta} X^{\alpha\beta}_A + \epsilon^A_{\alpha\beta\gamma} X^{\alpha\beta\gamma}_A + \epsilon^A_{\alpha\beta\gamma\delta} X^{\alpha\beta\gamma\delta}_A \tag{3.2}$$

where, as in (2.3)

$$X^\alpha_A = \theta^\alpha X_A; \quad X^{\alpha\beta}_A = \theta^\alpha \theta^\beta X_A, \text{ etc.}$$

Thus one can regard a local group element with generators $\left\{X_A\right\}$ and parameters $\left\{\epsilon^A(x, \theta)\right\}$ as a set of elements with generators $\left\{X_A^{\cdots\alpha}\right\}$ and parameters $\left\{\epsilon^{A}{}_{\cdots\alpha}(x)\right\}$. The new generators form an algebra of the type discussed in Section II.

The above correspondence is not limited to the specific example discussed and applies to any local group or supergroup in superspace. Since local gauge transformations in real space-time have definite meaning and physical implications, one is faced with the problem of providing a justification, from the physical point of view, for using local symmetry groups based on the new algebras, the full implications of this observation remains to be explored.

Conversations with F. Gursey and B. Baaquie are gratefully acknowledged. I would also like to express my appreciation to Professor Sidney Drell for the hospitality extended to him in the theory group at SLAC.

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