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ON LORENTZ INVARIANCE OF TWO-DIMENSIONAL QUANTUM CHROMODYNAMICS*

Namik K. Pak†

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

and

Department of Physics
University of California at San Diego, La Jolla, California 92093

P. Senjanovic**‡

Department of Physics
University of California, Berkeley, California 94720

ABSTRACT

We show that two-dimensional quantum chromodynamics is invariant under inhomogeneous Lorentz transformations only in the color singlet sector.

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‡Address after August 15, 1977: Laboratory for Theoretical Physics, Institute for Nuclear Sciences "Boris Kidric," 11001 Beograd, Yugoslavia; and after November 1, 1977: Department of Theoretical Physics, University of Oxford, 12 Parks Road, Oxford OX1 3PQ, ENGLAND.

I. INTRODUCTION

There exist no degrees of freedom corresponding to the gauge fields in two space-time dimensions. The most appropriate gauge choice is, then, the so-called axial gauge,[1] which not only eliminates all the nonlinear interactions among the gauge fields, but also yields a constraint equation which eventually is used to eliminate the unphysical gluon degrees of freedom (leaving only genuine dynamical degrees to work with). Of course, the desire to clarify the quantum nature of the system (a theory without ghosts) is not without an expense: manifest Lorentz invariance is given up. Lorentz invariance must be verified by explicit calculation.

The criterion for Lorentz invariance is the famous Schwinger algebra.[2] A sufficient condition for invariance under proper Lorentz transformations is that the Hermitean energy density operator $T_{00}(x)$ obeys the following equal-time commutator:

$$i\left[T^{00}(x), T^{00}(y)\right] = \left[T^{01}(x) + T^{01}(y)\right] \; \partial_x \delta(x-y) \tag{1}$$

which is supplemented by the condition that $T^{00}(x)$ should not have an explicit dependence on the coordinate x. With this supplemental condition, the equivalence of this relation to the inhomogeneous Lorentz algebra

$$[H, P] = 0 \tag{2a}$$

$$[H, K] = iP \tag{2b}$$

$$[P, K] = iH \tag{2c}$$

can easily be seen from the integrated form:

$$\iint dx\, dy\, \frac{(x-y)}{2} \left[T^{00}(x),\; T^{00}(y)\right] + iP = 0 \tag{3}$$

Here P, H and K are defined in the usual way

$$
\begin{aligned}
H &= \int dx\, T^{00}(x) \\
P &= \int dx\, T^{01}(x) \\
K &= x^{o}P - \int dx\, x\, T^{00}(x)
\end{aligned}
\tag{4}
$$

In a recent very ambitious paper,[3] on the color confinement properties of two-dimensional quantum chromodynamics (TDQCD), Li and Willemsen claimed that TDQCD is invariant under inhomogeneous Lorentz group in all sectors (in their phraseology, the Lorentz algebra and the conservation of the stress energy tensor is satisfied before taking matrix elements between the physical singlet states). We show in this note that TDQCD is Lorentz invariant only in the color singlet subspace and therefore reach the conclusion that the only meaningful sector of TDQCD is the color singlet sector (of course, for a reason different than (or rather additional to) those of Ref. 3).

II. LORENTZ INVARIANCE OF TDQCD

Restricting themselves to only symmetric Green's functions for the equations of motion for the A_0 component (after the gauge choice A_1=0)

$$\partial_x^2 A_0^a(x,t) = -g\, j_0^a(x,t) \tag{5}$$

they obtain an anomalous term on the right-hand side of Eq. (1):

$$\mathscr{A}(x,y) = ig^2 \left(B^2 - \frac{1}{4}\right) C^{abc} Q^c \left\{F_{01}^a(x,t),\ F_{01}^b(y,t)\right\} \tag{6}$$

Here, b is the coefficient of the x+y term in their symmetric Green's function:

$$V(x,y) = -\frac{1}{2}|x-y| + B(x+y) + C \tag{7}$$

We shall later comment on the implications of having a term B'(x-y) in V, clearly violating the property that V is symmetric in $x \longleftrightarrow y$.

The integrated anomaly (which contributes to the right-hand side of Eq. (2b) is

$$A = \iint dx\, dy\, y \mathscr{A}(x,y)$$

$$= i \frac{g^2}{2} \left(B^2 - \frac{1}{4}\right) C^{abc} Q^c \iint dx\, dy\, (y-x) \left\{F^a_{01}(x), F^b_{01}(y)\right\} \tag{8}$$

Let us introduce the dipole moment and quantum pole moment operators $D^a = \int dx\, x\, j^a_0(x)$, $q^a = \int dx\, x^2\, j^a_0(x)$ and calculate the anomaly. First note that by using V as given by Eq. (7), we obtain (quantizing in a box (-L, L))

$$\int_{-L}^{L} dx\, F^a_{01}(x) = g\, D^a + 2L\, Bg\, Q^q \tag{9}$$

$$\int dy\, y\, F^b_{01}(y) = \frac{1}{2} g\, q^b - \frac{1}{2} g\, L^2 Q^b \tag{10}$$

$$A = \frac{1}{2} ig^4 \left(B^2 - \frac{1}{4}\right) C^{abc} Q^c \Big[-2LB \left\{Q^a, Q^b\right\}$$

$$+ 2LB \left\{Q^a, q^b\right\} + \left\{D^a, q^b\right\} - L^2 \left\{D^a, Q^b\right\}\Big] \tag{11}$$

The first term in square brackets does not contribute, in view of the antisymmetry of the structure constants of the Lie algebra. The contribution of the fourth term also vanishes, as seen by application of the algebra of charges and dipole moments:

$$\left[Q^a, Q^b\right] = i\, C^{abc} Q^c \tag{12}$$

$$\left[Q^a, D^b\right] = i\, C^{abc} D^c \tag{13}$$

producing finally:

$$A = \frac{i}{2} ig^4 \left(B^2 - \frac{1}{4}\right) C^{abc} Q^c \left[2LB \left\{Q^a, q^b\right\} + \left\{D^a, q^b\right\}\right] \tag{14}$$

These two terms persist. Choosing $B = \pm 1/2$ to make this vanish, ruins translational invariance as is also noted in Ref. 3. On the other hand, the choice

$B=0$, designed to save translational invariance, will not make the anomalous terms vanish

$$A_{B=0} = -\frac{1}{8} ig^4 \, C^{abc} Q^c \left\{ D^a, q^b \right\} \tag{15}$$

Thus the only natural way to satisfy the Lorentz algebra is to restrict ourselves to the color-singlet sector.

Now let's go back to the question of translational invariance. With $V(x,y)$ given by Eq. (7), the change in the Hamiltonian under spatial displacements is

$$[P, H] = -\frac{i}{2} Bg^2 \, Q^a Q^a \tag{16}$$

and this vanishes only when $B=0$ (for color nonsinglets). The claim is made in Ref. 3 that, if we relax the restriction that V should be symmetric, then a term $B'(x-y)$ in V, instead of $B(x+y)$ would give rise to a translationally invariant theory if it was not for the violation of charge conservation. It may be encouraging to note that this new term in the propagator does not really violate charge conservation. This can be easily seen by noting that the additional terms in the Hamiltonian due to this new term in with B' are:

$$H_{B'} = LB'^2 g^2 \, Q^a Q^a - B' g^2 Q^a D^a \tag{17}$$

This commutes with Q^b, as can be seen by using the algebra (12)-(13).

The incorrect conclusion in Ref. 3 that this new piece in the propagator violates charge conservation was based on current conservation equations derived from the Lagrange equations of motion. These however do not follow from the operator formulation based on the Hamiltonian obtained by use of the new propagator. Now let us see whether this new propagator really guarantees translational invariance. First observe that the color-electric field now is

$$F^a_{01} = B' g Q^a + \frac{g}{2} \int dy \;\, \mathscr{E}(x-y) \, j^a_0(y) \tag{18}$$

Then, using

$$\left[P, F^a_{01}(x)\right] = -ig\ j^a_0(x) \tag{19}$$

we get

$$[P, H] = -i B' g^2\ Q^a Q^a \tag{20}$$

as can also be seen directly from Eq. (17). The only way this commutator can vanish along with the anomalous term in the Schwinger commutator is again by setting $Q^a=0$.

CONCLUSION

We have shown above that TDQCD is invariant under the inhomogeneous Lorentz group only in the color singlet sector. Of course, whether this represents an additional argument for confinement in the theory depends on whether one is willing to take the "kinematical" consideration of Lorentz invariance to be of importance in this context.

As a closing note, let us remind the reader that in Ref. 3 the authors remarkably enough also found confinement, although they thought Lorentz invariance held in all sectors of the Hilbert space. In their paper, the condition $Q^a=0$ followed as a consistency condition stemming from the axial current conservation anomaly, so it had a dynamical, rather than a "kinematical" origin.

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