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#### Abstract

Migdal's recursion equation proposed for the Wilson lattice gauge theory is studied for its weak and strong coupling behavior. The model is then solved numerically for the $\mathrm{SU}(2)$ gauge field, and it is shown that there is a continuous crossover from the weak coupling (asymptotically free) region to the strong coupling (quark confining) domain.

The Migdal recursion equation is a crude approximation for the behavior of the gauge field in $d=4$ dimension, and provides a theoretical example of how confinement of quarks could be achieved by the gauge field which exhibits asymptotic freedom.


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[^0]In this paper we study Migdal's recursion formula..$^{2,3}$ This formula gives a nontrivial model for the renormalization group equation of the lattice gauge field. The recursion equation is exact for the 2-dimensional gauge field, and only an approximation for higher dimensions. Migdal ${ }^{2}$ has given an intuitive picture of the approximations made, and Kadanoff ${ }^{3}$ has used variational techniques to "derive" the approximate recursion formula. We will make no attempt to justify the recursion formula. We will, however, give an interpretation of the recursion formula for higher than 2 dimensions. This interpretation is similar to the one given by Migdal. ${ }^{2}$

We will define the recursion equation using the Kadanoff-Wilson "blockspin" technique, ${ }^{5,6}$ as this makes the equations more transparent. We then study the equations for its weak and strong coupling behavior. Up until this point, one need not specify the gauge group. However, the most interesting feature of the recursion formula is that one can study the entire renormalization group transformation (i.e., the entire sequence of actions generated by the above transformation) if one uses a computer. For the numerical solution of the recursion formula, we used $\operatorname{SU}(2)$ for the gauge group.

## 1. The Recursion Formula

We briefly outline the basic features of the Wilson lattice gauge theory. 1,4 Consider a Euclidean spacetime lattice of infinite size. Let n specify the lattice site, and $\mu$ the directions on the lattice. The local gauge degrees of freedom are the finite group elements $\mathrm{U}_{\mathrm{n} \mu}$ belonging to the gauge group, G , which for definiteness, can be taken to be $\operatorname{SU}(\mathrm{n})$. One can picture each $\mathrm{U}_{\mathrm{n} \mu}$ corresponding to the unique link going from n to $\mathrm{n}+\hat{\mu}$ ( $\hat{\mu}=$ unit vector in the $\mu$ th direction).

The gauge field action functional is defined by (prime denotes $\mu \neq \nu$ )

$$
\mathrm{A}_{\mathrm{GF}}=\frac{1}{2 \mathrm{~g}_{0}^{2}} \sum_{\mathrm{n}} \sum_{\mu \nu}^{\prime} \operatorname{Tr}\left(\mathrm{W}_{\mathrm{n} \mu \nu}\right)
$$

where $g_{0}$ is the bare coupling constant and

$$
\mathrm{W}_{\mathrm{n} \mu \nu}=\mathrm{U}_{\mathrm{n} \mu} \mathrm{U}_{\mathrm{n}+\mu} \mathrm{U}_{\mathrm{n}+\nu}^{\dagger} \mathrm{U}_{\mathrm{n} \nu}^{\dagger}
$$

The theory is quantized by summing (i.e., performing the path integral of ) $e^{A_{G F}}$ over all possible values of all the degrees of freedom. Let $d U_{n \mu}$ be the invariant group measure of the group element $U_{n \mu}$. Then the path integral of $e^{A} G F$ is defined by

$$
\mathrm{Z}=\Pi_{\mu} \prod_{\mu} \int_{\mathrm{G}} \mathrm{dU}_{\mathrm{n} \mu} \mathrm{e}^{\mathrm{A}} \mathrm{GF}
$$

Note $A_{G F}$ is invariant under local gauge transformations, which, for the lattice theory, is defined by

$$
\mathrm{U}_{\mathrm{n} \mu} \rightarrow \mathrm{~V}_{\mathrm{n}} \mathrm{U}_{\mathrm{n} \mu} \mathrm{~V}_{\mathrm{n}+\hat{\mu}}^{\dagger}
$$

where $V_{n}$ is also an element of the gauge group $G$.
The simplicity of Migdal's formula lies in that it is defined on a twodimensional lattice. Going to higher dimension involves some approximations. So we first examine the 2-dimensional lattice gauge field. Consider an infinite 2-dimensional lattice. Then the Feynman path integral of the action functional is defined by

$$
\begin{align*}
\mathrm{Z} & =\prod_{\mathrm{i}=1,2} \prod_{\mathrm{n}} \int \mathrm{dU}_{\mathrm{ni}} \mathrm{e}^{\mathrm{A}_{\mathrm{GF}}[\mathrm{U}]}  \tag{1}\\
\mathrm{A}_{\mathrm{GF}} & =\frac{1}{2 \mathrm{~g}_{0}^{2}} \sum_{\mathrm{n}} \operatorname{Tr}\left(\mathrm{~W}_{\mathrm{n} 12}+\mathrm{W}_{\mathrm{n} 21}\right) \\
& =\frac{1}{2 \mathrm{~g}_{0}^{2}} \sum_{\mathrm{n}} \operatorname{Tr}\left(\mathrm{~W}_{\mathrm{n}}+\mathrm{W}_{\mathrm{n}}^{\dagger}\right) \tag{2}
\end{align*}
$$

where

$$
\begin{align*}
\mathrm{W}_{\mathrm{n}} & \equiv \mathrm{~W}_{\mathrm{n} 12}=\mathrm{W}_{\mathrm{n} 21}^{\dagger}  \tag{3}\\
& =\mathrm{U}_{\mathrm{n} 1} \mathrm{U}_{\mathrm{n}+\hat{1}, 2} \mathrm{U}_{\mathrm{n}+\hat{2}, 1}^{\dagger} \mathrm{U}_{\mathrm{n} 2}^{\dagger} \tag{4}
\end{align*}
$$

For clarity we want to define the renormalization group transformation in terms of the $\left\{\mathrm{W}_{\mathrm{n}}\right\}$, as they all are independent variables. (This is not true for $d>2$.) To do this, perform the gauge transformation

$$
\begin{align*}
U_{n i} & =V_{n} U_{n i}^{\prime} V_{n+\hat{i}}^{\dagger}  \tag{5a}\\
d U_{n 1}=d U_{n 1}^{\prime} & =d W_{n}^{\prime} ; \quad d U_{n 2}=d V_{n}  \tag{5b}\\
U_{n 2}^{\prime} & =1 \tag{5c}
\end{align*}
$$

(We are using the axial gauge.)
We briefly discuss the change of variables $U_{n 1} \rightarrow U_{n 1}^{\prime} \rightarrow W_{n}^{\prime}$. For the axial gauge, we have $W_{n}^{\prime}=U_{n 1}^{\prime} U_{n+2}^{\prime}+\hat{2}, 1$ Now $U_{n 1}^{\prime}$ and $U_{n+\hat{2}, 1}^{\prime}$ are both independent variables. Hence, when we vary $U_{n 1}^{\prime}$ through the gauge group, we consider $W_{n}^{\prime}$ to be doing the same, all the time holding $\mathrm{U}_{\mathrm{n}+\hat{2}, 1}^{+}$fixed. Then, from the invariance of the group measure, we have $d W_{n}^{\prime}=d U_{n 1}^{\prime}$. The change of variable from $\left\{\mathrm{U}_{\mathrm{n} 1}, \mathrm{U}_{\mathrm{n} 2}\right\}$ to $\left\{\mathrm{V}_{\mathrm{n}}, \mathrm{U}_{\mathrm{n} 1}^{\prime}\right\}$ is similar. Note the axial gauge means

$$
\begin{align*}
& \mathrm{U}_{\mathrm{n} 1}=\mathrm{V}_{\mathrm{n}} \mathrm{U}_{\mathrm{n} 1}^{\prime} \mathrm{V}_{\mathrm{n}+\hat{1}}^{\dagger} \\
& \mathrm{U}_{\mathrm{n} 2}=\mathrm{V}_{\mathrm{n}} \mathrm{~V}_{\mathrm{n}+\hat{1}}^{\dagger}
\end{align*}
$$

Since all the variables on the right-hand side are independent, we have $d U_{n 1}=d U_{n 1}^{r}$ and $d U_{n 2}=d V_{n}$. The change of variable in going to a specific gauge for a finite lattice is more complicated, and is discussed elsewhere. ${ }^{1,4}$

Then

$$
\begin{align*}
\mathrm{Z} & =\left\{\prod_{\mathrm{n}} \int \mathrm{~d} V_{\mathrm{n}}\right\}\left\{\prod_{\mathrm{n}} \int d W_{\mathrm{n}}^{\prime} \mathrm{e}^{1 / 2 \mathrm{~g}_{0}^{2} \operatorname{Tr}\left(W_{\mathrm{n}}^{\prime}+W_{\mathrm{n}}^{\prime \dagger}\right.}\right\} \\
& =\prod_{\mathrm{n}} \int \mathrm{~d} W_{\mathrm{n}} \mathrm{e}^{1 / 2 g_{0}^{2} \operatorname{Tr}\left(W_{\mathrm{n}}+W_{\mathrm{n}}^{\dagger}\right)} \tag{6}
\end{align*}
$$

We see that the system is totally decoupled, and that the integration variables are the $\left\{W_{n}\right\}$. Let

$$
\begin{equation*}
\mathrm{A}_{0} \equiv \frac{1}{2 \mathrm{~g}_{0}^{2}} \sum_{\mathrm{n}} \operatorname{Tr}\left(\mathrm{~W}_{\mathrm{n}}+\mathrm{W}_{\mathrm{n}}^{\dagger}\right): \text { completely local } \tag{7}
\end{equation*}
$$

Let the initial lattice spacing be $a_{0}$. The renormalization group defines iteratively a sequence of actions $\left\{A_{\ell}\right\}_{\ell=0}^{\infty}$ which describe the behavior of the gauge field on lattice of spacings $\left\{2^{\ell} \mathrm{a}_{0}\right\}$. All information of distances less than $2^{\ell} \mathrm{a}_{0}$ have been integrated out and the action $A_{\ell}$ can describe physics only for distances greater than $2^{\ell} a_{0}$. (The 2 in $2^{l} a_{0}$ is specific to Migdal's transformation.)

We now define the renormalization group transformation. We will specifically focus on how to obtain $A_{1}$ from $A_{0}$, and then generalize to obtaining $A_{\ell+1}$ from $A_{\ell}$. Consider the original lattice. We will combine blocks of four lattice sites to obtain one new lattice point, which is marked by a circle (see Fig. 1). There is a block-spin variable defined on the circled site, and the new action $A_{1}$ will be a function of these new block-spin variables. Since the new action $A_{1}$ is also completely local, we need consider only one set of four old lattice sites and how the transformation acts on them. Label the lattice sites as 1, 2, 3 and 4 (see Fig. 2). There is a variable $W_{n}$ (which is a matrix belonging to the gauge group at lattice site $n$ ) for each lattice site $n$. The piece
of $A_{0}$ belonging to the four lattice sites is

$$
\begin{equation*}
A_{c e l l}=\frac{1}{2 g_{0}^{2}} \sum_{i=1}^{4} \operatorname{Tr}\left(W_{i}+W_{i}^{\dagger}\right) \equiv \sum_{i} A\left(W_{i}\right) \tag{8}
\end{equation*}
$$

The piece of $A_{1}$ belonging to the circled site is then defined by integrating $e^{A}$ cell over all the $W_{i}$ 's with the constraint that the product of all the $W_{i}$ 's, i.e., $W_{1} W_{2} W_{3} W_{4}$ be kept fixed and equal the new block-spin variable $W$. Note W is also an element of the gauge group. The (nonlinear) transformation which maps $A_{0}$ into $A_{1}$ is called the renormalization group transformation. In symbols, we have

$$
\begin{equation*}
\mathrm{e}^{\mathrm{A}_{\text {block }}[\mathrm{W}]}=\prod_{\mathrm{i}=1}^{4} \int \mathrm{~d} W_{i} \delta\left[\mathrm{~W}_{-} \mathrm{W}_{1} W_{2} W_{3} W_{4}\right] \mathrm{e}^{\mathrm{A}} \text { cell } \tag{9}
\end{equation*}
$$

The $\delta$-function for the group clements is similar to the usual $\delta$-function, i.e., for $V$ in the group space, we have

$$
\begin{align*}
& \int d U \delta[U-V]=1 \\
& \int d U f(U) \delta[U-V]=f(V) \tag{10}
\end{align*}
$$

That (9) defines a renormalization group transformation which leaves Z invariant can be easily seen from (9) and (10), i.e.,

$$
\int d W e^{A} \text { block }^{[W]}=\prod_{i=1}^{4} \int d W_{i} e^{A} \operatorname{cell}^{[W]}
$$

since

$$
\int \mathrm{dW} \delta\left[\mathrm{~W}_{-} \mathrm{W}_{1} \mathrm{~W}_{2} \mathrm{~W}_{3} \mathrm{~W}_{4}\right]=1
$$

Let $U=e^{i B^{a} X^{a}}, V=e^{i C^{a} X^{a}} ;\left\{C^{a}\right\}$ is the coordinate of $V$ in the parameter space, and $\left\{\mathrm{X}_{\mathrm{a}}\right\}$ the generators of the gauge group. Let $\mathrm{dU}=\mu(\mathrm{B}) \prod_{\mathrm{a}} \mathrm{dB}^{\mathrm{a}}$; then an explicit representation of $\delta[\mathrm{U}-\mathrm{V}]$ is $\frac{1}{\mu(\mathrm{~B})} \prod_{\mathrm{a}} \delta\left(\mathrm{B}^{\mathrm{a}}-\mathrm{C}^{\mathrm{a}}\right)$, and $\delta[\mathrm{U}-\mathrm{V}]$ has the important property that, for $\sigma$ in the group space,

$$
\begin{equation*}
\delta\left[\sigma(\mathrm{U}-\mathrm{V}) \sigma^{+}\right]=\delta[\mathrm{U}-\mathrm{V}] \tag{11}
\end{equation*}
$$

We will use only (10) and (11) for our discussion. We return to (9); since A cell is a trace function of the $W_{i}$ 's, we have, for $\sigma$ in the gauge group

$$
\begin{equation*}
\mathrm{A}_{\mathrm{block}}\left[\sigma \mathrm{~W} \sigma^{+}\right]=\mathrm{A}_{\mathrm{block}}[\mathrm{~W}]: \text { gauge invariant } \tag{12}
\end{equation*}
$$

(where we have used (11) in obtaining (12)). Hence $A_{\text {block }}$ is also a trace function of the W , and in fact we can conclude by induction that the new action is always a trace function of the new block-spin variables, given that the original action is a trace function. Let the new (circled) lattice sites be labelled by m ; then

$$
\begin{equation*}
\mathrm{A}_{1}=\sum_{\mathrm{m}} \mathrm{~A}_{\text {block }}\left[\mathrm{W}_{\mathrm{m}}\right] \tag{13}
\end{equation*}
$$

and this completes the definition of the renormalization group transformation.
We now study (9) in more detail. First note that any product of all the $W_{i}$ 's gives the same $A_{b l o c k}[W]$ since $A_{c e l l}[W]$ is symmetric under any permutation of the four lattice sites. Performing the $\mathrm{W}_{4}$ integration gives

$$
e^{A_{\text {block }}(W)}=\int d W_{1} d W_{2} d W_{3} e^{\sum_{i=1}^{3} A\left(W_{i}\right)+A\left(W_{3}^{\dagger} W_{2}^{\dagger} W_{1}^{\dagger} W\right)}
$$

Do change of variables: $W_{2} W_{3}=V$; $\mathrm{dW}_{2}=\mathrm{dV}$. Then

$$
\begin{equation*}
e^{A} \text { block }(W)=\int d W_{1} d V d W_{3} e^{A\left(W_{1}\right)+A\left(N W_{3}^{\dagger}\right)+A\left(W_{3}\right)} e^{A\left(V^{\dagger} W_{1}^{\dagger} W\right)} \tag{14}
\end{equation*}
$$

Define convolution for functions $f(U)$ and $g(U)$ of the group elements $U$ by

$$
\begin{equation*}
(f * g)(V) \equiv \int d U f(U) g\left(U^{\dagger} V\right) \tag{15}
\end{equation*}
$$

Then, from (14) and using that $\mathrm{A}(\sigma)$ is a trace function of $\sigma$ gives

$$
\begin{align*}
e^{\mathrm{A}_{\text {block }}(W)} & =\int d V\left(e^{A^{*}} * e^{A}\right)(V) \cdot\left(e^{A^{A}} * e^{A}\right)\left(W V^{\dagger}\right)  \tag{16}\\
& =\left(e^{A} * e^{A} * e^{A} * e^{A}\right)(W) \tag{17}
\end{align*}
$$

In other words, $e^{A}$ block is given by a four-fold convolution of $e^{A}$ (which refers to a single site on the original lattice). From (16) we see that this four-fold convolution can itself be performed in two steps; the first step consists of performing the two-fold convolution (which corresponds to combining two lattice sites), and then convolving this function with itself to obtain $e^{A_{b l o c k}}$. This interpretation breaks the four-fold convolution into two identical steps of performing two-fold convolutions, and is more appropriate for numerical calculations.

The reason four lattice sites and not two lattice points were combined to define the new effective lattice was to preserve the symmetry of parity. Note in obtaining (17) no special assumptions were made about the initial action except that it be completely local and a trace function of the local variable. This property is true for each iteration, and hence (17) defines the general transformation. We note in passing that we could have defined the rather trivial renormalization group by integrating out variables at lattice sites 2, 3 and 4, and identifying W as $\mathrm{W}_{4}$. However, this W in no sense represents the combined "average value" of the fields at the four sites, and leads to trivial results. Also, this trivial transformation has no generalization to $\mathrm{d}>2$.

We now generalize Eq. (17) to arbitrary dimension $d$ which is larger than 2. To do so, we first discuss what the recursion formula means for dimension. The renormalization group transformation for $\mathrm{d}>2$ is assumed to produce a sequence of effective actions $A_{\ell}$ which are identical to the original action in all respects except that the action at any given site n can be an arbitrary function of the fundamental squares on the new lattice. More precisely, it is assumed that for each effective lattice with spacing $2^{\ell} a_{0}$, we have a set of gauge field variables which are defined on the effective lattice and can be pictured to
correspond uniquely to links connecting the effective lattice points. These variables range through the gauge group just like the original variables. The fundamental square is now defined to be the trace of the product of these matrices around a square (see Fig. 3). Then $A_{\ell}$ is assumed to be of the form

$$
\begin{equation*}
\mathrm{A}_{\ell}=\sum_{\substack{\mathrm{n} \in \text { effective } \\ \text { lattice }}} \sum_{\mu \nu}^{\prime} \mathrm{A}_{\mathrm{n} \mu \nu}^{(\ell)} \tag{18}
\end{equation*}
$$

where $A_{n \mu \nu}^{(\ell)}$ is an arbitrary function of the fundamental square $\operatorname{Tr}\left(W_{n \mu \nu}^{(\ell)}\right) . A$ lot of drastic approximations are involved in the ansatz of (18), and are discussed in Refs. 2 and 3.

We are now in a position to discuss Migdal's model for $A_{\ell}$ when $d>2$. Note for $\mathrm{d}=2$, the recursion equation combined the squares on four lattice sites to produce the effective square W , and $\mathrm{A}_{\text {block }}$ was an arbitrary trace function of this variable. Let $A_{b l o c k}^{(\ell)}$ be the effective action for a given lattice point, after $\ell$ iterations, for the 2-dimensional case. Then Migdal's model is defined, for all $n, \mu, \nu(\mu \neq \nu)$, by

$$
\begin{equation*}
\mathrm{e}^{A_{n \mu \nu}^{(\ell)}}=\left[\mathrm{e}^{\left.\mathrm{A}_{\mathrm{block}}^{(\ell)}\right]^{2^{\mathrm{d}-2}}}:\right. \text { Migdal's recursion formula } \tag{19}
\end{equation*}
$$

Note (19) guarantees translational invariance and symmetry under exchange of axes for $A_{l}$; the effective action $A_{l}$ is also gauge invariant with respect to gauge transformations defined on the effective lattice, since under such gauge transformations we have $\mathrm{W}_{\mathrm{n} \mu \nu}^{(\ell)} \rightarrow \mathrm{V}_{\mathrm{n}} \mathrm{W}_{\mathrm{n} \mu \nu}^{(\ell)} \mathrm{V}_{\mathrm{n}}^{\dagger}$ which leaves $\operatorname{Tr}\left(\mathrm{W}_{\mathrm{n} \mu \nu}^{(\ell)}\right)$ invariant.

In summary, we see that Migdal assumes, for any effective lattice of spacing $2^{\ell} a_{0}$, that $A_{l}$ is described purely by squares defined on this effective lattice. He then gives a prescription (which is nonlinear and nontrivial) of how to manufacture these squares for $\mathrm{d}>2$ dimensions from a 2 -dimensional lattice. This prescription defines a model for the renormalization group of the lattice gauge field.

## 2. Migdal's Model as an Approximation for the Exact Theory

We discuss the relation of Migdal's recursion formula to the exact renormalization group transformation for the d-dimensional lattice gauge theory. In particular, we will show how Migdal's prescription for introducing dimension arises from a crude approximation of the exact theory. We essentially repeat the arguments given by Migdal. ${ }^{2}$

For simplicity, start with the original action; let $U_{n \mu}$ be represented by the link $\underset{n}{\longrightarrow} \xrightarrow[n+\hat{\mu}]{ }$. A closed path (contour) is defined by the trace of product of $\mathrm{U}_{\mathrm{n} \mu}$ 's taken around the contour. The renormalization group transformation defined in terms of the W-variables in (9) (for $d=2$ ) can also be defined directly in terms of link integrations. For $\mathrm{d}=2$, this means combining four L-squares to form the new effective 2 L -square ( $\mathrm{L}=$ length of a side of the square) by integrating out from the action all the links which are interior for the $2 \mathrm{~L}-$ square (see Fig. 4). This procedure gives the 2L-square as a four-fold convolution of the L-square, which is our previous result.

To repeat this procedure for higher dimensions, we first consider the case of $d=3$. The lattice theory provides in the original action, for each lattice site, three L-squares with 2 links common between any two L-squares (see Fig. 5). In all, 9 links are needed to define these L-squares. To apply the above procedure, we first modify the lattice gauge theory by making all the L-squares independent of each other, in effect by introducing three new links (see Fig. 6). The renormalization group transformation is now defined by combining $2{ }^{3}=8$ L-cubes to form a 2 L -cube. (A L-cube is the three orthogonal contours defined on a cube, as shown in Fig. 6.) The way this is done is to integrate out all the links which are interior to the surface of the 2 L -cube. The resulting action for the 2 L -cube depends, in general, on all possible contours on the surface of the

2L-cube. Migdal makes the approximation that only the planar contours need be retained, and the rest can be simply ignored. Let us examine only the contours which lie in $x y-$ plane (see Fig. 7). We see that there are two such contours, with their $z$-axis being $\pm \mathrm{L} / 2$. Note each planar contour is simply the result of the 2-dimensional transformation, and is what we called $e^{\text {A }}$ block . However, we see that these xy-contours are also a function of the axis perpendicular to the plane $x y$, i.e., the effective action for the $x y$-contours is

$$
e^{A^{(1)}}=e^{A} \text { block }^{(+L / 2)} e^{A_{\text {block }}(-L / 2)}
$$

To recover the original form of the action, we can at most have a single contour in the xy-plane, and so Migdal simply ignores the splitting of the contours in the perpendicular direction, and obtains

$$
e^{A_{x y}^{(1)}} \simeq\left\{e^{A_{\text {block }}(0)}\right\}^{2}
$$

Hence this procedure reproduces the ansatz of (18), and the transformation can be iterated.

For the case of arbitrary d-dimensions, we simply need to evaluate the number of planar contours we get when we combine $2^{d}$ L-cubes to form a $2 L$ cube. This is because we ignore all other contours, plus because we ignore the separation of these planar contours in the directions perpendicular to the plane. Consider the origin of the coordinate to the unique point which is common to all the $2^{\mathrm{d}}$ L-cubes. Consider the plane defined by the $\mu \nu$ axes. The transverse coordinates of the $\mu \nu$-contours (planar) are specified by a (d-2) dimensional vector, which is of the form $x_{\perp}=(\underbrace{ \pm, \pm, \ldots, \pm}_{d-2}) \cdot \frac{L}{2}$. Hence, the number of planar-contours is $2^{d-2}$ (since this is the number of $x_{\perp}^{\prime} s$ ). Ignoring
the separation of these contours gives for the new effective action, for any site n

$$
\mathrm{e}^{\mathrm{A}_{\mathrm{n} \mu \nu}^{(1)}}=\left\{\mathrm{e}^{\mathrm{A}_{\mathrm{b}} \mathrm{block}}\right\}^{2^{\mathrm{d}-2}}
$$

We see this is simply Eq. (19) with $\ell=1$; since the ansatz is repeated for each iteration, this equation is valid for arbitrary number of iterations.

To recapitulate, the approximations of the exact theory made by Migdal are, for each iteration,
(a) Modification of the lattice gauge field action
(b) Ignoring all but the planar contours when evaluating the effective action
(c) Ignoring the splitting of the original planar contour into $2^{\mathrm{d}-2}$ planar contours.

Migdal claims that these approximations are reasonable when $L$ is less than the typical Compton wavelength of bound states, and that these approximations lead to a description of the critical behavior of the field (for $d=4$ ) which is good to $\sim 30 \%$.

## 3. Weak and Strong Coupling Approximations

We study Migdal's recursion formula for special values of the input coupling constant. We first study the strong coupling limit, as this is simpler than the weak coupling limit since, for $d=4$, the lowest order result suffices.

Let $G_{1}=\mathrm{g}_{0}^{2} / 2$; the strong coupling approximation means studying (19) for $G_{1} \gg 1$. Let

$$
\begin{equation*}
\mathrm{A}_{1} \equiv \frac{1}{4 \mathrm{G}_{1}} \sum_{\mathrm{n}} \operatorname{Tr}\left(\mathrm{~W}_{\mathrm{n}}+\mathrm{W}_{\mathrm{n}}^{\dagger}\right) \tag{20}
\end{equation*}
$$

We are going to generate the scquence of actions $\left\{A_{\ell}\right\}_{\ell=1}^{\infty}$. That is, the initial action is $A_{1}$. Let

$$
\begin{align*}
& V(1, W)=e^{\left(1 / 4 G_{1}\right) \operatorname{Tr}\left(W+W^{\dagger}\right)}  \tag{21}\\
& V(2, W)=e^{A_{n \mu \nu}^{(2)}} \tag{22}
\end{align*}
$$

Then, from (19)

$$
\begin{equation*}
\mathrm{V}(2, \mathrm{~W})=[\mathrm{V}(1, \cdot) * \mathrm{~V}(1, \cdot) * \mathrm{~V}(1, \cdot) * \mathrm{~V}(1, \cdot)(\mathrm{W})]^{2} \mathrm{~d}-2 \tag{23}
\end{equation*}
$$

We evaluate $V(2, W)$ in two steps; let

$$
\begin{align*}
\mathrm{VV}(1, \tau) & =(\mathrm{V}(1, \cdot \cdot) * \mathrm{~V}(1, \cdot))(\tau) \\
& =\int \mathrm{d} \sigma \mathrm{~V}(1, \sigma) \mathrm{V}\left(1, \sigma^{\dagger} \tau\right) \tag{24}
\end{align*}
$$

Then

$$
\begin{equation*}
\mathrm{V}(2, \mathrm{~W})=\{(\mathrm{VV}(1, \cdot) * \mathrm{VV}(1, \cdot))(\mathrm{W})\}^{2^{\mathrm{d}-2}} \tag{25}
\end{equation*}
$$

We now evaluate $\operatorname{VV}(1, \tau)$. Let the gauge group be $\mathrm{SU}(2)$; then $\operatorname{Tr}(\mathrm{U})=\operatorname{Tr}\left(\mathrm{U}^{\dagger}\right)$ giving

$$
\begin{equation*}
\mathrm{V}(1, \mathrm{~W})=\exp \left\{\frac{1}{2 \mathrm{G}_{1}} \operatorname{Tr}(\mathrm{~W})\right\} \tag{26}
\end{equation*}
$$

Letting $G_{1} \gg 1$ gives

$$
\begin{align*}
\operatorname{VV}(1, \tau)= & \int \mathrm{dW} \mathrm{e}^{1 / 2 \mathrm{G}_{1} \operatorname{Tr}(\mathrm{~W})+1 / 2 \mathrm{G}_{1} \operatorname{Tr}\left(\mathrm{~W}^{\dagger} \tau\right)} \\
= & \int \mathrm{dW}\left(1+\frac{1}{\mathrm{G}_{1}^{2}}(\text { const. })+\left(\frac{1}{2 \mathrm{G}_{1}}\right)^{2} \operatorname{Tr}(\mathrm{~W}) \operatorname{Tr}\left(\mathrm{W}^{\dagger} \tau\right)\right)+0\left(\mathrm{G}_{1}^{-3}\right) \\
& \left(\text { const. }=\frac{1}{4} \int \mathrm{dW} \operatorname{Tr}^{2}(\mathrm{~W})=0(1)\right) \tag{27}
\end{align*}
$$

From the orthogonality theore, we have ${ }^{7}$

$$
\begin{equation*}
\int \mathrm{dW} \mathrm{D}^{\ell}(\mathrm{W})_{\alpha \beta} \mathrm{D}^{\ell^{\prime}}\left(\mathrm{W}^{\dagger}\right)_{\gamma \delta}=\frac{\delta^{l l^{\prime}} \delta_{\alpha \delta^{\delta} \beta \gamma}}{\mathrm{d}_{\ell}} \tag{28}
\end{equation*}
$$

where $d_{l}=$ dimension of the $\ell^{\text {th }}$ representation. In our case, we are using the fundamental representation and hence $d_{l}=2$, giving

$$
\int \mathrm{dW} \operatorname{Tr}(\mathrm{~W}) \operatorname{Tr}\left(\mathrm{W}^{\dagger} \tau\right)=\frac{1}{2} \operatorname{Tr}(\tau)
$$

Therefore

$$
\begin{equation*}
\operatorname{VV}(1, \tau) \cong(\text { const. }) e^{\left(1 / 2 \mathrm{G}_{1}\right)^{2} \cdot \frac{1}{2} \operatorname{Tr}(\tau)}+0\left(\mathrm{G}_{1}^{-3}\right) \tag{29}
\end{equation*}
$$

and, repeating the convolution with $\mathrm{VV}(1, \tau)$ as the input function gives

$$
\begin{aligned}
\mathrm{V}(2, \mathrm{~W}) & =\text { (const. ) }\left\{\mathrm{e}^{\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2 \mathrm{G}_{1}}\right)^{2}\right)^{2} \operatorname{Tr}(\mathrm{~W})}\right)^{2 \mathrm{~d}-2} \\
& \equiv \text { (const.) } \mathrm{e}^{1 / 2 \mathrm{G}_{2} \operatorname{Tr}(\mathrm{~W})}
\end{aligned}
$$

Hence

$$
\begin{align*}
\mathrm{G}_{2} & =2^{8-\mathrm{d}} \mathrm{G}_{1}^{4}  \tag{31}\\
& : \text { strong coupling approximation }
\end{align*}
$$

The obvious generalization of (31) is

$$
\begin{equation*}
\mathrm{G}_{\mathrm{I}+1}=2^{8-\mathrm{d}} \mathrm{G}_{\mathrm{I}} \tag{32}
\end{equation*}
$$

We see that the sequence of actions, labeled by the coupling constant $G_{I}$ and having the functional form $V(I, W)=\exp \left\{\operatorname{Tr}(W) / 2 G_{1}\right\}$ goes to the strong coupling fixed point for which $\mathrm{G}^{*}=\infty$, i.e., $\mathrm{V}(\infty, \mathrm{W})=\mathrm{V}^{*}(\mathrm{~W})=1$ (provided we started with a. $G_{1} \gg 1$ ). We also see that this fixed point is stable for all $\mathrm{d} \leq 8$, and hence we need not go to higher order to determine the stability/instability of the fixed point for $\mathrm{d}=4$ (which is what we are interested in).

We now study the weak coupling sector. In this case, we have to do a second order calculation for the coupling constant renormalization, since to lowest order in $d=4$ dimensions, we shall find $G_{I+1}=G_{I}$ and hence cannot determine the stability/instability of the weak coupling fixed point. [The weak coupling fixed point is given by $\mathrm{V}^{*}(\mathrm{~W})=\delta[\mathrm{W}-1]$ which is equivalent to
$\left.\lim _{\mathrm{G}_{1} \rightarrow 0} \exp \left\{\frac{1}{2 \mathrm{G}_{1}} \operatorname{Tr}(\mathrm{~W}-1)\right\}\right]$. To do the second order calculation, we start with the action

$$
\begin{equation*}
\mathrm{V}(\mathrm{I}, \mathrm{~W})=\exp \left\{\frac{1}{2 \mathrm{G}_{\mathrm{I}}} \operatorname{Tr}(\mathrm{~W}-1)+\frac{\mathrm{H}_{\mathrm{I}}}{2 \mathrm{G}_{\mathrm{I}}} \operatorname{Tr}^{2}(\mathrm{~W}-1)\right\} \tag{33}
\end{equation*}
$$

and compute

$$
\begin{equation*}
\mathrm{V}(\mathrm{I}+1, \mathrm{~W})=\exp \left\{\frac{1}{2 \mathrm{G}_{\mathrm{I}+1}} \operatorname{Tr}(\mathrm{~W}-1)+\frac{\mathrm{H}_{\mathrm{I}+1}}{2 \mathrm{G}_{\mathrm{I}+1}} \operatorname{Tr}^{2}(\mathrm{~W}-1)\right\} \tag{34}
\end{equation*}
$$

The weak coupling approximation is valid for $G_{\mathrm{I}}, \mathrm{G}_{\mathrm{I}+1} \ll 1$, which in turn implies $\operatorname{Tr}(\mathrm{W}-1) \sim 0\left(\mathrm{G}_{\mathrm{I}}\right)$. We take $\mathrm{H}_{\mathrm{I}}=0(1)$. For notational simplicity, let $\mathrm{G}=\mathrm{G}_{\mathrm{I}}$ and $\mathrm{H}=\mathrm{H}_{\mathrm{I}}$. From (24)

$$
\begin{align*}
\mathrm{VV}(\mathrm{I}, \mathrm{~T}) & =\int \mathrm{dUV}(\mathrm{I}, \mathrm{U}) \mathrm{V}\left(\mathrm{I}, \mathrm{U}^{\dagger \mathrm{T}}\right) \\
& \equiv \int \mathrm{dU} \mathrm{e}^{\mathrm{E}(\mathrm{U}, \mathrm{~T})} \tag{35}
\end{align*}
$$

where

$$
\begin{align*}
\mathrm{E}(\mathrm{U}, \mathrm{~T})= & \frac{1}{2 \mathrm{G}} \operatorname{Tr}(\mathrm{U}-1)+\frac{\mathrm{H}}{2 \mathrm{G}} \operatorname{Tr}^{2}(\mathrm{U}-1) \\
& +\frac{1}{2 \mathrm{G}} \operatorname{Tr}\left(\mathrm{UT}^{\dagger}-1\right)+\frac{\mathrm{H}}{2 \mathrm{G}} \operatorname{Tr}^{2}\left(\mathrm{UT}^{\dagger}-1\right) \tag{36}
\end{align*}
$$

Make the change of variables

$$
\begin{align*}
\mathrm{U} \rightarrow \mathrm{UT}^{-1 / 2} & =\mathrm{U}^{\prime}  \tag{37}\\
\mathrm{dU} & =\mathrm{dU} \\
\therefore \mathrm{VV}(\mathrm{I}, \mathrm{~T}) & =\int \mathrm{dU} \mathrm{U}^{\mathrm{r}} \mathrm{e}^{\mathrm{E}\left(\mathrm{U}^{\top}, \mathrm{T}\right)} \tag{38}
\end{align*}
$$

and, dropping the prime on $U$ gives

$$
\begin{align*}
\mathrm{E}(\mathrm{U}, \mathrm{~T})= & \frac{1}{2 \mathrm{G}} \operatorname{Tr}\left(\mathrm{UT}^{1 / 2}+\mathrm{UT}^{-1 / 2}-2\right) \\
& +\frac{\mathrm{H}}{2 \mathrm{G}}\left[\operatorname{Tr}^{2}\left(\mathrm{UT}^{1 / 2}-1\right)+\operatorname{Tr}^{2}\left(\mathrm{UT}^{-1 / 2}-1\right)\right] \tag{39}
\end{align*}
$$

Let

$$
\begin{array}{rlrl}
\mathrm{U} & =\mathrm{e}^{\overrightarrow{\mathrm{u} \cdot \vec{\sigma}}} & \mathrm{~T}=\mathrm{e}^{\overrightarrow{\mathrm{i} \vec{\tau}} \cdot \vec{\sigma}} \\
& =\cos \mathrm{u}+\hat{\mathrm{u}} \hat{\mathrm{u}} \cdot \hat{\sigma} \sin \mathrm{u} & & (\vec{\sigma}: \text { Pauli matrices })
\end{array}
$$

Then

$$
\begin{equation*}
\mathrm{T}^{1 / 2}+\mathrm{T}^{-1 / 2}=2 \cos (\tau / 2) \tag{40}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Tr}\left(\mathrm{UT}^{ \pm 1 / 2}\right)=2\{\cos \mathrm{u} \cos (\tau / 2) \mp \hat{\mathrm{u}} \cdot \hat{\tau} \sin \mathrm{u} \sin (\tau / 2)\} \tag{41}
\end{equation*}
$$

We work in spherical coordinates for $\vec{u}$. Let us choose the $\vec{u}$ coordinate axes for $\overrightarrow{\mathrm{u}}$ and $\vec{\tau}$ such that $\hat{\tau}$ points in the 3 -direction, and let. $\mathrm{x}=\hat{\mathrm{u}} \cdot \hat{\tau}$. Then, the measure on $\operatorname{SU}(2)$ is given by

$$
\begin{align*}
\int \mathrm{dU} & =\frac{1}{2 \pi^{2}} \int_{0}^{\pi} \sin ^{2} u d u \int_{-1}^{+1} d x \int_{0}^{2 \pi} \mathrm{~d} \phi \quad(\phi=\text { azimuthal angle }) \\
& =\frac{1}{\pi} \int_{0}^{\pi} \sin ^{2} u d u \int_{-1}^{+1} d x \tag{42}
\end{align*}
$$

since $\mathrm{E}(\mathrm{U}, \tau)$ is not a function of $\phi$. Let $\sigma \equiv \tau / 2$; then from (39)-(41) we have

$$
\begin{align*}
E(U, \tau)= & \frac{2}{G}(\cos \sigma \cos u-1) \\
& +\frac{H}{2 G} \cdot 4\left[(\cos u \cos \sigma+x \sin u \sin \sigma-1)^{2}\right. \\
& \left.+(\cos u \cos \sigma-x \sin u \sin \sigma-1)^{2}\right] \\
= & \frac{2}{G}(\cos \sigma \cos u-1)+\frac{4 H}{G}\left[(\cos u \cos \sigma-1)^{2}+x^{2} \sin ^{2} u \sin ^{2} \sigma\right] \tag{43}
\end{align*}
$$

Note that $\operatorname{Tr}(W-1)=0(G)$ implies $u, \sigma=0(\sqrt{G})$; hence the second term in (43) is $0(G)$ and its exponential can be expanded in a power series, giving

$$
\begin{align*}
\mathrm{VV}(\mathrm{I}, \tau)= & \int \mathrm{dU} \mathrm{e}^{\mathrm{E}(\mathrm{U}, \tau)} \\
= & \frac{1}{\pi} \int_{0}^{\pi} d u \sin ^{2} \mathrm{ue}^{2(\cos \sigma \cos u-1) / \mathrm{G}} \\
& \times \int_{-1}^{+1} \mathrm{dx}\left[1+\frac{4 \mathrm{H}}{\mathrm{G}}(\cos \mathrm{u} \cos \sigma-1)^{2}+\mathrm{x}^{2} \frac{4 \mathrm{H}}{\mathrm{G}} \sin ^{2} u \sin ^{2} \sigma+0\left(\mathrm{G}^{2}\right)\right] \\
= & \frac{\text { (const.) }}{\pi} \int_{0}^{\pi} d u \sin ^{2} \mathrm{u}^{2} \cos \sigma \cos \mathrm{u} / \mathrm{G}\left[1+\frac{4 \mathrm{H}}{\mathrm{G}}(\cos \mathrm{u} \cos \sigma-1)^{2}\right. \\
& \left.+\frac{4 \mathrm{H}}{3 \mathrm{G}} \sin ^{2} u \sin ^{2} \sigma+0\left(\mathrm{G}^{2}\right)\right] \tag{44}
\end{align*}
$$

Let

$$
\begin{align*}
\mathrm{VV}(\mathrm{I}, \tau) & =\exp \{\mathrm{C}(\tau)\}  \tag{45}\\
& \equiv \exp \left\{\frac{1}{\mathrm{G}^{\dagger}}(\cos \tau-1)+\frac{2 \mathrm{H}^{\prime}}{\mathrm{G}^{\prime}}(\cos \tau-1)^{2}\right\} \tag{46}
\end{align*}
$$

Then, as shown in Appendix A (Eq. (A.13))

$$
\begin{align*}
& \frac{1}{\mathrm{G}^{\prime}}=\frac{1}{2 \mathrm{G}}-\frac{3}{2}-\frac{5}{2} \mathrm{H}+0(\mathrm{G}) \\
& 2 \mathrm{H}^{\prime} / \mathrm{G}^{\prime}=\left(\mathrm{H}-\frac{1}{4}\right) / 4 \mathrm{G}+0(\mathrm{G}) \tag{47}
\end{align*}
$$

Rewriting the above equations gives

$$
\begin{align*}
& \mathrm{G}^{\prime}=2 \mathrm{G}\left[1+\left(\frac{3}{4}+5 \mathrm{H}\right) \mathrm{G}\right]+0\left(\mathrm{G}^{3}\right) \\
& \mathrm{H}^{\prime}=\frac{1}{4}\left(\mathrm{H}-\frac{1}{4}\right)+0(\mathrm{G}) \tag{48}
\end{align*}
$$

: Recursion equation
We perform the convolution once more using $\operatorname{VV}(I, \tau)$ as the input function; let

$$
\begin{align*}
\operatorname{VA}(\mathrm{I}, \mathrm{~W}) & =(\mathrm{VV}(\mathrm{I}, \cdot)) *(\mathrm{VV}(\mathrm{I}, \cdot))(\mathrm{W}) \\
& \equiv \exp \left\{\frac{1}{\mathrm{G}^{\prime \prime}}(\cos \mathrm{W}-1)+\frac{2 \mathrm{H}^{\prime \prime}}{\mathrm{G}^{n}}(\cos \mathrm{~W}-1)^{2}\right\} \tag{49}
\end{align*}
$$

Then using (48) gives

$$
\begin{align*}
\mathrm{G}^{\prime \prime} & =2 \mathrm{G}^{\prime}\left[1+\left(\frac{3}{4}+5 \mathrm{H}^{\prime}\right) \mathrm{G}^{\prime}\right] \\
& =4 \mathrm{G}\left[1+\left(\frac{13}{8}+\frac{15}{2} \mathrm{H}\right) \mathrm{G}\right]+0\left(\mathrm{G}^{3}\right)  \tag{50}\\
\mathrm{H}^{\prime \prime} & =\frac{1}{4}\left(\mathrm{H}^{\prime}-\frac{1}{4}\right) \\
& =\frac{1}{16}\left(\mathrm{H}-\frac{5}{4}\right) \tag{51}
\end{align*}
$$

From Migdal's prescription for introducing dimension into the recursion formula, we have

$$
\begin{align*}
\mathrm{V}(\mathrm{I}+1, \mathrm{~W}) & =[\mathrm{VA}(\mathrm{I}, \mathrm{~W})]^{2^{\mathrm{d}-2}}  \tag{52}\\
& \equiv \exp \left\{\frac{1}{\mathrm{G}_{\mathrm{I}+1}}(\cos \mathrm{~W}-1)+\frac{2 \mathrm{H}_{\mathrm{I}+1}}{\mathrm{G}_{\mathrm{I}+1}}(\cos \mathrm{~W}-1)^{2}\right\} \tag{53}
\end{align*}
$$

Hence, from (49)-(53), we have

$$
\begin{align*}
& \mathrm{G}_{\mathrm{I}+1}=2^{2-\mathrm{d}} \mathrm{G}^{\prime \prime} \\
& 2 \mathrm{H}_{\mathrm{I}+1} / \mathrm{G}_{\mathrm{I}+1}=2^{\mathrm{d}-2}\left(2 \mathrm{H}^{\prime \prime} / \mathrm{G}^{\prime \prime}\right) \tag{54}
\end{align*}
$$

and using (50), (51) gives

$$
\begin{align*}
& \mathrm{G}_{\mathrm{I}+1}=2^{4-\mathrm{d}} \mathrm{G}_{\mathrm{I}}\left[1+\left(\frac{13}{8}+\frac{15}{2} \mathrm{H}_{\mathrm{I}}\right) \mathrm{G}_{\mathrm{I}}\right]+0\left(\mathrm{G}_{\mathrm{I}}^{3}\right) \\
& \mathrm{H}_{\mathrm{I}+1}=\frac{1}{16}\left(\mathrm{H}_{\mathrm{I}}-\frac{5}{4}\right)+0\left(\mathrm{G}_{\mathrm{I}}\right) \tag{55}
\end{align*}
$$

(using the fact that $G=G_{I}, H=H_{\mathrm{I}}$ ). We study (55) for $\mathrm{d}=4$; for $\mathrm{d}<4$, the coupling constant $G_{I}$ is increasing to lowest order, and the second order result is unnecessary. For $d=4$, we have

$$
\begin{align*}
& \mathrm{G}_{\mathrm{I}+1}=\mathrm{G}_{\mathrm{I}}+\left(\frac{13}{8}+\frac{15}{2} \mathrm{H}_{\mathrm{I}}\right) \mathrm{G}_{\mathrm{I}}^{2}+0\left(\mathrm{G}_{\mathrm{I}}^{3}\right)  \tag{56}\\
& \mathrm{H}_{\mathrm{I}+1}=\frac{1}{16}\left(\mathrm{H}_{\mathrm{I}}-\frac{5}{4}\right)+0\left(\mathrm{G}_{\mathrm{I}}\right) \tag{57}
\end{align*}
$$

[Note that our result does not agree with Migdal ${ }^{1}{ }^{16}$ result.]

We first examine what effect the $H_{I}$ term has on $G_{I+1}$. From (57) we have

$$
\begin{equation*}
\mathrm{H}_{\mathrm{I}+1}=\frac{1}{4}\left(\mathrm{H}_{\mathrm{I}}+\frac{1}{12}\right)-\frac{1}{12}+0\left(\mathrm{G}_{1}\right) \tag{58}
\end{equation*}
$$

Since $H_{1}=0(1)$, we see that $H_{I}$ very rapidly goes to the fixed point $H^{*}=-\frac{1}{12}$.
Let $\mathrm{H}_{\mathrm{I}} \simeq \mathrm{H}^{*}$; then from (56), we have

$$
\begin{equation*}
\mathrm{G}_{\mathrm{I}+1}=\mathrm{G}_{\mathrm{I}}+\mathrm{G}_{\mathrm{I}}^{2}>\mathrm{G}_{\mathrm{I}} \tag{59}
\end{equation*}
$$

Hence, we see that the instability of the weak coupling fixed point $G^{*}=0$ is not affected by the $H_{I}$ term present in (56). From (56), we see that $G^{*}=0$ is the only fixed point (using the fact that $G_{1}>0$, which implies $G_{I}>0$ ); the fixed point is independent of $H_{1}$, given that $H_{1}=0(1)$. The fixed point function for $G^{*}=0$ is $\mathrm{V}^{*}(\mathrm{~W})=\delta[\mathrm{W}-1]$. We conclude that, starting from the input initial action $A_{1}(W)=(\cos W-1) / G_{1}, G_{1} \ll 1$, we have $G_{I+1}>G_{I}$; that is, the $G^{*}=0$ fixed point is once unstable, and one leaves the fixed point with each iteration. We note in passing that the instability of the $G^{*}=0$ fixed point crucially hinges on the nonAbelian nature of the gauge group. This is clearly seen in this model by noting that the piece $\frac{13}{8} G_{I}^{2}$ in (56) arises from the invariant measure of the gauge group. If this term had been absent in (56), the $\mathrm{G}^{*}=0$ fixed point would no longer be unstable; in fact, if one does the second order calculation for the Abelian case, one finds that $G_{I}$ is still marginal, and no conclusions of the stability/instability of the $\mathrm{G}^{*}=0$ fixed point can be drawn.

Examining (56) or (59), we could naively expect that the sequence of coupling constants converge to the strong coupling fixed point (for which $G^{*}=\infty$ ). Our computer program shows that in fact this is exactly what happens; and that, for $\mathrm{d} \leq 4$, there are no other fixed points for Migdal's recursion equation except for the unstable weak coupling fixed point and the stable strong coupling fixed point.

The region for $G_{I}$ which lies in between the weak coupling region and the strong coupling region we call the intermediate coupling region. We will use this terminology in the next section.

We briefly discuss the recursion equation for $\mathrm{d}=4+\epsilon>4$ ( $\epsilon \simeq 0$ ). From (55), using $H_{\mathrm{I}}=\mathrm{H}^{*}=-\frac{1}{12}$, we have

$$
\begin{aligned}
G_{I+1} & =2^{4-d} G_{I}\left(1+G_{I}\right) \\
& =2^{-\epsilon} G_{I}\left(1+G_{I}\right)
\end{aligned}
$$

The fixed point equation is

$$
G^{*}=2^{-\epsilon} G^{*}\left(1+G^{*}\right)
$$

giving the following fixed points

$$
0, \quad 2^{\epsilon}-1, \quad \infty
$$

The fixed point $G_{c}^{*}=2^{\epsilon}-1$ is twice unstable. For $G_{1}<G_{c}^{*}, G_{I} \rightarrow 0$ as $I \rightarrow \infty$ and for $G_{1}>G_{c}^{*}, G_{I} \rightarrow \infty$ as $I \rightarrow \infty$. Both the fixed points $G^{*}=0$ or $\infty$ are now stable, and the system undergoes a phase transition at $G=G_{c}^{*}$. Hence, the asymptotically free domain of $\mathrm{G} \simeq 0$ is now separated by a phase transition from the strongly coupled domain of $G \simeq \infty$, and for $d=4+\epsilon$, we cannot continuously go from the asymptotically free theory to the strongly coupled theory; that is, for $d>4$, the gauge-field cannot simultaneously exhibit free-like behavior at short distances, and strong coupling behavior at large distance. However, as $\epsilon \rightarrow 0$, the $\mathrm{G}_{\mathrm{c}}^{*}=2^{\epsilon}-1$ fixed point coalesces with $G^{*}=0$ fixed point making it once unstable (since $G_{\mathrm{I}}>0$ ), and leaving us with a gauge-field which has no phase transition separating the weak from the strong coupling sector.
4. Numerical Solution of Migdal's Model

Recall from (24) and (25) we have

$$
\begin{align*}
\mathrm{VV}(\mathrm{I}, \tau) & =\int \mathrm{d} \sigma \mathrm{~V}(\mathrm{I}, \operatorname{Tr}(\sigma)) \mathrm{V}\left(\mathrm{I}, \operatorname{Tr}\left(\sigma^{\dagger} \tau\right)\right)  \tag{24}\\
& =\int \mathrm{d} \sigma \mathrm{~V}(\mathrm{I}, \operatorname{Tr}(\sigma)) \mathrm{V}(\mathrm{I}, \operatorname{Tr}(\sigma \tau))
\end{align*}
$$

and from (25)

$$
\mathrm{V}(\mathrm{I}+1, \tau)=\left\{\int \mathrm{dWVV}(\mathrm{I}, \operatorname{Tr}(\mathrm{~W})) \mathrm{VV}(\mathrm{I}, \operatorname{Tr}(\mathrm{~W} \tau))\right\}^{2 \mathrm{~d}-2}
$$

where, in obtaining (24') and (25 ), we have used (a) that $\mathrm{V}(\mathrm{I}, \tau)$ and $\mathrm{VV}(\mathrm{I}, \tau)$ are trace functions of $\tau$ and (b) for $\operatorname{SU}(2), \operatorname{Tr}\left(\sigma^{\dagger}\right)=\operatorname{Tr}(\sigma)$ and $\mathrm{d}\left(\sigma^{\dagger}\right)=\mathrm{d}\left(\sigma^{-1}\right)=\mathrm{d} \sigma$. Note also, since the invariant measure is always $\geq 0, \mathrm{~V}(\mathrm{I}+1, \tau) \geq 0$ if $\mathrm{V}(\mathrm{I}, \tau) \geq 0$. This positivity of $\mathrm{V}(\mathrm{I}, \tau)$ ensures that the effective action is always real, given that $\mathrm{V}(1, \tau) \geq 0$. This property is good for convergence of numerical calculations. We now study the equations (24') and (25') numerically. Since (25') is essentially the same as (24') except for the additional step of introducing dimension, we focus on (24'). For notational convenience, we consider $V(I, \tau)$ to be a function of $\frac{1}{2} \operatorname{Tr}(\tau)$. From (42) and with a simple rescaling, we have

$$
\begin{align*}
\int \mathrm{d} \sigma & =\frac{1}{2 \pi^{2}} \int_{0}^{\pi} \sin ^{2} \sigma \mathrm{~d} \sigma \int_{0}^{\pi} \mathrm{d} \phi \int_{0}^{\pi} \mathrm{d} \theta \sin \theta \\
& =\frac{1}{\pi} \int_{0}^{\pi} \sin ^{2} \sigma \mathrm{~d} \sigma \int_{-1}^{+1} \mathrm{dx} \tag{60}
\end{align*}
$$

(since the integrand is always independent of $\phi$ ). Note $x=\cos \theta$; choosing the spherical coordinate system for $\sigma$ such that $\hat{\tau}$ is in the 3 -direction, we have

$$
\begin{align*}
& \frac{1}{2} \operatorname{Tr}(\sigma)=\cos \sigma \\
& \frac{1}{2} \operatorname{Tr}(\sigma \tau)=\cos \sigma \cos \tau-\mathrm{x} \sin \sigma \sin \tau \tag{61}
\end{align*}
$$

$$
\begin{equation*}
\therefore \mathrm{VV}(\mathrm{I}, \tau)=\frac{1}{\pi} \int_{0}^{\pi} \sin ^{2} \sigma \mathrm{~d} \sigma \mathrm{~V}(\mathrm{I}, \cos \sigma) \int_{-1}^{+1} \mathrm{dxV}(\mathrm{I}, \cos \sigma \cos \tau-\mathrm{x} \sin \sigma \sin \tau) \tag{62}
\end{equation*}
$$

We make a change of variable; let

$$
\begin{align*}
& \cos \xi=\cos \sigma \cos \tau-\mathrm{x} \sin \sigma \sin \tau  \tag{63}\\
\therefore & -\sin \xi \mathrm{d} \xi=-(\sin \sigma \sin \tau) \mathrm{dx}
\end{align*}
$$

We now determine the upper and lower limits of integration for variable $\xi$.

$$
\begin{align*}
\mathrm{x}=+1: \quad \cos \xi & =\cos (\sigma+\tau) \\
\therefore \xi & =\sigma+\tau \bmod 2 \pi \\
& \equiv \min \{\sigma+\tau, 2 \pi-\sigma-\tau\}  \tag{64}\\
\mathrm{x}=-1: \quad \cos \xi & =\cos (\sigma+\tau) \\
\therefore \quad \xi & =|\sigma-\tau|
\end{align*}
$$

Note $\xi \in[0, \pi]$, since $\sigma, \tau \in[0, \pi]$

$$
\begin{equation*}
\therefore \mathrm{VV}(\mathrm{I}, \tau)=\frac{(1 / \pi)}{\sin \tau} \int_{0}^{\pi} \sin \sigma \mathrm{d} \sigma \mathrm{~V}(\mathrm{I}, \cos \sigma) \int_{\left|\sigma_{-\tau}\right|}^{\sigma+\tau \bmod 2 \pi} \sin \xi \mathrm{~d} \xi \mathrm{~V}(\mathrm{I}, \cos \xi) \tag{65}
\end{equation*}
$$

Let

$$
\begin{equation*}
\mathrm{W}(\mathrm{I} ; \sigma, \tau) \equiv \int_{|\sigma-\tau|}^{\sigma+\tau \bmod 2 \pi} \mathrm{~d} \xi \sin \xi \mathrm{~V}(\mathrm{I}, \cos \xi) \tag{66}
\end{equation*}
$$

Then

$$
\begin{equation*}
\mathrm{VV}(\mathrm{I}, \tau)=\frac{(\mathrm{I} / \pi)}{\sin \tau} \int_{0}^{\pi} \mathrm{d} \sigma \sin \sigma \mathrm{~V}(\mathrm{I}, \cos \sigma) \mathrm{W}(\mathrm{I} ; \sigma, \tau) \tag{67}
\end{equation*}
$$

The two integrations to be performed numerically are (66) and (67). The change of variable made in (63) is important for the numerical calculation. As things stand in (62), for the x integration we need the function $\mathrm{V}(1, \sigma)$ at points which are not the same as the points necessary to perform the $\sigma$-integration (given that we are performing the integration on a grid of points). Also, what

Eq. (62) gives us is the function $\operatorname{VV}(\mathrm{I}, \tau)$ at grid points which are to be used for performing the $\sigma$-integration. Hence, if we are to use values of the function $\mathrm{V}(\mathrm{I}, \sigma)$ or $\mathrm{VV}(\mathrm{I}, \sigma)$ at grid points which are computed by our recursion formula, then Eq. (62) cannot be used. And if we use some interpolated value of $V(I, \tau)$ to perform the x-integration, the errors introduced are uncomfortably large. The way out is to perform the change of variable from x to $\xi$; the resulting Eq. (65) uses the value of $V(I, \tau)$ only at the grid points which are fixed and which are the same as the ones used for performing the $\sigma$-integration. Hence, no interpolation for values of $\mathrm{V}(\mathrm{I}, \tau)$ are necessary. Also, in computing $\mathrm{W}(\mathrm{I} ; \sigma, \tau)$, the limits on the integral always fall on the fixed grid points, and no interpolation for the value of $\mathrm{V}(\mathrm{I}, \tau)$ is made. To summarize, the change of variable from $x$ to $\xi$ allows us to perform, for each iteration, both integrations (66) and (67) on the same fixed grid points for which the recursion formula gives values of the computed function.

We now discuss the computer program; the equations (24), (25) are ideally suitable for a numerical solution. Each step in the recursion is identical as far as the structure of the integrals go. What changes with each step is that the input function changes from $\mathrm{V}(\mathrm{I}, \sigma)$ to $\mathrm{V}(\mathrm{I}+1, \sigma)$. Since the function $\mathrm{V}(\mathrm{I}+1, \sigma)$ is itself computed from $\mathrm{V}(I, \sigma)$, this computed function is simply taken as the input in calculating $\mathrm{V}(\mathrm{I}+2, \sigma)$, etc. To start the recursion, we need the initial function $\mathrm{V}(1, \sigma)$. But $\mathrm{V}(1, \sigma)$ is simply the bare action exponentiated, i.e., $\mathrm{V}(1, \sigma)=\exp \left\{\frac{1}{2 \mathrm{G}_{1}} \operatorname{Tr}(\sigma-1)\right\}$. Hence, the single parameter that can be varied is the input coupling constant $G_{1}$. The only other variable in the recursion equation is the dimension d .

For the purpose of performing the integrations numerically, we have to discretize the variables $\sigma$ and $\xi$. The range of integration, for the first step
in the recursion, is divided into a total of $N$ number of lattice points. The total_number of lattice points may vary with each step in the recursion; the total number of lattice sites for the $\mathrm{I}^{\text {th }}$ step in the recursion is denoted by $\mathrm{N}(\mathrm{I})$. The number $N$ is input, whereas $N(I)$ is fixed by the computer program. The only other input for the computer program is the maximum number of iterations to be performed, denoted by M.

In summary, we see that for the computer program, we have four variables, namely $\mathrm{G}_{1}, \mathrm{~d}, \mathrm{~N}$ and M .

Consider the case of $\mathrm{G}_{1} \sim 10^{-3}$; then $\mathrm{V}(1, \sigma)=\exp \left\{\frac{1}{2 \mathrm{G}_{1}} \operatorname{Tr}(\sigma-1)\right\} \sim \exp \left\{-\frac{1}{2 \mathrm{G}_{1}} \sigma^{2}\right\}$. In other words, $\mathrm{V}(1, \sigma)$ rapidly goes to zero when $\sigma \gg \sqrt{\mathrm{G}_{1}}$. With this in mind, we choose, for each iteration $I$, the total space of integration $S(I)$. Choose $\mathrm{S}(\mathrm{I}=1)$ such that $\mathrm{V}(1, \sigma) \gtrsim \mathrm{G}_{1}^{2}$ on all points inside this space. The choice of $V(1, N) \sim G_{1}^{2}$ is necessary to do the second order calculation of $G_{I}$ for $d=4$; for $\mathrm{d}<4, \mathrm{G}_{1}{ }^{2}$ can be replaced by say. 001 without any large errors. For reasons to be explained later, $\mathrm{S}(\mathrm{I})$ must be of the form $\pi / 2^{\mathrm{n}}$ ( $\mathrm{n}=$ integer); for $\mathrm{G}_{1}$ small, $\mathrm{S}(1) \sim 3 \sqrt{\mathrm{G}_{1}}$. The computer program is slightly more complicated and $\mathrm{S}(\mathrm{I})$ may not be of the form $\pi / 2^{n}$ for the first few steps. Given $\mathrm{S}(\mathrm{I})$ and $N$, the lattice spacing for the integration points are then determined. Qualitatively, $\mathrm{S}(\mathrm{I})<\pi / 2$ for the weak coupling region, $\pi / 2<\mathrm{S}(\mathrm{I})<\pi$ for the intermediate coupling region.

As discussed above, the initial lattice spacing is fixed such that $V(1, N) \sim .001$. We choose to normalize the computed functions such that $\mathrm{V}(\mathrm{I}, 1)=\mathrm{VV}(\mathrm{I}, 1)=1$. As the iteration is performed to compute $\mathrm{V}(2, \sigma), \mathrm{V}(3, \sigma)$, etc., the interval of $\sigma$ on which these functions are $>\mathrm{V}(1, \mathrm{~N})$ also increases (since for $\mathrm{d} \leq 4$ the coupling constants $G_{I}$ are increasing). We hence extend the use of Eq. (67) to compute the function $\operatorname{VV}(\mathrm{I}, \tau)$ for values of $\tau$ until the point when $\mathrm{VV}(\mathrm{I}, \mathrm{N}(\mathrm{I})) \leq \mathrm{V}(1, \mathrm{~N})$; this defines $N(I)(\geq N)$. This procedure of increasing the total number of lattice points
is continued until $N(I) \geq 2 N$; when this happens, all the odd lattice points are dropped from the calculation, and the lattice spacing is consequently doubled. Also, the doubling of the total number of lattice points m times implies $S(I)=\pi / 2^{n-m}$. The procedure of increasing the total number of lattice points is continued until $S(I)=\pi$; when $S(I)=\pi$, the number of lattice points is permanently fixed, and the $\mathrm{S}(\mathrm{I})$ no longer is allowed to increase.

This method of doubling the lattice spacing is crucial in allowing one to go from the weak to the strong coupling domain. The reason being that if the lattice spacing was kept fixed at its initial value, we would eventually need about $2^{10}$ lattice points to cover the interval $[0, \pi]$ if we were starting from the weak coupling sector. This would make the calculation extremely expensive and from the economics point of view inaccessible.

The total number of lattice points $N(I)$ was always arranged to be even so that Simpson's rule could be used for integration purposes. Two different schemes were used for integration. The function $\mathrm{W}(\mathrm{I} ; \sigma, \tau)$ was computed using Simpson's rule. Let $f(\xi)=\sin \xi V(I, \cos \xi)$; then from (66)

$$
\begin{equation*}
\mathrm{W}(\mathrm{I} ; \sigma, \tau)=\int_{|\sigma-\tau|}^{(\sigma+\tau) \bmod 2 \pi} \mathrm{f}(\xi) \mathrm{d} \xi=\mathrm{W}(\mathbb{I} ; \tau, \sigma) \tag{68}
\end{equation*}
$$

Since $\mathrm{W}(\mathrm{I} ; \sigma, \tau)$ is symmetric under $\sigma \longleftrightarrow \tau$, we need only consider the case of $\sigma>\tau$. Then

$$
\mathrm{W}(\mathrm{I} ; \sigma, \tau)=\int_{\sigma-\tau}^{(\sigma+\tau) \bmod 2 \pi} \mathrm{f}(\xi) \mathrm{d} \xi
$$

which shows that the interval for $\xi$-integration is

$$
\left.\begin{array}{lll}
\text { (a) } & 2 \tau & \text { for } \\
\sigma+\tau<\pi  \tag{69}\\
\text { (b) } & 2(\pi-\sigma) & \text { for } \\
\sigma+\tau>\pi
\end{array}\right\}
$$

In other words, we always have an even number of intervals over which we need to perform the $\xi$-integration. This makes the $\xi$-integration ideal for using Simpson's rule. Consider the simple example of evaluating $\int_{1}^{3} f(\xi) d \xi$. Simpson's rule states that

$$
\begin{equation*}
\int_{1}^{3} \mathrm{f}(\xi) \mathrm{d} \xi=\text { (const.) }\left[\mathrm{f}_{1}+4 \mathrm{f}_{2}+\mathrm{f}_{3}\right] \tag{70}
\end{equation*}
$$

A straightforward extension can be made for arbitrary number of even intervals.
The trapezoidal rule was used for evaluating $\int_{0}^{\pi} \mathrm{d} \sigma \sin \sigma \mathrm{V}(\mathrm{I}, \cos \sigma) \mathrm{W}(\mathrm{I} ; \sigma, \tau)$. The reason being that the trapezoidal rule is more accurate for evaluating the integral of a periodic function over its period, than is using Simpson's rule to do the same. The integral being considered can easily be recast into an integral over $[-\pi, \pi]$. However, the gains of using the trapezoidal rule are minor, and we mostly preferred it due to its simplicity.

The reason for using Simpson's rule for $\mathrm{W}(1 ; \sigma, \tau)$ was that if one used the trapezoidal rule instead, then in going from the weak to the strong coupling domain, large systematic errors were introduced. The new scheme resolved this problem.

The definition of the coupling constant for the entire trajectory was made by assuming that the functional form of the effective action was

$$
\begin{equation*}
\operatorname{VE}(\mathrm{I}, \cos \alpha)=\exp \left\{\frac{1}{\mathrm{G}_{\mathrm{I}}}(\cos \alpha-1)+\frac{2 \mathrm{H}_{\mathrm{I}}}{\mathrm{G}_{\mathrm{I}}}(\cos \alpha-1)^{2}\right\} \tag{71}
\end{equation*}
$$

The coupling constant $G_{I}$ was then determined by using the above ansatz and could be evaluated by comparing the numerical value of the computed function $\mathrm{V}(\mathrm{I}, \cos \alpha)$ at three lattice points. The ansatz is exact for the weak and strong coupling limits, and is a natural interpolation for the intermediate region (where neither the weak nor strong coupling approximations are valid). For the
intermediate region, we evaluated $\mathrm{G}_{\mathrm{I}}$ for a number of adjacent points, and found that the variation was $\sim 10 \%$. We took the coupling constant $G_{I}$ to be the one evaluated using $V(I, 2), V(I, 3)$ and $V(I, 4)$.

The numerical accuracy of the computer program was checked with the weak and strong coupling analytic results. The program was accurate, for each iteration, to $1 \%$ when calculations were performed with $\sim 15$ lattice points.

The computer program produces the sequence of functions $V(I, \sigma)$ from which $G_{I}$ can be computed. Table I is a typical result. In Fig. 8 these results are plotted graphically.

Numerical Results
(I) For $\mathrm{d} \leq 4$, there are only two fixed points for the recursion equations, i.e., $G^{*}=0$ and $G^{*}=\infty$; there are no other fixed points. The $G^{*}=0$ fixed point is unstable for $\mathrm{d} \leq 4$ and the $\mathrm{G}^{*}=\infty$ fixed point is stable for $\mathrm{d} \leq 8$.
(II) The sequence of coupling constants for different initial values of $\mathrm{G}_{1}$ lie on the same trajectory; changing the value of $G_{1}$ simply shifts the sequence of coupling constants along the trajectory on which the other sequences lie. The renormalization group trajectory for the sequence of coupling constants $\left\{\mathrm{G}_{\mathrm{I}}\right\}_{\mathrm{I}=0}^{\infty}$ has three distinct regions, namely the weak coupling regime, the strong coupling regime and the intermediate region (where neither the weak or strong coupling approximations are valid). These three are smoothly connected and can be identified only qualitatively. By convention, we identify the intermediate region to be such that the deviation from the weak and strong approximations is $>10 \%$. We then numerically found that, for $d=4$ (the numbers depend on $d$ )
(a) Weak coupling regime is for $0 \leq G_{I} \leq 0.3$
(b) Intermediate regime is for $0.3 \leq G_{I} \leq 1.5$
(c) Strong coupling regime is for $1.5 \leq \mathrm{G}_{\mathrm{I}} \leq \infty$.
(III) The sequence of effective actions are monotonically increasing functions_(recall we normalized the effective actions such that $V(I, 1) \equiv 1$ ); that is $\mathrm{V}(\mathrm{I}+1, \alpha) \geq \mathrm{V}(\mathrm{I}, \alpha)$. Since $\mathrm{V}(\mathrm{I}=\infty, \alpha)=\mathrm{V}^{*}(\alpha)=1$, we see that the sequence of effective actions converge uniformly to the stable fixed point action. In fact, we found that to $\sim 15 \%$ accuracy

$$
\mathrm{V}(\mathrm{I}, \alpha)=\exp \left\{\frac{1}{\mathrm{G}_{\mathrm{I}}}(\cos \alpha-1)\right\}
$$

for all I .
(IV) The fixed point reached by the numerical calculation depends on the total range of integration used for the intermediate and strong coupling regimes. The exact theory gives $0 \leq \sigma \leq \pi$. In the numerical calculation, if the range $0 \leq \sigma \leq \mathrm{a}$ ( $\mathrm{a}>\pi$ or $\mathrm{a}<\pi$ ) was used, then a fixed point was reached which was different from $\mathrm{V}^{*}(\sigma)=1$. This is the reason why $\mathrm{S}(1)$ had to have the form $\pi / 2^{\mathrm{n}}$; because after doubling the lattice spacing $n$ times, we end up with $S(I)=\pi$. And calculations on this $S(I)$ lead to the correct fixed point. The range $0 \leq \sigma \leq \pi$ means that each group element is covered once and only once in the integration; and we see that the strong coupling fixed point is very sensitive to the entire structure of the non-Abelian gauge group. This fact is also apparent in the exact lattice gauge theory, where the strong coupling expansion involves the entire gauge group.

## 5. Conclusions

Migdal's recursion formula is, in essence, a scheme for coupling constant renormalization. The physical interpretation of $G_{I}$ is that it is strength of the gauge field felt by particles which couple to it, say quarks, when these quarks are separated by a distance of $2^{I} a_{0}$ ( $a_{0}=$ original spacetime lattice spacing). For the sake of discussion, let $\mathrm{G}_{1} \simeq 10^{-3}$ for $\mathrm{a}_{0} \simeq 10^{-16} \mathrm{~cm}$. (Asymptotic
freedom tells us that $G_{1} \rightarrow 0$ as $a_{0} \rightarrow 0$.) Then the recursion equation shows that as we separate the quarks to larger distances, the strength of gauge field increases continuously. The fact that the model has an unstable weak coupling fixed point and a stable strong coupling fixed point (for $d=4$ ) implies that the quarks behave almost like free particles at short distances, and become strongly coupled at large distances. The absence of any other fixed points shows that the weak and strong coupling behavior of the gauge field is not separated by any phase transition, and the quarks go continuously from their short distance weak coupling behavior to their strongly coupled behavior (which gives rise to the bound states of the quarks). Also, since the coupling constant becomes arbitrarily large for an arbitrarily large distance, we see that the quark-antiquark separate to a definite distance, after which the quark/gauge field system produces pions since this becomes energetically more favorable than any further separation of the quark-antiquark in question. This explains why we cannot separate quarks in a bound state to arbitrary, macroscopic distances.

Of course, in the above discussion, we have assumed that the qualitative behavior of the pure gauge field is not destroyed when the quark field is coupled to it. Given this assumption, we see that Migdal's model provides a theoretical example of our physical ideas of quark confinement, and gives a simple example of an asymptotically free field going over to a strongly coupled system. As is well known, this rather remarkable behavior is peculiar to non-Abelian gauge fields and is shown by no other known quantum field.

## APPENDIX A

Here we compute

$$
\begin{align*}
\operatorname{VV}(I, \tau)= & \int_{0}^{\pi} \sin ^{2} u d u e^{2(\cos \sigma \cos u-1) / G} \\
& \times\left[1+\frac{4 H}{G}(\cos u \cos \sigma-1)^{2}+\frac{4 \mathrm{H}}{3 \mathrm{G}} \sin ^{2} u \sin ^{2} \sigma+0\left(\mathrm{G}^{2}\right)\right] \tag{A.1}
\end{align*}
$$

This computation will give the result stated in (46). (Recall $\sigma=\tau / 2$.) Let

$$
\begin{align*}
\mathrm{a} & =\cos \sigma / \mathrm{G} \\
& : 0(1 / \mathrm{G}) \tag{A.2}
\end{align*}
$$

Since $\sigma, u=0(\sqrt{G})$ we have
(i) $2(\cos \sigma \cos u-1) / \mathrm{G}=2(\cos \sigma-1) / \mathrm{G}-\mathrm{au}^{2}+($ const. $) \mathrm{G}+0\left(\mathrm{G}^{2}\right)$

$$
\begin{align*}
\sin ^{2} \mathrm{u} & =\mathrm{u}^{2}+(\text { const. }) \mathrm{G}+0\left(\mathrm{G}^{3}\right)  \tag{A.3}\\
& =\mathrm{u}^{2}\left[1+0\left(\mathrm{G}^{2}\right)\right] \tag{ii}
\end{align*}
$$

(iii)

$$
\begin{align*}
(\cos u \cos \sigma-1)^{2} & =(\cos \sigma-1)^{2}-(\cos \sigma-1) u^{2}+0\left(G^{3}\right) \\
\sin ^{2} \sigma \sin ^{2} u & =\left(\sin ^{2} \sigma\right) u^{2}+0\left(G^{3}\right) \tag{iv}
\end{align*}
$$

Let

$$
\begin{align*}
& P=1+\frac{4 I I}{G}(\cos \sigma-1)^{2}  \tag{A.4}\\
& Q=\frac{4 H}{G}\left\{\frac{1}{3} \sin ^{2} \sigma-(\cos \sigma-1)\right\} \tag{A.5}
\end{align*}
$$

Then, using (A.1)-(A.5) gives

$$
\begin{align*}
\operatorname{VV}(\mathrm{I}, \tau)= & \mathrm{e}^{2(\cos \sigma-1) / \mathrm{G}} \int_{0}^{\pi} \mathrm{du} \mathrm{u}^{2} \mathrm{e}^{-\mathrm{au}^{2}}\left[\mathrm{P}+\mathrm{Qu}^{2}\right]  \tag{A.6}\\
\simeq & \mathrm{e}^{2(\cos \sigma-1) \mathrm{G}} \frac{1}{2} \int_{-\infty}^{+\infty} \mathrm{du} \mathrm{u}^{2} \mathrm{e}^{-\mathrm{au}^{2}}\left[\mathrm{P}+\mathrm{Qu}^{2}\right] \\
= & \text { (const.) } \frac{\mathrm{e}^{2(\cos \sigma-1) / \mathrm{G}}}{\cos ^{3 / 2} \sigma}\left[1+\frac{4 \mathrm{H}}{\mathrm{G}}(\cos \sigma-1)^{2}\right. \\
& \left.+6 \mathrm{H}\left\{\frac{1}{3} \sin ^{2} \sigma-(\cos \sigma-1)\right\}+0\left(\mathrm{G}^{2}\right)\right] \tag{A.7}
\end{align*}
$$

But

$$
V V(I, \tau)=\exp \{C(\tau)\}
$$

Therefore, up to a constant

$$
\begin{array}{r}
\mathrm{C}(\tau)=2(\cos \sigma-1) / \mathrm{G}-\frac{3}{2} \ln (\cos \sigma)+\frac{4 \mathrm{H}}{\mathrm{G}}(\cos \sigma-1)^{2} \\
+6 \mathrm{H}\left\{\frac{1}{3} \sin ^{2} \sigma-(\cos \sigma-1)\right\}+0\left(\mathrm{G}^{2}\right) \tag{A.8}
\end{array}
$$

Recall

$$
\begin{align*}
\sigma & =\tau / 2=0(\sqrt{\mathrm{G}}) \\
\therefore \quad \sin ^{2} v & =-\frac{1}{2}(\cos \tau-1)  \tag{A.9}\\
\cos \sigma & =\sqrt{1+\frac{1}{2}(\cos \tau-1)} \tag{A.10}
\end{align*}
$$

$$
\begin{equation*}
\ln \cos \sigma=\frac{1}{4}(\cos \tau-1)+0\left(\mathrm{G}^{2}\right) \tag{A.11}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
\mathrm{C}(\tau)=\left[\frac{1}{2 \mathrm{G}}-\frac{3}{8}-\frac{5}{2} \mathrm{H}\right](\cos \tau-1)+\frac{1}{4 \mathrm{G}}\left[\mathrm{H}-\frac{1}{4}\right](\cos \tau-1)^{2} \tag{A.12}
\end{equation*}
$$

But

$$
\mathrm{C}(\tau)=\frac{1}{\mathrm{G}^{\prime}}(\cos \tau-1)+\frac{2 \mathrm{H}^{\prime}}{\mathrm{G}^{\prime}}(\cos \tau-1)^{2}
$$

Therefore we have

$$
\begin{align*}
& \frac{1}{\mathrm{G}^{\prime}}=\frac{1}{2 \mathrm{G}}-\frac{3}{8}-\frac{5}{2} \mathrm{H} \\
& 2 \mathrm{H}^{\prime} / \mathrm{G}^{\prime}=\left(\mathrm{H}-\frac{1}{4}\right) / 4 \mathrm{G} \tag{A.13}
\end{align*}
$$

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TABLE I
Coupling Constant Renormalization in Migdal's Model

$$
\begin{array}{ll}
G_{1}=.015, & H_{1}=H^{*}=-1 / 12, \quad d=2 \\
N \sim 16 & G_{I+1}^{w k}=G_{I}^{w k} 1+G_{I}^{w k}
\end{array}
$$

| $\mathrm{I}=$ No. Iterations | $\mathrm{G}_{\mathrm{I}}$ | $\mathrm{G}_{\mathrm{I}}^{\mathrm{wk} / \mathrm{G}_{\mathrm{I}}}$ |
| :---: | :---: | :---: |
| 1 | .01500 | 1.0000 |
| 2 | .0152 | 1.0091 |
| 3 | .0154 | 1.0098 |
| 4 | .0157 | 1.0098 |
| 5 | .0159 | 1.0097 |
| 6 | .0162 | 1.0097 |
| 7 | .0165 | 1.0099 |
| 8 | .0167 | 1.0106 |
| 9 | .0170 | 1.0106 |
| 10 | .0173 | 1.0105 |
| 11 | .0176 | 1.0105 |
| 12 | .0179 | 1.0108 |
| 13 | .0183 | 1.0116 |
| 14 | .0185 | 1.0116 |
| 15 | .0189 | 1.0116 |
| 16 | .0193 | 1.0115 |
| 17 | .0197 | 1.0114 |
| 18 | .0201 | 1.0120 |
| 19 | .0205 | 1.0119 |
| 20 | .0209 | 1.0117 |
| 21 | .0214 | 1.0115 |
| 23 | .0218 | 1.0126 |
| 25 | .0223 | 1.0131 |
|  |  |  |
| 2 |  |  |


| I | $\mathrm{G}_{\text {I }}$ | $\mathrm{G}_{\mathrm{I}}^{\mathrm{wk}} / \mathrm{G}_{\mathrm{I}}$ |
| :---: | :---: | :---: |
| 26 | . 0239 | 1.0125 |
| 27 | . 0245 | 1.0068 |
| 28 | . 0252 | 1.0108 |
| 29 | . 0258 | 1.0131 |
| 30 | . 0264 | 1.0152 |
| 31 | . 0271 | 1.0172 |
| 32 | . 0278 | 1.0095 |
| 33 | . 0288 | 1.0128 |
| 34 | . 0295 | 1.0144 |
| 35 | . 0304 | 1.0169 |
| 36 | . 0323 | 1.0142 |
| 37 | . 0335 | 1.0158 |
| 38 | . 0347 | 1.0151 |
| 39 | . 0359 | 1.0146 |
| 40 | . 0373 | 1.0139 |
| 41 | . 0387 | 1.0180 |
| 42 | . 0404 | 1.0192 |
| 43 | . 0421 | 1.0185 |
| 44 | . 0439 | 1.0173 |
| 45 | . 0459 | 1.0159 |
| 46 | . 0482 | 1.0143 |
| 47 | . 0506 | 1.0124 |
| 48 | . 0533 | 1.0265 |
| 49 | . 0564 | 1.0265 |
| 50 | . 0598 | 1.0241 |
| 51 | . 0637 | 1.0210 |
| 52 | . 0680 | 1.0174 |
| 53 | . 0731 | 1.0326 |
| 54 | . 0791 | 1.0305 |
| 55 | . 0860 | 1.0248 |
| 56 | . 0943 | 1.0171 |
| 57 | . 1044 | 1. 0070 |


| I | $\mathrm{G}_{\mathrm{I}}$ | $\mathrm{G}_{\mathrm{I}}^{\mathrm{Wk}} / \mathrm{G}_{\mathrm{I}}$ |
| :---: | ---: | :---: |
| 58 | .1176 | 0.9908 |
| 59 | .1337 | 0.9731 |
| 60 | .1550 | 1.0403 |
| 61 | .1882 | 1.0181 |
| 62 | .2342 | 0.9740 |
| 63 | . .3113 | 0.8978 |
| 64 | 1.0432 | 0.7391 |
| 65 | 11.4017 | 0.4734 |
| 66 | 251 | 218.1458 |

(All the numbers above have been rounded off)

In the graph of $G_{I}$ vs. $I$, we've plotted the figures of the table starting from $\mathrm{I}=39$ (see Fig. 9).

Table I was calculated by making the program run six steps at a time. The second last coupling constant was then used as input for the next six steps.

FIGURE CAPTIONS

1. Block-spin construction for Migdal's model. Four old lattice points are combined to form a new lattice point; circled sites form the new effective lattice.
2. Block-spin construction for a single new block-spin variable at the new lattice site (which is circled).
3. The effective fundamental square for the gauge field after $\ell$ iterations; the square has side $2 l a_{0}$, where $a_{0}$ is the original lattice spacings.
4. Link picture for forming a 2L-square from four L-squares.
5. Configuration of squares for the exact lattice gauge theory.
6. Approximation of the exact theory made by Migdal.
7. The planar contours formed by combining $2^{3}$ L-cubes to form a 2 L -cube.
8. Graph of coupling constant renormalization trajectory, with $G_{I}$ vs. I, where $I=$ number of iterations, and $G_{I}$ is the effective coupling constant for the gauge field.


Fig. 1


Fig. 2


Fig. 3


Fig. 4


Fig. 5


Fig. 6


Fig 7


Fig. 8


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