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INSTANTONS WITH FRACTIONAL TOPOLOGICAL CHARGE*

Lay-Nam Chang†

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

Ngee-Pong Chang

Physics Department
City College of the City, University of New York
New York, New York 10031

ABSTRACT

We introduce and discuss the properties of multivalent gauges for Yang-Mills fields, and demonstrate the existence of instantons with fractional topological charges in such gauges.

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†On leave of absence from the University of Pennsylvania, Philadelphia, Pennsylvania.

The study of Yang-Mills gauge fields has so far been carried out in univalent gauges. Under a rotation in physical space by 2π, the field strengths return to their original value. There exist, however, multivalent gauges for non-Abelian groups where the field strengths do not return to their original value but are, of course, gauge equivalent to their original value; i.e., for the field strength matrix, $f_{\mu\nu}$,

$$f_{\mu\nu}(\phi + 2\pi) = \Omega_0(\phi)\, f_{\mu\nu}(\phi)\, \Omega_0^\dagger(\phi) \tag{1}$$

where $\Omega_0(\phi) \in G$, the gauge group, but is not the identity. For an Abelian group, such multivalent gauges are physically meaningless, since the $f_{\mu\nu}$ are gauge-invariant and directly measurable. For non-Abelian groups since the $f_{\mu\nu}$ are only gauge-covariant, such multivalence is, in principle, admissible. The action density remains single-valued for these gauges.

In view of the existence of these new gauges, we must now distinguish between invariance under single and multiple-valued gauge transformations. The action, energy, and topological charge densities are all generally gauge invariant. However, the closed loop integral

$$\Phi_c = \mathrm{Tr}\left(\exp\left(\frac{1}{2} \oint b_\mu \, dx_\mu\right)\right) \tag{2}$$

is only invariant under single-valued gauge transformations. In (2), an ordering of the matrix b_μ is implied. In a quantum theory, beyond the tree level, the allowed gauge transformations which leave the S-matrix invariant are the single-valued gauge transformations. But, as is evident in the functional integral formalism, the integral over b_μ must include, besides the univalent gauges of different homotopy classes, these multivalent gauges as well. They would contribute to the structure of the vacuum.

In this paper we present a remarkable feature of such multivalent gauges. They give rise to instantons of fractional topological charge. We shall exhibit this for the SU(2) group.

Consider the generating matrix, S [1)]

$$S = \cos\chi + i \sin\chi \cos\theta\, \sigma_3 + i \sin\chi \sin\theta(\sigma_1 \cos n\phi + \sigma_2 \sin n\phi) \tag{3}$$

where, in Euclidean space, $x_2 = R \cos\chi$, $x_3 = R \sin\chi \cos\theta$, $x_2 = R \sin\chi \sin\theta \sin\phi$, $x_1 = x_2 \cot\phi$. For $n \neq 1, 2, 3, \ldots$ S generates a multi-valued gauge

$$b_\mu = 2f\, \partial_\mu S \cdot S^\dagger \quad . \tag{4}$$

Suppose that f is a function of R only, with $f \to 1$ as $R \to \infty$, and $f \to 0$ as $R \to 0$. If f has no singularities in between these limits, the Chern-Pontrygin number, q, is given by

$$q = \frac{1}{128\pi^2} \int d^4x\, \mathrm{Tr}(f_{\mu\nu} f_{\lambda\rho})\, \epsilon^{\mu\nu\lambda\rho} = n \quad . \tag{5}$$

For n equal to a fraction, S generates a fractional topological charge. The result is independent of whether (4) satisfied the Yang-Mills field equation. [2)]

The field strengths associated with (4) are not invariant under a rotation in ϕ by 2π, but satisfies eq. (1) with

$$\Omega_0(\phi) = \cos \pi n - i\sigma_3 \sin \pi n \qquad n \neq 1, 2, 3, \ldots \tag{6}$$

It is convenient for our calculation to perform a multiple-valued gauge transformation to a univalent gauge. Of course this does not change the topological charge nor the Yang-Mills field equations.

The field strengths obtained from (4) are not self-dual with respect to a flat Euclidean metric. They are instead anti-self-dual with respect to the metric

$$g_{RR} = 1 \qquad g_{\chi\chi} = \frac{R^2}{n} \qquad g_{\theta\theta} = \frac{R^2 \sin^2\chi}{n}$$

$$g_{\phi\phi} = nR^2 \sin^2\chi \ \sin^2\theta \tag{7}$$

if f is given by

$$f = \frac{R^\gamma}{\lambda^\gamma + R^\gamma} \qquad \gamma = 2\sqrt{n} \tag{8}$$

and λ is a scale parameter.

The field strengths in the univalent gauge are given by the tensorial components

$$\begin{aligned}
f_{\chi\theta} &= -\,4i\,\sigma_3 \sin^2\chi \ \sin\theta \ f(1-f) \\
&\quad + 4i \sin^2\chi \ \cos\theta \ f(1-f) \left[\sigma_1 \cos\phi + \sigma_2 \sin\phi\right] \\
&\quad - 4i \sin\chi \ \cos\chi \ f(1-f) \left[\sigma_1 \sin\phi - \sigma_2 \cos\phi\right] \\
f_{\chi\phi} &= +\,4i\,\sigma_3 \sin\chi \ \cos\chi \ \sin^2\theta \cdot nf(1-f) \\
&\quad - 4i\, nf(1-f) \sin\chi \ \cos\chi \ \sin\theta \ \cos\theta \ (\sigma_1 \cos\phi + \sigma_2 \sin\phi) \\
&\quad - 4i\, nf(1-f) \sin^2\chi \ \sin\theta \ (\sigma_1 \sin\phi - \sigma_2 \cos\phi) \\
f_{\theta\phi} &= 4i\, nf(1-f) \sin^2\chi \ \sin\theta \ \cos\theta \ \sigma_3 \\
&\quad + 4i\, nf(1-f) \sin^2\chi \ \sin^2\theta \ (\sigma_1 \cos\phi + \sigma_2 \sin\phi)
\end{aligned} \tag{9}$$

with the other three components simply given by

$$\begin{aligned}
f_{\mu\nu} &= -\frac{1}{2}\,\eta_{\mu\nu\lambda\rho}\, f^{\lambda\rho} \\
\eta_{\mu\nu\lambda\rho} &= \sqrt{g}\ \epsilon_{\mu\nu\lambda\rho} \quad .
\end{aligned} \tag{10}$$

Since the field strengths are anti-self-dual, the gauge is sourceless.

The space has negative curvature with the Riemann curvature tensor given by

$$R^\chi{}_{\theta\chi\theta} = \frac{n-1}{n} \sin^2\chi$$

$$R^{\chi}{}_{\phi\chi\phi} = n(n-1) \sin^2\chi \sin^2\theta$$

$$R^{\theta}{}_{\phi\theta\phi} = n(n-1) \sin^2\chi \sin^2\theta$$

and

$$R_{\theta\theta} = \frac{2(n-1)}{R^2} g_{\theta\theta}$$

$$R_{\chi\chi} = \frac{2(n-1)}{R^2} g_{\chi\chi}$$

$$R_{\phi\phi} = \frac{2(n-1)}{R^2} g_{\phi\phi} \qquad (11)$$

This space is not conformally flat. It also has a vanishing Euler characteristic, and zero topological index.

The fractional Chern-Pontrygin number may come as a surprise. However, the mapping defined by S is not from S_3 to S_3, but to a Riemannian manifold and the Chern number which is not necessarily integral [2]. Our solution f in the complex C_4 space of Yang [3] is not everywhere holomorphic.

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Footnotes

1) This parametrization of SU(2) has been considered previously in ref. [1]. However, no attempt was made there to minimize the Yang-Mills action density.

2) Non-integral values of q can also be obtained by modifying the angular dependence of S on θ and χ. But the resulting field strengths are neither self-dual nor anti-self-dual with respect to any non-singular metric.

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