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NEW EQUATIONS FOR FOUR-BODY SCATTERING*

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ABSTRACT

We present new equations for four-body scattering, obtained by generalizing our three-body formalism to the four-body case. These equations, although equivalent to those of Faddeev-Yakubovskii, are expressed in terms of singularity-free physical transition amplitudes, and their energy-independent effective potentials require only half-on-shell subsystem transition amplitudes (and bound state wavefunctions) as input. However, due to the detailed index structure of the Faddeev-Yakubovskii formalism, the result of our generalization is considerably more complicated than in the three-body case.

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I. INTRODUCTION

The treatments of the four-body problem that exist in the literature provide a variety of solutions to the problem of finding appropriate four-body scattering equations. Of those obtained by generalizing Faddeev's three-body theory, [1] the approach due to Yakubovskii[2] is the most well-established, in particular because its equivalence with the Schrœdinger equation has been demonstrated. The most characteristic feature of this formalism, and also its main weakness, is its very detailed classification of the clustering properties of the four-body system.

In some alternative approaches (such as that due to Sloan[3]), a less detailed index structure is considered, for instance using only a two-cluster classification of the four particles. As compared to Yakubovskii's, the resulting equations exhibit in general a more complicated structure, and their connection with the Schrœdinger equation remains unclear.

A common feature of all these formalisms is that they have been developed almost exclusively at the formal operator level: the actual complexity involved (such as the singularity structure of the considered entities) is therefore not explicitly shown.

In the present work, we seek to establish a four-body formalism based on the Faddeev-Yakubovskii (FY) theory in a way that makes the actual structure of the formalism more evident. For this purpose we follow a method suggested by our previously developed three-body formalism, [4] in which a thorough singularity analysis of the Faddeev kernel led us to singularity-free physical amplitudes that obey dynamical equations with a considerably simplified input.

With these results in mind, we carry out a similar singularity analysis of the four-body kernel. As in the three-body case, this task is considerably simplified by using the complete sets of eigenstates of the channel Hamiltonians. The analysis turns out to be particularly straightforward for FY entities labeled by two-cluster indices only—such as the wavefunction components $\Psi^{\sigma} = \sum_{\beta} \Psi^{\sigma}_{\beta}$, Ψ^{σ}_{β} being the conventional four-body FY component—and leads very naturally to new singularity-free amplitudes components for four-body scattering.

In order to obtain equations for such amplitudes, however, the FY formalism requires that we also analyze the wavefunction component Ψ^{σ}_{β} itself; i.e., it requires that the singularity analysis be made taking into account the full index structure of the formalism. Unfortunately, this more detailed analysis turns out to be less straightforward than the first; in addition to the physical transition amplitudes, we are forced to introduce a nonphysical amplitude which, although not present in the full four-body wavefunction, still appears in the dynamical equations.

Nevertheless, the set of equations we are led to exhibit essentially the same features as our corresponding three-body equations: namely, a multichannel Lippmann-Schwinger structure with energy-independent effective potentials that require a simplified subsystem input (i.e., only half-on-shell subsystem scattering amplitudes and bound state wavefunctions).

In Section II we review the main techniques and results of our three-body formalism. In Section III we introduce the four-body notation that will be used throughout the paper, some basic aspects of the FY formalism, and the appropriate complete sets of the channel Hamiltonians. The singularity analysis of the component Ψ^{σ} is carried out in Section IV, where the physical scattering amplitudes are identified.

The fully-split FY components Ψ^{σ}_{β} are analyzed in Section V, and the equations that the scattering amplitude components satisfy are obtained in Section VI.

Finally in Section VII, we generalize our formalism to the fully-off-shell case and connect our amplitudes to the operator formalism.

In the Appendix we confirm that our amplitudes are indeed components of the physical scattering amplitudes.

II. THE THREE-BODY CASE: A REVIEW

The main feature of our three-body formalism[4] is the analysis of the singularity structure of the kernel $G_0 t_\beta \bar{\delta}_{\beta\gamma}$ of the Faddeev equations

$$|\Psi^+_{\beta(\alpha)}\rangle = \delta_{\beta\alpha} |\vec{p}^{(0)}_\alpha \, \phi^\alpha_\kappa\rangle - G_0(E+i0)\, t_\beta(E+i0) \sum_{\gamma\neq\beta} |\Psi^+_{\gamma(\alpha)}\rangle \quad . \tag{2.1}$$

In (2.1), $|\Psi^+_{\beta(\alpha)}\rangle$ is the Faddeev component of the three-body wavefunction corresponding to an initial state $|\vec{p}^{(0)}_\alpha \, \phi^\alpha_\kappa\rangle$ of a bound pair in channel α and a third free particle; $E = \tilde{p}^{(0)2}_\alpha - \kappa^2_\alpha$ is the total energy, $t_\beta(E+i0)$ is the two-body transition operator in channel β, $G_0(E+i0) = (\tilde{p}^2+\tilde{q}^2-E-i0)^{-1}$, $\bar{\delta}_{\beta\gamma} = 1-\delta_{\beta\gamma}$, and $\tilde{p}^2_\alpha = p^2_\alpha/2\eta_\alpha$, $\tilde{q}^2_\alpha = q^2_\alpha/2\mu_\alpha$, with $\eta_\alpha = \left[m_\alpha(m_\beta+m_\gamma)\right]/(m_\alpha+m_\beta+m_\gamma)$ and $\mu_\alpha = (m_\beta m_\gamma)/(m_\beta+m_\gamma)$.

Since $G_0 t_\beta = G_\beta V_\beta$, where $G_\beta = (\tilde{p}^2+\tilde{q}^2+V_\beta-E-i0)^{-1}$, the singularity structure of the Faddeev kernel is best exposed using the spectral decomposition of G_β, or equivalently, by considering projections onto channel eigenstates, i.e., onto the complete set of eigenstates $\left\{|\vec{p}_\beta \phi^\beta_\kappa\rangle, |\vec{p}_\beta \psi^-_{\vec{q}_\beta}\rangle\right\}$ of the channel Hamiltonian $H_\beta = \tilde{p}^2+\tilde{q}^2+V_\beta$.[5]

In this way, we obtained for the three-body wavefunction components the representation[4]

$$\langle\vec{p}_\beta\vec{q}_\beta|\Psi^+_{\beta(\alpha)}\rangle = \delta_{\beta\alpha}\,\delta(\vec{p}_\alpha-\vec{p}^{(0)}_\alpha)\,\phi^\alpha_\kappa(\vec{q}_\alpha) - \frac{\phi^\beta_\kappa(\vec{q}_\beta)}{\tilde{p}^2_\beta-\kappa^2_\beta-E-i0}\,\mathscr{H}_{\beta\alpha}(\vec{p}_\beta;\vec{p}^{(0)}_\alpha;E+i0)$$

$$-\int d\vec{q}\,'_\beta\,\psi^-_{\vec{q}\,'_\beta}(\vec{q}_\beta)\,\frac{1}{\tilde{p}^2_\beta+\tilde{q}\,'^2_\beta-E-i0}\,\mathscr{E}_{\beta\alpha}(\vec{p}_\beta\vec{q}\,'_\beta;\vec{p}^{(0)}_\alpha;E+i0) \tag{2.2}$$

where the elastic/rearrangement and breakup poles occur explicitly in separate terms, and $\mathscr{H}_{\beta\alpha}$ and $\sum_\beta \mathscr{E}_{\beta\alpha}$ are the corresponding physical scattering amplitudes. It is important to note that these amplitudes are free from primary singularities.[6] We then proceeded to show that, when expressed in terms of $\mathscr{H}_{\beta\alpha}$ and $\mathscr{E}_{\beta\alpha}$, the three-body equations (2.1) take a particularly simple form: they become coupled multichannel Lippmann-Schwinger-type equations, whose "effective potentials" are energy independent, and require only half-on-shell two-body input (in contrast, in all previous exact formulations of the three-body problem such effective potentials are energy-dependent and require fully-off-shell two-body input).

III. THE FOUR-BODY CASE: PRELIMINARY CONSIDERATIONS

For our treatment of four-body scattering we make use of the FY equations for the FY wavefunction components,[7,8] i.e.,

$$|\Psi_\beta^{\sigma(\tau)}\rangle = \delta^{\sigma\tau}\,|\Phi_\beta^{(\tau)}\rangle - \sum_{\gamma\subset\sigma} G_0(E+i0)\,K_{\beta\gamma}^{\sigma}(E+i0)\sum_{\rho\supset\gamma}\bar{\delta}^{\sigma\rho}\,|\Psi_\gamma^{\rho(\tau)}\rangle \quad . \tag{3.1}$$

The wavefunction components are labeled <u>both</u> by two-cluster indices σ, ρ, τ, etc. (i.e., of the type (123)(4) or (12)(34)), and by three-cluster indices α, β, γ, etc. (of the type (12)(3)(4), i.e., pair indices). The decomposition is such that $\sum_\sigma \sum_{\beta\subset\sigma} \Psi_\beta^{\sigma(\tau)}$ is the full four-body wavefunction. A three-cluster index below a two-cluster index (as in $\Psi_\beta^{\sigma(\tau)}$) indicates that the three clusters have been obtained by further splitting one of the two clusters (as in $\sigma = (123)(4) \rightarrow (12)(3)(4) = \beta$).[9] This is also described by writing $\beta\subset\sigma$.

In Eq. (3.1), $\Phi_\beta^{(\tau)}$ denotes the β-component of the initial state wavefunction;[10] the operator $K_{\beta\gamma}^{\sigma}$ is the three-body kernel operator of subsystem σ (more precisely, it is the two-cluster subsystem kernel operator, since σ can be either of the 3+1 or the 2+2 type), defined as

$$K^{\sigma}_{\beta\gamma} = \sum_{\lambda\subset\sigma} \left\{ V_{\beta}\delta_{\beta\lambda} - V_{\beta} G^{\sigma} V_{\lambda} \right\} \bar{\delta}_{\lambda\gamma} \ , \tag{3.2}$$

where $G^{\sigma} = (H^{\sigma} - E - i0)^{-1} = \left(H_0 + \sum_{\gamma\subset\sigma} V_{\gamma} - E - i0 \right)^{-1}$.

In order to proceed with our treatment of the four-body case, we need to define the appropriate complete sets of eigenstates of the channel Hamiltonians H^{σ}. For σ of the 3+1 type, the complete set of eigenstates of the three-body Hamiltonian $\tilde{p}^2 + \tilde{q}^2 + \sum_{\gamma\subset\sigma} V_{\gamma}$ is given by Faddeev[11] as being,

$$\left\{ |\Phi\rangle, \ |\Psi^{\pm}_{(\delta)\vec{p}}\rangle \ , \ |\Psi^{\pm}_{\vec{p}\,\vec{q}}\rangle \right\} \ , \qquad \text{all } \delta\subset\sigma \tag{3.3}$$

where $|\Phi\rangle$ is a three-body bound state (we only consider one three-body bound state per channel) of energy $-\kappa^2_{\sigma}$; $|\Psi^{+}_{(\delta)\vec{p}}\rangle$ is the (outgoing wave) scattering state corresponding to an initial state of a bound pair δ and a third free particle with relative momentum $\vec{p}$, and $|\Psi^{+}_{\vec{p}\,\vec{q}}\rangle$ is the (outgoing wave) scattering state corresponding to an initial state of three free particles of relative momenta $\vec{p}, \vec{q}$.

Therefore, in the 3+1 case, the complete set of four-body channel eigenstates can be written as

$$\left\{ |\vec{r}\,\Phi^{(\sigma)}\rangle \ , \ |\vec{r}\Psi^{(\sigma)\pm}_{(\delta)\vec{p}}\rangle \ , \ |\vec{r}\,\Psi^{(\sigma)\pm}_{\vec{p}\,\vec{q}}\rangle \right\} \ , \qquad \text{all } \delta\subset\sigma \tag{3.4}$$

where if, say, $\sigma = (123)(4)$, $\vec{r}_{\sigma}$ is the momentum of the fourth particle relative to the center-of-mass of the other three. (Note that we suppress the channel indices of all variables.)

On the other hand, if σ is of the 2+2 type, the complete set of channel eigenstates is given by

$$\left\{ |\vec{s}\,\Phi^{(\sigma)}\rangle \ , \ |\vec{s}\,\Psi^{(\sigma)\pm}_{(\delta)\vec{q}}\rangle \ , \ |\vec{s}\,\Psi^{(\sigma)\pm}_{\vec{q}\,\vec{q}'}\rangle \right\} \ . \tag{3.5}$$

In (3.5), if we let δ, γ label the two subsystems of σ (i.e., if $\sigma = (12)(34)$ and $\delta = (12)$, then $\gamma = (34)$), $|\vec{s}\,\Phi^{(\sigma)}\rangle = |\vec{s}\,\phi^{\delta}_{\kappa}\phi^{\gamma}_{\kappa}\rangle$ represents a state of two bound pairs moving with relative momentum $\vec{s}$ and corresponding to a total energy $E = \tilde{s}^2 - \kappa^2_{\delta} - \kappa^2_{\gamma}$, where $\tilde{s}^2_{\sigma} = s^2_{\sigma}/2\eta_{\sigma}$, with $\eta_{\sigma} = [(m_1+m_2)(m_3+m_4)]/(m_1+m_2+m_3+m_4)$ if $\sigma = (12)(34)$. Similarly, $|\vec{s}\,\Psi^{(\sigma)\pm}_{(\delta)\vec{q}}\rangle = |\vec{s}\,\phi^{\delta}_{\kappa}\,\psi^{\pm}_{\vec{q}_{\gamma}}\rangle$ represents a state where the δ-pair is bound, while the γ-pair is in a scattering state of initial momentum $\vec{q}_{\gamma}$, and so forth.

In what follows, we will in general not treat the two kinds of indices σ separately, but use only the set (3.4), with the understanding that when σ is of the 2+2 type, the labels $\vec{r}, \vec{p}, \vec{q}$ of (3.4) should be replaced by the labels $\vec{s}, \vec{q}, \vec{q}\,'$ of (3.5).

IV. SINGULARITY ANALYSIS OF $\Psi^{\sigma(\tau)}$: THE SCATTERING AMPLITUDES

The most natural generalization of our three-body formalism would be to consider four-body wavefunction components labeled only by a two-cluster index σ. As we have seen in Eq. (3.1), however, the FY components $\Psi^{\sigma(\tau)}_{\beta}$ represent a more detailed splitting of the full wavefunction, since in them not only the last interacting subsystem is specified (labeled by σ), but also the last interacting pair (within the subsystem labeled by σ).

Therefore, we first consider the singularity structure of the "partially summed" wavefunction component $\Psi^{\sigma(\tau)} = \sum_{\beta\subset\sigma} \Psi^{\sigma(\tau)}_{\beta}$. Using Eq. (3.1), we find

$$|\Psi^{\sigma(\tau)}\rangle = \delta^{\sigma\tau}\,|\vec{r}^{(0)}\,\Phi^{(\tau)}\rangle - G^{\sigma}(E+i0)\sum_{\gamma\subset\sigma}\sum_{\rho\supset\gamma}\bar{V}^{(\sigma)}_{\gamma}\,\bar{\delta}^{\sigma\rho}\,|\Psi^{\rho(\tau)}_{\gamma}\rangle \quad , \tag{4.1}$$

where $\bar{V}^{(\sigma)}_{\gamma} = \sum_{\lambda\subset\sigma} V_{\lambda}\,\bar{\delta}_{\lambda\gamma}$ (it is understood that $\gamma\subset\sigma$), and we have used the relation

$$G_0\sum_{\beta} K^{\sigma}_{\beta\gamma} = G^{\sigma}\bar{V}^{(\sigma)}_{\gamma} \quad , \tag{4.2}$$

which follows from (3.2).

With the explicit appearance of the channel Green's function G^σ in (4.1), the singularity analysis of $\Psi^{\sigma(\tau)}$ becomes straightforward. Using the complete set of channel eigenstates (3.4) or (3.5), we obtain

$$G^\sigma(E+i0) = \int |\vec{r}\,\Phi^{(\sigma)}> \frac{d\vec{r}}{\tilde{r}^2 - \kappa_\sigma^2 - E - i0} <\vec{r}\,\Phi^{(\sigma)}|$$

$$+ \sum_{\delta\subset\sigma} \int |\vec{r}\,\Psi^{(\sigma)-}_{(\delta)\vec{p}}> \frac{d\vec{r}\,d\vec{p}}{\tilde{r}^2 + \tilde{p}^2 - \kappa_\delta^2 - E - i0} <\vec{r}\,\Psi^{(\sigma)-}_{(\delta)\vec{p}}|$$

$$+ \int |\vec{r}\,\Psi^{(\sigma)-}_{\vec{p}\,\vec{q}}> \frac{d\vec{r}\,d\vec{p}\,d\vec{q}}{\tilde{r}^2 + \tilde{p}^2 + \tilde{q}^2 - E - i0} <\vec{r}\,\Psi^{(\sigma)-}_{\vec{p}\,\vec{q}}| \quad , \tag{4.3}$$

where $\tilde{p}^2$ and $\tilde{q}^2$ are defined in Section II, and $\tilde{r}_\sigma^2 = r_\sigma^2/2\eta_\sigma$, with $\eta_\sigma = \left[m_4(m_1+m_2+m_3)\right]/(m_1+m_2+m_3+m_4)$ if $\sigma=(123)(4)$. With the aid of Eq. (4.3), (4.1) can now be written as

$$<\vec{r}\,\vec{p}\,\vec{q}\,|\Psi^{\sigma(\tau)}> = \delta^{\sigma\tau}\,\delta(\vec{r}-\vec{r}^{(0)})\,\Phi^{(\tau)}(\vec{p},\vec{q}) - \Phi^{(\sigma)}(\vec{p},\vec{q})\,\frac{\mathscr{H}^{\sigma\tau}(\vec{r},\vec{r}^{(0)},\ E+i0)}{\tilde{r}^2 - \kappa_\sigma^2 - E - i0}$$

$$- \sum_{\delta\subset\sigma} \int \Psi^{(\sigma)-}_{(\delta)\vec{p}'}(\vec{p}\,\vec{q})\,\frac{d\vec{p}'}{\tilde{r}^2 + \tilde{p}'^2 - \kappa_\delta^2 - E - i0}\,\mathscr{F}^{\sigma\tau}_{(\delta)}(\vec{r},\vec{p}';\vec{r}^{(0)};E+i0)$$

$$- \int \Psi^{(\sigma)-}_{\vec{p}'\vec{q}'}(\vec{p}\,\vec{q})\,\frac{d\vec{p}'\,d\vec{q}'}{\tilde{r}^2 + \tilde{p}'^2 + \tilde{q}'^2 - E - i0}\,\mathscr{E}^{\sigma\tau}(\vec{r},\vec{p}',\vec{q}';\vec{r}^{(0)};E+i0)\ , \tag{4.4}$$

where

$$\mathscr{H}^{\sigma\tau}(\vec{r},\vec{r}^{(0)};E+i0) = <\vec{r}\,\Phi^{(\sigma)}|\sum_{\gamma\subset\sigma}\ \sum_{\rho\supset\gamma} \bar{\delta}^{\sigma\rho}\,\bar{V}^{(\sigma)}_\gamma\,|\Psi^{\rho(\tau)}_\gamma>$$

$$\mathscr{F}^{\sigma\tau}_{(\delta)}(\vec{r},\vec{p}';\vec{r}^{(0)};E+i0) = <\vec{r}\,\Psi^{(\sigma)-}_{(\delta)\vec{p}'}|\sum_{\gamma\subset\sigma}\ \sum_{\rho\supset\gamma} \bar{\delta}^{\sigma\rho}\,\bar{V}^{(\sigma)}_\gamma\,|\Psi^{\rho(\tau)}_\gamma> \tag{4.5}$$

$$\mathscr{E}^{\sigma\tau}(\vec{r},\vec{p}'\vec{q}';\vec{r}^{(0)};E+i0) = <\vec{r}\,\Psi^{(\sigma)-}_{\vec{p}'\vec{q}'}|\sum_{\gamma\subset\sigma}\ \sum_{\rho\supset\gamma} \bar{\delta}^{\sigma\rho}\,\bar{V}^{(\sigma)}_\gamma\,|\Psi^{\rho(\tau)}_\gamma> \ .$$

Equation (4.4) constitutes a four-body analog of Eqs. (2.2); i.e., it explicitly exhibits all the physical poles of the wavefunction components $\Psi^{\sigma(\tau)}$ in separate terms. The residues at these poles—i.e., the amplitudes (4.5)—are free from primary singularities (just as in our three-body formalism), and are the components of the physical scattering amplitudes: As is shown in the Appendix, the on-shell values of $\mathscr{H}^{\sigma\tau}$, $\sum_{\sigma\supset\delta} \mathscr{F}^{\sigma\tau}_{(\delta)}$ and $\sum_{\sigma} \mathscr{E}^{\sigma\tau}$ are the amplitudes for elastic/rearrangement, partial breakup and full breakup, respectively.[12]

The remaining step in the generalization would now be to find equations for these amplitudes. Unfortunately, as can be seen from Eqs. (4.1), $\Psi^{\sigma(\tau)}$ is coupled to all the FY components $\Psi^{\sigma(\tau)}_{\beta}$, and not simply to the remaining $\Psi^{\rho(\tau)}$. As a result, no equations for the wavefunction components $\Psi^{\sigma(\tau)}$ are available within the FY formalism, and it is therefore not possible to obtain dynamical equations for the amplitudes (4.5) at this stage.

To proceed within the FY formalism, it is also necessary to perform a singularity analysis of the FY components $\Psi^{\sigma(\tau)}_{\beta}$ (for which, of course, Eqs. (3.1) are available). This however is not straightforward, as will be seen in the next sections, and is certain to lead to a larger number of amplitude components (this being the weak point of the FY formalism in general).

At this point one could therefore abandon the FY formalism and use other dynamical equations for the components $\Psi^{\sigma(\tau)}$, for example those discussed in Refs. 3 and 13. However, all such alternatives we are aware of lead to dynamical equations with effective potentials that are not only energy dependent, but also require fully-off-shell subsystem input. In addition, these alternative equations may possibly admit spurious solutions. For these reasons, we choose to remain within the FY formalism for the present work.

V. SINGULARITY ANALYSIS OF THE FY COMPONENTS $\Psi_\beta^{\sigma(\tau)}$

Recalling Eqs. (3.1) and (3.2), we see that the kernel that must be now analyzed for singularities is $G_0 K^\sigma_{\beta\gamma}$. In analogy with (4.2), we write

$$G_0 K^\sigma_{\beta\gamma} = \sum_{\lambda \subset \sigma} G^\sigma_{\beta\lambda} V_\lambda \bar{\delta}_{\lambda\gamma} \quad , \tag{5.1}$$

where

$$G^\sigma_{\beta\lambda} = \delta_{\beta\lambda} G_0 - G_0 V_\beta G^\sigma \tag{5.2}$$

is the Faddeev component of the Green's function G^σ, with the property that $\sum_\beta G^\sigma_{\beta\lambda} = G^\sigma$. Therefore, we see that for the pole decomposition of $\Psi_\beta^{\sigma(\tau)}$ it is necessary to analyze the Green's function <u>components</u> G^σ_β, rather than G^σ itself. As is evident from (5.2), use of the spectral decomposition of G^σ (Eq. (4.3)) is not sufficient, since there is also a pole in G_0. This pole is accounted for in the following way: In each term that results from applying the spectral decomposition (4.3) to the product $G_0(E+i0)V_\beta G^\sigma(E+i0)$ of (5.2) we use the resolvent identity

$$G_0(E+i\epsilon) = G_0(z') + (E+i\epsilon - z')\, G_0(E+i\epsilon)\, G_0(z') \quad , \tag{5.3}$$

with z' equal to the energy of the corresponding channel eigenstate (with an imaginary part ϵ' that is always understood to go to zero before ϵ). Then, the $G_0(z')V_\beta$ factors in (5.2) can be eliminated using the three-body relations

$$\begin{aligned}
G_0(\tilde{r}^2 - \kappa_\sigma^2)\, V_\beta\, |\vec{r}\,\Phi^{(\sigma)}\rangle &= -|\vec{r}\,\Phi^{(\sigma)}_\beta\rangle \\
G_0(\tilde{r}^2 + \tilde{p}^2 - \kappa_\delta^2 - i\epsilon')\, V_\beta\, |\vec{r}\,\Psi^{(\sigma)-}_{(\delta)\vec{p}}\rangle &= -|\vec{r}\,\Psi^{(\sigma)-}_{\beta;(\delta)\vec{p}}\rangle \quad , \\
G_0(\tilde{r}^2 + \tilde{p}^2 + \tilde{q}^2 - i\epsilon')\, V_\beta\, |\vec{r}\,\Psi^{(\sigma)-}_{\vec{p}\,\vec{q}}\rangle &= -|\vec{r}\,\chi^{(\sigma)-}_{\beta;\vec{p}\,\vec{q}}\rangle
\end{aligned} \tag{5.4}$$

where $\chi_\beta^{(\sigma)-}$ is what remains of $\Psi_\beta^{(\sigma)-}$ once the initial-state plane wave has been subtracted.

As a result, we obtain a "pole decomposition" of the Green's function components given by

$$G^{\sigma}_{\beta\lambda}(E+i0) = \int |\vec{r}\,\Phi^{(\sigma)}_{\beta}> \frac{d\vec{r}}{\tilde{r}^2 - \kappa^2_{\sigma} - E - i0} <\vec{r}\,\Phi^{(\sigma)}|$$

$$+ \sum_{\delta\subset\sigma} \int |\vec{r}\,\Psi^{(\sigma)-}_{\beta;(\delta)\vec{p}}> \frac{d\vec{r}\,d\vec{p}}{\tilde{r}^2 + \tilde{p}^2 - \kappa^2_{\delta} - E - i0} <\vec{r}\,\Psi^{(\sigma)-}_{(\delta)\vec{p}}|$$

$$+ \int |\vec{r}\,\Psi^{(\sigma)-}_{\beta(\lambda);\vec{p}\,\vec{q}}> \frac{d\vec{r}\,d\vec{p}\,d\vec{q}}{\tilde{r}^2 + \tilde{p}^2 + \tilde{q}^2 - E - i0} <\vec{r}\,\Psi^{(\sigma)-}_{\vec{p}\,\vec{q}}|$$

$$+ G_0(E+i0) \Big\{ \delta_{\beta\lambda} - \int |\vec{r}\,\Phi^{(\sigma)}_{\beta}> d\vec{r} <\vec{r}\,\Phi^{(\sigma)}| -$$

$$- \sum_{\delta\subset\sigma} \int |\vec{r}\,\Psi^{(\sigma)-}_{\beta;(\delta)\vec{p}}> d\vec{r}\,d\vec{p} <\vec{r}\,\Psi^{(\sigma)-}_{(\delta)\vec{p}}| -$$

$$- \int |\vec{r}\,\Psi^{(\sigma)-}_{\beta(\lambda);\vec{p}\,\vec{q}}> d\vec{r}\,d\vec{p}\,d\vec{q} <\vec{r}\,\Psi^{(\sigma)-}_{\vec{p}\,\vec{q}}| \Big\}, \qquad (5.5)$$

where we have also replaced $|\vec{r}\,\chi^{(\sigma)}_{\beta;\vec{p}\,\vec{q}}>$ by $|\vec{r}\,\Psi^{(\sigma)-}_{\beta(\lambda);\vec{p}\,\vec{q}}> - \delta_{\beta\lambda}|\vec{r}\,\vec{p}\,\vec{q}>$ and made use of the fact that G_0 is diagonal in an $|\vec{r}\,\vec{p}\,\vec{q}>$ representation.

In (5.5) we see that upon summation over $\beta\subset\sigma$, the factor multiplying $G_0(E+i0)$ vanishes identically. In addition, the first three terms become equal to the expression (4.3) for G^{σ}, since the Faddeev components of (5.5) add up to the full channel eigenstates.

Using (5.5), we finally obtain the sought-for pole decomposition of the FY kernel (5.1), and also of the FY wavefunction components (3.1):

$$
\begin{aligned}
<\vec{r}\,\vec{p}\,\vec{q}\,|\Psi_\beta^{\sigma(\tau)}> = \delta^{\sigma\tau}\,\delta(\vec{r}-\vec{r}^{(0)})\,\Phi_\beta^{(\tau)}(\vec{p}\,\vec{q})
&- \frac{\Phi_\beta^{(\sigma)}(\vec{p}\,\vec{q})}{\tilde{r}^2-\kappa_\sigma^2-E-i0}\,\mathscr{H}^{\sigma\tau}(\vec{r},\vec{r}^{(0)};E+i0) \\
&- \sum_{\delta\subset\sigma}\int \Psi^{(\sigma)-}_{\beta;(\delta)\vec{p}'}(\vec{p}\,\vec{q})\,\frac{d\vec{p}'}{\tilde{r}^2+\tilde{p}'^2-\kappa_\delta^2-E-i0}\,\mathscr{F}^{\sigma\tau}_{(\delta)}(\vec{r}\,\vec{p}';\vec{r}^{(0)};E+i0) \\
&- \sum_{\lambda\subset\sigma}\int \Psi^{(\sigma)-}_{\beta(\lambda);\vec{p}'\vec{q}'}(\vec{p}\,\vec{q})\,\frac{d\vec{p}'\,d\vec{q}'}{\tilde{r}^2+\tilde{p}'^2+\tilde{q}'^2-E-i0}\,\mathscr{E}^{\sigma\tau}_\lambda(\vec{r}\,\vec{p}'\vec{q}';\vec{r}^{(0)};E+i0) \\
&- \frac{1}{\tilde{r}^2+\tilde{p}^2+\tilde{q}^2-E-i0}\,\mathscr{Y}^{\sigma\tau}_\beta(\vec{r}\,\vec{p}\,\vec{q};\vec{r}^{(0)};E+i0)\quad, \qquad (5.6)
\end{aligned}
$$

where $\mathscr{H}^{\sigma\tau}$ and $\mathscr{F}^{\sigma\tau}_{(\delta)}$ have already been defined in (4.5), and $\mathscr{E}^{\sigma\tau}_\lambda$ is a decomposition of the amplitude $\mathscr{E}^{\sigma\tau}$ of (4.5), i.e.,

$$
\mathscr{E}^{\sigma\tau}_\lambda(\vec{r}\;\vec{p}'\vec{q}';\vec{r}^{(0)};E+i0) = <\vec{r}\,\Psi^{(\sigma)-}_{\vec{p}'\vec{q}'}|V_\lambda \sum_{\gamma\subset\sigma}\bar{\delta}_{\lambda\gamma}\sum_{\rho\supset\gamma}\bar{\delta}^{\sigma\rho}\,|\Psi_\gamma^{\rho(\tau)}>\quad, \qquad (5.7)
$$

with $\sum_{\lambda\subset\sigma}\mathscr{E}^{\sigma\tau}_\lambda = \mathscr{E}^{\sigma\tau}$. The remaining amplitude $\mathscr{Y}^{\sigma\tau}_\rho$ is given by

$$
\begin{aligned}
\mathscr{Y}^{\sigma\tau}_\beta\,(\vec{r}\,\vec{p}\,\vec{q};\vec{r}^{(0)};E+i0) = <\vec{r}\,\vec{p}\,\vec{q}\,|\sum_{\lambda\subset\sigma}\Big\{\delta_{\beta\lambda}
&- \int|\vec{r}'\,\Phi_\beta^{(\sigma)}>d\vec{r}'<\vec{r}'\,\Phi^{(\sigma)}| - \sum_{\delta\subset\sigma}\int|\vec{r}'\,\Psi^{(\sigma)-}_{\beta;(\delta)\vec{p}'}>d\vec{r}'\,d\vec{p}'<\vec{r}'\,\Psi^{(\sigma)-}_{(\delta)\vec{p}'}| \\
&- \int|\vec{r}'\,\Psi^{(\sigma)-}_{\beta(\lambda);\vec{p}'\vec{q}'}>d\vec{r}'\,d\vec{p}'\,d\vec{q}'<\vec{r}'\,\Psi^{(\sigma)-}_{\vec{p}'\vec{q}'}|\Big\}\,V_\lambda\sum_{\gamma\subset\sigma}\bar{\delta}_{\lambda\gamma}\sum_{\rho\supset\gamma}\bar{\delta}^{\sigma\rho}\,|\Psi_\gamma^{\rho(\tau)}>\;.
\end{aligned}
$$

(5.8)

Equation (5.6) constitutes a further generalization of our previous decomposition (4.4), where now all physical singularities of the FY component $\Psi_{\beta}^{\sigma(\tau)}$ are explicitly exhibited in separate terms. It is a remarkable fact that in (5.6) the β-dependence in the terms containing $\mathscr{H}$ and $\mathscr{F}$ factorizes, so that these scattering amplitudes still depend only on the two-cluster index σ of the wavefunction. In other words, further splitting of $\Psi^{\sigma(\tau)}$ in (4.4) into $\Psi_{\beta}^{\sigma(\tau)}$ in (5.6) only produces a splitting of the amplitude $\mathscr{E}^{\sigma\tau}$.

In addition, the amplitude $\mathscr{Y}_{\beta}^{\sigma\tau}$ must now be introduced. Just as in (5.5), this amplitude vanishes identically upon summation of $\Psi_{\beta}^{\sigma(\tau)}$ over all $\beta\subset\sigma$ (as did the last term in (5.5)), and is therefore also absent from the full wavefunction. Consequently, $\mathscr{Y}_{\beta}^{\sigma\tau}$ is not a physical scattering amplitude.

VI. EQUATIONS FOR THE SCATTERING AMPLITUDES

Let us now derive the equations that our amplitudes $\mathscr{H}^{\sigma\tau}$, $\mathscr{F}_{(\delta)}^{\sigma\tau}$ and $\mathscr{E}_{\beta}^{\sigma\tau}$ satisfy.

Replacing the pole decomposition (5.6) for $|\Psi_{\gamma}^{\rho(\tau)}\rangle$ in the definitions (4.5) and (5.7) for these amplitudes, and in the definition (5.8) for $\mathscr{Y}_{\beta}^{\sigma\tau}$, the following (half-on-shell) equations are immediately obtained:

$$\begin{aligned}
\mathscr{H}^{\sigma\tau}(\vec{r};\vec{r}^{(0)};E+i0) &= \bar{\delta}^{\sigma\tau}\,\mathscr{V}^{(\mathscr{H}\mathscr{H})\sigma\tau}(\vec{r};\vec{r}^{(0)}) \\
&- \sum_{\rho\neq\sigma}\int \mathscr{V}^{(\mathscr{H}\mathscr{H})\sigma\rho}(\vec{r};\vec{r}\,')\,\frac{d\vec{r}\,'}{\tilde{r}'^{2}-\kappa_{\rho}^{2}-E-i0}\,\mathscr{H}^{\rho\tau}(\vec{r}\,';\vec{r}^{(0)};E+i0) \\
&- \sum_{\rho\neq\sigma}\sum_{\delta\subset\rho}\int \mathscr{V}_{\delta}^{(\mathscr{H}\mathscr{F})\sigma\rho}(\vec{r};\vec{r}\,'\,\vec{p}\,')\,\frac{d\vec{r}\,'\,d\vec{p}\,'}{\tilde{r}'^{2}+\tilde{p}'^{2}-\kappa_{\delta}^{2}-E-i0}\,\mathscr{F}_{(\delta)}^{\rho\tau}(\vec{r}\,'\,\vec{p}\,';\vec{r}^{(0)};E+i0) \\
&- \sum_{\rho\neq\sigma}\sum_{\lambda\subset\rho}\int \mathscr{V}_{\lambda}^{(\mathscr{H}\mathscr{E})\sigma\rho}(\vec{r};\vec{r}\,'\,\vec{p}\,'\,\vec{q}\,')\,\frac{d\vec{r}\,'\,d\vec{p}\,'\,d\vec{q}\,'}{\tilde{r}'^{2}+\tilde{p}'^{2}+\tilde{q}'^{2}-E-i0}\,\mathscr{E}_{\lambda}^{\rho\tau}(\vec{r}\,'\,\vec{p}\,'\,\vec{q}\,';\vec{r}^{(0)};E+i0) \\
&- \sum_{\rho\neq\sigma}\sum_{\lambda\subset\rho}\int \mathscr{V}_{\lambda}^{(\mathscr{H}\mathscr{Y})\sigma\rho}(\vec{r};\vec{r}\,'\,\vec{p}\,'\,\vec{q}\,')\,\frac{d\vec{r}\,'\,d\vec{p}\,'\,d\vec{q}\,'}{\tilde{r}'^{2}+\tilde{p}'^{2}+\tilde{q}'^{2}-E-i0}\,\mathscr{Y}_{\lambda}^{\rho\tau}(\vec{r}\,'\,\vec{p}\,'\,\vec{q}\,';\vec{r}^{(0)};E+i0)
\end{aligned} \tag{6.1}$$

where $E = \tilde{r}^{(0)2} - \kappa_\tau^2$. The corresponding equations for $\mathscr{F}^{\sigma\tau}_{(\beta)}$, $\mathscr{E}^{\sigma\tau}_{\beta}$ and $\mathscr{Y}^{\sigma\tau}_{\beta}$ are obtained from (6.1) by replacing, respectively, $\mathscr{V}^{(\mathscr{H}\mathscr{H})}$ by $\mathscr{V}^{(\mathscr{F}\mathscr{H})}$, $\mathscr{V}^{(\mathscr{E}\mathscr{H})}$ and $\mathscr{V}^{(\mathscr{Y}\mathscr{H})}$, and so on.

Examples of the potentials appearing in (6.1) are,

$$\mathscr{V}^{(\mathscr{H}\mathscr{H})\sigma\rho}(\vec{r};\vec{r}\,') = \sum_{\beta\subset\sigma} <\vec{r}\,\Phi^{(\sigma)}|V_\beta \bar{\delta}_{\beta\gamma}|\vec{r}\,'\,\Phi^{(\rho)}_\gamma>$$

$$\mathscr{V}^{(\mathscr{H}\mathscr{F})\sigma\rho}_{\lambda}(\vec{r};\vec{r}\,'\,\vec{p}\,') = \sum_{\beta\subset\sigma} <\vec{r}\,\Phi^{(\sigma)}|V_\beta \bar{\delta}_{\beta\gamma}|\vec{r}\,'\Psi^{(\rho)-}_{\gamma(\lambda);\vec{p}\,'}>$$

$$\mathscr{V}^{(\mathscr{F}\mathscr{F})\sigma\rho}_{\delta\lambda}(\vec{r}\,\vec{p}\,;\vec{r}\,'\vec{p}\,') = \sum_{\beta\subset\sigma} <\vec{r}\,\Psi^{(\sigma)-}_{(\delta);\vec{p}}|V_\beta \bar{\delta}_{\beta\gamma}|\vec{r}\,'\Psi^{(\rho)-}_{\gamma(\lambda);\vec{p}\,'}> \qquad (6.2)$$

where the index γ is uniquely determined by the conditions $\gamma\subset\sigma$ and $\gamma\subset\rho$, $(\sigma\neq\rho)$. (Note that when both σ and ρ are of the 2+2 type, $\sigma\cap\rho\equiv 0$, so the corresponding potentials vanish.)

In spite of the fact that two-body potentials appear in (6.2), all effective potentials in (6.1) can be expressed in terms of half-on-shell subsystem scattering amplitudes and bound state wavefunctions, with no two-body potentials remaining explicitly. For example, $\mathscr{V}^{(\mathscr{H}\mathscr{F})}$ in (6.2) can be written as

$$\mathscr{V}^{(\mathscr{H}\mathscr{F})\sigma\rho}_{\lambda}(\vec{r}_\sigma;\vec{r}\,'_\rho\,\vec{p}\,'_\lambda) = -<\Phi^{(\sigma)}|\vec{p}^{(1)}_\gamma\,\phi^\gamma_\kappa>(\tilde{p}^{(1)2}_\gamma - \kappa_\gamma^2 + \kappa_\sigma^2)\times$$
$$\times\left\{\delta_{\gamma\lambda}\,\delta(\vec{p}^{(2)}_\lambda - \vec{p}\,'_\lambda) - \frac{\mathscr{H}^{\rho}_{\gamma\lambda}(\vec{p}^{(2)}_\gamma,\vec{p}\,'_\lambda;\tilde{p}'^{2}_\lambda - \kappa_\lambda^2 - i0)}{\tilde{p}^{(2)2}_\gamma - \kappa_\gamma^2 - \tilde{p}'^{2}_\lambda + \kappa_\lambda^2 + i0}\right\}$$
$$-\int d\vec{q}_\gamma <\Phi^\sigma|\vec{p}^{(1)}_\gamma\psi^+_{\vec{q}_\gamma}>(\tilde{p}^{(1)2}_\gamma + \tilde{q}^2_\gamma + \kappa_\sigma^2)\frac{1}{\tilde{p}^{(2)2}_\gamma + \tilde{q}^2_\gamma - \tilde{p}'^{2}_\lambda + \kappa^2_\lambda + i0}\,\mathscr{E}^{\rho}_{\gamma\lambda}(\vec{p}^{(2)}_\gamma,\vec{q}_\gamma;\vec{p}\,'_\lambda;\tilde{p}'^{2}_\lambda - \kappa^2_\lambda - i0)$$
$$(6.3)$$

where, as in (6.2), $\gamma=\sigma\cap\rho$ is uniquely determined by σ and ρ $(\sigma\neq\rho)$. Also,

$$\vec{p}^{(1)}_\gamma = \frac{M_\rho}{M_\gamma+M_\rho}\vec{r}_\sigma + \vec{r}\,'_\rho \qquad \text{and} \qquad \vec{p}^{(2)}_\gamma = \vec{r}_\sigma + \frac{M_\sigma}{M_\gamma+M_\sigma}\vec{r}\,'_\rho \quad ,$$

where if, say, σ=(123)(4) and ρ=(124)(3), γ=12, $M_\gamma = M_{12} = m_1 + m_2$, $M_\sigma = m_4$ and $M_\rho = m_3$.

The factors appearing to the left in (6.3) are projections of the three-body bound state wavefunction onto the complete set of two-body channel eigenstates. The amplitudes $\mathscr{H}$ and $\mathscr{E}$ are the scattering amplitudes of our three-body formalism, taken half-on-shell.

The potentials coupling $\mathscr{Y}_\beta^{\sigma\tau}$ to the physical amplitudes differ somewhat from those in (6.2); e.g.,

$$\begin{aligned}\mathscr{V}^{(\mathscr{Y}\mathscr{Y})\sigma\rho}_{\beta\lambda}(\vec{r}\,\vec{p}\,\vec{q}\,,\vec{r}'\vec{p}'\vec{q}') = \langle\vec{r}\;\vec{p}\;\vec{q}\,|\Big\{\delta_{\beta\gamma} &- \int|\vec{r}''\,\Phi^{(\sigma)}_\beta\rangle\, d\vec{r}''\,\langle\vec{r}''\,\Phi^{(\sigma)}| \\ &- \sum_{\delta\subset\sigma}\int|\vec{r}''\Psi^{(\sigma)-}_{\beta;(\delta)\vec{p}''}\rangle\, d\vec{r}''\, d\vec{p}''\,\langle\vec{r}''\,\Psi^{(\sigma)-}_{(\delta)\vec{p}''}| \\ &- \int|\vec{r}''\,\Psi^{(\sigma)-}_{\beta(\lambda);\vec{p}''\vec{q}''}\rangle\, d\vec{r}''\, d\vec{p}''\, d\vec{q}''\,\langle\vec{r}''\,\Psi^{(\sigma)-}_{\vec{p}''\vec{q}''}|\Big\}\, V_\gamma\,\bar{\delta}_{\gamma\lambda}|\vec{r}'\vec{p}'\vec{q}'\rangle\;. \qquad (6.4)\end{aligned}$$

As expected, all these potentials vanish upon summation over $\beta\subset\sigma$. Again, all two-body potentials that appear explicitly in (6.4) can be eliminated in favor of half-on-shell subsystem amplitudes and bound state wavefunctions (the first term $\delta_{\beta\gamma}$ in (6.4) is actually cancelled by a piece of the fourth term).

The coupled integral equations (6.1) constitute a generalization of our three-body equations to the four-body case. We obtain in this way a formalism with advantages similar to those present in our three-body theory, namely:

(i) The dynamical equations are expressed in terms of components of the physical scattering amplitudes;

(ii) The amplitude components defined in the formalism are free from primary singularities, i.e., from poles (in the off-shell variables);

(iii) The equations have the structure of a multichannel Lippmann-Schwinger formulation, with effective potentials that are independent of the four-body energy;

(iv) The equations require as input only half-on-shell subsystem transition amplitudes and bound state wavefunctions.

As pointed out before, however, the equations also include a nonphysical amplitude $\mathscr{Y}_{\beta}^{\sigma\tau}$, and our goal is therefore not fully achieved. The presence of this nonphysical amplitude can be understood as follows:

The FY equations are obtained from the four-body Lippmann-Schwinger equations by means of a two-step procedure[8]: the two-body disconnected pieces are first removed from the kernel, and only then are three-body disconnected pieces removed. (This is done in such a way that the resulting FY kernel connects three particles after one iteration and all four particles after two iterations.[8]) As a consequence, the full wavefunction is split <u>first</u> according to three-cluster indices, and <u>then</u> split further according to two-cluster indices.

On the other hand, as we have seen, the singularity structure of the full wavefunction is most naturally exhibited by considering the wavefunction components $\Psi^{\sigma(\tau)}$, split only according to the two-cluster index σ. The (prior) additional splitting according to three-cluster indices required by the FY formalism (in order to achieve connectedness of the kernel) appears thus far less natural from the point of view of the singularities of the kernel (or from the point of view of asymptotic channels).

The FY formalism nevertheless requires that we perform the more complicated singularity analysis of the fully-split wavefunction components $\Psi_{\beta}^{\sigma(\tau)}$, i.e., that we retain the full index context of the FY equations. In choosing to remain within the FY formalism, and insisting on energy-independent half-on-shell input, we are not only required to split the breakup amplitude $\mathscr{E}^{\sigma\tau}$ further into components $\mathscr{E}_{\beta}^{\sigma\tau}$ (an expected complication) but also to introduce the nonphysical amplitudes $\mathscr{Y}_{\beta}^{\sigma\tau}$.

VII. GENERALIZATION TO THE FULLY-OFF-SHELL CASE

In the previous sections we constructed our four-body formalism keeping the use of four-body operators and operator relations to a minimum; i.e., staying essentially within the wavefunction approach. It is illustrative however to consider how our formulation relates to the four-body transition operators, and how a fully-off-shell version of our amplitudes can be obtained from these operators.

To do so, we first recall that in our three-body formalism the fully-off-shell amplitudes are defined using the three-body operator[4]

$$T_{\beta\alpha}(z) = V_{\beta} G_0(z)\, U_{\beta\alpha}(z)\, G_0(z)\, V_{\alpha} \quad , \tag{7.1}$$

where $U_{\beta\alpha}(z)$ is the three-body AGS transition operator.[14] The on-shell matrix elements of the operator (7.1) between appropriate channel eigenstates give the various three-body physical transition amplitude components.

In order to obtain the corresponding four-body operators, it is convenient to make use of the matrix formalism[7]: We first define a matrix version of (7.1) by means of the four-body matrix of operators $\hat{\mathbf{T}}^{\sigma\tau} = \left\{\hat{T}_{\beta\alpha}^{\sigma\tau}\right\}$, according to

$$\hat{\mathbf{T}}^{\sigma\tau} = \mathbf{V}^{(\sigma)} \mathbf{G}_0^{(\sigma)}\, \mathbf{T}^{\sigma\tau}\, \mathbf{G}_0^{(\tau)} \mathbf{V}^{(\tau)} \quad , \tag{7.2}$$

where $\mathbf{V}^{(\sigma)} = \left\{-\bar{\delta}_{\beta\alpha}\, G_0^{-1}\right\}$, $\mathbf{G}_0^{(\sigma)} = \left\{-\delta_{\beta\alpha}\, G_0 t_{\beta} G_0\right\}$, etc. (with $\beta, \alpha \subset \sigma$), and $\mathbf{T}^{\sigma\tau} = \left\{U_{\beta\alpha}^{\sigma\tau}\right\}$ stands for the matrix of four-body AGS operators.[7]

Next, as in (7.1), we define

$$T_{\beta\alpha}^{\sigma\tau} = V_{\beta} G_0\, \hat{T}_{\beta\alpha}^{\sigma\tau}\, G_0 V_{\alpha} \quad , \tag{7.2}$$

or, more explicitly,

$$T^{\sigma\tau}_{\beta\alpha} = V_\beta G_0 \left(\sum_{\gamma\subset\sigma} \sum_{\lambda\subset\tau} \bar{\delta}_{\beta\gamma} t_\gamma G_0 \, U^{\sigma\tau}_{\gamma\lambda} G_0 t_\lambda \bar{\delta}_{\lambda\alpha} \right) G_0 V_\alpha \quad . \tag{7.3}$$

The equations these operators satisfy are easily obtained using the four-body equations for $U^{\sigma\tau}_{\beta\alpha}$ [7]:

$$T^{\sigma\tau}_{\beta\alpha}(z) = \bar{\delta}^{\sigma\tau} \, \bar{\delta}_{\beta\gamma} V_\beta G_0(z) \, t_\gamma(z) \, G_0(z) \, V_\alpha \bar{\delta}_{\gamma\alpha}$$

$$- \sum_{\rho\neq\sigma} \sum_{\lambda\subset\rho} V_\beta \bar{\delta}_{\beta\gamma'} G^\rho_{\gamma'\lambda}(z) \, T^{\rho\tau}_{\lambda\alpha}(z) \quad , \tag{7.4}$$

where $G^\rho_{\gamma'\lambda}$ has been defined in (5.2) (recall also (5.1)), and γ, γ' are determined by the conditions $\gamma = \sigma\cap\tau$ and $\gamma' = \sigma\cap\rho$.

By analogy with the three-body case, we expect matrix elements of the operators (7.3) (rather than matrix elements of just $U^{\sigma\tau}_{\beta\alpha}$) to be closely related to the amplitudes of the previous sections. Indeed, by applying the pole decomposition (5.5) of $G^\rho_{\gamma'\lambda}$ (with $E+i0\to z$) to (7.4), and projecting onto channel eigenstates, we easily verify that the resulting kernels are identical to the kernels of Eqs. (6.1). Moreover, when z is chosen to be the energy of the initial state, also the resulting driving terms become identical to the driving terms of (6.1).

We can therefore identify the half-on-shell matrix elements of $T^{\sigma\tau}_{\beta\alpha}$ between appropriate initial and final states with our previously defined scattering amplitudes $\mathscr{H}^{\sigma\tau}$, $\mathscr{F}^{\sigma\tau}_{(\delta)}$ of (4.5) and $\mathscr{E}^{\sigma\tau}_\beta$ of (5.7) (recall that $\mathscr{E}^{\sigma\tau} = \sum_{\beta\subset\sigma} \mathscr{E}^{\sigma\tau}_\beta$).

With this identification it is straightforward to define the corresponding fully-off-shell versions of our amplitudes as

$$\mathscr{H}^{\sigma\tau}(\vec{r};\vec{r}^{(0)};z) = \langle \vec{r}\, \Phi^{(\sigma)} | T^{\sigma\tau}(z) | \vec{r}^{(0)} \Phi^{(\tau)} \rangle$$

$$\mathscr{F}^{\sigma\tau}_{(\delta)}(\vec{r}\,\vec{p};\vec{r}^{(0)};z) = \langle \vec{r}\, \Psi^{(\sigma)-}_{(\delta)\vec{p}} | T^{\sigma\tau}(z) | \vec{r}^{(0)} \Phi^{(\tau)} \rangle$$

$$\mathscr{E}_{\beta}^{\sigma\tau}(\vec{r}\;\vec{p}\;\vec{q};\vec{r}^{(0)};z) = \langle \vec{r}\,\Psi^{(\sigma)-}_{\vec{p}\,\vec{q}} \,|\, \sum_{\alpha\subset\tau} T^{\sigma\tau}_{\beta\alpha}(z)\,|\vec{r}^{(0)}\Phi^{(\tau)}\rangle \tag{7.5}$$

where

$$T^{\sigma\tau} = \sum_{\beta\subset\sigma}\sum_{\alpha\subset\tau} T^{\sigma\tau}_{\beta\alpha} \quad . \tag{7.6}$$

It is important to note that it is from the appropriately "dressed" operator (7.3) that we can obtain singularity-free scattering amplitudes. This is in analogy with the three-body case, where the factor $V_{\beta}G_0$ in $T_{\beta\alpha}$ (Eq. (7.1)) is present to eliminate the primary singularities of the matrix elements of $U_{\beta\alpha}$. In the four-body case, the factor $V_{\beta}G_0\bar{\delta}_{\beta\lambda}t_{\lambda}G_0$ in (7.3) performs a similar function.

The equations satisfied by the amplitudes (7.5) can be directly obtained from the operator equation (7.4), using (5.5) with E+i0 replaced by z. The effective potentials in the resulting equations are identical to those of Eqs. (6.1), but the driving terms are slightly different.

At this point, in view of the complications we have encountered in generalizing the three-body formalism of Ref. 4 (in particular the appearance of the nonphysical amplitude $\mathscr{Y}_{\beta}^{\sigma\tau}$), one may ask whether the off-shell four-body amplitudes have really been chosen properly. We therefore conclude this section by giving another argument in favor of our choice.

For this we turn to the full four-body Green's function G, and note that in terms of the transition operators we have defined, it is straightforward to write

$$G = G_0 - G_0\,T\,G_0 =$$

$$= G_0 - \sum_{\gamma} G_0 t_{\gamma} G_0 - \sum_{\sigma}\sum_{\substack{\beta\subset\sigma\\ \alpha\subset\sigma}} G_{\beta}T^{(\sigma)}_{\beta\alpha}G_{\alpha} - \sum_{\sigma,\tau} G^{\sigma}T^{\sigma\tau}G^{\tau} \quad , \tag{7.7}$$

where $T^{(\sigma)}_{\beta\alpha}$ is the three-body (i.e., two-cluster) transition operator (Eq. (7.1)), and $T^{\sigma\tau}$ has been defined in (7.6).

In (7.7) we observe that the four-, three- and two-cluster disconnected pieces of $\underset{\sim}{G}$ have been separated from the true one-cluster (i.e., four-body connected) piece in a very natural manner. In addition, it is easy to verify that the four-body connected pieces of G can be written as

$$G^{\sigma} T^{\sigma\tau} G^{\tau} = \sum_{\substack{\beta\subset\sigma \\ \gamma\subset\sigma}} G^{\sigma}_{\beta\gamma} T^{\sigma\tau}_{\gamma\lambda} \tilde{G}^{\tau}_{\lambda\alpha} \quad , \tag{7.8}$$

where $G^{\sigma}_{\beta\gamma}$ is the "left-hand" splitting of G^{σ} as defined in (5.2), and $\tilde{G}^{\tau}_{\lambda\alpha} = \delta_{\lambda\alpha} G_0 - G^{\tau} V_{\alpha} G_0$ is the corresponding "right-hand" splitting of G^{τ}.

We thus see that both the operators $T^{\sigma\tau}$ of (7.6) and $T^{\sigma\tau}_{\beta\alpha}$ of (7.3) appear in the cluster decomposition of the four-body Green's function in a very natural manner, suggesting that they are indeed the proper choice of transition operators in this formalism.

VIII. CONCLUSIONS

In a previous paper, we have shown how a thorough singularity analysis of the Faddeev kernel leads to a three-body formalism that holds several advantages over Faddeev's formulations, although remaining completely equivalent to it.

In the present work we have carried out a generalization of this method to the four-body case, by performing an analogous singularity analysis of the Faddeev-Yakubovskii four-body kernel. When performing such an analysis on the wavefunction components $\Psi^{\sigma(\tau)}$—where σ is a two-cluster index—we find, as expected, a natural expansion of $\Psi^{\sigma(\tau)}$ in terms of singularity-free scattering amplitudes that exhibits all the physical singularities of the full wavefunction.

In addition, we also find a corresponding natural separation of the four-body Green's function into pieces of increasing degree of connectedness.

However, since this analysis is carried out on objects that are labeled only by two-cluster indices, while the FY formalism involves objects labeled by both two- and three-cluster indices, no dynamical equations within the FY formalism can be obtained in this manner; it becomes necessary to carry out a more detailed and much less transparent singularity analysis of the FY components $\Psi_{\beta}^{\sigma(\tau)}$.

Such an analysis does yield dynamical equations that exhibit advantages analogous to those obtained in our three-body formalism, namely,

(i) The equations are expressed in terms of components of the physical amplitudes;

(ii) The amplitude components defined are free from primary singularities, i.e., from poles (in the off-shell variables) that correspond to physical singularities;

(iii) The equations have the structure of a multichannel Lippmann-Schwinger formulation, with effective potentials that are independent of the four-body energy;

(iv) The equations require as input only half-on-shell subsystem transition amplitudes and bound state wavefunctions.

However, the equations also include a nonphysical amplitude $\mathscr{Y}_{\beta}^{\sigma\tau}$, which is an unexpected complication. This additional amplitude is the result of a lack of correspondence between the singularity structure of the FY equations and their detailed index structure: In fact, to our present understanding, the connectedness of the (twice iterated) FY kernel has been obtained through a procedure that is incompatible with a straightforward singularity analysis. The nonphysical amplitude $\mathscr{Y}_{\beta}^{\sigma\tau}$ serves to compensate for this incompatibility, in a way that allows the desired features (i) to (iv) to be carried over directly from the three-body case.

Whether or not to remain within the FY formalism becomes therefore a matter of deciding which characteristics of the four-body equations one chooses to emphasize. As was pointed out, we could have chosen to consider formalisms other than that of FY to obtain equations for the components $\Psi^{\sigma(\tau)}$. None of these formalisms, however, are clearly free from spurious solutions; and, more importantly for our present treatment, all the alternative formalisms we are aware of lead to equations with an input that is not only energy-dependent, but also fully-off-shell. In keeping with our aim of obtaining a theory without such features, we have chosen for the present work to remain within the FY formalism. Nevertheless, further work on alternative formulations of the four-body theory is clearly called for.

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APPENDIX

We show here that the on-shell values of our amplitudes $\mathscr{H}^{\sigma\tau}$, $\mathscr{F}^{\sigma\tau}_{(\delta)}$ and $\mathscr{E}^{\sigma\tau}$ yield the transition amplitudes for all physical processes starting from an initial state of the 3+1 type.

In order to do so we first establish some intermediate results, such as the relationship between the three-body initial state wavefunction and its Faddeev components. Combining the relations $|\Phi_\lambda^{(\tau)}> = -G_0 V_\lambda |\Phi^{(\tau)}>$ with the Faddeev equations $|\Phi_\gamma^{(\tau)}> = -G_0 t_\gamma \sum_{\lambda\subset\tau} \bar{\delta}_{\gamma\lambda} |\Phi_\lambda^{(\tau)}>$, we get

$$|\Phi_\gamma^{(\tau)}> = G_0 t_\gamma \sum_{\lambda\subset\tau} \bar{\delta}_{\gamma\lambda} G_0 V_\lambda |\Phi^{(\tau)}> \quad , \tag{A.1}$$

where it is understood that all operators are to be taken on-shell.

Combining now relations (7.5), (7.3) and (A.1), we get for the half-on-shell amplitude $\mathscr{H}^{\sigma\tau}$ the expression

$$\mathscr{H}^{\sigma\tau}(\vec{r};\vec{r}^{(0)};E+i0) = <\vec{r}\,\Phi^{(\sigma)}|\sum_{\gamma\subset\sigma}\sum_{\lambda\subset\sigma} V_\lambda \bar{\delta}_{\lambda\gamma} \sum_{\alpha\subset\tau} G_0 t_\gamma G_0 U_{\gamma\alpha}^{\sigma\tau} |\vec{r}^{(0)}\Phi_\alpha^{(\tau)}> \;. \tag{A.2}$$

If we now take (A.2) fully-on-shell, we can again use (A.1) to obtain

$$\mathscr{H}^{\sigma\tau}(\vec{r};\vec{r}^{(0)};E+i0) = \sum_{\gamma\subset\sigma}\sum_{\alpha\subset\tau} <\vec{r}\,\Phi_\gamma^{(\sigma)}|U_{\gamma\alpha}^{\sigma\tau}|\vec{r}^{(0)}\Phi_\alpha^{(\tau)}> \quad , \tag{A.3}$$

which is known to be the expression for the elastic and rearrangement scattering amplitudes.[8]

Next we turn to the full breakup amplitude. Taking the expression for $\mathscr{E}^{\sigma\tau}$ in (7.5) fully-on-shell, and applying (A.1), we get

$$\mathscr{E}^{\sigma\tau}(\vec{r}\,\vec{p}\,\vec{q};\vec{r}^{(0)};E+i0) = <\vec{r}\,\Psi_{\vec{p}\,\vec{q}}^{(\sigma)-}|\sum_{\gamma\subset\sigma}\sum_{\lambda\subset\sigma} V_\lambda \bar{\delta}_{\lambda\gamma} \sum_{\alpha\subset\tau} G_0 t_\gamma G_0 U_{\gamma\alpha}^{\sigma\tau} |\vec{r}^{(0)}\Phi_\alpha^{(\tau)}> \;. \tag{A.4}$$

In order to proceed we need the expression for $|\vec{r}\,\Psi^{(\sigma)-}_{\vec{p}\,\vec{q}}>$ in terms of the initial state $|\vec{r}\;\vec{p}\;\vec{q}>$. This is obtained from three-body theory by recalling that

$$|\chi^{(\sigma)-}_{\beta;\vec{p}\,\vec{q}}> = -G_0(E-i0)\sum_{\lambda\subset\sigma} M^{\sigma}_{\beta\lambda}(E-i0)\,|\vec{p}\,\vec{q}> \quad , \tag{A.5}$$

where $M^{\sigma}_{\beta\lambda} = V_{\beta}\delta_{\beta\lambda} - V_{\beta}G^{\sigma}V_{\lambda}$ is the three-body Faddeev operator in subsystem σ. Combining (A.5) with the last of Eqs. (5.4) we obtain

$$G_0(E-i0)\,V_{\gamma}\,|\vec{r}\;\Psi^{(\sigma)-}_{\vec{p}\,\vec{q}}> = G_0(E-i0)\sum_{\lambda\subset\sigma} M^{\sigma}_{\gamma\lambda}(E-i0)\,|\vec{p}\;\vec{q}> \; . \tag{A.6}$$

With (A.6), the on-shell amplitude $\mathscr{E}^{\sigma\tau}$ can be written (recall that $G_0^{+}(E-i0) = G_0(E+i0)$, etc.),

$$\mathscr{E}^{\sigma\tau} = \sum_{\beta\subset\sigma}\;\sum_{\gamma\subset\sigma}\;\sum_{\lambda\subset\sigma}\;\sum_{\alpha\subset\tau} <\vec{r}\;\vec{p}\;\vec{q}\,|M^{\sigma}_{\beta\lambda}\bar{\delta}_{\lambda\alpha}G_0 t_{\gamma} G_0 U^{\sigma\tau}_{\gamma\alpha}\,|\,\vec{r}^{\,(0)}\Phi^{(\tau)}_{\alpha}> . \tag{A.7}$$

To simplify this expression we recall from the matrix notation[7] that $\mathbf{G}_0^{(4)} = \left\{-\delta^{\sigma\tau}G_0 W^{\sigma}_{\beta\alpha}G_0\right\}$, where $W^{\sigma}_{\beta\alpha}$ is the connected part $M^{\sigma}_{\beta\alpha}$; i.e.,

$$W^{\sigma}_{\beta\alpha} = M^{\sigma}_{\beta\alpha} - \delta_{\beta\alpha}t_{\beta} = -\sum_{\gamma\subset\sigma} M^{\sigma}_{\beta\gamma}\bar{\delta}_{\gamma\alpha}G_0 t_{\alpha} \quad . \tag{A.8}$$

Using the fact that $\mathbf{G}_0^{(4)}\mathbf{T}^{(4)} = \mathbf{N}^{(4)} = \left\{G_0 K^{\sigma\tau}_{\beta\alpha}\right\}$, where $K^{\sigma\tau}_{\beta\alpha}$ is the four-body kernel operator, we can now write instead of (A.7),

$$\mathscr{E}^{\sigma\tau} = \sum_{\lambda\subset\sigma}\;\sum_{\alpha\subset\tau} <\vec{r}\;\vec{p}\;\vec{q}\,|\,K^{\sigma\tau}_{\lambda\alpha}\,|\,\vec{r}^{\,(0)}\Phi^{(\tau)}_{\alpha}> \; . \tag{A.9}$$

When summed over σ, (A.9) becomes identical to the expression for the full breakup scattering amplitude given in Ref. 8.

We conclude by considering the partial breakup amplitude. We proceed as before, and take expression (7.5) for $\mathscr{F}^{\sigma\tau}_{(\delta)}$ fully on-shell,

$$\mathscr{F}^{\sigma\tau}_{(\delta)} = \sum_{\gamma\subset\sigma}\;\sum_{\lambda\subset\sigma}\;\sum_{\alpha\subset\tau} <\vec{r}\;\Psi^{(\sigma)-}_{(\delta)\vec{p}}\,|\,V_{\lambda}\bar{\delta}_{\lambda\gamma}\,G_0 t_{\gamma} G_0 U^{\sigma\tau}_{\gamma\alpha}\;|\vec{r}^{\,(0)}\Phi^{(\tau)}_{\alpha}> \tag{A.10}$$

again using (A.1). Further, we recall from three-body theory that

$$|\Psi^{(\sigma)-}_{\beta;(\delta)\vec{p}}> = \left(\delta_{\beta\delta} - G_0(E-i0)\, K^{\sigma}_{\beta\delta}(E-i0)\right) |\vec{p}\ \phi^{\delta}_{\kappa}> \quad . \tag{A.11}$$

If this expression is multiplied by $t_{\beta}(E-i0)\,\bar{\delta}_{\beta\lambda}$, the Faddeev equation for $K^{\sigma}_{\beta\delta}$ can be used to simplify the right-hand side. Using in addition the second of Eqs. (5.4) on the left-hand side, we get

$$-\sum_{\lambda\subset\sigma} t_{\beta}(E-i0)\,\bar{\delta}_{\beta\lambda} G_0(E-i0)\, V_{\lambda}\,|\Psi^{(\sigma)-}_{(\delta)\vec{p}}> = K^{\sigma}_{\beta\delta}(E-i0)|\vec{p}\ \phi^{\delta}_{\kappa}> \quad . \tag{A.12}$$

Finally, with the relation $K^{\sigma}_{\beta\delta} = -t_{\beta} G_0 U^{\sigma}_{\beta\delta}$ we get for the on-shell value of (A.10),

$$\mathscr{F}^{\sigma\tau}_{(\delta)} = \sum_{\gamma\subset\sigma}\ \sum_{\alpha\subset\tau} <\vec{r}\ \vec{p}\ \phi^{\delta}_{\kappa}|\sum_{\lambda\subset\sigma} U^{\sigma}_{\gamma\lambda} G_0 t_{\lambda} G_0 U^{\sigma\tau}_{\lambda\alpha}|\ \vec{r}^{\,(0)} \Phi^{(\tau)}_{\alpha}> \quad . \tag{A.13}$$

We compare this with the expression obtained in Ref. 8 for the partial breakup amplitude, i.e., with

$$\sum_{b_2} <\boldsymbol{\Phi}^{[b_3]}|\mathbf{B}^{(3,2)}|\boldsymbol{\Phi}^{[a_2]}> \ ; \tag{A.14}$$

with the definitions $\boldsymbol{\Phi}^{[b_3]} = \left\{\delta_{\beta\gamma}\,\Phi_{(\beta)}\right\}$, $\boldsymbol{\Phi}^{[a_2]} = \left\{\delta^{\sigma\tau}\,\Phi^{(\tau)}_{\gamma}\right\}$ and $\mathbf{B}^{(3,2)} = \left\{B^{\sigma\tau}_{\beta\alpha}\right\} = \left\{\sum_{\gamma\subset\sigma} U^{\sigma}_{\beta\gamma} G_0 t_{\gamma} G_0 U^{\sigma\tau}_{\gamma\alpha}\right\}$, (A.14) becomes identical to (A.13) when the latter is summed over all $\sigma\subset\delta$.

FOOTNOTES AND REFERENCES

1. L. D. Faddeev, Mathematical Aspects of the Three-Body Problem in Quantum Scattering Theory (Davey, New York, 1965).
2. O. A. Yakubovskii, Yad. Fiz. 5, 1312 (1967) [Sov. J. Nucl. Phys. 5, 937 (1967)].
3. I. H. Sloan, Phys. Rev. C 6, 1945 (1972).
4. B. R. Karlsson and E. M. Zeiger, Phys. Rev. D 11, 939 (1975).
5. The choice of incoming wave scattering states was motivated in Ref. 4.
6. Primary singularities (i.e., elastic, rearrangement, or breakup poles) can be distinguished from secondary singularities by the fact that the former are present in all terms of an iterative series of the amplitudes, while the latter get progressively weaker and disappear after a few iterations.
7. B. R. Karlsson and E. M. Zeiger, Phys. Rev. D 9, 1761 (1974).
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9. More details on this four-body index notation can be found in Ref. 7.
10. For definiteness, we will always consider a two-cluster initial state of the type (123)(4), i.e., a three-body bound state and a fourth free particle.
11. Ref. 1, Chapter 9.
12. As in the three-body case, we must use incoming wave scattering states in (4.3) in order that the entities defined in (4.5) be the scattering amplitude components.
13. T. Sasakawa, in Few Body Dynamics, edited by Asoke N. Mitra et al. (North-Holland, Amsterdam, 1976), p. 116; Phys. Rev. C 13, 1801 (1976).
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