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CORRECT ANALYTIC EXTRAPOLATIONS TO SMALL ω^2 AND t II: THE $\pi\pi$ SYSTEM^{*}

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ABSTRACT

Pion-pion elastic scattering amplitudes below threshold have been computed from fixed-t dispersion relations correctly formulated in the sense of Hadamard. The results are fully consistent with the features predicted by current algebra and PCAC.

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I. Introduction

Pion-pion elastic scattering, among all hadronic processes, enjoys a high degree of symmetry and a remarkable simplicity. Apart from the modest complication of three different isospin channels, pion-pion elastic scattering is the only hadronic process which crosses to itself, becoming thus an invaluable field for the study of strong interactions; it is indeed expected that even the knowledge of few properties of the latter could be enough to determine the structure of pion-pion amplitudes at low momenta. This has been largely demonstrated by the celebrated derivation by Weinberg¹ of a linear expansion of these amplitudes based on $SU_2 \times SU_2$ current algebra and PCAC, remarkably stable with respect to subsequent inclusions of unitary corrections and higher order terms.^{2,3}

Usually the amplitudes at low momenta are expressed in terms of scattering lengths and effective range parameters; however, these parameters cannot be extracted from experimental data in the physical region in a "stable" way. Indeed, it can be proved that, in order for a finite error bound to exist, we must treat only parameters averaged over a finite segment of the boundary. As a consequence, whenever the effective range parameters were sizeable, a test of current algebra results for scattering lengths would be particularly difficult (note that this happens to be exactly the case for isoscalar $\pi\pi$ S-wave).

Such a difficulty is automatically overcome if, more wisely, predictions on the low-momentum expansions are compared with extrapolations of the amplitudes from the physical region to the interior of their analyticity domain.

Even this extrapolation requires the adoption of a correct technique (in the sense of Hadamard), to reach the maximum of stability, particularly in the $\pi\pi$

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case, where, despite the recent increase in experimental information, large uncertainties still exist at c.m. energies below 500 MeV. Such a technique was reached, exploiting the infinite topological forms in which Cauchy integrals can be written, by Ciulli, Fisher and Nenciu⁴, and has been applied in an accompany-ing paper⁵ (to which we refer for more details) to a systematic investigation of elastic π^{\pm} p scattering.

Such a method weights statistically the experimental input used—so that it does not give undue relevance to poorly known amplitude parameters—and leads to an extrapolation error bound which automatically saturates the Nevanlinna lower bound^{4, 5}, and it is therefore guaranteed to be correct (in the sense of Hadamard) and to be optimally stable.

II. The Method and the Input

The method, in the case of pion-pion elastic scattering, is rather simple. Since at the present level of information and in the absence of any rapidly varying structure at low momenta we can neglect completely all electromagnetic and SU_2 -noninvariance effects, the singularity structure of pion-pion amplitudes is reduced to a right-hand cut in the ω^2 plane from $\omega_0^2 = \mu^2 (1 + t/4\mu^2)^2$ to infinity, which can be mapped into the unit circle in the plane of the variable

$$z(\omega^{2}) = \frac{\omega_{0} - \sqrt{\omega_{0}^{2} - \omega^{2}}}{\omega_{0} + \sqrt{\omega_{0}^{2} - \omega^{2}}} , \qquad (1)$$

such that z(0) = 0. Due to the peculiar crossing symmetry of the process, it is not necessary to compute both ω^2 and t dependence for all isospin-invariant amplitudes, since the two are obviously and simply correlated. The preoccupying

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features which emerge at once from the mapping (1) are the relatively large arc Γ_{a} on which the very low $\pi\pi$ mass regions $w \leq 0.5$ GeV is mapped (from $2 \times 156.9^{\circ}$ at $t = -2\mu^{2}$ to $2 \times 160.2^{\circ}$ at $t = 2\mu^{2}$), and the small two arcs Γ_{b} on which our best knowledge of $\pi\pi$ interaction features is concentrated, from a c.m. energy of 0.5 GeV to one of 1.9 GeV; this latter energy corresponds to an angle of 178.7° on the unit circle (almost independent of t in the range $|t| \leq 2\mu^{2}$), limiting each arc (both in the upper and lower half of the unit circle) covered by our most detailed information to a scarce 21.8° at $t = -2\mu^{2}$, decreasing to 18.5° at $t = 2\mu^{2}$. This leaves out only a very small $2 \times 1.3^{\circ}$ for the arc Γ_{c} on which the "asymptotic region" w > 1.9 is mapped.

This means that any substantial improvement on our check of lowmomentum predictions can come only from a better analysis of $\pi\pi$ interaction at c.m. energies below 500 MeV, particularly from K_{e4} decay, since low-mass di-pion production cannot be analysed in a model-independent way, and often the model includes the very elements we would like to test against our extrapolations.⁶

A first evaluation of the three amplitudes $A^{I}(\omega,t)$ —where I is the t-channel isospin—was attempted some years ago⁷ as soon as the first statistically good analysis of di-pion production⁸ was available. Since then there has been a considerable improvement both in our knowledge of the intermediate energy region⁹⁻¹¹ and of the low-mass di-pion system, both in production^{6, 12} and decay^{13, 14} processes which can certainly allow a better extrapolation to very low momenta.

As in our previous works^{5,7}, the amplitudes are modified so that their behaviour as z - 1 is as uniform as possible, and such as to allow the

introduction of a constant bound on Γ_c :

$$F^{(0)}(z) = (1+z)^{3/2} A^{(0)}[\omega(z), t]$$
(2)

$$F^{(1)}(z) = A^{(1)}[\omega(z), t]/\omega$$
 (3)

$$F^{(2)}(z) = (1+z)^{1/2} A^{(2)}[\omega(z), t]$$
(4)

and, once their "error corridors" around the measured hystograms $\hat{F}^{(I)}(z)$

$$\epsilon^{(I)}(z) \ge |F^{(I)}(z) - \hat{F}^{(I)}(z)|, \ z \ \epsilon \Gamma_{a} \cup \Gamma_{b}$$
(5)

and their bounds

$$M^{(I)} \ge |F^{(I)}(z)|, z \in \Gamma_c$$
 (6)

are fixed, the weight functions

$$\mathscr{G}^{(\mathbf{I})}(z) = \exp \frac{1}{2\pi \mathbf{i}} \left[\int_{\Gamma_{\mathbf{a}} \cup \Gamma_{\mathbf{b}}} \ln \frac{\lambda}{\epsilon^{(\mathbf{I})}(\zeta)} \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} + \int_{\Gamma_{\mathbf{c}}} \ln \frac{\lambda}{\mathbf{M}^{(\mathbf{I})}} \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} \right]$$
(7)

can be built, leading to the optimal estimates

$$\hat{\mathbf{F}}^{(\mathbf{I})}(0) = \frac{1}{2\pi \mathscr{G}^{(\mathbf{I})}(0)} \int_{\Gamma_{\mathbf{a}} \cup \Gamma_{\mathbf{b}}} \operatorname{Re}\left[\hat{\mathbf{F}}^{(\mathbf{I})}(\theta) \mathscr{G}^{(\mathbf{I})}(\theta)\right] d\theta$$
(8)

with the corresponding Nevanlinna bounds

$$\Delta^{(\mathbf{I})} = \lambda \mathscr{G}^{(\mathbf{I})}(0)^{-1} .$$
(9)

The input selected for the present analysis are the energy-independent analysis of Hyams et al.⁹ for the intermediate energy region Γ_b , and a smoothed interpolation of the more recent analysis on K_{e4} decay^{13,14} and low mass di-pion production¹² together with results from older analyses of forward-backward asymmetry zeros¹⁵ for the "threshold region" Γ_a . Since these latter data are rather consistent with each other, they have been interpolated by an effective range approximation, smoothly joined to pion-pion scattering data in the region 500-600 MeV.

To ensure good analyticity properties with these approximations we have computed the integrals

$$\delta^{(\mathbf{I})} = \frac{1}{2\pi \mathbf{i} \mathscr{G}^{\mathbf{I}}(0)} \int_{\Gamma_{\mathbf{a}} \cup \Gamma_{\mathbf{b}}} \hat{\mathbf{F}}^{(\mathbf{I})}(z) \mathscr{G}^{(\mathbf{I})}(z) dz \qquad (10)$$

which have been systematically checked to satisfy the inequality 6

$$\delta^{(I)^2} << \Delta^{(I)^2}$$
 (11)

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Note that as a result of our choice of amplitudes the rather poorly understood P-wave scattering length is much less important here than for other dispersive computations¹⁶; moreover, since our low-energy parameters (and their errors) are just a smooth interpolation used to translate rather unrelated data into an hystogram and an error corridor, those problems are completely beyond our concern, as long as inequalities (11) are satisfied.

To assure finiteness of arg $\mathscr{G}^{(I)}$, the integrand in relation (7) has to be continuous on Γ ; therefore $\epsilon^{(I)}$ is joined continuously to the constant bound $M^{(I)}$ just beyond the boundary point z_M between Γ_b and Γ_c . Such a bound can be fixed equal to $|\hat{F}^{(I)}(z_M)|$ without any serious problem, due to the smallness of $1 - \theta_M / \pi$ (where $\theta_M = \arg z_M$), and the essentially non-increasing behaviour of our $|\hat{F}^{(I)}(z)|$ beyond the ρ peak.

Due to the much larger length of Γ_a relative to Γ_b one could fear that the relatively poorly known low-momentum data influence too deeply the extrapolation results: however, from relation (8), the weight $\mathscr{G}^{(I)}(z)$ can suppress at will the poor amplitude measurements on Γ_a , once a sufficiently large error bound on the low-momentum parameters is introduced, and this is what is needed for inequalities (11) to be adequately satisfied. What in fact happens with the use of relations (8)-(9) and a self-consistent input—satisfying relation (11)—is that while Γ_b determines (almost completely) $\hat{\mathbf{F}}^{(I)}(0)$, we can see, rewriting (9) with $\lambda = \epsilon_M^{(I)} = \max_{\substack{\theta \in \Gamma_a \cup \Gamma_b}} \epsilon_{\substack{(I) \\ \theta \in \Gamma_b}} \epsilon_{\substack{(I) \\ \theta \in$

$$\Delta^{(I)} = \epsilon_{M}^{(I)} M^{(I)} M^{(I)} \times \exp\left[-\frac{1}{\pi} \int_{0}^{\theta} M \ln \frac{\epsilon_{M}^{(I)}}{\epsilon^{(I)}(\theta)} d\theta\right]$$
(12)

and therefore Γ_{a} , via $\epsilon_{M}^{(I)}$, determines largely the size of $\Delta^{(I)}$.

III. Output and Comments

Results for the three pion-pion amplitudes $A^{(0)}(\omega^2 = 0, t)$, $A^{(2)}(\omega^2 = 0, t)$ and $A^{(1)}(0,t) = \lim_{\omega \to 0} A^{(1)}(\omega,t)/\omega$ are listed in Table I. Errors, as a consequence of the extent of Γ_a , are very large, particularly for the first amplitude where the threshold region is further enhanced by the factor $(1+z)^{3/2}$ necessary to make the asymptotic bound on Γ_c finite. Since our early determinations,⁷ our knowledge of the low-momentum region $w \leq 500$ MeV has been considerably improved, reducing the possible set of parameters essentially to the one which, being closer to current algebra predictions¹⁻³, we then called the "optimist's set." This has somewhat reduced errors on $A^{(2)}$ and $A^{(1)'}$ to a 20-25% level, making a test of low-energy predictions on these amplitudes much more significant than in our previous attempt. Despite this, error levels are still such that an investigation of the zeros' positions is not yet feasible with sufficient accuracy.

To get a clearer picture, much more statistics have to be accumulated below 500 MeV, mainly from accurate and large studies of both K_{e4} and doublecharged, low-mass di-pion production. However, once possible on-massshell corrections of $\mathcal{O}(2m_{\pi}^2/m_{\rho}^2)$ are considered, the values for $A^{(1)'}$ give a good test of the π - π Adler-Weisberger relation⁷ $A^{(1)'}(q^2 = \omega^2 = t = 0) = 4/f_{\pi}^2$.

We can also compare predictions of Refs. 1-3 with our extrapolations. Neglecting contributions from cut singularities and higher powers in momenta, the extrapolation of Ref. 1, linear in the invariants s,t,u, together with its assumptions on the expectation values of sigma-commutators, can be fitted to values of Table I, giving

$$f_{\pi} = 122.2 \pm 4.8 \text{ MeV}$$

with a χ^2 of 1.66 for 26 degrees of freedom, only 7% less than the experimental value¹⁸ $f_{exp} = 131.78 \pm 0.11$ MeV for the pion decay constant. Keeping the same linear approximation, but releasing any hypothesis about sigma commutators, we get from current algebra and PCAC the simple expansions

$$A^{(1)'}(0,t) = \lim_{\omega \to 0} A^{(1)}(\omega,t)/\omega = 4/\tilde{f}_{\pi}^{2},$$

$$A^{(0)}(0,t) = (2t + 5\sigma_{2} - m_{\pi}^{2})/\tilde{f}_{\pi}^{2},$$

$$A^{(2)}(0,t) = (2m_{\pi}^{2} + 2\sigma_{2} - t)/\tilde{f}_{\pi}^{2},$$
(13)

since in a linear expansion crossing symmetry imposes the constraint

$$\sigma_0 - \sigma_2 = m_\pi^2 . \tag{14}$$

A least square fit gives then

and

$$f_{\pi} = 122.3 \pm 6.5 \text{ MeV}$$

 $\sigma_2/m_{\pi}^2 = 0.16 \pm 0.46$,

confirming the smallness of exotic pieces in the sigma commutator. However, possible sizeable curvature effects cannot be excluded by our large errors, and such effects could affect rather critically the transition between off-mass-shell current algebra conditions and on-mass-shell amplitudes (unless supplementary conditions were added³).

We have also listed in Table I the predictions of the model (quartic in the momenta) developed by Morgan and Shaw³, who add to the current-algebra conditions additional information on the two-body intermediate states and unitarity conditions to determine the needed parameters. The accord of their predictions with our extrapolations is very good, and all expected systematic deviations from a simple linear model where f_{π} is fixed to its experimental value f_{exp} , as the systematic increase in $A^{(1)}$ and in the slopes of $A^{(0)}$ and $A^{(2)}$ amplitudes as a consequence of unitarity effects, seem to be confirmed, as is their smallness, measured on the average by $(f_{exp}^2 - \tilde{f}_{\pi}^2)/f_{exp}^2 \simeq 0.14 \pm 0.09$, which compares well with their expected size $\mathcal{O}(2m_{\pi}^2/m_{\rho}^2)$; note that in the $\pi - \pi$ case effects of this size come both from the mass extrapolation and from unitarity corrections from the t-channel physical region³.

We can therefore conclude saying that no evidence has been uncovered against the conclusions originally drawn from $SU_2 \times SU_2$ current algebra¹, or their subsequent improvements²⁻³; stronger conclusions could only be

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reached, rather than by more extended measurement of $\pi N \rightarrow \pi \pi N(\Delta)$ processes (since these latter cannot be very well analysed in a model independent way at low di-pion masses⁶), by studies of low-mass di-pion production in purely electromagnetic or weak processes, like $e^+e^- \rightarrow \pi^+\pi^{-19}$ (for the isovector P-wave), $e^+e^- \rightarrow e^+e^-\pi^+\pi^-$ or $e^+e^-\pi^0\pi^0$ (for isoscalar and isotensor S-waves) and K_{e4} decay.

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After completion of this work, we received a copy of a somewhat different analysis of the I = $1 \pi \pi$ channel²¹, whose results essentially confirm our conclusions on the validity of Adler-Weisberger relation in the $\pi \pi$ system.

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Table Caption

Table I: Results of our extrapolation compared with current algebraic predictions based on expansions for $\pi\pi$ amplitudes respectively linear in the invariants (Weinberg, Ref. 1) and quartic in the momenta (Morgan and Shaw, Ref. 3).

| | | Extrapolation | Weinberg | Morgan-Shaw |
|--------------------------|------|------------------|----------|-------------|
| | t | | | |
| $A^{(0)}$: | -2.0 | -5.89 ± 3.25 | -5.608 | -5.702 |
| | -1.5 | -4.22 ± 3.35 | -4.487 | -4.601 |
| | -1.0 | -2.86 ± 3.45 | -3.365 | -3.473 |
| | -0.5 | -1.52 ± 3.56 | -2.243 | -2.316 |
| | 0.0 | -0.19 ± 3.67 | -1.122 | -1.132 |
| | 0.5 | 1.10 ± 3.80 | 0 | 0.082 |
| | 1.0 | 2.37 ± 3.93 | 1.122 | 1.329 |
| | 1.5 | 3.61 ± 4.07 | 2.243 | 2.616 |
| | 2.0 | 4.82 ± 4.22 | 3.365 | 3.952 |
| (1)1 | | | | |
| A : | -2.0 | 5.08 ± 2.12 | | 5.674 |
| | -1.5 | 5.01 ± 1.74 | | 5.626 |
| | -1.0 | 4.99 ± 1.50 | | 5.607 |
| | -0.5 | 5.01 ± 1.32 | | 5.608 |
| | 0.0 | 5.04 ± 1.19 | 4.487 | 5.622 |
| | 0.5 | 5.07 ± 1.09 | | 5.648 |
| | 1.0 | 5.11 ± 1.02 | | 5.682 |
| | 1.5 | 5.16 ± 0.96 | | 5.722 |
| | 2.0 | 5.20 ± 0.91 | | 5.768 |
| $\langle \alpha \rangle$ | | | | |
| $A^{(2)}$: | -2.0 | 6.20 ± 1.62 | 4.487 | 5.854 |
| | -1.5 | 5.37 ± 1.67 | 3.926 | 5.011 |
| | -1.0 | 4.55 ± 1.72 | 3.365 | 4.036 |
| | -0.5 | 3.74 ± 1.78 | 2,804 | 3.432 |
| | 0.0 | 2.95 ± 1.83 | 2.243 | 2.690 |
| | 0.5 | 2.17 ± 1.89 | 1.683 | 1.977 |
| | 1.0 | 1.41 ± 1.96 | 1.122 | 1.293 |
| | 1.5 | 0.68 ± 2.03 | 0.561 | 0.636 |
| | 2.0 | -0.05 ± 2.10 | 0 | 0.007 |

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TABLE I

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