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Correct Analytic Extrapolations to Small ω^2 and t
I: The π-N System*

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Abstract

Fixed t-dispersion relations, correct in the sense of Hadamard, are applied to the determination of $\pi^{\pm}$p elastic scattering amplitudes below threshold. The relevant results are a pion-nucleon coupling constant $G^2/4\pi \simeq 13.16$ and a sigma commutator matrix element $\Sigma \simeq 41$ MeV, considerably lower than previous estimates. They result from both the inclusion of mass-splitting in the nucleon doublet and the extremely singular nature of the nucleon exchange contributions in the proximity of $\omega^2 = t = 0$. Consequently a $(3,\bar{3}) \otimes (\bar{3},3)$ mechanism for the "explicit" breaking of chiral $SU_3 \otimes SU_3$ symmetry "à la" Gell-Mann, Oakes and Renner does not meet any difficulty in accommodating these values.

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1. Analytic extrapolations for the π-N system: the state of the art

It has long been known[1] that the "conventional" use of dispersion relations, as extrapolation tools either to the interior of the holomorphy domain $\mathscr{D}$ of the scattering amplitude F or to its boundary Γ, constitutes a classical case of an incorrectly posed problem, in the sense of Hadamard, and that, as a consequence, the calculated values are unstable.

Stability is often reached (as for the "discrepancy method") following the phenomenological principle ("whenever in doubt, expand in series and retain lowest terms only") often attributed to Fermi,[2] or some more sophisticated and less easily identifiable version of the same.

However, the apparently naive observation that Cauchy integrals can be written in infinite, tautological forms, simply multiplying F by any function $\mathscr{G}$ holomorphic in $\mathscr{D}$, allowed Ciulli, Fischer and Nenciu[3] to reach a simple minimum condition on the extrapolation error, saturating the Nevanlinna lower bound.[1,3]

Their analytic extrapolation technique, correct in the sense of Hadamard, is additionally free, as a bonus, of some problems, like subtractions and guesswork on asymptopia, which usually plague the more traditional approaches.

Yet a large fraction of the most credited results on πN low-energy parameters is based on an incorrect use of analyticity. The situation is particularly bad for scattering lengths, since they require a "boundary-to-boundary" extrapolation, which can be correctly attempted only and only if we extrapolate to a finite arc, or "in the mean," and not to a single point on the boundary.[1,4]

Furthermore, SU_2-breaking effects are present, above and below the elastic threshold. While these effects could be neglected as long as they were dwarfed by bigger uncertainties (as is still the case for most elastic processes),

$\pi^{\pm}p$ elastic scattering has now considerable evidence for such a breaking, at least in the P_{33} resonant wave.[5]

The naive expectation that such effects, as those generated by unphysical ranges present in $\pi^{-}p$ scattering,[6] could be comparable to the ratio

$$\delta = (m_n - m_p)/(m_n + m_p)$$

is destroyed by properly introducing the correlations induced by fixed-t dispersion relations between the πN coupling constant G and the S-wave scattering lengths.[7] As a result of the neglect of SU_2 breaking in the nucleon doublet, we may expect, once we solve these correlations, changes in $\Delta G^2/G^2$ comparable to the ratio $\Delta\omega_B/\omega_B \simeq \left(m_n^2 - m_p^2\right)/\mu_{\pm}^2$ (here $\mu_{\pm}$ and μ_0 will indicate the pion masses).

Nucleon exchange terms are furthermore strongly varying around $\omega^2 = t = 0$: in the important limit[8] $t \to 2\mu_{\pm}^2$, $\omega^2 \to 0$, none of the invariant emplitudes has a change in the nucleon exchange contribution of order δ, when the SU_2-symmetric limit $m_n \to m_p$ is assumed, and only in the combination $A^{(+)} + \omega \cdot B^{(+)}$ a complete cancellation occurs between terms of order δ^0. All other combinations have nucleon exchange terms which change either by terms of order δ^0, or, worse, by terms of order δ^{-1}, as it is in the case for the amplitudes $A^{(-1)}/\omega$ and $B^{(-)}$.

This paper will apply the correct technique developed by Ciulli, Fischer and Nenciu[3] to fixed-t extrapolation of elastic $\pi^{\pm}p$ scattering amplitudes; though the next section will deal with the technique in detail, we do not claim any original development, and apologize to the more learned readers for this piece of advertising. But we feel our choice justified by the inadequate popularity enjoyed by so powerful a method. The following sections will discuss the details of the $\pi^{\pm}p$ scattering analysis, our choice for the inputs and the results

obtained, trying whenever possible to discuss their implications on the symmetries of strong interactions.

In this we have displayed a marked empathy with the authors of the so-called GMOR model,[9] which in our opinion conforms best to a very old principle of "logical economy."[10]

2. A correct formulation for fixed-t analytic extrapolation

Let us assume that an amplitude $F(\nu^2)$, real analytic and even under crossing, is known to lie inside a finite "error corridor" on a finite portion Γ_1 of the boundary Γ of its holomorphy domain $\mathscr{D}$,

$$|F(\nu^2) - \hat{F}(\nu^2)| \leq \epsilon(\nu^2), \quad \nu^2 \in \Gamma_1 \tag{1}$$

(where F and ϵ are continuous on Γ_1), and to be bounded on the remainder Γ_2 of the boundary by some finite, continuous function

$$|\hat{F}(\nu^2)| \leq M(\nu^2), \quad \nu^2 \in \Gamma_2 \quad . \tag{2}$$

To us, and the authors of Refs. 1 and 3, this is the simplest way of smoothly interpolating between the actual, isolated measurements, for which the problem of continuation would have no stable solution. The details of such an interpolation influence the actual numerical result, but <u>not its stability</u>; since only continuity is needed to build a workable algorithm, only the simplest continuous point-to-point interpolation will be used on Γ_1.

All problems related to integration contours of infinite measure may be conveniently eliminated by mapping $\mathscr{D}$, the cut ν^2 complex plane, onto the unit disk D, and the map can be chosen so that the point ω^2, internal to $\mathscr{D}$, where we are going to extrapolate F, will fall at the center of D.

For ω^2 on the real axis (the only points we shall be interested in), this is realized by the z-map

$$z(\omega^2, \nu^2) = \frac{\sqrt{\nu_t^2 - \omega^2} - \sqrt{\nu_t^2 - \nu^2}}{\sqrt{\nu_t^2 - \omega^2} + \sqrt{\nu_t^2 - \nu^2}} \tag{3}$$

where ν_t^2 is the lowest branch point in the ν^2 plane, and we choose $\nu_t^2 \geq \omega^2 \geq 0$, neglecting the unphysical cut[11] due to radiative capture.

If Γ_1 extends in the ν^2 plane from ν_t^2 to N^2, after the z-map Γ_1 will span on the unit circle an arc, symmetric around $z(\omega^2, \nu_t^2) = 1$, of length

$$2\theta_M = 4 \tan^{-1} \sqrt{\frac{N^2 - \nu_t^2}{\nu_t^2 - \omega^2}} \quad .$$

We can write in D the Cauchy theorem as

$$F(z = 0) = F(\omega^2) = \frac{1}{2\pi i\, \mathcal{G}(0)} \oint_{\Gamma} \frac{F(z)\, \mathcal{G}(z) dz}{z} \tag{4}$$

where $\mathcal{G}(z)$ is any function holomorphic in D; Ciulli, Fischer and Nenciu[3] solved the problem of finding a $\mathcal{G}(z)$ such that the actually computable integral

$$\hat{F}(z = 0) = \hat{F}(\omega^2) = \frac{1}{2\pi i\, \mathcal{G}(0)} \int_{\Gamma_1} \frac{\hat{F}(z)\, \mathcal{G}(z) dz}{z} \tag{5}$$

has an actually computable error bound

$$| F(\omega^2) - \hat{F}(\omega^2) | \leq \Delta(\omega^2) < \infty \tag{6}$$

which saturates Nevanlinna lower bound.

The construction of such a $\mathcal{G}(z)$ is rather simple: let us first decompose

$$F(z) = \gamma^{-1}(z)\, f(z) \tag{7}$$

where f has a constant width λ of the error corridor and a constant bound μ

on Γ_2; it is easy to derive then

$$\left.\begin{aligned} |\gamma(z)| &= \lambda/\epsilon(z) \ , \quad z \in \Gamma_1 \\ &= \mu/M(z) \ , \quad z \in \Gamma_2 \end{aligned}\right\} \tag{8}$$

and to construct $\gamma(z)$ with the Schwarz-Villat formula, as

$$\gamma(z) = \exp \frac{1}{2\pi i} \oint_{\Gamma} \ln |\gamma(\zeta)| \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} \ . \tag{9}$$

Let us now consider the problem of determining f(0) from $\hat{f}(z) = \gamma(z) \cdot \hat{F}(z)$ with a minimum error bound, using a weight function g(z). Since all points of Γ_1 have now the same statistical weight, being the error corridor of uniform width, and being the bound on Γ_2 a constant, we can choose, up to an arbitrary multiplicative constant,

$$\left.\begin{aligned} |g(z)| &= 1 \ , \quad z \in \Gamma_1 \\ &= \text{constant} < 1, \quad z \in \Gamma_2 \ . \end{aligned}\right\} \tag{10}$$

Introducing the harmonic measure ϕ, vanishing on Γ_1, and it conjugate $\tilde{\phi}$, we have then

$$g(z) = \exp - \rho \left[\phi(z) + i\,\tilde{\phi}(z)\right] \tag{11}$$

and the extrapolation error bound is easily computed as

$$|f(0) - \hat{f}(0)| \leq \left\{\lambda\left[1 - \phi(0)\right] + \mu\,\phi(0) \exp - \rho\right\} \cdot \exp \rho\,\phi(0)$$

which has the absolute minimum

$$\mu^{\phi(0)} \lambda^{1-\phi(0)} = \lambda g(0)^{-1} \tag{12}$$

for $\rho = \ln \mu/\lambda$. This minimum saturates the Nevanlinna lower bound: our choice for g(z) was then "natural" indeed, since it did not affect the information

content of the data and reached the "optimal" error bound.

Returning now to the amplitude F(z), the absolute minimum for the extrapolation error Δ will then be reached by the weight function

$$\mathcal{G}(z) = g(z)\,\gamma(z)\,, \tag{13}$$

as can be easily seen writing Cauchy integral (4) for f(z) and then using definition (7); the weight is readily constructed, using Schwarz-Villat formula (9), as

$$\mathcal{G}(z) = \exp\frac{1}{2\pi i}\left[\int_{\Gamma_1} \ln\frac{\lambda}{\epsilon(\zeta)}\,\frac{\zeta+z}{\zeta-z}\,\frac{d\zeta}{\zeta} + \int_{\Gamma_2} \ln\frac{\lambda}{M(\zeta)}\,\frac{\zeta+z}{\zeta-z}\,\frac{d\zeta}{\zeta}\right], \tag{14}$$

and will give a minimum error bound

$$\tilde{\Delta} = \delta\gamma(0)^{-1} = \lambda\,\mathcal{G}(0)^{-1}\,, \tag{15}$$

independent of λ, which again saturates the Nevanlinna lower bound.

To demonstrate the main advantage of this technique, let us compare it with the most favorable "conventional" case, when $F(\nu^2)$ obeys an unsubtracted dispersion relation, and let be, for simplicity,

$$\epsilon(\nu^2) = \epsilon \quad , \qquad \nu_t^2 \le \nu^2 \le N^2$$

and

$$M(\nu^2) = M_0\left(\nu^2/\nu_t^2\right)^{-\alpha} \quad (\alpha > 0), \qquad \nu^2 > N^2\,.$$

The extrapolation to the points $\omega^2 < \nu_t^2$ has then a computable error bound, which for the simple case $\omega^2 = 0$ is

$$\Delta_{conv} = \frac{\epsilon}{\pi}\ln N^2/\nu_t^2 + \frac{1}{\pi\alpha}\,M_0\left(N^2/\nu_t^2\right)^{-\alpha}\;;$$

note that the error is no longer computable either for $\omega^2 \ge \nu_t^2$ or for any physical-region subtraction. Δ_{conv} will have an absolute minimum for

$M_0 \left(N^2/\nu_t^2\right)^{-\alpha} = \epsilon$, since any other choice for N leads to a loss of information , and we shall then have

$$\Delta_{conv} = \frac{\epsilon}{\pi\alpha} \left[1 + \ln \frac{M_0}{\epsilon} \right] \tag{16}$$

for the choice $N^2 = \nu_t (M_0/\epsilon)^{1/(2\alpha)}$.

In the z-map (3) the bound M becomes on the circle $z = e^{i\theta}$

$$M(\theta) \simeq M_0 (\cos\theta/2)^{\alpha}$$

and, for $\pi - \theta_M \ll \pi$, we can compute the Nevanlinna bound, saturated by the choice (14) for $\mathcal{G}(z)$, as

$$\tilde{\Delta} \simeq \epsilon^{\theta_M/\pi} \left[M_0 \left(\frac{\pi - \theta_M}{2e} \right)^{2\alpha} \right]^{1 - \theta_M/\pi} \tag{17}$$

where, to make a consistent comparison with the conventional estimate (16), we have to choose

$$\theta_M = 2 \tan^{-1} \sqrt{(M_0/\epsilon)^{1/\alpha} - 1} \quad ;$$

such a comparison is displayed in Table I and demonstrates dramatically the pathological behavior of Δ/ϵ in the limit $\epsilon \to 0$ typical of an incorrectly posed problem.

3. Correct techniques and $\pi^{\pm}p$ elastic amplitudes.

How can this formalism be adapted to elastic $\pi^{\pm}p$ scattering? Let us review its assumptions and see how they fit or have to be modified to fit such a process.

We first assumed a symmetric error corridor in (1), whose sections on

the F-plane are circles of radius $\epsilon(z)$. The most general case, however, would at least require, for suitably small errors, an elliptical section on the F-plane, whose orientation will be specified by some real phase $\alpha(z)$. To treat just the simple case $d\alpha/dz = 0$ one already needs boundary value techniques, replacing Cauchy theorem with the differential monogeneity conditions on the F projections on the ellipse axes.[12] This is, however, adequate only for $t = 0$, where ImF and ReF are separately measureable, and since we can expect $d\alpha/dz \neq 0$ at any $t \neq 0$, where one has to use either partial-wave or amplitude analyses, we prefer the much simpler algorithm based on a symmetric error corridor.

Of course this implies an overly conservative estimate of error bounds, which, in view of the unknown systematic effects hidden in all analyses, we prefer to a more optimistic attitude; we also remind the reader that the conventional standard deviation has been replaced in (1) by an absolute bound, which has to be larger (we choose to fix it at the 95% probability level), though still of the same order of magnitude, and this is indeed the highest price we have to pay for our correct and workable algorithm.

A finite bound M(z) can easily be reached within the z-map multiplying the amplitude F(z) by a simple power $(1 + z)^{\beta}$, which has a cut running from -1 to $-\infty$ and does not introduce then any additional singularity inside D; β can also be adjusted so that we can conservatively replace M(z) with a constant M, making the construction of $\mathcal{G}(z)$ considerably easier. It must also be noted that, unless $|\mathcal{G}|$ is continuous on Γ, $\mathcal{G}(z)$ develops infinite oscillations around the points of discontinuity; assuming $\epsilon(z)$ and M(z) to be continuous on Γ_1 and Γ_2, respectively, this can be avoided everywhere but at $z_M = \exp \pm i\,\theta_M$, unless $\epsilon(z_M) = M(z_M)$. According to the common-sense expectation that our knowledge of any process is rather melting into increasing ignorance than

suddenly vanishing at a point, we have joined $\epsilon(z)$ continuously to the constant $M(z) = M$ at a point $|\theta'| > \theta_M$, such that $|\theta'| - \theta_M \ll \pi - \theta_M$.

Given these modifications, we can convert any given amplitude, measured on a set of points so dense that a smooth hystogram $\hat{F}(z)$ can be drawn through them, into the functions $G(z) = F(z)(1 + z)^{\beta}$, with its corresponding hystogram $\hat{G}(z) = \hat{F}(z)(1 + z)^{\beta}$, and $\mathcal{G}(z)$, built with Schwarz-Villat formula (14) from errors and bounds on $G(z)$, and then we have all the pieces required to build the algorithm proposed in Ref. 3.

However, this is rigorously applicable only to a function holomorphic in D (such as $\pi\pi$ elastic amplitudes). We could eliminate the pole at $z_B = z(\omega^2, \omega_B^2)$ multiplying $G(z)$ and its hystogram by the additional factor $(z - z_B)$. But if our purpose is to gain insight on the dynamical features of πN interactions, other than the πN coupling constant, we are much more interested in $\tilde{F}(\omega^2)$, the amplitude minus its nucleon exchange contribution, than in $F(\omega^2)$ itself. Then this approach is useless, since, calling $G'(z) = G(z)(z - z_B)$, we have

$$\tilde{F}(\omega^2) = F(\omega^2) - F(\omega^2)_B = -G'(0)/z_B - F(\omega^2)_B$$

and, for ω^2 close to the pole position ω_B^2, $\tilde{F}(\omega^2)$ would be given as the difference of two very large numbers, close to each other.

Let us instead consider the integral on the right-hand side of identity (4) for the function $G(z)$, and define the generalized Cauchy integrals

$$I_{i,n} = \frac{1}{2\pi i\, \mathcal{G}_i(0)} \oint_{\Gamma} z^{n-1} F_i(z)(1+z)^{\beta_i} \mathcal{G}_i(z)\, dz \tag{18}$$

of which the previous one is just the particular case $n = 0$. Here i labels the

different invariant amplitudes for the process $\pi^{\pm} p \rightarrow \pi^{\pm} p$ and their combinations we may be interested in; note that, from the definition (14) of the weight functions $\mathcal{G}_i(z)$, integrals $I_{i,n}$ derived for a linear combination of amplitudes F_j cannot be obtained by the <u>same</u> linear combination of the integrals $I_{j,n}$. Through the technique we already described, when $\mathcal{G}_i(z)$ are given by (14) each of these integrals is computed with the optimal error bound $\tilde{\Delta}_i = \lambda \mathcal{G}_i(0)^{-1}$, independent of n.

Cauchy theorem then gives the identity equivalent to (4) for πN scattering, namely,

$$I_{i,n} = \tilde{F}_i(\omega^2)\delta_{n,0} + \left[F_i(\omega^2)_B \delta_{n,0} + \right.$$

$$\left. + \lim_{z \rightarrow z_B} \frac{(z - z_B) F_i(\omega^2)_B \; \mathcal{G}_i(z) \; z^{n-1} (1 + z)^{\beta_i}}{\mathcal{G}_i(0)} \right]$$

which can be written, using the general form for the nucleon exchange contributions

$$F_i(\omega^2)_B = \frac{G^2}{4\pi} \left(\frac{r_i}{\omega_B^2 - \omega^2} + d_i \right) \quad , \tag{19}$$

as

$$I_{i,n} = \tilde{F}_i(\omega^2)\delta_{n,0} + \frac{G^2}{4\pi} \left\{ d_i \delta_{n,0} + \right.$$

$$\left. + \frac{r_i}{\omega_B^2 - \omega^2} \left[\delta_{n,0} - (1 + z_B)^{\beta_i} z_B^n \left(\frac{\nu_t^2 - \omega^2}{\nu_t^2 - \omega^2} \right)^{1/2} \frac{\mathcal{G}_i(z_B)}{\mathcal{G}_i(0)} \right] \right\} . \tag{20}$$

With respect to "conventional" techniques, there is a small price to pay, i.e., the cumbersome second term on the right-hand side of Eq. (20) for n = 0. As we

can easily check, such a term is smoothly varying even in the limit $\omega^2 \to \omega_B^2$; furthermore in our range of t this correction is always small compared to $I_{i,0}$, eliminating unwelcome strong corrections between $\tilde{F}_i(\omega^2)$ and $G^2/4\pi$. Indeed we have, for constant $\epsilon_i(z)$, up to a scale factor,

$$\mathcal{G}_i(z) = (\epsilon_i/M_i)^{\phi(z)+i\tilde{\phi}(z)}$$

and then, for $N^2 \gg \nu_t^2$,

$$\mathcal{G}_i'(0)/\mathcal{G}_i(0) = \ln \frac{\epsilon_i}{M_i}\left(\frac{d\phi}{dz}\right)_{z=0} \simeq \frac{4}{\pi}\ln\frac{\epsilon_i}{M_i}\left(\frac{\nu_t^2-\omega^2}{N^2}\right)^{1/2} \tag{21}$$

Integrals $I_{i,n}$ with $n \geq 1$ allow us to determine $G^2/4\pi$ with the same algorithm used for amplitude extrapolation just increasing by $n\theta$ the argument of the integrand used to compute $I_{i,0}$. All these integrals are optimally evaluated, since their errors still saturate the Nevanlinna bound.

A serious problem is posed by the fact that measurements do not extend down to ν_t^2, since the asymptotic region Γ_2 is contracted by the z-map at the expense of an expansion of the threshold region, and this makes the integrals (18) sensitive to the low-energy continuation of the amplitudes. We can, however, see that for the functions $G_i'(z) = (z - z_B)G_i(z)$ and their weights $\mathcal{G}_i'(z)$, the integrals

$$I_{i,m}' = \frac{1}{2\pi i\, \mathcal{G}_i'(0)} \oint_\Gamma G_i'(z)\, \mathcal{G}_i'(z)\, z^m dz \tag{22}$$

should vanish for any $m \geq 0$, or be consistent with zero within their Nevanlinna bounds $\tilde{\Delta}_i' = \lambda\, \mathcal{G}_i'(0)^{-1}$; this, however, is not always true if we use the values derived from incorrect approaches which may be found in current compilations.[13]

The correct solution to the problem has been pointed out by Lichard[14]: functions $\chi_i^2\left(a^I_{\ell\pm}, r^I_{\ell\pm}, \ldots; \Delta a^I_{\ell\pm}, \Delta r^I_{\ell\pm}, \ldots\right)$ of the low-energy parameters and their errors can be built as

$$\chi_i^2 = \sum_{m=0}^{M} \left(\hat{I}'_{i,m}\right)^2 \tag{23}$$

and then standard minimum procedures are applied to χ_i^2; note that in this case we have a criterion for the "acceptability" of a minimum solution, since we may restrict our search to values of χ_i^2 satisfying the inequality

$$\chi_i^2 < (M+1)\,\tilde{\Delta}_i'^2 \quad .$$

Needless to say, the method determines an effective parametrization on the energy range just above threshold ($T_\pi < 21$ MeV with our choice of πN data), not the coefficients for the momentum expansion of $K_{cm}^{2\ell+1} \cdot \text{cotan}\, \delta^I_{\ell\pm}(K_{cm})$ around threshold!

4. πN system, above and below threshold.

The accurate analysis by Carter, Bugg and Carter,[5] commonly dubbed "CBC analysis," covers only the first resonance region, $310 \geq T_\pi \geq 88$ MeV. To use also the information collected at lower and higher energies, we have to join it as smoothly as possible to some other analysis. Of the three analyses most advertised and readily available, the so-called "theoretical" CERN analysis by Almehed and Lovelace[15] does not quote any error for its partial amplitudes, and is therefore useless for the present approach; since the analysis makes heavy use of partial-wave conventional dispersion relations, it is also probably not "kosher" at all for a correct analytic extrapolation.

Of the two energy-independent analyses giving both parameters and errors,

the recent Saclay analysis,[16] having only very few points below the second resonance, is very hard to be joined to the CBC analysis at $T_\pi \simeq 300$ MeV and furthermore leaves the region $T_\pi \leq 88$ MeV, which is mapped into a sizable arc by the z-map, completely uncovered. Thus we are left only with the rather old "experimental" CERN analysis of Donnachie et al.,[17] covering the energy range $1940 \geq T_\pi \geq 21$ MeV, which joins much better to the analysis of Ref. 5.

Once this selection is completed, we have to derive the effective parameters describing the amplitudes just above threshold at $21 \text{ MeV} \geq T_\pi \geq 0$ and low momentum transfer $|t| \leq 2\mu_\pm^2$. It must be recalled that, if we wish to take properly into account the mass-splittings of hadron multiplets, $\pi^- p$ elastic scattering has two unphysical cuts in the energy plane: a rather long one arising from the virtual radiative capture process $\pi^- p \to \gamma n$ from $\omega_B = \left(m_n^2 - m_p^2 - \mu_\pm^2 + t/2\right)/2m_p$ to $\omega_{el} = \mu_\pm + t/4m_p$, and a much shorter one arising from the charge-exchange process $\pi^- p \to \pi^0 n$ from $\omega_{chex} = \left[(m_n + \mu_0)^2 - m_p^2 - \mu_\pm^2 + t/2\right]/2m_p$ to ω_{el}. The first gives everywhere a contribution[6] of $0(\alpha)$ which may be absorbed, together with Coulomb corrections, into the electromagnetic corrections[5] and then eliminated, leaving us with a workable algorithm, since we can extrapolate in a cut ω^2 plane rather than in the upper, or lower, half of the ω plane, with a troublesome boundary-to-boundary problem.[1,4] We then use $\nu_t = \omega_{chex}$, and this leads to a rather large unknown low-energy region, since for instance we have at $\omega^2 = t = 0$ an arc $|\theta| \lesssim 0.47$ for the unphysical region and two arcs $1.13 \lesssim |\theta| \lesssim 0.47$ for the low T_π region above threshold. On this arc, the amplitudes F_i have to be represented by the effective parameters $a^I_{\ell\pm}$, $r^I_{\ell\pm}$, ... and their relative errors, with the adequate continuations[18] to the unphysical cut, and an accurate minimization of the χ_i^2 defined by (23) cannot be overemphasized.

It must be noted that since at energies $88 \geq T_\pi \geq 21$ MeV, we still have to rely

on rather old measurements of $\pi^{\pm}p$ elastic scattering, our errors will be rather large, and no improvement will be reached without a better systematic study of the πN system at very low energies, such as those available at meson factories.

We have limited ourselves to a linear dependence on t close to threshold, which should be adequate in our small range $|t| \leq 2\mu_{\pm}^2$, limiting then the analysis to an S+P-wave approximation; the "optimal" values are rather insensitive to the inclusion of D-waves, and since the χ_i^2 minima are largely independent both from D-wave scattering lengths and from effective-range parameters, we have used, as in Ref. 14, a simple scattering length parametrization. It is remarkable that, even taking into account non-physical ranges, still our results reproduce quite well Lichard's ones,[14] confirming the smallness of SU_2-violations[5,6,19] outside the single-particle or resonant contributions. This is due partly to the small shift in the pole position, of order δ, in the z-map, and partly to the large weight given by $\mathcal{G}_i'(z)$ to the first resonance region.

Before going to the actual evaluation of the integrals $\hat{I}_{i,n}$ we have to choose the exponents β_i entering the definition of the functions $G_i(z)$ and, consequently, the asymptotic bounds $M_i(z) = M_i$. The exponents have been chosen so that all functions, up to small exponent and possible logarithms, may have almost the same decreasing power behavior as $z \to -1$ that amplitude $B^{(-)}(\omega^2, t)$ has. We have then assumed M_i to be comparable to $|\hat{G}_i(z_M)|$; since for our data selection $\pi - \theta_M \ll \pi$, the particular choice for $M_i / |\hat{G}_i(z_M)|$ does not appreciably influence $I_{i,n}$ and Δ_i even when increased by orders of magnitude. Being $|\hat{G}_i(z)|$, on the average, decreasing functions of z as $z \to z_M$ beyond the second resonance, we fix, assuming such a behavior to continue beyond x_M, $M_i / |\hat{G}_i(z_M)| = 1$.

The free scale λ_i contained in the weights $\mathcal{G}_i(z)$, which is actually arbitrary,

can be fixed, to factorize explicitly the error scale, to be

$$\lambda_i = \tilde{\epsilon}_i = \max_{z \in \Gamma_1} \epsilon_i(z)$$

so that the optimal error saturating Nevanlinna bound becomes

$$\tilde{\Delta}_i = \tilde{\epsilon}_i^{\,\theta_M/\pi} \, M_i^{1-\theta_M/\pi} \exp\left[-\frac{1}{\pi} \int_0^{\theta_M} \ln \frac{\tilde{\epsilon}_i}{\epsilon_i(\theta)} \, d\theta \right] \tag{24}$$

where the error scale is now appearing explicitly.

To derive the amplitudes $\tilde{F}_i$ from relations (18) and (20), we still need explicit forms for the nucleon exchange terms $F_i(\omega^2)_B$; with a pseudo-vector coupling, and including $\Delta m = m_n - m_p \neq 0$, they are

$$A_B^{(+)}(\omega^2, t) = \frac{G^2}{m_p} \left[1 - \delta - \frac{\Delta m \, \omega_B}{\omega_B^2 - \omega^2} \right] \tag{25}$$

$$B_B^{(+)}(\omega, t)/\omega = \frac{G^2}{m_p} \frac{1}{\omega_B^2 - \omega^2} \tag{26}$$

$$A_B^{(-)}(\omega, t)/\omega = -\frac{G^2}{m_p} \frac{\Delta m}{\omega_B^2 - \omega^2} \tag{27}$$

$$B_B^{(-)}(\omega^2, t) = -\frac{G^2}{m_p} \left[\frac{(1-\delta)^2}{2\, m_p} - \frac{\omega_B}{\omega_B^2 - \omega^2} \right] . \tag{28}$$

From these formulae, we can derive the parameters r_i to be used in Eq. (20); note, however, that we are free in our choice for the d_i's, which depend on our particular choice for the amplitudes $\tilde{F}_i(\omega^2, t)$. To reproduce the most conventional

forms of the zero-energy theorems of current algebra, like Adler's PCAC condition,[20] Adler-Weisberger relation[21] and the on-mass-shell approximation to πN sigma term,[8] we shall choose

$$\tilde{F}_1 = \tilde{A}^{(+)} = A^{(+)} - A_B^{(+)} + G^2/m_p \tag{29}$$

$$\tilde{F}_2 = \tilde{B}^{(+)}/\omega = \left(B^{(+)} - B_B^{(+)}\right)/\omega \tag{30}$$

$$\tilde{F}_3 = \tilde{A}^{(-)}/\omega = \left(A^{(-)} - A_B^{(-)}\right)/\omega \tag{31}$$

$$\tilde{F}_4 = \tilde{B}^{(-)} = B^{(-)} - B_B^{(-)} - 2G^2/(m_p + m_n)^2 \quad , \tag{32}$$

to which we shall add the spin-averaged amplitudes

$$C^{(\pm)} = A^{(\pm)} + \omega B^{(\pm)}$$

and their counterparts

$$\tilde{F}_5 = \tilde{C}^{(+)} = C^{(+)} - C_B^{(+)} \tag{33}$$

$$\tilde{F}_6 = \tilde{C}^{(-)}/\omega = \left(C^{(-)} - C_B^{(-)}\right)/\omega - 2G^2/(m_p + m_n)^2 \quad . \tag{34}$$

In terms of these six amplitudes, we can use current algebra and PCAC at $q_1^2 = q_2^2 = t = \omega^2 = 0$, and, extrapolating in $q_1^2 = q_2^2 = t/2$ at $\omega = 0$, get the on-mass-shell results

$$\tilde{F}_1(0, 2\mu_\pm^2) = 2\sum(1 + \delta_1)/f_\pi^2 + G^2/m_p \tag{35}$$

$$\tilde{F}_2(0, 2\mu_\pm^2) = 4G^2 m_p \, \delta_2/\Delta m^2 \tag{36}$$

$$\tilde{F}_3(0, 2\mu_\pm^2) = (1 + \delta_3)/f_\pi^2 - 4 m_p G^2 \delta_3/\left[(m_p + m_n)^2 \Delta m\right] \tag{37}$$

$$\tilde{F}_4(0, 2\mu_\pm^2) = -2G^2(1 + \delta_4)/(m_n + m_p)^2 + 2G^2 \delta_4/(m_n^2 - m_p^2) \tag{38}$$

and

$$F_5\left(0,\ 2\mu_\pm^2\right) = 2\Sigma(1+\delta_5)/f_\pi^2 \tag{39}$$

$$F_6\left(0,\ 2\mu_\pm^2\right) = (1+\delta_6)/f_\pi^2 - 2G^2/(m_n+m_p)^2\ , \tag{40}$$

in terms of the pion decay constant f_π and the πN sigma term Σ.

Correction factors δ_i are expected not to exceed a maximum of the order of the ratio $2\mu_\pm^2/m_\rho^2$ from t-channel unitarity arguments[22] (in the absence of anomalous thresholds) and to obey relations

$$\delta_1 = \delta_5 \qquad \delta_3 - \delta_4 \simeq \frac{m_p\,\Delta m}{f_\pi^2\,G^2}\,(\delta_3 - \delta_6)\ .$$

Only two of these relations are really testable without further knowledge of the δ_i: relation (35), or Adler's condition[20] and relation (40), the Adler-Weisberger relation,[21] while relation (39), with a reasonable guess on δ_5 from t-channel unitarity, allows a "determination" of the πN sigma term.[8]

All other relations, due to the powers of Δm in the denominators, can offer only a check of the smallness of the on-mass-shell corrections δ_i, since these latter dominate over the zero-energy limit.

5. Evaluation and interpretation of numerical results

Let us begin with $n \geq 1$ in integrals $I_{i,n}$; now the only unknown on the right-hand side of Eq. (20) is the coupling constant $G^2/4\pi$, and the equation may be written as

$$I_{i,n} = Z_i(\omega^2,t)\, z_B^{n-1}\,\frac{G^2}{4\pi}\ ,\quad n \geq 1 \tag{41}$$

and since $z_B \ll 1$ the coefficients of $G^2/4\pi$ are rapidly decreasing with increasing n, so that we soon get nothing more than analyticity checks for all n but n = 1.

Even then, only $B^{(+)}/\omega$ has a large enough residue r_2 to allow a good determination of the coupling constant. Due to its small extrapolation error, also $C^{(-)}/\omega$ could afford an independent, though much less accurate determination, once one tries to reduce systematic effects coming from truncation at $\theta = \theta_M$ of the integral at the expense of an error increase by a factor $4/\pi$.[23]

We have computed $\hat{I}_{2,1}$ and Z_2 in the range $1/2\,\mu_\pm^2 \geq \omega^2 \geq 0$, $|t| \leq 2\mu_\pm^2$, obtaining error bounds at a level of 8 - 10% and values for $\hat{I}_{2,1}$, consistent with each other and relation (41) to better than 1%. The resulting average value for the coupling constant is

$$G^2/4\pi = 13.161 \pm 0.137 \tag{42}$$

(where we have given the standard error of the average over 54 values for $\hat{I}_{2,1}$).

When compared with all previous determinations, we find our value in very good agreement with the "conventional" determination by Samaranayake and Woolcock,[6] which carefully includes all SU_2-symmetry violations, but only marginally consistent with all other widely advertised[13] values derived either via conventional methods of continuation[24] or via expansions in series of orthogonal polynomials[25] or rational functions[26]; as expected from the influence of mutual correlations, the deviations from these values are systematically comparable with $\Delta\omega_B/\omega_B \simeq \left(m_n^2 - m_p^2\right)/\mu_\pm^2$. Note also that our value is much closer than many previous results[24-26] to central values obtained both from isovector exchanges in pion photoproduction[27] and from pion exchange in nucleon-nucleon scattering.[28]

Since G is not only important as a measure of the pion-nucleon interaction, but also it enters a check of the existence of a spontaneously broken, chiral symmetry in Goldberger-Treiman relation[29]

$$(m_p + m_n)\, g_A(0) = -\sqrt{2}\; f_\pi\, G(1 - \epsilon) \;, \tag{43}$$

let us expand a little on the implications of result (42).

If the low pion mass is an indication of an approximate chiral $SU_2 \times SU_2$ symmetry, spontaneously broken, we expect $\epsilon \simeq 0\left(m_\pi^2/M^2\right)$, with M a "typically hadronic" mass ($\simeq 1$ GeV/c^2); using the experimental values[13] $g_A(0) = -1.260 \pm 0.012$ and $f_\pi/\mu_\pm = 0.9442 \pm 0.0008$, the "conventional" estimate[13] for $G^2/4\pi$ would give $\epsilon = (6.0 \pm 1.2) \times 10^{-2}$, which requires some additional pain[30] to be accommodated in a simple model of chiral symmetry breaking,[9] while value (42) gives $\epsilon = (1.3 \pm 1.1) \times 10^{-2}$, which fits snugly into the frame of Ref. 9, without giving any headache at all.

With the coupling constant (42), we can now correct integrals $\hat{I}_{i,0}$ and obtain the estimates for the "non-Born" parts $\tilde{F}_i(\omega^2, t)$ of amplitudes $F_i(\omega^2, t)$. Due to the smallness of r_i and d_i for all $i \neq 2$, corrections are small but for $\tilde{B}^{(+)}/\omega$, and errors are in general not increased too much over the Nevanlinna bounds. Tables IV-IX present the numerical results in the same range used to determine $G^2/4\pi$; due to the independence of weight functions $\mathcal{G}_i$ on each other, we can use the definitions

$$\tilde{A}^{(+)} + \omega \tilde{B}^{(+)} = G^2/m_p + \tilde{C}^{(+)} \qquad \text{and} \qquad \tilde{A}^{(-)}/\omega + \tilde{B}^{(-)} = \tilde{C}^{(-)}/\omega$$

(trivial in any "conventional"approach) as consistency checks of our numerical computations.

The first one, becoming independent of $\tilde{B}^{(+)}/\omega$ at $\omega^2 = 0$, allows a third check of our coupling constant determination, better than the use of $\hat{I}_{6,1} + \hat{I}_{6,2}$ made in Ref 23; we can in fact derive

$$G^2/4\pi = m_p \left[\tilde{A}^{(+)}(0,t) - \tilde{C}^{(+)}(0,t)\right]/4\pi \tag{44}$$

which give an average over 9 points

$$G^2/4\pi = 13.510 \pm 0.171 \tag{45}$$

(where the error is again the standard error of the average), in good agreement with value (42) within the typical error bounds of (44).

We can further try to compare our extrapolations to the zero-energy theorems (35) - (40); reasonable estimates for $\delta_1 = \delta_5$ and δ_6 may be obtained writing a phase representation for $C^{(+)}(t) = C^{(+)}(\omega = 0,\ t;\ q_1^2 = q_2^2 = t/2)$ and $C^{(-)}(t) = \lim_{\omega \to 0} C^{(-)}(\omega, t;\ q_1^2 = q_2^2 = t/2)/\omega$, assuming the only zeros at low t to be Adler zeros and extending Watson's theorem at least up to $t \simeq 1\ \mathrm{GeV}^2$. From $\pi\pi$ P-wave and I = 0, S-wave elastic phase-shifts[31] we get the values (not very different from those derivable in a naive $\rho + \epsilon$ dominance model) $\delta_5 \simeq 0.10$ and $\delta_6 \simeq 0.07$.

Zero-energy theorems become then

$$\mu_\pm \tilde{A}^{(+)}(0, 2\mu_\pm^2) \simeq 24.60 + \Sigma/(56.5\ \mathrm{MeV}) \tag{46}$$

$$\mu_\pm \tilde{C}^{(+)}(9, 2\mu_\pm^2) \simeq \Sigma/(56.5\ \mathrm{MeV}) \tag{47}$$

and

$$\mu_\pm^2 \lim_{\omega \to 0} \tilde{C}^{(-)}(\omega, 2\mu_\pm^2)/\omega \simeq -0.63\ , \tag{48}$$

fully consistent with our determinations. Values for the Σ term may be constrained, in the frame of GOR model for chiral symmetry breaking,[9] between a minimum of $\simeq 17$ MeV, given by a unitarity condition on $C^{(+)}(\omega^2, t = q_1^2 = q_2^2 = 0)$ for $\pi\Xi$ elastic scattering, and a maximum of $\simeq 40$ MeV, derived assuming the unitary singlet piece of $SU_3 \times SU_3$ breaking Hamiltonian to contribute not more than one-third to the average baryon mass, to keep perturbations around the chiral limit physically meaningful. These bounds may also be compared to the more sophisticated estimates given by Renner.[33]

Unfortunately any conclusion on the symmetry breaking mechanism is

prevented both by the large error on $\tilde{C}^{(+)}(0, 2\mu_{\pm}^2)$, consequence of our poor knowledge of $C^{(+)}$ at pion energies less than 88 MeV, and by the fact that the value derivable for Σ with our estimate for δ_5, consistent with chiral perturbation theory, is only marginal to the expectations of GOR model, since from relation 47 and Table VIII, we get

$$\Sigma \simeq 41 \pm 23 \text{ MeV} \tag{49}$$

where now the error is an absolute error bound. This of course reduces to a purely academic question the extreme sensitivity of Σ to additional pieces in the symmetry-breaking Hamiltonian (behaving, for instance, as an (8,8) representation of $SU_3 \otimes SU_3$).[34]

Note also that our central value is in perfect agreement with many previous determinations which used the non-spin-flip amplitude $F^{(+)}$ at $t = 2\mu_{\pm}^2$, and found $\Sigma \simeq 60$ MeV.[26,35-38] Treating correctly the nucleon pole, we derive (with $m_n \neq m_p$)

$$\tilde{F}^{(+)}(0, 2\mu_{\pm}^2) = \tilde{C}^{(+)}(0, 2\mu_{\pm}^2) \tag{50}$$

instead of the result obtained with $m_n - m_p = 0$

$$\tilde{F}^{(+)}(0, 2\mu_{\pm}^2) = \tilde{C}^{(+)}(0, 2\mu_{\pm}^2) - \frac{G^2}{m_p} \frac{\mu_{\pm}^2}{2m_p^2 - \mu_{\pm}^2} . \tag{50'}$$

Using the zero-energy theorem (39), the correction required to go from (50') to (50) is rather large and represents a change in Σ of $\simeq -20$ MeV, which brings the results obtained by Nielsen and Oades,[35] Lichard,[36] Langbein,[26] Chao et al.[37] and Höhler et al.[38] (which all use $\tilde{T}^{(+)}$ and $m_n - m_p = 0$ instead of $\tilde{C}^{(+)}$ and the correct mass spectrum) to agree remarkably well with relation (49). Our value agrees only marginally with the "internal dispersion relation" result of

Moir et al.,[39] but in view of their sensitivity to a "best-fit"-selection criterion[40] we feel such a difference not compelling at all.

We can then conclude that, once the result of Ref. 39 is taken with the necessary caution and the huge value of Carter et al.[41] suitably reduced for systematic effects in higher partial waves,[35] all "on-mass-shell" determinations of Σ agree with a central value $\simeq 40$ MeV,[42] unfortunately only marginal to the domain constant with an "orthodox" $(3,\bar{3}) + (\bar{3},3)$ model.[9]

All other features of amplitudes $\tilde{F}_i(\omega^2,t)$ do not seem to require further complications[43] than one-particle exchanges with constant couplings (normalized, when possible, to zero-energy theorems at $\omega^2 = 0$, $t = 2\mu_\pm^2$), given by $\Delta(1231)$ in both s and u channels and $\rho(770)$ and ϵ (990) in the t channel. These exchanges, once the mass of the ϵ is fixed, contain only one free parameter, the product $G_{\epsilon\pi\pi}G_{\epsilon\bar{N}N}$ for the amplitudes $\tilde{A}^{(+)}$ and $\tilde{C}^{(+)}$, since Δ and ρ couplings may be fixed respectively by a fit to $\delta_{1+}^{3/2}$ around the resonance[5,13] (giving $G^2_{\Delta N\pi}/4\pi \simeq 15$) and by ρ-meson dominance, which requires

$$G_{\rho\pi\pi} \simeq G^V_{\rho\bar{N}N} \simeq f_\rho \simeq 2.26$$

$$G^T_{\rho\bar{N}N}/G^V_{\rho\bar{N}N} \simeq \mu_p - \mu_n \simeq 4.70 \quad .$$

With respect to systematic analyses by Lichard[36] and Höhler et al.[38] around $\omega^2 = t = 0$ the major differences are a downward shift in $\tilde{A}^{(+)}$, consistent with the decreased value for $G^2/4\pi$, and a substantially steeper $\tilde{C}^{(-)}/\omega$, consistent with the ratio $G^T_{\rho\bar{N}N}/G^V_{\rho\bar{N}N}$ suggested by ρ-meson dominance.

The results exhibit also a strange hump around $t = 0$ in $\tilde{C}^{(-)}/\omega$ and a much less pronounced shoulder in $\tilde{C}^{(+)}$ at the same position (better visibility if smooth Δ and ϵ exchanges are subtracted); its presence in both $\tilde{C}^{(\pm)}$ amplitudes (and not elsewhere) with the same sign and its position allow to track it back to an underestimate of the errors on S_{11} wave around the first resonance in Ref. 5; though we

tested that such an effect is indeed washed out by an increase in S_{11} errors in this energy range, our philosophy prevents us from tampering any further with our input. Note that analyses like those of Ref. 36 and 38 eliminate a priori any such structure by assuming a linear or quasi-linear expansion around $\omega^2 = t = 0$ and then computing its coefficients from smooth interpolations in the physical region.

6. Summary and conclusions

At the present level of information on $\pi^{\pm}p$ elastic scattering

(i) we have found an appreciable reduction in $G^2/4\pi$ due to a systematic description of SU_2-invariance violations, and the new value $G^2/4\pi \simeq 13.16$ is consistent with a very small correction $\epsilon \lesssim 2\%$ to the Goldberger-Treiman relation;

(ii) there is no evidence for large corrections to the zero-energy theorems when going from zero pion four-momenta to the on-mass-shell point $\omega^2 = 0$, $t = 2\mu_{\pm}^2$;

(iii) the Σ-commutator is estimated to have a value of $\simeq 41$ MeV (with an absolute error bound of 23 MeV), in agreement with previous determinations once these are corrected for SU_2-non-invariance effects in the nucleon pole, but neither reinforcing nor weakening the case for a pure $(3, \bar{3})$ model of chiral symmetry breaking;

(iv) apart from a hump in $\tilde{C}^{(\pm)}$ amplitudes, very probably caused by an underestimate in the errors of the input, we do not see any structure beyond very simple one-particle exchanges, normalized to zero-energy theorems at $\omega^2 = 0$ and $t = 2\mu_{\pm}^2$.

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After completion of this work, we received copies of two remarkable papers by S. Ciulli and coworkers,[44] which present in a concise and practical form the basic algorithms for optimal analytic extrapolations, and which we strongly recommend to any interested reader.

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Table Captions

I. Comparison between error bounds in a conventional and a correct dispersive approach.

II. The essential parameters in deriving the low-energy amplitudes with the optimally evaluated integrals $\hat{I}_{i,n}$.

III. Effective scattering lengths $a^{I}_{\ell\pm}$ derived with Lichard's statistical method,[10] with $r^{I}_{\ell\pm} = 0$ and $a^{I}_{\ell\pm} = 0$ ($\ell \geq 2$).

IV. The amplitude $\tilde{A}^{(+)}(\omega^2, t)$.

V. The amplitude $\tilde{B}^{(+)}(\omega, t)/\omega$.

VI. The amplitude $\tilde{A}^{(-)}(\omega, t)/\omega$.

VII. The amplitude $\tilde{B}^{(-)}(\omega^2, t)$.

VIII. The amplitude $\tilde{C}^{(+)}(\omega^2, t)$.

IX. The amplitude $\tilde{C}^{(-)}(\omega, t)/\omega$.

TABLE I

	$\alpha = 1/4$		$\alpha = 1/10$	
M_0/ε	$\Delta_{conv}/\varepsilon-1$	$\tilde{\Delta}/\varepsilon-1$	$\Delta_{conv}/\varepsilon-1$	$\tilde{\Delta}/\varepsilon-1$
2.5	1.4	-5.0×10^{-2}	5.1	-1.3×10^{-3}
5	2.3	-1.3×10^{-2}	7.3	-4.1×10^{-5}
10	3.2	-3.2×10^{-3}	9.5	-1.3×10^{-6}
10^2	6.1	-3.2×10^{-5}	16.8	-1.3×10^{-11}
10^3	9.1	-3.2×10^{-7}	24.2	-1.3×10^{-16}
10^4	12.1	-3.2×10^{-9}	31.5	-1.3×10^{-21}
10^5	14.9	-3.2×10^{-11}	38.8	-1.3×10^{-26}
10^6	17.9	-3.2×10^{-13}	46.2	-1.3×10^{-31}

TABLE II

i	F_i	β_i	r_i	d_i
1	$A^{(+)}$	3/2	$-\frac{4\pi\omega_B(m_n - m_p)}{m_p}$	$-\frac{4\pi(m_n - m_p)}{m_n + m_p}$
2	$B^{(+)}/\nu$	-1/2	$\frac{4\pi}{m_p}$	0
3	$A^{(-)}$	0	$-\frac{4\pi(m_n - m_p)}{m_p}$	0
4	$B^{(-)}$	0	$\frac{4\pi\omega_B}{m_p}$	0
5	$C^{(+)}$	3/2	$-\frac{4\pi\omega_B(m_n - m_p - \omega_B)}{m_p}$	$-\frac{4\pi(m_n - m_p)}{m_n + m_p}$
6	$C^{(-)}/\nu$	0	$-\frac{4\pi(m_n - m_p - \omega_B)}{m_p}$	0

TABLE III

$a^{1/2}_{0+}$	=	0.208 ± 0.020	$a^{3/2}_{0+}$	=	−0.092 ± 0.017
$a^{1/2}_{1-}$	=	−0.109 ± 0.036	$a^{3/2}_{1-}$	=	−0.063 ± 0.022
$a^{1/2}_{1+}$	=	−0.045 ± 0.035	$a^{3/2}_{1+}$	=	0.187 ± 0.022

TABLE IV

ω^2	0.0	0.1	0.2
t			
-2.0	20.06 ± 1.73	20.58 ± 1.77	21.14 ± 1.81
-1.5	20.66 ± 1.78	21.18 ± 1.81	21.74 ± 1.86
-1.0	21.27 ± 1.83	21.80 ± 1.86	22.36 ± 1.90
-0.5	21.90 ± 1.88	22.43 ± 1.91	22.99 ± 1.95
0.0	22.55 ± 1.93	23.08 ± 1.96	23.64 ± 2.00
0.5	23.22 ± 1.98	23.75 ± 2.01	24.32 ± 2.05
1.0	23.92 ± 2.04	24.45 ± 2.07	25.02 ± 2.10
1.5	24.65 ± 2.09	25.18 ± 2.12	25.74 ± 2.15
2.0	25.41 ± 2.15	25.94 ± 2.18	26.50 ± 2.21

ω^2	0.3	0.4	0.5
t			
-2.0	21.74 ± 1.87	22.40 ± 1.95	23.14 ± 2.05
-1.5	22.35 ± 1.91	23.01 ± 1.98	23.74 ± 2.08
-1.0	22.97 ± 1.96	23.63 ± 2.02	24.35 ± 2.11
-0.5	23.60 ± 2.00	24.26 ± 2.06	24.98 ± 2.15
0.0	24.25 ± 2.05	24.90 ± 2.11	25.62 ± 2.18
0.5	24.92 ± 2.10	25.57 ± 2.15	26.28 ± 2.23
1.0	25.62 ± 2.15	26.26 ± 2.20	26.96 ± 2.27
1.5	26.34 ± 2.20	26.98 ± 2.25	27.67 ± 2.31
2.0	27.10 ± 2.25	27.74 ± 2.30	28.42 ± 2.36

TABLE V

ω^2	0.0	0.1	0.2
t			
-2.0	-3.97 ± 0.52	-4.09 ± 0.56	-4.22 ± 0.62
-1.5	-3.85 ± 0.50	-3.96 ± 0.55	-4.09 ± 0.60
-1.0	-3.73 ± 0.49	-3.84 ± 0.53	-3.96 ± 0.58
-0.5	-3.62 ± 0.48	-3.73 ± 0.52	-3.84 ± 0.56
0.0	-3.51 ± 0.47	-3.61 ± 0.50	-3.72 ± 0.54
0.5	-3.40 ± 0.46	-3.50 ± 0.49	-3.60 ± 0.53
1.0	-3.29 ± 0.45	-3.39 ± 0.48	-3.49 ± 0.52
1.5	-3.19 ± 0.44	-3.28 ± 0.47	-3.37 ± 0.50
2.0	-3.09 ± 0.43	-3.17 ± 0.46	-3.26 ± 0.49

ω^2	0.3	0.4	0.5
t			
-2.0	-4.36 ± 0.69	-4.51 ± 0.78	-4.67 ± 0.91
-1.5	-4.22 ± 0.66	-4.37 ± 0.74	-4.52 ± 0.86
-1.0	-4.09 ± 0.64	-4.23 ± 0.71	-4.38 ± 0.81
-0.5	-3.96 ± 0.61	-4.09 ± 0.68	-4.24 ± 0.77
0.0	-3.84 ± 0.59	-3.96 ± 0.66	-4.10 ± 0.73
0.5	-3.71 ± 0.57	-3.83 ± 0.63	-3.96 ± 0.70
1.0	-3.59 ± 0.56	-3.71 ± 0.61	-3.83 ± 0.67
1.5	-3.47 ± 0.54	-3.58 ± 0.59	-3.70 ± 0.65
2.0	-3.36 ± 0.53	-3.46 ± 0.57	-3.57 ± 0.62

TABLE VI

ω^2	0.0	0.1	0.2
t			
-2.0	-7.83 ± 0.73	-7.95 ± 0.77	-8.07 ± 0.82
-1.5	-8.00 ± 0.74	-8.12 ± 0.77	-8.24 ± 0.82
-1.0	-8.17 ± 0.74	-8.28 ± 0.77	-8.41 ± 0.81
-0.5	-8.33 ± 0.74	-8.45 ± 0.78	-8.58 ± 0.82
0.0	-8.49 ± 0.75	-8.62 ± 0.78	-8.74 ± 0.82
0.5	-8.65 ± 0.75	-8.78 ± 0.78	-8.74 ± 0.82
1.0	-8.82 ± 0.76	-8.94 ± 0.79	-8.91 ± 0.82
1.5	-8.98 ± 0.76	-9.11 ± 0.79	-9.07 ± 0.82
2.0	-9.15 ± 0.77	-9.27 ± 0.79	-9.24 ± 0.82

ω^2	0.3	0.4	0.5
t			
-2.0	-8.18 ± 0.87	-8.31 ± 0.95	-8.42 ± 1.04
-1.5	-8.36 ± 0.87	-8.49 ± 0.94	-8.62 ± 1.02
-1.0	-8.54 ± 0.86	-8.67 ± 0.39	-8.81 ± 1.01
-0.5	-8.71 ± 0.86	-8.85 ± 0.92	-9.00 ± 0.99
0.0	-8.88 ± 0.86	-9.02 ± 0.91	-9.17 ± 0.98
0.5	-9.05 ± 0.86	-9.19 ± 0.91	-9.35 ± 0.97
1.0	-9.21 ± 0.86	-9.36 ± 0.91	-9.52 ± 0.97
1.5	-9.38 ± 0.86	-9.53 ± 0.91	-9.69 ± 0.96
2.0	-9.54 ± 0.86	-9.69 ± 0.90	-9.85 ± 0.96

TABLE VII

ω^2	0.0	0.1	0.2
t			
-2.0	8.21 ± 0.75	8.32 ± 0.79	8.42 ± 0.84
-1.5	8.29 ± 0.75	8.39 ± 0.79	8.50 ± 0.83
-1.0	8.36 ± 0.75	8.47 ± 0.78	8.58 ± 0.82
-0.5	8.43 ± 0.75	8.53 ± 0.78	8.64 ± 0.82
0.0	8.49 ± 0.74	8.60 ± 0.78	8.71 ± 0.81
0.5	8.55 ± 0.74	8.66 ± 0.77	8.77 ± 0.81
1.0	8.61 ± 0.74	8.72 ± 0.77	8.83 ± 0.80
1.5	8.67 ± 0.74	8.78 ± 0.77	8.88 ± 0.80
2.0	8.74 ± 0.74	8.84 ± 0.77	8.94 ± 0.80

ω^2	0.3	0.4	0.5
t			
-2.0	8.54 ± 0.90	8.66 ± 0.97	8.80 ± 1.07
-1.5	8.62 ± 0.89	8.75 ± 0.95	8.88 ± 1.05
-1.0	8.69 ± 0.87	8.82 ± 0.94	8.96 ± 1.02
-0.5	8.76 ± 0.87	8.89 ± 0.92	9.02 ± 1.00
0.0	8.82 ± 0.86	8.95 ± 0.91	9.08 ± 0.98
0.5	8.88 ± 0.85	9.01 ± 0.90	9.14 ± 0.96
1.0	8.94 ± 0.84	9.06 ± 0.89	9.20 ± 0.95
1.5	9.00 ± 0.84	9.12 ± 0.88	9.25 ± 0.93
2.0	9.05 ± 0.83	9.17 ± 0.87	9.29 ± 0.92

TABLE VIII

ω^2	0.0	0.1	0.2
t			
-2.0	-3.797 ± 0.228	-3.730 ± 0.231	-3.660 ± 0.236
-1.5	-3.134 ± 0.203	-3.056 ± 0.204	-2.976 ± 0.207
-1.0	-2.490 ± 0.178	-2.402 ± 0.178	-2.311 ± 0.178
-0.5	-1.851 ± 0.166	-1.753 ± 0.164	-1.650 ± 0.162
0.0	-1.347 ± 0.184	-1.239 ± 0.182	-1.126 ± 0.179
0.5	-0.862 ± 0.233	-0.745 ± 0.231	-0.624 ± 0.229
1.0	-0.360 ± 0.292	-0.233 ± 0.290	-0.102 ± 0.290
1.5	0.169 ± 0.351	0.305 ± 0.350	0.446 ± 0.351
2.0	0.719 ± 0.409	0.864 ± 0.409	1.014 ± 0.410

ω^2	0.3	0.4	0.5
t			
-2.0	-3.588 ± 0.243	-3.511 ± 0.252	-3.429 ± 0.266
-1.5	-2.891 ± 0.210	-2.803 ± 0.215	-2.709 ± 0.223
-1.0	-2.216 ± 0.179	-2.115 ± 0.180	-2.009 ± 0.183
-0.5	-1.543 ± 0.161	-1.430 ± 0.160	-1.310 ± 0.160
0.0	-1.009 ± 0.178	-0.885 ± 0.175	-0.754 ± 0.173
0.5	-0.497 ± 0.227	-0.364 ± 0.226	-0.225 ± 0.225
1.0	0.035 ± 0.289	0.177 ± 0.289	0.326 ± 0.290
1.5	0.592 ± 0.351	0.744 ± 0.353	0.903 ± 0.356
2.0	1.170 ± 0.412	1.333 ± 0.415	1.501 ± 0.419

TABLE IX

ω^2	0.0	0.1	0.2
t			
-2.0	0.439 ± 0.088	0.402 ± 0.093	0.358 ± 0.098
-1.5	0.723 ± 0.077	0.708 ± 0.080	0.692 ± 0.083
-1.0	0.961 ± 0.065	0.963 ± 0.067	0.968 ± 0.070
-0.5	0.974 ± 0.059	0.985 ± 0.060	0.999 ± 0.061
0.0	0.684 ± 0.063	0.692 ± 0.064	0.704 ± 0.065
0.5	0.222 ± 0.078	0.218 ± 0.079	0.215 ± 0.081
1.0	-0.140 ± 0.096	-0.155 ± 0.098	-0.170 ± 0.101
1.5	-0.393 ± 0.113	-0.415 ± 0.116	-0.438 ± 0.120
2.0	-0.547 ± 0.130	-0.574 ± 0.133	-0.602 ± 0.137

ω^2	0.3	0.4	0.5
t			
-2.0	0.308 ± 0.105	0.251 ± 0.114	0.186 ± 0.127
-1.5	0.675 ± 0.088	0.656 ± 0.094	0.640 ± 0.102
-1.0	0.974 ± 0.072	0.984 ± 0.076	1.001 ± 0.081
-0.5	1.018 ± 0.063	1.041 ± 0.065	1.074 ± 0.068
0.0	0.719 ± 0.066	0.739 ± 0.068	0.768 ± 0.070
0.5	0.214 ± 0.083	0.216 ± 0.086	0.222 ± 0.089
1.0	-0.185 ± 0.104	-0.200 ± 0.108	-0.213 ± 0.112
1.5	-0.461 ± 0.124	-0.486 ± 0.128	-0.513 ± 0.134
2.0	-0.631 ± 0.142	-0.663 ± 0.148	-0.697 ± 0.154