PUB-1920
April 1977
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A REALIZATION OF NAMBU MECHANICS: A PARTICLE INTERACTING WITH AN SU(2) MONOPOLE*

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ABSTRACT

We study the system of a particle bearing the isospin-degrees of freedom interacting with an SU(2) 't Hooft-Polyakov monopole. We show that its equation of motion can be cast into the form of Nambu's generalized mechanics.

(Submitted to Phys. Rev.)
Comments and Addenda

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*Work supported in part by the Energy Research and Development Administration.
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Some time ago, Nambu suggested some possible generalizations of classical Hamiltonian mechanics.[1] As the simplest extension, he proposed the replacement of the conventional canonical doublet (p_n, q_n) by a set of three variables (P_n, Q_n, R_n). The usual Poisson bracket was generalized to the Nambu bracket $[A, B, C]$ containing three quantities:

$$[A, B, C] = \sum_n \frac{\partial(A, B, C)}{\partial(Q_n, P_n, R_n)} \quad . \tag{1}$$

The time evolution of a dynamical quantity $f(P, Q, R)$ was assumed to be determined by

$$\frac{df}{dt} = [f, F, G] \quad , \tag{2}$$

where $F(P, Q, R)$ and $H(P, Q, R)$ are alternatives of the Hamiltonian function in the conventional scheme.

The appearance of the third variable R makes it difficult to conceive systems which obey Nambu's equations of motion. It was pointed out that the Euler equation for a rigid rotator can be written in the form of (2).[1] Several authors have shown that some systems with constraints can be described by Nambu's mechanics.[2] In these examples, the variable R was constructed from the conventional position and momentum variables. In this note, we put forth another example of Nambu's mechanics where the variable R cannot be expressed solely as a function of position and momentum variables.

We consider the classical motion of a point particle with mass m and isospin T_i $(i = 1, 2, 3)$ interacting with an SU(2) magnetic monopole.[3] According to Hasenfratz and 't Hooft, the equations of motion are[5]

$$\begin{aligned} \dot{x}_i &= \frac{1}{m}\left(p_i - eA_i^a(x)\, T_a\right) \ , \\ \dot{p}_i &= \frac{1}{m}\left(p_j - eA_j^a(x)\, T_a\right) \frac{\partial A_j^b}{\partial x_i}\, eT_b - \frac{\partial V(r)}{\partial x_i} \quad , \end{aligned} \tag{3}$$

and

$$\dot{T}_a = -\epsilon_{abc} \frac{1}{m}\left(p_i - eA_i^d(x)\, T_d\right) eA_i^b T^c \quad ,$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, x_i and p_i's are the Cartesian coordinates and linear momentum of the particle, respectively, e is the coupling constant, $A_i^a(x)$ is the potential due to the monopole, V(r) is some spherically symmetric potential which may provide the binding force. ϵ_{abc} is the Levi-Civita tensor and the summation over the repeated indices is assumed throughout. These equations can be derived from the following ones:

$$\frac{df(x,p,T)}{dt} = [f, H] \quad , \tag{4}$$

$$[A, B] = \frac{\partial A}{\partial x_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial x_i} + \epsilon_{abc}\frac{\partial A}{\partial T_a}\frac{\partial B}{\partial T_b} T_c \quad , \tag{5}$$

and

$$H = \frac{1}{2m}\left(p_j - eA_j^a(x)\, T_a\right)^2 + V(r) \quad , \tag{6}$$

where all of the x_i, p_i and T_i's are regarded as c-numbers. The gauge potential $A_i^a(x)$ is of the form

$$A_i^a(x) = \epsilon_{ia\ell}\, x_\ell W(r) \quad , \tag{7}$$

where W(r) should be the solution of a complicated nonlinear differential equation with the boundary condition $-er^2 W(r) \to 1$ $(r \to \infty)$.[3] For simplicity and concreteness, we consider the limiting case that

$$W(r) = -\frac{1}{er^2} \tag{8}$$

for any value of r.[4]

Our purpose is to case (3) or (4)-(6) into the form of (1) and (2) by suitably choosing P_i, Q_i, R_i, F and G. It was observed in Ref. 5 that

$$J_i = T_i + \epsilon_{ijk} x_j p_k \quad , \qquad (i=1,2,3) \tag{9}$$

are conserved. We now define θ and ϕ by

$$\cos\theta = \frac{J_i x_i}{Jr} \quad , \qquad J = \sqrt{J_1^2 + J_2^2 + J_3^2} \tag{10}$$

and

$$r \sin\theta\, \dot{\phi} = \frac{1}{Jr\sin\theta}\, \epsilon_{ijk} \dot{x}_i J_j x_k \quad . \tag{11}$$

We next define u_1, u_2 and u_3 by

$$u_1 + J\sin\theta = \frac{1}{J\sin\theta}\, \epsilon_{ijk} p_i J_j x_k \quad ,$$

$$u_2 = \frac{1}{Jr\sin\theta}\, \epsilon_{ijk} p_i (\epsilon_{j\ell m} J_\ell x_m) x_k \quad , \tag{12}$$

and[6]

$$u_3 = p_i x_i \quad .$$

The nine equations of motion for x_i, p_i and T_i $(i=1,2,3)$ are then equivalent to

$$\dot{r} = \frac{u_3}{mr} \quad , \qquad \dot{\theta} = 0 \quad , \qquad \dot{\phi} = \frac{J}{mr^2} \quad ,$$

$$\dot{J}_1 = 0 \quad , \qquad \dot{J}_2 = 0 \quad , \qquad \dot{J}_3 = 0 \quad , \tag{13}$$

$$\dot{u}_1 = -\frac{J\cos\theta}{mr^2} u_2 \;, \quad \dot{u}_2 = \frac{J\cos\theta}{mr^2} u_1 \;\text{ and }\; \dot{u}_3 = 2H - (2V + rV') \quad ,$$

where $V' = \frac{dV(r)}{dr}$. If we further define variables Φ, u, σ and S by

$$\Phi = \phi - J\int^r \frac{dr'}{r' f(r')} \quad ,$$

$$u = \sqrt{u_1^2 + u_2^2} \quad ,$$

$$\sigma = \tan^{-1} \frac{u_2}{u_1} - \phi \cos \theta \tag{14}$$

and

$$S = m \int^{r} \frac{r'dr'}{f(r')} \quad ,$$

where $f(r) = u_3$ is given by

$$\{f(r)\}^2 = 2mr^2 \{H - V(r)\} - J^2 \sin^2 \theta \quad , \tag{15}$$

then it follows readily that Eqs. (13) are equivalent to

$$\dot{\Phi} = \dot{u} = \dot{\sigma} = \dot{H} = \dot{\theta} = \dot{J}_1 = \dot{J}_2 = \dot{J}_3 = 0 \tag{16}$$

and

$$\dot{S} = 1 \quad . \tag{17}$$

To make contact with Nambu's mechanics, we proceed to identify the eight variables Q_2, Q_3, P_1, P_2, P_3, R_1, R_2 and R_3 with any independent eight functions of Φ, u, σ, H, θ, J_1, J_2 and J_3. Through identification of Q_1 with S, F with P_1 and G with R_1, we find that any dynamical quantity f(P, Q, R) in this system satisfies (2).

The above analysis was made for a very special dynamical system. It is, however, apparent that the system with 3N fundamental variables can be described by (1) and (2) if 3N-1 integrals are known. We have only to identify $Q_2, \ldots, Q_N$, $P_1, \ldots, P_N$, $R_1, \ldots, R_N$ with 3N-1 independent functions of 3N-1 integrals, F with P_1, G with R_1 and Q_1 with a certain quantity S which is so constructed as to satisfy $\dot{S}=1$.[7] Nevertheless, we offer this special example because it suggests the potential relevance of Nambu's mechanics for systems with internal degrees of freedom nontrivially coupled to space-time ones.

Acknowledgments

The author wishes to express his appreciation to Professor S. D. Drell for his hospitality at SLAC. He thanks Dr. H. C. Tze for helpful comments and careful reading of the manuscript. Thanks are also due to Drs. P. Y. Pac and M. J. Hayashi for encouragements.

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