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## I. INTRODUCTION

The value of luminosity, synchrotron light source brightness, quantum lifetime, etc., for an electron storage ring is directly dependent tipon the natural beam size and shape in the transverse phase space. These transverse beam parameters can be determined from the stationary particle distribution, $\psi$, which depends upon (a) quantum excitations determined by the horizontal and vertical energy dispersion functions $\eta_{\mathrm{x}} \mathrm{y}$ and $\eta_{\mathrm{x}, \mathrm{y}}^{\prime}$ in the machine, (b) radiation damping provided by the RF acceleration, and (c) coupling between the transverse betatron motions caused by the skew quadrupole and solenoid magnetic fields. A straightforward method to find $\psi$ is by solving the Fokker-Planck equation, ${ }^{1}$ which conveniently takes into account these factors.

In this approach the quantum diffusion effects are described by three quantities, $\mathrm{H}_{\mathrm{xx}}, \mathrm{H}_{\mathrm{xy}}$, and $\mathrm{H}_{\mathrm{yy}}$, which are integrals of the $\beta$ - and $\eta$-functions and their derivatives cvaluated over the bending magnets in the machine; the radiation damping effects are characterized by the radiation damping constants $\alpha_{x, y}$ provided by an RF system. ${ }^{2}$ The coupling effects are represented by a coupling coefficient, Q, assuming smooth coupling between the betatron motions. ${ }^{3}$ Under these assumptions, $\psi$ cian be found analytically and the expressions for transverse beam parameters in terms of $Q, H_{X X}, H_{X X}$ $\mathrm{H}_{\mathrm{yy}}, \alpha_{\mathrm{x}}$, and $\alpha_{\mathrm{y}}$ can be obtained. From these expressions, invariant conditions between some of the beam parameters can easily be shown. These results have been used to estimate the effects in PFP and SPEAR due to magnet alignment and vertical closed-orbit errors.

## II. COMPUTATIONAL PROCEDURE

The analysis which leads to the expressions for the beam dist:ibution parameters in the transverse betatron phase space ( $x_{\beta}, x_{\beta}^{1}, y_{\beta}$, and $y_{\beta}$ ) has been described elsewhere. 4 The stationary particle distribution was found to be gaussian. An outline of the procedure for finding $\left\langle\mathrm{x}_{\beta}^{2}\right\rangle,\left\langle\mathrm{x}_{\beta} \mathrm{y}_{\beta}\right\rangle$, etc., is shown in Fig. 1. The computations involved will be deserbibed in this section.


Fig. 1. Flow chart of the method.
A. Diffusion Integrals and Coupling Coefficient

First the values of $\eta_{\mathrm{x}, \mathrm{y}}$ and $\eta_{\mathrm{x}, \mathrm{y}}^{\prime}$ are computed for a given storage ring with known distrlbution of linear coupling

[^0]elements and sextupole magnets. Then the diffusion integrals are evaluated over all of the bending magnets:
$H_{i j}=C_{\mathscr{L}} \frac{\gamma^{5}}{2 \pi R} \oint \frac{\mathrm{ds}}{|\rho| \sqrt{\beta_{i} \beta_{j}}}-\left[\eta_{\mathrm{i}} \eta_{\mathrm{j}}+\left(\beta_{\mathrm{i}} \eta_{\mathrm{i}}^{\prime}-\frac{1}{2}{ }^{\prime}{ }_{\mathrm{i}}^{\prime} \eta_{\mathrm{i}}\right)\left(\beta_{\mathrm{j}} \eta_{\mathrm{j}}^{\prime}-\frac{1}{2} \beta_{\mathrm{j}}^{\prime} \eta_{\mathrm{j}}\right)\right]$,
with $\mathrm{i}, \mathrm{j}=\mathrm{x}, \mathrm{y}, \beta_{\mathrm{i}}$ the betatron function, R the average machine radius, $\rho$ the radius of curvature in a bending magnet, $\gamma$ the beam energy in units of rest energy, and $C_{\mathscr{L}}=55 \mathrm{r} \mathrm{e}^{\mathrm{K} /}$ $48 \sqrt{3} \mathrm{~m}_{\mathrm{e}}=2.16 \times 10^{-19} \mathrm{~m}^{3} / \mathrm{s}$.

The coupling coefficient at a reference point in the attice, $\theta_{r}$, is computed by integrating the strength of the skew quadrupole field, $S_{Q}$, and the strength of the solenoid field, $\mathrm{S}_{\mathrm{M}}$, over the coupling elements: ${ }^{3}$

$$
\begin{align*}
\mathrm{Q}= & \left.\left.\mathrm{Q}_{1}+\mathrm{i} Q_{2}=\int_{\theta_{\mathrm{r}}}^{\theta_{\mathrm{r}}^{+2 \pi} \mathrm{~d} \theta\{\exp [\mathrm{i}} \int_{\theta_{\mathrm{r}}}^{\theta} \mathrm{d} \theta\left(\frac{\mathrm{R}}{\beta_{\mathrm{x}}}-\frac{\mathrm{R}}{\beta_{\mathrm{y}}}-\Delta \nu\right)\right]\right\}  \tag{2}\\
& \cdot \frac{\sqrt{\beta_{\mathrm{x}} \beta_{\mathrm{y}}}}{4 \pi \mathrm{R}}\left[\mathrm{~S}_{\mathrm{Q}}-\frac{\mathrm{S}_{\mathrm{M}} \mathrm{R}}{4}\left(\frac{\beta_{\mathrm{x}}^{\prime}}{\beta_{\mathrm{x}}}-\frac{\beta_{\mathrm{y}}^{\prime}}{\beta_{\mathrm{y}}}\right)-\frac{\mathrm{i}^{2}}{2} \mathrm{~S}_{\mathrm{M}} \mathrm{R}\left(\frac{1}{\beta_{\mathrm{x}}}+\frac{1}{\beta_{\mathrm{y}}}\right)\right]
\end{align*}
$$

where $S_{Q}(\theta)=\left(\partial B_{x} / \partial x\right) R^{2} / B \rho$ and $S_{M}(\theta)=B_{z} R / B \rho$ with $B \rho$ the particle rigidity and $\Delta \nu=\nu_{\mathrm{x}}-\nu_{\mathrm{y}}-\mathrm{m}$ the distance from the nearest coupling resonance with ${ }^{y} m$ an integer. ${ }^{5}$

## B. Beam Distribution Parameters

The expected values at the reference point for the beam coordinates in the $x-y$ plane can be computed by:

$$
\left[\begin{array}{c}
\frac{\left\langle x_{\beta}^{2}\right\rangle}{\beta_{x}}  \tag{3}\\
\frac{\left(x_{P} y_{\beta}\right\rangle}{\sqrt{\beta_{j}{ }_{y}}} \\
\frac{\left\langle y_{\beta}^{2}\right\rangle}{\beta_{y}}
\end{array}\right]=\mathscr{y}\left[\begin{array}{ccc}
\alpha_{y} \Delta \nu^{2}+\left(\alpha_{x}+\alpha_{y}\right)|Q|^{2} & 2 \alpha_{y} Q_{1} \Delta \nu & \left(\alpha_{x}+\alpha_{y}\right)|Q|^{2} \\
\alpha_{y} Q_{1} \Delta \nu & 2\left(\alpha_{x}+\alpha_{y}\right) Q_{1}^{2} & -\alpha_{x} Q_{1} \Delta \nu \\
\left(\alpha_{x}+\alpha_{y}\right)|Q|^{2} & -2 \alpha_{x} Q_{1} \Delta \nu & \alpha_{x} \Delta \nu^{2}+\left(\alpha_{x}+\alpha_{y}\right)|Q|^{2}
\end{array}\right]\left[\begin{array}{l}
H_{x x} \\
H_{x y} \\
H_{y y}
\end{array}\right]
$$

where $D=\frac{1}{2\left[\left(\alpha_{\mathrm{x}}+\alpha_{\mathrm{y}}\right)^{2}|Q|^{2}+\alpha_{\mathrm{x}} \alpha_{\mathrm{y}} \Delta \nu^{2}\right]}$
with $\beta_{\mathrm{x}, \mathrm{y}}$ evaluated at the reference point.
It is interesting to note that these parameters obey the invariant condition

$$
\begin{equation*}
\alpha_{\mathrm{x}} \frac{\left\langle\mathrm{x}_{\beta}^{2}\right\rangle}{\beta_{\mathrm{x}}}+\alpha_{\mathrm{y}} \frac{\left\langle\mathrm{y}_{\beta}^{2}\right\rangle}{\beta_{\mathrm{y}}}=\frac{1}{2}\left(\mathrm{H}_{\mathrm{xx}}+\mathrm{H}_{\mathrm{yy}}\right) \tag{4}
\end{equation*}
$$

independent of the coupling strength. The other expected values $\left\langle x_{\beta} x_{\beta}^{\prime}\right\rangle\left\langle x_{\beta}^{\prime 2}\right\rangle,\left\langle x_{\beta}^{\prime} y_{\beta}\right\rangle,\left\langle x_{\beta} y_{\beta}^{\prime}\right\rangle,\left\langle x_{\beta}^{\prime} y_{\beta}^{\prime}\right\rangle,\left\langle y_{\beta} y_{\beta}^{\prime}\right\rangle$, and $\left\langle y_{\beta}^{\prime 2}\right\rangle$ are computed by the expressions:

$$
\begin{align*}
& \frac{\left\langle x_{\beta}^{2}\right.}{\gamma_{\mathrm{x}}}=\frac{2\left\langle\mathrm{x}_{\beta} \mathrm{x}_{\beta}^{\prime}\right\rangle}{\beta_{\mathrm{x}}^{\prime}}=\frac{\left\langle\mathrm{x}_{\beta}^{2}\right\rangle}{\beta_{\mathrm{x}}}, \quad \frac{\left\langle\mathrm{y}_{\beta}^{\prime 2}\right\rangle}{\gamma_{\mathrm{y}}}=\frac{2\left\langle y_{\beta} \mathrm{y}_{\beta}^{\prime}\right\rangle}{\beta_{\mathrm{y}}^{1}}=\frac{\left\langle\mathrm{y}_{\beta}^{2}\right\rangle}{\beta_{\mathrm{y}}}, \\
& \left\langle\left(\beta_{\mathrm{x}} \mathrm{x}_{\beta}^{\prime}-\frac{1}{2} \beta_{\mathrm{x}}^{\prime} \mathrm{x}_{\beta}\right)\left(\beta_{\mathrm{y}} \mathrm{y}_{\beta}^{\prime}-\frac{1}{2} \beta_{\mathrm{y}}^{\beta^{\prime}} \mathrm{y}_{\beta}\right)\right\rangle=\left\langle\mathrm{x}_{\beta} \mathrm{y}_{\beta}\right\rangle . \tag{5}
\end{align*}
$$

and
$\left\langle x_{\beta}\left(\beta_{y} y_{\beta}^{\prime}-\frac{1}{2} \beta_{y}^{\prime} y_{\beta}\right)\right\rangle=-\left\langle y_{\beta}\left(\beta_{\mathrm{x}} \mathrm{x}_{\beta}^{\prime}-\frac{1}{2} \beta_{\mathrm{x}}^{\prime} \mathrm{x}_{\beta}\right)\right\rangle=\frac{\sqrt{\beta \beta^{\beta}}{ }^{2}}{\alpha_{\mathrm{x}}+\alpha_{\mathrm{y}}} \mathrm{H}_{\mathrm{xy}}^{\prime}$,
where $\gamma_{\mathrm{i}}=\left(1+\beta_{\mathrm{i}}^{2} / 4\right) / \beta_{\mathrm{i}}$ and $\mathrm{H}_{\mathrm{xy}}^{\prime}$ is an additional diffusion in-
tegral defined to be


## C. Beam Proflle

A sketch of the beam profile in the $x-y$ plane is shown in Fig. 2. The measurable quantities are:


Fig. 2. Beam profile.

$$
\begin{aligned}
& \sigma_{x}^{2}=\left\langle\mathrm{x}_{\beta}^{2}\right\rangle+\eta_{\mathrm{x}}^{2}\left\langle\delta^{2}\right\rangle, \\
& \sigma_{y}^{2}=\left\langle\mathrm{y}_{\beta}^{2}\right\rangle+\eta_{\mathrm{y}}^{2}\left\langle\delta^{2}\right\rangle, \\
& \text { and } \\
& \sigma_{\mathrm{xy}}=\left\langle\mathrm{x}_{\beta} \mathrm{y}_{\beta}\right\rangle+\eta_{\mathrm{x}} \eta_{\mathrm{y}}\left\langle\delta^{2}\right\rangle(1
\end{aligned}
$$

where $\eta_{\mathrm{x}, \mathrm{y}}$ are evaluated at the reference point $\theta_{r}$ and $\left\langle\delta^{2}\right\rangle^{\frac{1}{2}}$ is the rms energy spread in the beam. The profile tilt angle is

$$
\phi=\frac{1}{2} \tan ^{-1}\left[2 \sigma_{\mathrm{xy}} /\left(\sigma_{\mathrm{x}}^{2}-\sigma_{\mathrm{y}}^{2}\right)\right] .
$$

For a special case in which $\alpha_{x}=\alpha_{y}=\alpha$ and $\eta_{y}=0 \mathrm{ev}$ erywhere in the lattice, the profile parameters of an upright beam can be written as:
$\frac{\left\langle x_{\beta}^{2}\right\rangle}{\beta_{x}}=\frac{1}{1+A^{2}} \frac{\left\langle x_{\beta_{0}^{2}}^{\beta_{x}}{ }_{x}\right.}{} ; \quad \frac{\left\langle y_{\beta}^{2}\right\rangle}{\beta_{y}}=\frac{A^{2}}{1+A^{2}} \frac{\left\langle x_{\beta_{0}}^{2}\right.}{\beta_{x}}$
with $<\mathrm{x}_{\beta}^{2}>/ \beta_{\mathrm{x}}=\mathrm{H}_{\mathrm{xx}} / 2 \alpha$ the horizontal beam emittance in the absence of coupling. The aspect ratio A is found to be

$$
\begin{equation*}
A=\left(\frac{2|Q|^{2}}{2|Q|^{2}+\Delta \nu^{2}}\right)^{\frac{1}{2}} \tag{10}
\end{equation*}
$$

## III. APPLICATION

The procedure described in the previous section is used to cairulate the beam profile in PEP and SPEAR due to magnet atymment and closed orbit errors. Since there are no vertial bending magnets in these machines, we take $\eta_{x}$ to be given i, y the unperturbed lattice solution and $\eta_{\mathrm{y}}$ to be the valtes $u r$ duced by errors. The integral $\mathrm{H}_{\mathrm{xx}}$ is tound by Eq.
(1). II. we assume that the causes of the errors are uncorrelated, ${ }^{\text {und }}$ are sufficiently small, we have $\mathrm{H}_{\mathrm{yy}} \approx 2 \mathrm{C}_{\mathscr{y}} \gamma^{5}\left\langle\eta_{\mathrm{y}}^{2}\right\rangle$ $\beta_{\mathrm{y}} \mathrm{R}_{1},-$, with the expectation value $\left\langle\eta_{\mathrm{y}}^{2}\right\rangle$ in the bending magnets given by

$$
\begin{align*}
\frac{\left\langle\eta_{y}^{2}\right\rangle}{\beta_{y}} \approx & \frac{1}{2 \sin ^{2} \pi \nu}\left[\frac{1}{4} \sum_{\mathrm{b}} \beta_{\mathrm{yb}} \mathrm{~S}_{\mathrm{b}}^{2} \Delta \theta_{\mathrm{b}}^{2}+\frac{1}{4} \sum_{\mathrm{q}} \beta_{\mathrm{yq}} \mathrm{~S}_{\mathrm{q}}^{2} \Delta y_{\mathrm{q}}^{2}\right. \\
& \left.+\sum_{\mathrm{q}} \beta_{\mathrm{yq}} \eta_{\mathrm{xq}} \mathrm{~s}_{\mathrm{q}}^{2} \Delta \theta_{\mathrm{q}}^{2}+\sum_{\mathrm{s}} \beta_{\mathrm{ys}} \eta_{\mathrm{xs}}^{2} \mathrm{~S}_{\mathrm{s}}^{2} \Delta \mathrm{y}_{\mathrm{s}}^{2}\right] \tag{11}
\end{align*}
$$

Under the same condition, $\mathrm{H}_{\mathrm{X}}$ has an expectation value zero hut can vary between $\pm \sqrt{\mathrm{H}_{\mathrm{xx}} \mathrm{H}_{\mathrm{yy}}} / 2$. In the expression for $\left\langle\eta_{>}^{2}\right\rangle / \beta_{\mathrm{y}}$, the first and second sums come from dipole kick: " ilue to rotational errors in bends ( $\Delta \theta_{b}$ ) and orbit errors in quials ( $\Delta \mathrm{y}_{q}$ ), respectively. The third and fourth sums come from coupling to $\eta_{x}$ due to rotational errors in quads $\left(\Delta \theta_{\mathrm{C}}\right)$ and orbit errors in sextupoles ( $\Delta \mathrm{y}_{\mathrm{s}}$ ).

IThe reference point is taken to be the interaction point with the coupling coefficient given approximately by

$$
\begin{equation*}
Q^{2} \approx \frac{1}{4 \pi^{2}}\left[\sum_{q} \beta_{x q} \beta_{y q} S_{q}^{2} \Delta 0_{\mathrm{q}}^{2}+\sum_{\mathrm{s}} \beta_{\mathrm{xs}} \beta_{\mathrm{ys}} \mathrm{~S}_{\mathrm{s}}^{2} \Delta \mathrm{y}_{\mathrm{s}}^{2}\right] \tag{12}
\end{equation*}
$$

The strength of the bending, quadrupole, and sextupole marnets is denoted by $\mathrm{S}_{\mathrm{b}}=\mathrm{B}_{\mathrm{y}} \ell / \mathrm{B} \rho, \mathrm{S}_{\mathrm{d}}=\left(\partial \mathrm{B}_{\mathrm{y}} / \partial \mathrm{x}\right) \ell / \mathrm{B} \rho$, anc: $\mathrm{S}_{\mathrm{s}}=$ $\left(\partial^{2} \mathrm{~B}_{\mathrm{y}} / \partial \mathrm{x}^{2}\right) \ell / 2 \mathrm{~B} \rho$, respectively, with $\ell$ the magnet lengths:

The vertical beam sizes are computed from Eqs. (3) and (7) for three typical cases assuming $\Delta \theta=0.5 \times 10^{-3}$ in all magnets; $\langle\Delta y\rangle=2 \mathrm{~mm}$ in the interaction region quadrupole magnets, and $\left\langle\frac{\Delta y^{2}}{\beta_{y}}\right\rangle^{\frac{1}{2}}=.5 \times 10^{-3} \mathrm{~m}^{\frac{1}{2}}$ in the rest of the lattice. Some of the relevant machine parameters are: energy $=3$ $\mathrm{GeV}, \nu_{\mathrm{x}}=5.28, \nu_{v}=5.18, \eta_{\mathrm{X}}^{*}=0.0, \beta_{\mathrm{X}}^{*}=1.2 \mathrm{~m}$, and $\beta_{\mathrm{Y}}^{*}=$ 0.07 m for the SPEAR configuration; energy $=15 \mathrm{GeV}, \nu_{\mathrm{x}}=$ $18.7, \nu_{\mathrm{y}}=19.2, \eta_{\mathrm{X}}^{*}=-.67 \mathrm{~m}, \beta_{\mathrm{X}}^{*}=3.8 \mathrm{~m}$, and $\beta_{\mathrm{X}}^{*}=.155^{x} \mathrm{~m}$ for the PEP "normal" configuration, and $\beta_{\mathrm{y}}^{*}=.11 \mathrm{~m}$ for the "low- $\beta_{\bar{Y}}^{*}$ " configuration.

In Table I the quantities $Q_{o p t}$ and $\sigma_{\mathrm{y}}^{*}$,opt denote the values of coupling constant and vertical beam size at the interaction point for the so-called "optimum" luminosity operation. ${ }^{6}$ The quantity $Q / Q_{o p t}$ is therefore a measure of the

Table I. Order of magnitude estimates of vertical beam size in PEP and SPEAR for some assumed errors.

| Parameter | SPEAR | PEP "Normal" | PEP ${ }^{\prime \prime}$ Low $-\beta_{y}^{* \prime \prime}$ |
| :--- | :--- | :---: | :---: |
| $\left\langle\eta_{\mathrm{y}}^{2}\right\rangle / \beta_{\mathrm{y}}$ | .9 mm | 5 mm | 10 mm |
| $\left\langle\eta_{\mathrm{y}}^{*}\right\rangle$ | 0.8 cm | 3 cm | 3.3 cm |
| $\langle Q\rangle$ | 0.006 | .03 | .06 |
| $\sigma_{\mathrm{y}}^{*}$ | 0.015 mm | .09 mm | .16 mm |
| $\langle Q\rangle /\langle Q\rangle_{\text {opt }}$ | .33 | .3 | .8 |
| $\sigma_{\mathrm{y}}^{*} / \sigma_{\mathrm{y}, \text { opt }}^{*}$ | 0.36 | .33 | .84 |

relative strength of residual coupling effect due to errors as compared with the coupling effect due to beam-beam interaction; $\sigma_{\mathrm{Y}}^{*} / \sigma_{\mathrm{Y}}^{*}$, opt is a measure of the relative vertical beam size. Note that the PEP "low- $\beta_{\mathrm{y}}^{*}$ " configuration has a large residual coupling coefficient leading to a large beam sizc. Since most of the residual coupling comes from the interaction region quadrupoles and the strong sextupoles used for chromaticity corrections, it is suggested that vertical closed orbit be controlled to an accuracy of $\pm .2 \mathrm{~mm}$ in these magnets in order to reduce the beam size to an acceptable level. For the PEP configuration $\sigma_{\mathrm{y}}^{*}$ has been estimated for a maximum $\Delta \nu$ value equal to 0.5 . Since the aspect ratio is lawrer for $\Delta \nu<0.5$ (Eq. (10)), the values of $\sigma_{\mathrm{y}}^{*}$ given in Table I rcpresent a conservative estimate.

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