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CHROMATICITY CORRECTION IN LARGE STORAGE RINGS*

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Introduction

As the size of storage rings proposed and under construction increases, the chromaticity reaches values which require correction schemes with a new order of sophistication involving sextupoles.

In small storage rings (the 500-MeV Princeton-Stanford e^--e^- rings, VEPP-1, VEEP-2 and ACO), the chromaticity was so small that no correction was needed, although careful study of the behavior of the beam in ACO indicated that a chromaticity correction would be useful. In the 1.5-GeV storage ring, ADONE, it was discovered that the chromaticity, although still small (ξ = ≃ -3), caused the so-called head-tail instability which limited storable currents to very low values. Because there was insufficient space for sextupoles, this instability was cured by using a feedback system. However, it became clear that chromaticity correction with sextupoles would have to be incorporated into the design of the next generation of storage rings, especially since all storage rings after ADONE used the low-β insertion which pushes the chromaticity to higher values (-20 in SPEAR and -100 to -160 in PEP). Chromaticity correction to avoid the head-tail instability by using sextupoles rather than a feedback system is necessary where large chromaticity is encountered since the tune spread caused by the momentum spread in the beam would be intolerably large (±0.1 in SPEAR and ±0.6 to ±1.0 in PEP).

In determining the chromaticity correction in PEP it was found, however, that further sophistication of the correction was needed over that required for SPEAR. In SPEAR only two families of sextupoles are required to give the vertical and horizontal chromaticity the desired values. While this can be accomplished in principle by as few as two sextupoles, in practice, the linear beam dynamics would be perturbed too much by the strong non-linear fields required. Tracking studies for SPEAR have shown that distribution of the sextupoles around the ring is necessary to ensure stable particle trajectories within the beam-stay-clear and apart from that, no further sophistication of the sextupole correction was found to be necessary.

The very large chromaticities encountered in storage rings like PEP, PETRA and the "Very Big Storage Rings" produce a new set of undesirable characteristics in the beam dynamics which require a more complicated means of correction.

The method of attack used at PEP to solve the problem of both satisfactory chromaticity correction and stable non-linear motion is discussed in this paper. The method consists first of analytically determining the effect of the sextupoles on the equilibrium orbit x_e through second order in $\delta = \Delta p/p$ and the tune variation through both third order in δ and second order in the transverse betatron amplitude x_β and y_β. Next the sextupole arrangements are determined to minimize these variations. The sextupole arrangements determined in this manner are then included in defining the complete PEP lattice. Using this lattice various tracking programs are used to determine the dependence of the equilibrium orbit, tune shift, and betatron functions upon the particle momentum and the stability as a function of betatron amplitude. We have included the synchrotron motion in several of the tracking programs and obtain the frequency spectrum of the particle motion in addition to the usual phase space plots.

Equations of Motion

It is useful to define the "reduced" variables of Courant and Snyder[1] such that the actual variable is denoted by a tilda, i.e., the reduced horizontal dimension $x = \tilde{x}/\sqrt{\beta_x}$. We separate the horizontal motion into the sum of the betatron and synchrotron motion, $\tilde{x} = \tilde{x}_\beta + \tilde{\eta}\delta$ so that in terms of the reduced variables the equations of motion can be written as:

$$\ddot{\eta} + \nu_{ox}^2\eta = \frac{1}{\rho}(1-\delta+\delta^2-\dots)\,\nu_{ox}^2\beta_x^{3/2} + k(\delta-\delta^2+\dots)\nu_{ox}^2\beta_x^2\eta - \frac{1}{2}m\,(\delta-\delta^2+\dots)\,\nu_{ox}^2\beta_x^{3/2}\eta^2 + \dots \tag{1a}$$

$$\ddot{x}_\beta + \nu_{ox}^2\,x_\beta = (k-m\tilde{\eta})\,(\delta-\delta^2+\delta^3-\dots)\,\nu_{ox}^2\,\beta_x^2\,x_\beta - \frac{1}{2}m\,(1-\delta+\delta^2-\dots)\,\nu_{ox}^2\beta_x^{5/2}\,x_\beta^2 \tag{1b}$$

$$\ddot{y}_\beta + \nu_{oy}^2\,y_\beta = -(k-m\tilde{\eta})(\delta-\delta^2+\dots)\,\nu_{oy}^2\,\beta_y^2\,y_\beta + \dots \tag{1c}$$

Where the derivatives are taken with respect to the unperturbed betatron phase, ν_o and β are the unperturbed tune and betatron function, ρ is the bending radius, k is the usual focusing parameter equal to the ratio of the quadrupole strength to the magentic rigidity, and m is the ratio of the field gradient in a sextupole to the magnetic rigidity. Both k and m are positive for horizontally-focusing magnets.

In storage rings built so far it was sufficient to solve these equations to the lowest order of pertubation on the right-hand side. For the betatron motion it was sufficient to consider only the zero-th harmonic of $(k-m\tilde{\eta})\beta^2$ which is proportional to the chromaticity. Apart from that, the higher harmonics which could cause resonances were reduced by distributing the sextupoles around the ring.

In order to obtain valid results for the range of momenta of interest in PEP it has been necessary to solve for η through second order in δ and to solve for the tune shift through third order in δ. To determine the tune dependence upon betatron amplitude due to the sextupoles requires a large number of harmonies of $(k-m\tilde{\eta})$ and it is necessary to use second order theory.

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Variation of Eta Function with Momentum

The solution of Eq. (1a) is found by substituting into the differential equation:

$$\eta = \eta_0 + \eta_1 \delta + \eta_2 \delta_2 + \ldots \tag{2}$$

and using a successive approximation technique to obtain:

$$\eta = \sum_{n=o} (F_{on} + \delta(F_{1n} - F_{on}) + \delta^2(F_{2n} - F_{1n} + F_{2n}) + \ldots) \frac{\cos n\phi}{\nu_{ox}^2 - n^2} \tag{3}$$

with F_{on}, F_{1n}, F_{2n} the Fourier harmonics of $(\nu_{ox}^2 \beta_x^2/\rho)$, $(\nu_{ox}^2 \beta_x^2 \tilde{\eta}_o [k - \frac{1}{2} m\tilde{\eta}_o])$ and $(\nu_{ox} \beta_x{}^2 \tilde{\eta}_1 [k - \frac{1}{2} m\tilde{\eta}_o])$, respectively. Figure 1 shows the comparison of the eta function at the interaction point ($\phi = o$) calculated by Eq. (3) and with the matrix formalism.

Variation of Linear Tune with Momentum

To determine the variation of the linear tune with momentum we start with Eqs. (1b) and (1c) and neglect the quadratic terms in x_β and y_β. The differential equations can be written in the form:

$$\ddot{u} + (\nu_{oo}^2 + A_o + \sum_{n=1}^{\infty} A_n \cos n\phi)\, u = 0 \tag{4a}$$

where
$$A_n = a_n \delta + b_n \delta^2 + c_n \delta^3 \tag{4b}$$

with a_n, b_n and c_n are the Fourier harmonics of $\pm\nu_{oo}^2 \beta\, (m\tilde{\eta}_o - k)$, $\pm\nu_{oo}^2 \beta\, (m\tilde{\eta}_1 - m\tilde{\eta}_o + k)$ and $\pm\nu_{oo} \beta\, (m\tilde{\eta}_2 - m\tilde{\eta}_1 + m\tilde{\eta}_o - k)$ respectively for the x motion and y motion. We have also made use of the fact that the ring is symmetric about the interaction region, $\phi = 0$.

In order to obtain the expression for the tune valid through third order in δ, it is necessary to solve Eq. (4a) through third order in A. We have used three methods to obtain the necessary solution. The first is the "smooth approximation" which uses a successive approximation technique and gives valid results in the case where the tune shift is small compared to the distance to the half interger resonance.(2) The other two methods are based on a formalism developed by Vogt Nielsen which shows that the tune is given by:[3]

$$\cos 2\pi\nu = \tfrac{1}{2} \left[V(2\pi) - V(-2\pi) \right] \tag{5}$$

where $V(\phi)$ is a solution of Eq. (4a). The solution $V(\phi)$ is expanded in the base functions $V_k(\phi)$ with $V_o(\phi) = \cos \nu_o \phi$ and

$$V_{k+1}(\phi) = -\frac{1}{\nu_o} \int V_k(\alpha) A(\alpha) \sin \nu_o(\alpha - \phi)\, d\alpha \tag{6}$$

These expressions are inserted into Eq. (5) to obtain successive approximations for the tune variation with momentum. The quadratic term in δ contains double integrals, the cubic term triple integrals, etc. One of our computer programs, HARMON, is used to find sextupole distributions that minimize these integrals and is very efficient in finding a correct solution once the proper sextupoles are chosen. To help determine the most efficient sextupoles, a third approach was developed. In this approach the perturbation was developed in Fourier components as in the "smooth approximation", Eq. (4b) and inserted into the integrals of Eq. (6) to solve for ν explicitly. This method is straightforward but somewhat tedious. In the limit of small tune shift this last method gives the same results as the smooth approximation, but has the advantage of being valid for the case where the tune shift is large enough to shift one into a half integral stop band. In Fig. 2 the tune shift with momentum as obtained from the smooth approximation is compared to the tune shift obtained by matrix multiplication.

Variation of Tune with Betatron Amplitude

Because the betatron phases are the independent variables in Eqs. (1b) and (1c) and are different between the x and y motion, it is easier to start directly from the Hamiltonian:

$$\mathcal{H} = \frac{Px^2}{2} + k_x(\theta)\frac{x^2}{2} + \frac{P_y^2}{2} + k_y(\theta)\frac{y^2}{2} + \frac{m(\theta)}{6}(x^3 - 3xy^2) \tag{8}$$

This Hamiltonian is first transformed into action angle variables in such a way as to remove the theta dependence in the linear equations of motion and then the new coefficients of the non-linear terms are expanded in a Fourier series. In the usual approximation one neglects the rapidly oscillating terms and retains only the terms which are either resonant or correspond to a tune shift with betatron amplitude. All present day storage rings are designed to avoid the third order resonances and to first order in the sextupole strength, m, there is no tune shift with amplitude. However, the new generation low-beta storage rings require such large sextupole strengths to control the chromaticity that it is necessary to consider effects which are second order in the sextupole strength. For this case one must first transform the cubic terms away into higher-order terms before neglecting the rapidly oscillating terms. This is a tedious and lengthly process and the only result presented here is the tune shift with amplitude.

$$\begin{aligned} \Delta\nu_x &= G_{xx}\, \hat{x}_\beta^2 + G_{xy}\, \hat{y}_\beta^2 \\ \Delta\nu_y &= G_{yx}\, \hat{x}_\beta^2 + G_{yy}\, \hat{y}_\beta^2 \end{aligned} \tag{9}$$

where the coefficients G contain quadratic terms in the Fourier harmonics of $m(\theta)$ weighted by the appropriate powers of the unperturbed betatron functions, and $G_{xy} = G_{yx}$. It has been possible to replace the sum over the quadratic terms with an integral that is more convenient to evaluate. A typical coefficient G_{xx} is expressed in integral form below.

$$G_{xx} = -\frac{1}{64\pi} \int_0^C \int_0^{s_2} ds_1\, ds_2\, \beta_x^{3/2}(s_2)\, \beta_x^{3/2}(s_1)\, m(s_1)\, m(s_2) \cdot \left\{ \frac{\cos 3\psi_x(s_1, s_2)}{\sin 3\pi\nu_x} + 3\, \frac{\cos \psi_x(s_1, s_2)}{\sin \pi\nu_x} \right\} \tag{10}$$

where $\psi_x(s_1, s_2) = \nu_x \left[\pi + \phi_x(s_1) - \phi_x(s_2) \right]$

In Fig 3 we show a typical spectrum for non-linear betatron motion as obtained from a tracking program which includes the synchrotron motion. The tune shifts as obtained from this program are in good agreement with that calculated from Eq. (9).

Conclusion

The analytic expressions which have been developed for calculating the dispersion function through second order in δ and for the tune shifts through third order in δ, agree well with the results of the matrix multiplication method over the range in δ of interest.

We also have developed analytic expressions for the tune shift with betatron amplitude which agree well with the results from the tracking programs and can be used to determine for which amplitudes the tune will be shifted to a resonance. With the aid of our computer programs we first determine the sextupole families that minimize the tune and dispersion variations using these analytic expressions. Then particles are tracked around the ring with this sextupole distribution to determine stability of the betatron motion. A phase space diagram for a typical PEP configuration is shown in Fig. 4.

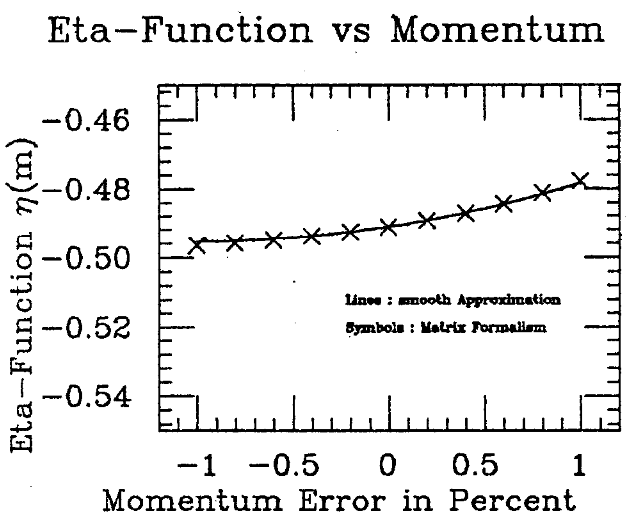

Fig. 1

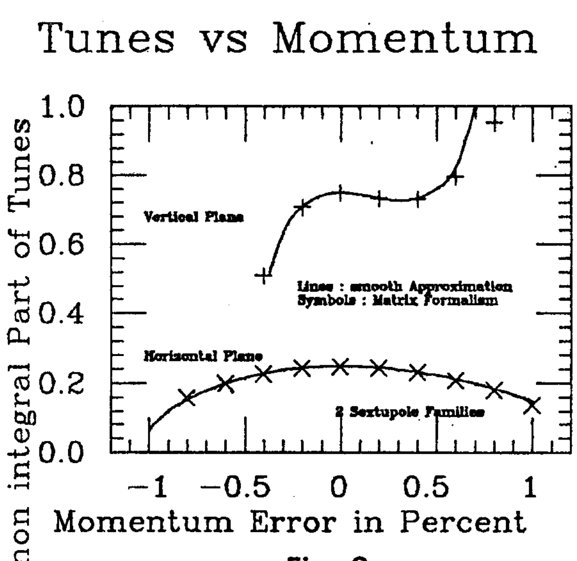

Fig. 2a

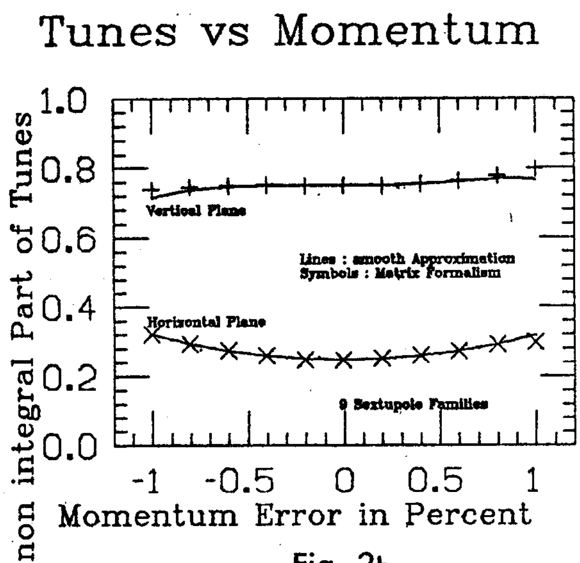

Fig. 2b

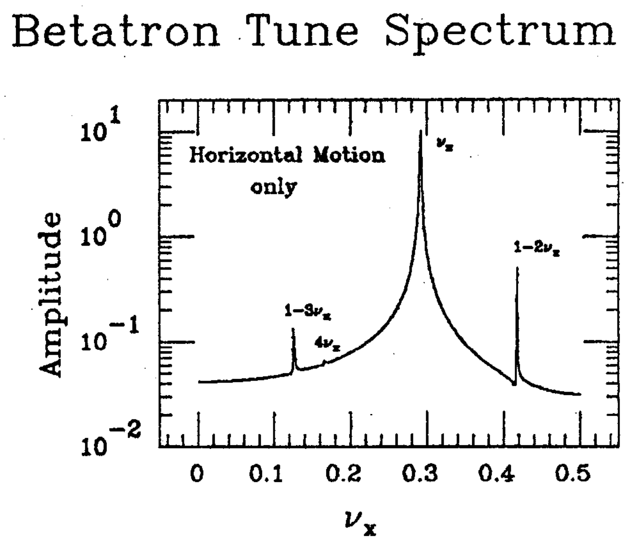

Fig. 3

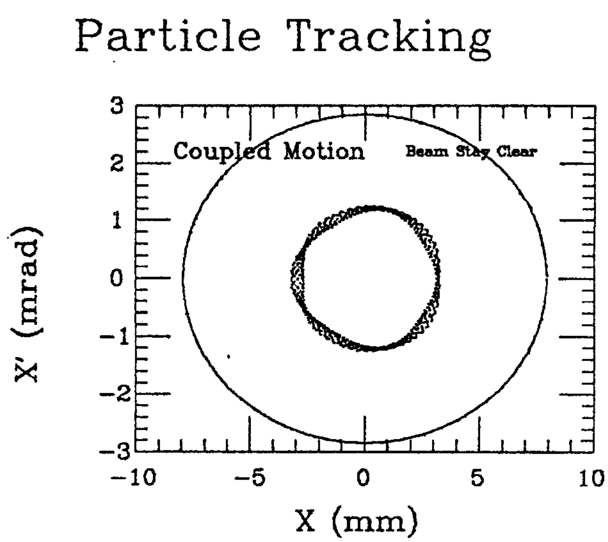

Fig. 4

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