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# RIGOROUS BOUNDS FOR FORM FACTORS AT ALL $q^2$

## AND THE DRELL-YAN-WEST RELATION\*

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RIGOROUS BOUNDS FOR FORM FACTORS

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## ABSTRACT

Rigorous bounds are described for form factors for scalar and spinor particles using sidewise dispersion relations. Bounds on the large  $q^2$  behavior and the Drell-Yan-West relation are discussed and related to the propagator weight function for possible interpolating fields for the particle involved.

#### I. INTRODUCTION

The possibility of a relationship between electromagnetic form factors : large momentum transfer and deep inelastic electron scattering structure fur. tions near the elastic threshold was first pointed out by Drell and Yan [1] and shortly thereafter by West [2]. This so-called Drell-Yan-West (DYW) relation was derived within the framework of a modified field theory model with a transverse momentum cutoff and has also been discussed in parton models [3]. Although the relation is plausible and various model dependent rather handwaving arguments have been given [4], no really convincing derivation has been made.

We shall discuss the problem from the standpoint of a sidewise dispersion relation for the form factor in which, by the use of the Schwarz inequality, the imaginary part of the form factor is related to the structure functions. We will show that under certain circumstances, the DYW relation emerges as an <u>inequality</u>. The critical issue is the high energy behavior of the weight function of the propagator associated with the interpolating field used to describe a physical particle.

Although we were unaware of it at the time we began this work, a similar dispersion-Schwarz inequality approach had been taken some time ago by Cooper and Pagels and by West [5]. These authors, however, did not have our purpose in mind. For the sake of making this paper reasonably self-contained we shall repeat much of their formal analysis. Our use of the Schwarz inequality is rather different and somewhat more refined. One remarkable feature of the inequality is that it connects a quantity defined only in terms of an (arbitrary) interpolating field to purely S-matrix quantities. The full consequences of such relations is not clear to us, but the freedom of choice of interpolating fields will be important to our results.

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In Section II we discuss the electromagnetic form factor of a scalar particle (a pion, for example) and its relation to the structure functions and the propagator weight function. Then in Section III the large momentum transfer limit is treated under various assumptions. In Section IV, a model theory involving a transverse momentum cutoff is described. This discussion is repeated for the nucleon in Sections V and VI. A brief summary is given in Section VII. In the appendix some results on propagators and interpolating fields within the framework of a soluble model are given.

#### **II. SCALAR PARTICLES**

In this section we shall discuss the relation between the form factor of a scalar particle and the corresponding deep-inelastic structure functions analogous to that given in the next section for a spin one-half particle. It will be shown that the asymptotic behavior of the form factor can be related to the threshold behavior of the structure functions only if the relevant propagator weight function falls off sufficiently rapidly. We also show later how a model field theory with a transverse momentum cutoff in fact leads to the weight function decreasing more rapidly than the same theory without the cutoff.

The form factor of a scalar particle is defined in terms of the matrix element between one particle states (corresponding to mass m,  $p^2 = p'^2 = -m^2$ ) of the electromagnetic current operator  $j_{\mu}$ , namely by writing

$$M_{\mu}(p',p) = \chi^{*}(p') T_{\mu}(p',p) \chi(p) = (4p_{0}'p_{0})^{1/2} < p' |j_{\mu}()|p\rangle$$

or in terms of the conventional form factor  $F(q^2)$ , with  $q \equiv p-p^{\dagger}$ ,

$$M_{\mu}(p', p) = (p'+p)_{\mu} F(q^2)$$
.

In these expressions we regard  $T_{\mu}(p',p)$  as the irreducible vertex function formed from the current  $j_{\mu}$  and two scalar field operators and is the analogue of

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 $q^2 \Delta_{F_c}(q^2)\Gamma_{\mu}(p',p)$  in the spin one-half case. The  $\chi(p)$  are "wave functions" whose only role is to indicate the on-shell projection operation and are the analogues of the spinors u for spin one-half. It will be necessary for us to deal with the half off-shell quantity  $R_{\mu}(p',p)$  given by

$$R_{\mu}(p',p) = \chi^{*}(p') T_{\mu}(p',p) \Delta_{F_{c}}(p^{2}) (p^{2}+m^{2}) ,$$

where  $\Delta_{F_c}(p^2)$  is the fully dressed propagator of the scalar particle and, of course,  $p^2 \neq -m^2$ .

The structure of  $R_{\mu}(p',p)$  is more complicated than that of  $M_{\mu}(p',p)$  since p is off-shell. We have, in general,

$$R_{\mu}(p',p) = \chi^{*}(p') \left[ (p'+p)_{\mu} F(q^{2},-p^{2}) + (p'-p)_{\mu} G(q^{2},-p^{2}) \right]$$

where we know that on-shell,  $p^2 = -m^2$ , G must vanish. This is a consequence of the generalized Ward identity [6] which takes the form

$$(p' - p)_{\mu} T_{\mu}(p', p) = e \left[ \Delta_{F_{c}}^{-1}(p') - \Delta_{F_{c}}^{-1}(p) \right]$$
  
or in terms of  $R_{\mu}(p', p)$ , using  $\chi^{*}(p') \Delta_{F_{c}}^{-1}(p') = 0$ ,  
 $(p' - p)_{\mu} R_{\mu}(p', p) = -e\chi^{*}(p') (p^{2} + m^{2})$ 

from which we find

$$G(q^2, -p^2) = \frac{p^2 + m^2}{q^2} \left[ F(q^2, -p^2) - e \right]$$

We write then finally

$$R_{\mu}(p',p) - \frac{e(p^{2}+m^{2})}{q^{2}}q_{\mu} = \left[(p'+p)_{\mu} - \frac{(p^{2}+m^{2})q_{\mu}}{q^{2}}\right]F(q^{2},-p^{2})$$

where we have dropped the wave function  $\chi^*(p')$  for ease of writing and have introduced  $q_{\mu} \equiv (p-p')_{\mu}$ .

Remembering that  $p'^2 = -m^2$  we see that the coefficient of  $F(q^2, -p^2)$  is orthogonal to  $q_{\mu}$  so that we may easily project F from  $R_{\mu}$  essentially by multiplying by

$$V_{\mu} \equiv (p'+p)_{\mu} - \frac{(p^2 - {p'}^2)q_{\mu}}{q^2}$$

Thus

$$F(q^2, -p^2) = V \cdot R/V^2$$

where

$$V^{2} = 4 \left[ p^{2} p'^{2} - (p, p')^{2} \right]$$

We note in passing that  $V^2$  may be expressed in terms of the variables  $q^2$  and  $\nu$  (= -p' \cdot q/m) as

$$V^2 = -4m^2 (\nu^2 + q^2)/q^2$$

The next step is to write a representation for  $R_{\mu}$  in terms of field operators. It is then elementary, following Bincer [7], to derive a dispersion relation for  $F(q^2, -p^2)$  in the variable  $s \equiv -p^2$ , for fixed positive  $q^2$ . By standard methods we have

$$\mathbf{R}_{\mu}(\mathbf{p',p}) = i(2\mathbf{p}_{0}')^{1/2} \int d^{4}x \ e^{i\mathbf{p \cdot x}} \langle \mathbf{p'} | \left[ \mathbf{j}_{\mu}(0), \mathbf{J}^{\dagger}(x) \right] | \mathbf{0} \rangle \theta(-\mathbf{x}_{0})$$

Here J(x) is the "source" of the spin zero field which we may imagine to be given in terms of suitable interpolating field  $\phi(x)$  by  $(m^2 - \Box) \phi(x) \equiv J(x)$ . We ignore equal-time commutators. The absorptive part of  $R_{\mu}$  from which the imaginary part of  $F(q^2, -p^2)$  is to be computed is

$$\operatorname{Im} \mathbf{R}_{\mu}(\mathbf{p}', \mathbf{p}) = \frac{1}{2} (2\mathbf{p}_{0}')^{1/2} \int d^{4}x \ e^{i\mathbf{p}\cdot \mathbf{x}} < \mathbf{p}' | \left[ \mathbf{j}_{\mu}(0), \mathbf{J}^{\dagger}(\mathbf{x}) \right] | 0 >$$
$$= \frac{1}{2} (2\mathbf{p}_{0}')^{1/2} \sum_{\mathbf{n}} (2\pi)^{4} \delta(\mathbf{p} - \mathbf{p}_{\mathbf{n}}) < \mathbf{p}' | \mathbf{j}_{\mu} | \mathbf{n} > < \mathbf{n} | \mathbf{J}^{\dagger} | 0 >$$

Because J is a scalar operator, all of the states  $|n\rangle$  have total angular momentum zero. Using the relation between  $F(q^2, -p^2)$  and  $R_{\mu}$  we find

Im F (q<sup>2</sup>, -p<sup>2</sup>) = 
$$\frac{1}{2} (2p'_0)^{1/2} \frac{1}{V^2} \sum_{n} (2\pi)^4 \delta(p-p_n) < p' |V \cdot j|n > < n |J^{\dagger}|0 >$$

We use this form to bound Im F by using the Schwarz inequality:

$$|\operatorname{Im} F|^{2} \leq \frac{1}{4(V^{2})^{2}} \sum_{n} (2\pi)^{4} \delta(p-p_{n})(2p_{0}^{\prime}) |\langle p^{\prime}|V \cdot j|n \rangle|^{2}$$
$$\times \sum_{n} (2\pi)^{4} \delta(p-p_{n}) |\langle n|J^{\dagger}|0 \rangle|^{2}$$

The first factor can be expressed in terms of structure functions which we define as follows:

$$\sum_{n} (2\pi)^{3} \delta(p-p_{n})(2p_{0}') |\langle p'|V \cdot j|0 \rangle|^{2} \equiv e^{2}m V_{\mu} \widetilde{W}_{\mu\nu} V_{\nu}$$

where

$$\widetilde{\mathbf{W}}_{\mu\nu} = \left( \delta_{\mu\nu} - \frac{\mathbf{q}_{\mu}\mathbf{q}_{\nu}}{\mathbf{q}^2} \right) \ \widetilde{\mathbf{W}}_1 + \left( \mathbf{p}_{\mu}^{\dagger} - \mathbf{q}_{\mu} \frac{\mathbf{q} \cdot \mathbf{p}^{\dagger}}{\mathbf{q}^2} \right) \ \left( \mathbf{p}_{\nu}^{\dagger} - \mathbf{q}_{\nu} \frac{\mathbf{q} \cdot \mathbf{p}^{\dagger}}{\mathbf{q}^2} \right) \ \frac{\mathbf{W}_2}{\mathbf{m}^2}$$

and  $e^2 = 4\pi\alpha$ . Note that  $\widetilde{W}_1, \widetilde{W}_2$  are not the true full structure functions because of the restriction of zero angular momentum on the states |n>.

The second factor in the inequality for  $|\text{Im F}|^2$  is related to the weight function in the Källén-Lehmann [8] representation for the propagator for a spinzero field. We write

$$S \equiv \sum_{n} (2\pi)^{4} \delta(p_{n}-p) < 0 |J| n > < n |J^{\dagger}| 0 > = 2\pi \widetilde{\rho} (-p^{2})^{4}$$

and remark that this is related to the usual weight function that appears as

$$\Delta_{\mathbf{F}_{\mathbf{C}}}(\mathbf{p}^2) = \int_0^\infty d\kappa^2 \frac{\rho(\kappa^2)}{\kappa^2 + \mathbf{p}^2 - i\epsilon}$$

with

$$\rho(-p^{2}) = \sum_{n} (2\pi)^{3} \delta(p_{n}-p) < 0 |\phi| n > < n |\phi^{\dagger}| 0 >$$

The relation between  $\widetilde{\rho}$  and  $\rho$  is simply

$$\tilde{\rho}(-p^2) = (m^2 + p^2)^2 \rho(-p^2)$$

since  $J = (m^2 - D)\phi$ . Note that while  $\rho(-p^2)$  contains a single particle contribution  $\delta(p^2 + m^2)$ , this is absent from  $\tilde{\rho}(-p^2)$ .

Putting all the pieces together and writing  $p^2 = -s$ , we find

$$|\operatorname{Im} F(q^{2}, s)|^{2} \leq \frac{\pi^{2} e^{2} m}{(V^{2})^{2}} \widetilde{\rho}(s) \ V \cdot \widetilde{W} \cdot V$$

Note that

$$V \cdot W \cdot V = (-V^2) \left\{ \frac{\nu^2 + q^2}{q^2} \widetilde{W}_2 - \widetilde{W}_1 \right\} ,$$

and therefore

$$|\operatorname{Im} F(q^{2}, s)|^{2} \leq \frac{\pi^{2} e^{2} q^{2} m}{4(\nu^{2} + q^{2}) m^{2}} \left\{ \frac{\nu^{2} + q^{2}}{q^{2}} \widetilde{W}_{2} - \widetilde{W}_{1} \right\} \widetilde{\rho}(s)$$

In terms of the longitudinal structure function  $\widetilde{W}^{\phantom{\dagger}}_{L}$  defined by

$$\widetilde{W}_{L} = \frac{\nu^{2} + q^{2}}{q^{2}} \widetilde{W}_{2} - \widetilde{W}_{1} ,$$

the inequality may be expressed as

$$|\operatorname{Im} F(q^2, s)|^2 \leq \frac{\pi^2 e^2 q^2 m}{4m^2(\nu^2 + q^2)} \widetilde{W}_{L} \widetilde{\rho}(s)$$

Finally, one may introduce the usual structure functions  $\widetilde{\mathscr{F}}$  and the variable  $\omega = 2m\nu/q^2$  and write

$$|\operatorname{Im} F(q^{2}, s)|^{2} \leq \frac{\pi^{2} e^{2}}{\omega^{2} q^{2} + 4m^{2}} \widetilde{\mathscr{F}}_{L}(\omega, q^{2}) \widetilde{\rho}(s)$$
$$\widetilde{\mathscr{F}}_{L} = m \widetilde{W}_{L} = \frac{1}{2} \left\{ \left( \omega + \frac{2m}{\nu} \right) \widetilde{\mathscr{F}}_{2} - 2 \widetilde{\mathscr{F}}_{1} \right\}$$

We assume that  $F(q^2, s)$  satisfies an unsubtracted dispersion relation:

$$F(q^{2},s) = \frac{1}{\pi} \int_{9m^{2}}^{\infty} ds' \frac{\operatorname{Im} F(q^{2},s')}{s'-s-i\epsilon}$$

The physical form factor corresponds to setting  $s=m^2$  and we write this as simply  $F(q^2)$ :

$$|F(q^{2})| \leq \frac{1}{\pi} \int_{9m^{2}}^{\infty} \frac{ds}{s - m^{2}} |Im F(q^{2}, s)|$$
$$\leq e \int_{9m^{2}}^{\infty} \frac{ds}{s - m^{2}} \frac{\widetilde{\mathscr{F}}_{L}^{1/2} \widetilde{\rho}^{1/2}(s)}{(\omega^{2}q^{2} + 4m^{2})^{1/2}}$$

Recall that

$$s = -p^2 = -(p'+q)^2 = m^2 - q^2 + 2m\nu = m^2 + q^2(\omega - 1)$$

This inequality relates two quantities defined as S-matrix elements and hence measurable, to the field theoretic quantity  $\tilde{\rho}(s)$  which is defined in terms of an arbitrary interpolating field. We shall try to use this arbitrariness in the weight function  $\tilde{\rho}$  to achieve the most restrictive inequality possible. As an aside, note that since we cannot restrict the sum over intermediate states to spinor states only, it is not possible to discuss the extra power of ( $\omega$ -1) found in models in which the pion is a bound state of spinor quarks compared to scalar quarks [9].

# III. LARGE MOMENTUM TRANSFER

In order to discuss the behavior of  $|F(q^2)|$  for large  $q^2$  we must make some assumptions about  $\widetilde{\mathscr{F}}_L$  and about  $\widetilde{\rho}$ . We note first that in the scaling limit  $(q^2 \rightarrow \infty, \omega \text{ fixed})$  the so-called Callen-Gross relation [10]  $\omega \widetilde{\mathscr{F}}_2 = 2 \widetilde{\mathscr{F}}_1$  is expected to hold, at least for the true structure functions which involve contributions from states of all angular momentum. The deviations from this relation can be expected to be of order  $1/q^2$  and thus of the same order as the term  $2m \widetilde{\mathscr{F}}_2/\nu$ that appears in  $\widetilde{\mathscr{F}}_L$ . We shall first take this, in fact, as representative of  $\widetilde{\mathscr{F}}_L$ in the large  $q^2$  limit in order to have something concrete to work with. We have then for large  $q^2$  and with the above assumptions

$$|\mathbf{F}(\mathbf{q}^{2})| \leq \sqrt{2} \text{ me} \int_{9m^{2}}^{\infty} \frac{ds}{s-m^{2}} \left[ \frac{q^{2}}{(s-m^{2}+q^{2})+4m^{2}q^{2}} \right]^{1/2} \left[ \frac{\widetilde{\mathscr{F}}_{2}(\omega)\widetilde{\rho}(s)}{s-m^{2}+q^{2}} \right]^{1/2}$$

where we assume that  $\widetilde{\mathscr{F}}_2$  indeed scales and is a function of  $\omega$  only; we have also expressed the integrand in terms of s. We assume that

$$\widetilde{\mathscr{F}}_{2}(\omega) \sim (\omega - 1)^{p} \qquad \omega \rightarrow 1 \quad .$$

Dropping the  $4m^2q^2$  term in the first bracket in the limit of large  $q^2$  we have

$$|F(q^{2})| \leq \frac{\sqrt{2} \text{ me}}{(q^{2}) \frac{p}{2} + 1} \int_{9m^{2}}^{\infty} ds(s-m^{2})^{\frac{p}{2}-1} \left[\frac{q^{2}}{s-m^{2}+q^{2}}\right]^{3/2} \left[\frac{\widetilde{\mathscr{F}}_{2}(\omega)}{(\omega-1)^{p}}\right]^{1/2} [\widetilde{\rho}(s)]^{1/2}$$

Since  $\widetilde{\mathscr{F}}_{2}(\omega)/(\omega-1)^{p}$  is finite at  $\omega=1$  we conclude

$$|F(q^2)| \leq const/(q^2)^{\frac{p}{2}+1}$$

provided

$$\int_{9m^2}^{\infty} ds \ (s-m^2)^{\frac{p}{2}-1} [\widetilde{\rho}(s)]^{1/2} < \infty$$

If  $\tilde{\rho} \sim s^{-r}$  for large s, the integral will converge for r > p. For the pion, it is believed that p=1 so that this limit contradicts the usually accepted quark counting lore on form factors which predicts  $F \sim (q^2)^{-1}$  whereas we have the bound  $F < (q^2)^{-3/2}$ .

Alternatively, it is easy to verify that if for some reason  $\mathscr{F}_{L}(\omega, q^{2}) \to G(\omega)$ , rather than  $\mathfrak{M}_{2}(\omega)/\nu$  as  $q^{2} \to \infty$ , and if  $G(\omega) \sim (\omega-1)^{p}$  near  $\omega=1$ , the limit would be  $|F(q^{2})| < (q^{2})^{-(p+1)/2}$  in agreement with the DYW relation. We shall see shortly that in a model field theory with a transverse cutoff that one can expect a  $\tilde{\rho}(s)$  that goes like 1/s or better so that our optimistic estimates may be valid.

If however the integral over  $\rho$  does not converge we must proceed in a slightly different fashion. We go back to our first case,  $\widetilde{\mathscr{F}}_{L} \sim m \widetilde{\mathscr{F}}_{2} / \nu$  and write

$$|\mathbf{F}(\mathbf{q}^2)| \leq \frac{\sqrt{2} \operatorname{me}}{\mathbf{q}^2} \int_{9\mathrm{m}}^{\infty} \frac{\mathrm{ds}}{\mathrm{s-m}^2} \left[ \frac{\mathbf{q}^2}{\mathrm{s-m}^2 + \mathbf{q}^2} \right]^{3/2} \widetilde{\mathscr{F}}_2^{1/2}(\omega) [\widetilde{\rho}(\mathbf{s})]^{1/2}$$

and scale the s-variable according to  $s-m^2=xq^2$ ,  $\omega-1=x$  so that we may write

$$|\mathbf{F}(\mathbf{q}^{2})| \leq \frac{\sqrt{2} \text{ me}}{\mathbf{q}^{2}} \int_{8m^{2}/\mathbf{q}^{2}}^{\infty} \frac{d\mathbf{x}}{\mathbf{x}} \left(\frac{1}{1+\mathbf{x}}\right)^{3/2} \widetilde{\mathscr{F}}_{2}^{1/2} (1+\mathbf{x}) \left[\widetilde{\rho}(\mathbf{x}\mathbf{q}^{2})\right]^{1/2}$$

and if we again say

$$\mathcal{F}^{1/2}(1+x) \sim x^{p/2} \qquad x \to 0$$

$$\widetilde{\rho}(xq^{2}) \sim (xq^{2})^{-r} \quad \text{as } xq^{2} \to \infty$$

$$|F(q^{2})| \leq \frac{\text{const}}{(q^{2})^{\frac{r}{2}+1}} \int_{8m^{2}/q^{2}}^{\infty} \frac{dx}{x} \left(\frac{1}{1+x}\right)^{3/2} x^{\frac{p-r}{2}}$$

so that if r < p we may set the lower limit to zero and obtain

$$|F(q^2)| \leq const/(q^2)^{\frac{r}{2}+1}$$

which is a fine limit but has nothing to do with the structure function threshold behavior. Note also that we recover our previous result by imagining that r > pin which case the lower limit cannot be set equal to zero and one must integrate by parts a sufficient number of times before passing to the limit.

#### IV. A TRANSVERSE CUTOFF MODEL

It is interesting to look at the propagator spectral weight function,  $\rho$  (m<sup>2</sup>), in a theory which resembles some of the softened field theoretical models frequently discussed in connection with parton models [3]. In particular we want to study modifications in  $\rho$  (m<sup>2</sup>) for large m<sup>2</sup> induced by the introduction of a transverse momentum cutoff.

As an example we consider a trilinear scalar coupling and the following familiar approximation to the Dyson equation for the unrenormalized propagator  $\Delta'(p^2)$  for a field with bare mass  $M_0$  interacting with another of bare mass  $\mu$ :

$$\left[\Delta^{\prime}(p^{2})\right]^{-1} = p^{2} + M_{0}^{2} - \Pi(p^{2})$$
$$\Pi(p^{2}) = \frac{g_{0}^{2}}{i} \int \frac{d^{4}k}{(2\pi)^{4}} \frac{\Delta^{\prime}[(p-k)^{2}]}{k^{2} + \mu^{2} - i\epsilon}$$

Our approximation consists of neglecting vertex corrections and propagator corrections for the field of mass  $\mu$ . We have then only the wave function renormalization, Z, of the mass  $M_0$  field and a mass renormalization.

Renormalized quantities are introduced in the standard way:

$$\Delta(p^{2}) = Z_{2}^{-1} \Delta'(p^{2}) ,$$

$$Z_{2}^{-1} = 1 - \frac{d\Pi(p^{2})}{dp^{2}} \Big|_{p^{2} = -M^{2}}$$

$$g^{2} = Z_{0}^{2} g_{0}^{2} ,$$

$$M^2 = M_0^2 - \Pi(-M^2)$$

It is further useful to introduce  $\widetilde{\Pi}(p^2) \equiv Z \Pi(p^2)$  which then involves only the renormalized quantities  $g^2$  and  $\Delta$ . In terms of these we have

$$\Delta^{-1}(p^2) = p^2 + M^2 - \widetilde{\Pi}(p^2) - \widetilde{\Pi}(-M^2) - (p^2 + M^2) \widetilde{\Pi}'(-M^2)$$
$$Z_2 = 1 + \frac{d\widetilde{\Pi}}{dp^2} \Big|_{p^2 = -M^2} = 1 + \widetilde{\Pi}'(-M^2) \quad .$$

It has been shown by Saegner [11] that  $\Delta(p^2)$  has a Källen-Lehmann representation [8]

$$\Delta(p^{2}) = \int_{0}^{\infty} dm^{2} \frac{\rho(m^{2})}{p^{2} + m^{2} - i\epsilon} ,$$
  
$$\rho(m^{2}) = \delta(M^{2} - m^{2}) + \rho_{c}(m^{2}) \theta(m - M - \mu)$$

Writing  $p^2 = -s$ , we can deduce a very complicated nonlinear integral equation for  $\rho_c(s)$  using on the one hand the above representation for  $\Delta(p^2)$  and on the other hand the relation between  $\Delta^{-1}$  and  $\widetilde{\Pi}$ . Thus we have

$$\operatorname{Im} \Delta^{-1}(-s) = -\frac{\operatorname{Im} \Delta(-s)}{|\Delta(-s)|^2} = -\frac{\Pi_{\rho_c}(s)}{|\Delta(-s)|^2} = -\operatorname{Im} \widetilde{\Pi}(-s)$$

from which it follows that

$$\rho_{\rm c}({\rm s}) = \frac{|\Delta(-{\rm s})|^2}{\Pi} \operatorname{Im} \widetilde{\Pi}(-{\rm s})$$

To proceed, we shall express  $\widetilde{\Pi}(-s)$  in such a way that we may easily insert a transverse momentum cutoff. We introduce an arbitrary quantity P which serves to define the longitudinal direction of the four vector p and write for p and the integration variable k the following  $(p^2=-s)$ :

$$p = \left(0_{\perp}, P - \frac{s}{4P}; i\left(P + \frac{s}{4P}\right)\right)$$

$$\begin{split} \mathbf{k} &= \left( \mathbf{k}_{\perp}, \, \mathbf{x}\mathbf{P} - \frac{\mathbf{k}_{\perp}^2 - \mathbf{k}^2}{4\mathbf{x}\mathbf{P}}; \, \mathbf{i} \left( \mathbf{x}\mathbf{P} + \frac{\mathbf{k}_{\perp}^2 - \mathbf{k}^2}{4\mathbf{x}\mathbf{P}} \right) \right) \quad \text{,} \\ \mathbf{d}^4 \mathbf{k} &= \mathbf{d}^2 \mathbf{k}_{\perp} \, \mathbf{d} \mathbf{k}^2 \, \frac{\mathbf{d} \mathbf{x}}{2 \, |\mathbf{x}|} \quad \text{.} \end{split}$$

We note also

$$(p-k)^2 = \frac{k^2}{x} - s(1-x) - \frac{k^2(1-x)}{x}$$

The limits on  $k^2$  and x are  $-\infty$  to  $+\infty$ . We find then for II(-s)

$$\begin{split} \widetilde{\Pi}(-\mathbf{s}) &= \frac{g^2}{i} \int \frac{dm^2}{(2\pi)^4} \rho\left(m^2\right) \int d^2\mathbf{k}_{\perp} \int_{-\infty}^{\infty} \frac{dx}{2 \cdot \mathbf{x}} \int_{-\infty}^{\infty} d\mathbf{k}^2 \left[\mathbf{k}^2 + \mu^2 - i\epsilon\right]^{-1} \\ &\times \left[\frac{\mathbf{k}_{\perp}^2}{\mathbf{x}} - \mathbf{s}(1-\mathbf{x}) - \mathbf{k}^2 \left(\frac{1-\mathbf{x}}{\mathbf{x}}\right) + \mathbf{m}^2 - i\epsilon\right]^{-1} \end{split}$$

Evidently, if x > 1 or x < 0 both poles in the  $k^2$ -plane lie in the upper half-plane and the  $k^2$  integral is zero. This restricts x to the interval 0 to +1 and the  $k^2$ integral may be done by closing the contour in the upper half-plane, say, picking up the pole at  $k^2 = -\mu^2 + i\epsilon$ . We find

$$\widetilde{H}(-s) = \frac{g^2}{(2\pi)^3} \int dm^2 \rho(m^2) \int d^2k_{\perp} \int_0^1 \frac{dx}{2x} \left[ \frac{k_{\perp}^2}{x} + \mu^2 \frac{1-x}{x} - s(1-x) + m^2 - i\epsilon \right]$$
$$= \frac{g^2}{(2\pi)^3} \int d^2k_{\perp} \int_0^1 \frac{dx}{2x} \Delta \left[ \frac{k_{\perp}^2}{x} + \mu^2 \frac{1-x}{x} - s(1-x) \right]$$

Without a cutoff,  $\widetilde{\Pi}$  is, of course, logarithmically divergent. However, the imaginary part is finite and that is the quantity of interest:

$$\operatorname{Im} \widetilde{\mathrm{II}}(-s) = \frac{\pi g^2}{(2\pi)^3} \int \mathrm{dm}^2 \rho(\mathrm{m}^2) \int \mathrm{d}^2 \mathrm{k}_{\perp} \int_0^1 \frac{\mathrm{dx}}{2\mathrm{x}} \, \delta \left[ \frac{\mathrm{k}_{\perp}^2}{\mathrm{x}} + \frac{1-\mathrm{x}}{\mathrm{x}} \, \mu^2 - \mathrm{s}(1-\mathrm{x}) + \mathrm{m}^2 \right]$$
$$= \frac{\pi^2 g^2}{(2\pi)^3} \int \mathrm{dm}^2 \rho(\mathrm{m}^2) \int_0^{\mathrm{p}_{\mathbf{r}}^2} \mathrm{dk}_{\perp}^2 \, \frac{1}{\sqrt{4\mathrm{s}} \left(\mathrm{p}_{\mathbf{r}}^2 - \mathrm{k}_{\perp}^2\right)}$$

where we use the  $\delta$ -function to do the x integration (both roots lie in the interval 0 to 1 and the upper limit on the  $k_{\perp}^2$  integration is a reflection of the physically obvious fact that  $s > \left[ \left( m^2 + k_{\perp}^2 \right)^{1/2} + \left( \mu^2 + k_{\perp}^2 \right)^{1/2} \right]^2$ ). Here  $p_r$  is the relative momentum given by

$$4sp_{r}^{2} = \left[s - (m+\mu)^{2}\right] \left[s - (m-\mu)^{2}\right]$$

The fact that we must have  $p_r^2 > 0$  implies the upper limit  $(\sqrt{s} - \mu)^2$  for the m<sup>2</sup> integration.

We are particularly interested in the behavior of Im  $\widetilde{\Pi}(-s)$  for large s with and without a transverse momentum cutoff. Let us study then the  $k_{\perp}$  integral and introduce a cutoff function  $f(k_T^2)$ . The quantity of interest is

$$F(t) = \int_0^t d\tau \frac{f(\tau)}{\sqrt{t-\tau}}$$

where we have written  $t=p_r^2$ ,  $\tau=k_{\perp}^2$ . With this notation,

Im 
$$\widetilde{II}(-s) = \frac{g^2}{16\pi\sqrt{s}} \int_0^{(\sqrt{s}-\mu)^2} dm^2 \rho (m^2) F(p_r^2)$$
.

If there is no cutoff,  $f(\tau)=1$ , the  $\tau$  integral is trivial,

$$F_1 = \int_0^t d\tau \frac{1}{\sqrt{t-\tau}} = 2 \sqrt{t}$$

and

Im 
$$\widetilde{\Pi}_{1}(-s) = \frac{g^{2}}{8\pi\sqrt{s}} \int_{0}^{(\sqrt{s}-\mu)^{2}} dm^{2}\rho(m^{2}) p_{r}$$
  

$$\rightarrow \frac{g^{2}}{16\pi} \int_{0}^{\infty} dm^{2}\rho(m^{2})$$

independent of s provided the integral over  $\sigma$  exists. If  $\rho(m^2) \sim 1/m^2$  for large  $m^2$ , Im  $\widetilde{il}(-s)$  would increase logarithmically in (-s). If we assume  $\rho$  goes to

zero more rapidly so that  $\text{Im}\,\widetilde{\Pi}_1(-s)$  approaches a constant, we can use our relation between  $\rho_c$  and  $\text{Im}\,\widetilde{\Pi}$  to discuss the asymptotic behavior of  $\rho_c$ . With this assumption we conclude

$$\sigma_{c1} = \frac{|\Delta(-s)|^2}{\pi} \operatorname{Im} \widetilde{\Pi}_1(-s) \to \operatorname{const/s}^2$$

since  $\Delta(-s) \sim 1/s$  for large s. The constant of proportionality can be expressed in terms of the wave function renormalization Z using

$$z^{-1} = \int dm^2 \sigma(m^2) \quad ,$$

 $\mathbf{as}$ 

$$\sigma_{c1} \rightarrow \frac{g^2}{16\pi^2 Z^3} \cdot \frac{1}{s^2}$$

Next we consider what happens if there is a transverse momentum cutoff. It is obvious that if  $f(\tau)$  is positive and goes to zero rapidly for large  $\tau$  we will have

$$F(t) \rightarrow \frac{1}{\sqrt{t}} \int_0^\infty d\tau f(\tau)$$

This leads to

$$\operatorname{Im} \widetilde{\mathrm{II}}(-\mathrm{s}) = \frac{\mathrm{g}^2}{16\pi\sqrt{\mathrm{s}}} \left( \int_0^\infty \mathrm{d}\tau \ \mathrm{f}(\tau) \right) \int_0^{\sqrt{\mathrm{s}}-\mu} \mathrm{d}\mathrm{m}^2 \frac{\rho(\mathrm{m}^2)}{\mathrm{p}_{\mathrm{r}}}$$
$$\rightarrow \frac{\mathrm{g}^2}{8\pi \mathrm{s}} \left( \int_0^\infty \mathrm{d}\tau \ \mathrm{f}(\tau) \right) \int_0^\infty \mathrm{d}\mathrm{m}^2 \rho(\mathrm{m}^2)$$

and finally, for the asymptotic form of  $\rho_{\mathbf{C}}$  to

$$\rho_{\rm c}({\rm s}) \rightarrow \frac{{\rm g}^2}{8\pi^2 {\rm Z}^3} \frac{\int_0^\infty {\rm d}\tau \ {\rm f}(\tau)}{{\rm s}^3}$$

We see that the transverse momentum cutoff has gained us a full power of s in the rate of decrease of  $\rho_{c}(s)$ .

If we lift the restriction on the positivity of the cutoff function  $f(\tau)$  we can, of course, cause the integral F(t) to fall off much more rapidly than the  $1/\sqrt{t}$ we found above. We have studied some examples just to illustrate that one can obtain almost any desired behavior for F(t).

Given F(t), f(t) can be immediately found by quadratures:

$$f(t) = \frac{1}{\pi} \frac{d}{dt} \int_0^t d\tau \frac{F(\tau)}{\sqrt{t-\tau}}$$
$$\rightarrow \frac{-1}{2\pi t^{3/2}} \int_0^\infty d\tau F(\tau), \qquad t \rightarrow \infty$$

The formula for F(t) is called Abel's equation for f(t); the inversion is easily carried out by Laplace transform methods. The limiting form for large t holds for any positive F(t) such that the integral exists. We give two examples:

(1) 
$$F(t) = \frac{e^{-\lambda t}}{\lambda \sqrt{t}}$$
,  $f(t) = \delta(t) + \frac{1}{\lambda} \frac{d}{dt} e^{-\lambda t/2} I_0(\frac{\lambda t}{2})$ 

(2) 
$$F(t) = \frac{a^4 e^{-a^2/4t}}{2\sqrt{\pi} t^{3/2}}$$
,  $f(t) = \frac{a^3(a^2-2t) e^{-a^2/4t}}{4\pi t^{5/2}}$ 

There are many more examples that can be constructed using tables of Laplace transform but the general pattern of the cutoff functions, f(t), is clear: They are positive for small argument, go through zero and then approach zero for large agreement like  $-t^{-3/2}$  or faster. Although these cutoffs are admittedly somewhat weird, since one does not understand the mechanism of transverse momentum limitation they should not be necessarily discarded out of hand, and one should be prepared for the possibility that the spectral weight function of the propagator corresponding to some choice of interpolating field might decrease rapidly. Note that an oscillating cutoff function does not force the weight function in the propagator to go negative. In our examples they are always positive so

that the theory is perfectly consistent at this level. This is to be contrasted with the regulation procedure of Pauli and Villars [12].

#### V. NUCLEON ELECTROMAGNETIC FORM FACTOR

The sidewise dispersion relation for a nucleon were given long ago by Bincer [6] and we shall lean heavily on his results and reproduce many of them for ease of reference. The fundamental quantity of physical interest is the matrix element of the electromagnetic current density operator  $j_{\mu}(x)$  between physical nucleon states of four-momenta p', p and spin s', s:

$$M_{\mu}(p', s'; p, s) = \left(\frac{p'_0 p_0}{M^2}\right)^{1/2} < p', s'|j_{\mu}(0)|p, s>$$

which is generally written in terms of form factors  $F_1$ ,  $F_2$  as

$$\mathbf{M}_{\mu} = \overline{\mathbf{u}}(\mathbf{p's'}) \left[ i\gamma_{\mu} \mathbf{F}_{1}(\mathbf{q}^{2}) + i\sigma_{\mu\nu} \mathbf{q}^{\nu} \mathbf{F}_{2}(\mathbf{q}^{2}) \right] \mathbf{u}(\mathbf{ps})$$

where q=p-p' and u,  $\bar{u}$  are spinors describing the initial and final nucleon states [13]. The Pauli form factors are related to the more popular form factors  $G_E$ and  $G_M$  defined by Sachs [14] by

$$G_{E} = F_{1} - \frac{q^{2}}{2M} F_{2}$$
$$G_{M} = F_{1} + 2MF_{2}$$

We shall be interested in writing a dispersion relation in what is essentially the mass of the nucleon which entails taking it off its mass shell. To this end it is useful to introduce the proper vertex function  $\Gamma_{\mu}(p^{*}, p)$  which describes the interaction of off-shell nucleons and photons and define a new quantity  $R_{\mu}$  in terms of it:

$$R_{\mu}(p's';p) \equiv \overline{u}(p's') \ i\Gamma_{\mu}(p',p) \ S_{F_{c}}(p)(i\gamma \cdot p + M)$$

where  $S_{F_{c}}$  is the renormalized nucleon propagator. The precise relation between  $M_{\mu}$  and  $R_{\mu}$  is

$$\begin{split} \mathbf{M}_{\mu}(\mathbf{p}^{\dagger}\mathbf{s}^{\dagger},\mathbf{p}\,\mathbf{s}) &= \mathbf{q}^{2} \Delta_{\mathbf{F}_{\mathbf{c}}}(\mathbf{q}^{2}) \, \mathbf{R}_{\mu}(\mathbf{p}^{\dagger}\mathbf{s}^{\dagger};\mathbf{p}) \, \mathbf{u}(\mathbf{p}\mathbf{s}) \\ &= \mathbf{i}\mathbf{q}^{2} \, \Delta_{\mathbf{F}_{\mathbf{c}}}(\mathbf{q}^{2}) \, \overline{\mathbf{u}}(\mathbf{p}^{\dagger}\mathbf{s}^{\dagger}) \, \Gamma_{\mu}(\mathbf{p}^{\dagger},\mathbf{p}) \, \mathbf{u}(\mathbf{p}\mathbf{s}) \end{split}$$

where  $\Delta_{F_{c}}(q^2)$  is the renormalized photon propagator. We will drop the product  $q^2 \Delta_{F_{c}}(q^2)$  since it is unity to lowest order of electromagnetic interactions. The vertex function  $\Gamma_{\mu}(p',p)$  satisfies the Ward-Takahashi identity [6]

$$i(p'-p)_{\mu}\Gamma_{\mu}(p',p) = e\left[S_{F_{c}}^{-1}(p') - S_{F_{c}}^{-1}(p)\right]$$
,

where e is the nucleon charge, which leads to

$$q_{\mu}R_{\mu}(p's';p) = e \bar{u}(p's') (i\gamma \cdot p + M) \quad .$$

This condition enables us to write the structure of the spinor  $\mathbf{R}_{\mu}$  as

$$\begin{split} \mathbf{R}_{\mu}(\mathbf{p}^{*}\mathbf{s}^{*};\mathbf{p}) &- \mathbf{e}\,\overline{\mathbf{u}}(\mathbf{p}^{*}\mathbf{s}^{*})\,\mathbf{q}_{\mu}(\mathbf{M}+\mathbf{i}\gamma\cdot\mathbf{p})/\mathbf{q}^{2} = \\ &= \overline{\mathbf{u}}(\mathbf{p}^{*}\mathbf{s}^{*})\,\sum_{\lambda=\pm}\,\left[\left(\mathbf{i}\gamma_{\mu}-\mathbf{q}_{\mu}\mathbf{i}\gamma\cdot\mathbf{q}/\mathbf{q}^{2}\right)\mathbf{F}_{1}^{\lambda}+\mathbf{i}\gamma_{\mu\nu}\mathbf{q}_{\nu}\mathbf{F}_{2}^{\lambda}\right]\Lambda_{\lambda} \quad, \end{split}$$

where

$$\Lambda_{\lambda} = (W - \lambda i\gamma \cdot p)/2W$$
,  $p^2 = -W^2$ ,  $p'^2 = -M^2$ 

and

$$\mathbf{F}_{i}^{\pm} \equiv \mathbf{F}_{i}(\mathbf{q}^{2}, \mathbf{M}, \pm \mathbf{W}) \quad .$$

The form factors  $F_i^+$  evidently coincide with the physical Pauli quantities  $F_1(q^2)$ ,  $F_2(q^2)$  if we go onto the mass shell W=+M.

The next step in our analysis is to write a representation for  $R_{\mu}$  in terms of field operators from which, as Bincer [7] has shown, dispersion relations for

the  $F_i^{\pm}$  as functions of W may be derived, holding q<sup>2</sup> fixed and positive (space-like). Standard reduction formalism leads us immediately to

$$R_{\mu}(p's',p) = i \frac{p'_{0}}{M} \int d^{4}x e^{ip \cdot x} \langle p's'| [j_{\mu}(0),\bar{f}(x)] |0\rangle \theta(-x_{0}) ,$$

where we have dropped possible equal-time commutator terms and  $\overline{f}(x)$  is related to the interpolating field for the nucleon,  $\psi$ , by

$$\left(-\gamma_{\mu}^{\mathrm{T}} \frac{\partial}{\partial x_{\mu}} + \mathrm{M}\right) \overline{\psi}(\mathbf{x}) \equiv \overline{\mathbf{f}}(\mathbf{x}) \quad .$$

The absorptive part of  $\mathtt{R}_{\mu}$  from which the imaginary parts of the  $\mathtt{F}_{i}^{\lambda}$  are to be computed is

$$\operatorname{Im} R_{\mu}(p's', p) = \frac{1}{2} \left(\frac{p'_{0}}{M}\right)^{1/2} \int d^{4}x \, e^{ip \cdot x} < p's' | \left[j_{\mu}(0), \overline{f}(x)\right] | 0 >$$
$$= \frac{1}{2} \left(\frac{p'_{0}}{M}\right)^{1/2} \sum_{n} (2\pi)^{4} \, \delta(p - p_{n}) < p's' | j_{\mu} | n > < n | \overline{f}(0) | 0 >$$

It is important to note that the states  $|n\rangle$  are all of total angular momentum 1/2.

We must now project out the imaginary parts of the  $F_i^{\lambda}$  (it is at this point our work begins to deviate from that of Ref. [5]). To do this we multiply by a "polarization vector"  $\epsilon_{\mu}$  and form

$$\begin{split} \sum_{\mathbf{s}'} \epsilon_{\mu} \operatorname{Im} \mathbf{R}_{\mu} \Lambda_{\lambda} \mathbf{u}(\mathbf{p}'\mathbf{s}') &= \\ &= \operatorname{Tr} \left\{ \frac{M - i\gamma \cdot \mathbf{p}'}{2M} \left[ i\gamma \cdot \left( \epsilon - q \frac{\epsilon \cdot q}{q^2} \right) \operatorname{Im} \mathbf{F}_1^{\lambda} + i\epsilon_{\mu} \sigma_{\mu\nu} q_{\nu} \operatorname{Im} \mathbf{F}_2^{\lambda} \right] \frac{W - i\lambda\gamma \cdot \mathbf{p}}{2W} \right\} \\ &= \frac{W + \lambda M}{MW} \left[ \operatorname{Im} \mathbf{F}_1^{\lambda} - \frac{q^2}{M + \lambda W} \operatorname{Im} \mathbf{F}_2^{\lambda} \right] \left( \epsilon \cdot \mathbf{p}' - q \frac{q \cdot \mathbf{p}'}{q^2} \right) \,. \end{split}$$

In order to be able to solve for  $\text{Im F}_1$  and  $\text{Im F}_2$  separately we must find another combination; there are no other vectors in the problem so we bring one in from

the outside, call it t. We then form

$$\begin{split} \sum_{\mathbf{s}^{\dagger}} \epsilon_{\mu} \, \operatorname{Im} \mathbf{R}_{\mu} \Lambda_{\lambda} \mathbf{i} \gamma \cdot \mathbf{t} \gamma_{5} \mathbf{u}(\mathbf{p}^{\dagger} \mathbf{s}^{\dagger}) \\ &= \operatorname{Tr} \Biggl\{ \frac{\mathbf{M} - \mathbf{i} \gamma \cdot \mathbf{p}^{\dagger}}{2\mathbf{M}} \Biggl[ \mathbf{i} \gamma \cdot \left( \epsilon - \mathbf{q} \frac{\epsilon \cdot \mathbf{q}}{\mathbf{q}^{2}} \right) \operatorname{Im} \mathbf{F}_{1}^{\lambda} + \mathbf{i} \epsilon_{\mu} \sigma_{\mu\nu} \mathbf{q}_{\nu} \operatorname{Im} \mathbf{F}_{2}^{\lambda} \Biggr] \frac{\mathbf{W} - \mathbf{i} \lambda \gamma \cdot \mathbf{p}}{2\mathbf{W}} \, \mathbf{i} \gamma \cdot \mathbf{t} \gamma_{5} \Biggr\} \\ &= - \frac{\lambda}{\mathbf{MW}} \, \epsilon_{\mu\nu\lambda\sigma} \mathbf{p}_{\mu}^{\dagger} \mathbf{p}_{\nu} \epsilon_{\lambda} \mathbf{t}_{\sigma} \left[ \operatorname{Im} \mathbf{F}_{1}^{\lambda} + (\mathbf{M} + \lambda \mathbf{W}) \, \operatorname{Im} \mathbf{F}_{2}^{\lambda} \right] \quad , \end{split}$$

and find

$$\operatorname{Im} \mathbf{F}_{j}^{\lambda} = \sum_{s'} \epsilon_{\mu} \operatorname{Im} \mathbf{R}_{\mu} \Lambda_{\lambda} \left[ \mathbf{A}_{j}^{\lambda} + \mathbf{B}_{j}^{\lambda} \mathbf{i} \gamma \cdot \mathbf{t} \gamma_{5} \right] \mathbf{u}(p's')$$

where

$$\begin{split} \mathbf{A}_{1}^{\lambda} &= \lambda \frac{\mathbf{MW}(\mathbf{M} + \lambda \mathbf{W})}{(\mathbf{M} + \lambda \mathbf{W})^{2} + \mathbf{q}^{2}} \left[ \boldsymbol{\epsilon} \cdot \left( \mathbf{p}^{\prime} - \mathbf{q} \frac{\mathbf{q} \cdot \mathbf{p}^{\prime}}{\mathbf{q}^{2}} \right) \right]^{-1} \\ \mathbf{B}_{1}^{\lambda} &= -\lambda \frac{\mathbf{MW}\mathbf{q}^{2}}{(\mathbf{M} + \lambda \mathbf{W})^{2} + \mathbf{q}^{2}} \left[ \boldsymbol{\epsilon}_{\mu\nu\lambda\sigma} \mathbf{p}_{\mu}^{\prime} \mathbf{p}_{\nu} \boldsymbol{\epsilon}_{\lambda} \mathbf{t}_{\sigma} \right]^{-1} \\ \mathbf{A}_{2}^{\lambda} &= -\lambda \frac{\mathbf{MW}}{(\mathbf{M} + \lambda \mathbf{W})^{2} + \mathbf{q}^{2}} \left[ \boldsymbol{\epsilon} \cdot \left( \mathbf{p}^{\prime} - \mathbf{q} \frac{\mathbf{q} \cdot \mathbf{p}^{\prime}}{\mathbf{q}^{2}} \right) \right]^{-1} \\ \mathbf{B}_{2}^{\lambda} &= -\lambda \frac{\mathbf{MW}(\mathbf{M} + \lambda \mathbf{W})}{(\mathbf{M} + \lambda \mathbf{W})^{2} + \mathbf{q}^{2}} \left[ \boldsymbol{\epsilon}_{\mu\nu\lambda\sigma} \mathbf{p}_{\mu}^{\prime} \mathbf{p}_{\nu} \boldsymbol{\epsilon}_{\lambda} \mathbf{t}_{\sigma} \right]^{-1} \end{split}$$

We may choose the vectors  $\epsilon_{\mu}$  and  $t_{\mu}$  at our convenience though neither of them may be parallel to  $q_{\mu}$ . We will make a definite choice later.

We now use our expression for  ${\rm Im}\,{\rm R}_{\mu}$  in terms of the sum over states and find

$$\operatorname{Im} \mathbf{F}_{j}^{\lambda} = \frac{1}{2} \left( \frac{\mathbf{p}_{0}^{\prime}}{\mathbf{M}} \right)^{1/2} \sum_{\mathbf{s}^{\prime}, \mathbf{n}} (2\pi)^{4} \delta(\mathbf{p} - \mathbf{p}_{n}) < \mathbf{p}^{\prime} \mathbf{s}^{\prime} | \epsilon \cdot \mathbf{j} | \mathbf{n} > < \mathbf{n} | \mathbf{f} | \mathbf{0} > \Lambda_{\lambda} \left[ \mathbf{A}_{j}^{\lambda} + \mathbf{B}_{j}^{\lambda} \mathbf{i} \gamma \cdot \mathbf{t} \gamma_{5} \right] \mathbf{u}(\mathbf{p}^{\prime} \mathbf{s}^{\prime})$$

This expression will be bounded by using the Schwarz inequality on the double sum on the right:

$$\begin{aligned} |\operatorname{Im} \mathbf{F}_{j}^{\lambda}|^{2} &\leq \frac{1}{4} \sum_{\mathbf{s'} \mathbf{n}} (2\pi)^{4} \,\delta(\mathbf{p} - \mathbf{p}_{\mathbf{n}}) \binom{\mathbf{p}_{0}^{\prime}}{\mathbf{M}} \,|<\mathbf{p's'}| \,\epsilon \cdot \mathbf{j}|\mathbf{n} > |^{2} \\ &\times \sum_{\mathbf{s'} \mathbf{n}} (2\pi)^{4} \,\delta(\mathbf{p}_{\mathbf{n}}^{\prime} - \mathbf{p}) \,|<\mathbf{n}|\mathbf{f}| \,|\mathbf{0} > \Lambda_{\lambda} \left[A_{j}^{\lambda} + B_{j}^{\lambda} \mathbf{i}\gamma \cdot \mathbf{t}\gamma_{5}\right] \,\mathbf{u}(\mathbf{p's'})|^{2} \end{aligned}$$

The first factor can be expressed immediately in terms of deep inelastic electron structure functions defined by

$$\frac{1}{2} \sum_{\mathbf{s'n}} (2\pi)^3 \,\delta(\mathbf{p_n} - \mathbf{p}) \left(\frac{\mathbf{p'_0}}{\mathbf{M}}\right) |\langle \mathbf{p's'}| \epsilon \cdot \mathbf{j} |\mathbf{n} \rangle|^2 \equiv e^2 \epsilon_{\mu}^* \widetilde{W}_{\mu\nu} \epsilon_{\nu}$$

with

$$\widetilde{W}_{\mu\nu} = \left(\delta_{\mu\nu} - q_{\mu}q_{\nu}/q^{2}\right)\widetilde{W}_{1} + \left(p_{\mu}^{\dagger} - q_{\mu}\frac{q \cdot p^{\dagger}}{q^{2}}\right)\left(p_{\nu}^{\dagger} - q_{\nu}\frac{q \cdot p^{\dagger}}{q^{2}}\right)\frac{W_{2}}{M^{2}}$$

and  $e^2 = 4\pi \alpha \simeq 4\pi/137$ . We should note that in view of the original restriction on the states  $|n\rangle$  to be of total angular momentum 1/2,  $\widetilde{W}_1$  and  $\widetilde{W}_2$  are not the customary structure functions which include states of all angular momenta. We can of course weaken our inequality by allowing the sum to include all states and make the replacement  $\widetilde{W}_1 \rightarrow W_1$  and  $\widetilde{W}_2 \rightarrow W_2$  where  $W_1$ ,  $W_2$  are the familiar quantities. We will return to this point later on.

The second factor in  $|\text{Im F}^{\lambda}|^2$  is related to the weight functions in the Källén-Lehmann [10] representation of the propagator of a spin-one half field. We define

$$\begin{split} \mathscr{S}_{j}^{\lambda} &\equiv \sum_{\mathbf{s'n}} \overline{\mathbf{u}}(\mathbf{p's'}) \left[ \mathbf{A}_{j}^{\lambda*} + i\gamma \cdot \mathbf{t} \gamma_{5} \mathbf{B}_{j}^{\lambda*} \right] \Lambda_{\lambda} (2\pi)^{3} \delta(\mathbf{p_{n}} - \mathbf{p}) \\ &\times \langle 0|\mathbf{f}|\mathbf{n} \rangle \langle \mathbf{n}|\mathbf{\overline{f}}|0\rangle \Lambda_{\lambda} \left[ \mathbf{A}_{j}^{\lambda} + i\gamma \cdot \mathbf{t} \gamma_{5} \mathbf{B}_{j}^{\lambda} \right] \mathbf{u}(\mathbf{p's'}) \quad , \end{split}$$

and

$$\sum_{\mathbf{n}} (2\pi)^3 \delta(\mathbf{p}_{\mathbf{n}} - \mathbf{p}) < 0 |\mathbf{f}| \mathbf{n} > < \mathbf{n} |\mathbf{\bar{f}}| \mathbf{0} > \equiv \sum_{\lambda = \pm} \lambda \widetilde{\rho}_{\lambda} (-\mathbf{p}^2) \Lambda_{\lambda}$$

To see the precise relation of the  $\tilde{\rho}$ 's to the conventionally defined propagator weight functions, we recall that

$$S_{F_{c}}(p) = (i\gamma \cdot p + M)^{-1} \int_{(M+\mu)^{2}}^{\infty} dW'^{2} \frac{\left[ (-i\gamma \cdot p + W') \rho_{1}(W'^{2}) - \rho_{2}(W'^{2}) \right]}{(W'^{2} - W^{2} - i\epsilon)}$$

where

$$p^2 = -W^2$$

and

$$\sum_{n} (2\pi)^{3} \delta(p_{n}-p) <0 |\psi| n > < n |\overline{\psi}| 0 > = (-i\gamma \cdot p + W) \rho_{1}(W^{2}) - \rho_{2}(W^{2})$$
$$= \sum_{\lambda=\pm} \lambda \rho_{\lambda}(W^{2}) \Lambda_{\lambda} \quad .$$

We find

$$\rho_{+} = 2W\rho_{1} - \rho_{2}$$
$$\rho_{-} = \rho_{2}$$

and so when we use the relation between  $\psi$  and f, namely  $(\gamma_{\mu}\partial/\partial x_{\mu}+M)\psi=f$ , we see immediately that

$$\widetilde{\rho}_{\lambda} = (W - \lambda M)^2 \rho_{\lambda}$$

Finally we recall [10] that

$$\begin{split} & 2 W \rho_1 - \rho_2 \geq 0 \\ & \rho_2 \geq 0 \\ & \rho_1 \geq 0 \end{split} .$$

Thus

$$\widetilde{
ho}_{\lambda} \geq 0$$
 ,

and in particular

$$\widetilde{\rho}_{+} \leq (W-M)^2 2W\rho_1$$

The latter relation is used in Ref. [5] together with the definition of the wave function renormalization constant

$$Z_{1}^{-1} = 1 + \int_{(M+\mu)^{2}}^{\infty} dW^{2} \rho_{1}(W^{2})$$
$$= 1 + \int_{(M+\mu)^{2}}^{\infty} dW \left[\rho_{+} + \rho_{-}\right]$$

The sum over the spin s in the definition of  $\mathscr{S}_{i}^{\lambda}$  is trivial and we find

$$\mathscr{P}_{j}^{\lambda} = \frac{\rho_{\lambda}}{2MW} \left\{ \left[ (W + \lambda M)^{2} + q^{2} \right] \left[ A_{j}^{\lambda*} A_{j}^{\lambda} + B_{j}^{\lambda*} B_{j}^{\lambda} t \cdot t \right] + 4B_{j}^{\lambda*} B_{j}^{\lambda} p' \cdot t t \cdot p \right\}$$

The vector t enters only in the denominator of the  $B_j^{\lambda}$ 's and explicitly in  $\mathscr{P}_j^{\lambda}$ . It is not difficult to show by going to the rest frame of p' that  $\mathscr{P}_j^{\lambda}$  is minimized by choosing p' $\cdot t = p \cdot t = 0$ . In addition the choice  $t \cdot \epsilon = 0$  maximizes the  $B_j^{\lambda}$  denominator.

We have then

$$\mathscr{P}_{j}^{\lambda} = \frac{\rho_{\lambda}}{2MW} \left[ \left( W + \lambda M \right)^{2} + q^{2} \right] \left[ A_{j}^{\lambda*} A_{j}^{\lambda} + B_{j}^{\lambda*} B_{j}^{\lambda} t \cdot t \right] ,$$

and in terms of  $\mathscr{S}_{j}^{\lambda}$  and the tensor  $\widetilde{W}_{\mu\nu}$  we have

$$|\operatorname{Im} \mathbf{F}_{\mathbf{j}}^{\lambda}|^{2} \leq 2\pi^{2} \mathbf{e}^{2} \epsilon^{*} \cdot \widetilde{\mathbf{W}} \cdot \epsilon \mathscr{P}_{\mathbf{j}}^{\lambda} .$$

We shall now choose the vector  $\epsilon$  in such a way as to minimize the product  $\epsilon^* \cdot W \cdot \epsilon \mathscr{S}_j^{\lambda}$ . To this end work in the rest frame of p', and note first that we may as well choose  $\epsilon \cdot q = 0$  because any component of  $\epsilon$  parallel to q annihilates  $\widetilde{W}_{\mu\nu}$ . The following kinematic relations obtain:

$$\mathbf{p'} = (\overline{\mathbf{0}}, \mathbf{iM})$$
$$\mathbf{q} = (\mathbf{0}_{\perp}, \mathbf{q}_{z}, \mathbf{i}(\mathbf{p}_{0} - \mathbf{M}))$$
$$\mathbf{p} = (\mathbf{0}_{\perp}, \mathbf{q}_{z}, \mathbf{i}(\mathbf{W}^{2} + \mathbf{q}_{z}^{2})^{1/2})$$

$$\nu \equiv -p' \cdot q/M = q_0 = p_0 - M$$

$$q_z^2 = q^2 + \nu^2 = \left[ (W+M)^2 + q^2 \right] \left[ (W-M)^2 + q^2 \right] / 4M^2$$

$$W^2 = M^2 - q^2 + 2M\nu = M^2 + q^2(\omega - 1)$$

$$\omega \equiv 2M\nu/q^2 \quad , \qquad 1 \le \omega \le \infty$$

In our chosen frame,  $t_0=0$ ,  $\epsilon_0=q_z \epsilon_z/\nu$ ,  $\epsilon \cdot t=t \cdot p=0$  and we may choose  $\epsilon_{\perp}$  parallel to the x axis,  $t_{\perp}$  parallel to y.

Then we have

$$\begin{split} \epsilon \cdot \widetilde{\mathbf{W}} \cdot \epsilon &= \widetilde{\mathbf{W}}_{1} \left( \epsilon_{\perp}^{2} - \epsilon_{z}^{2} \mathbf{q}^{2} / \nu^{2} \right) + \widetilde{\mathbf{W}}_{2} (1 + \mathbf{q}^{2} / \nu^{2}) \epsilon_{z}^{2} \\ \mathcal{S}_{1}^{\lambda} &= \frac{\rho_{\lambda}}{2\mathbf{M}} \frac{\mathbf{W} \mathbf{q}^{4}}{\left[ (\mathbf{W} + \lambda \mathbf{M})^{2} + \mathbf{q}^{2} \right]} \frac{1}{\nu^{2} + \mathbf{q}^{2}} \left[ \frac{(\mathbf{W} + \lambda \mathbf{M})^{2} \nu^{2}}{\mathbf{q}^{4} \epsilon_{z}^{2}} + \frac{1}{\epsilon_{\perp}^{2}} \right] \\ \mathcal{S}_{2}^{\lambda} &= \frac{\rho_{\lambda}}{2\mathbf{M}} \frac{\mathbf{W} (\mathbf{W} + \lambda \mathbf{M})^{2}}{\left[ (\mathbf{W} + \lambda \mathbf{M})^{2} + \mathbf{q}^{2} \right]} \frac{1}{\nu^{2} + \mathbf{q}^{2}} \left[ \frac{\nu^{2}}{(\mathbf{W} + \lambda \mathbf{M})^{2} \epsilon_{z}^{2}} + \frac{1}{\epsilon_{\perp}^{2}} \right] \end{split}$$

and writing  $\epsilon_z = \cos \theta$ ,  $\epsilon_{\perp} = \sin \theta$ , our job is to choose the angle  $\theta$  to minimize  $|\operatorname{Im} F_j^{\lambda}|$ . The result of this elementary calculation is most conveniently expressed in terms of what might be called the longitudinal structure function  $\widetilde{W}_L$  defined by

$$\widetilde{W}_{L} \equiv \frac{\nu^{2} + q^{2}}{q^{2}} \widetilde{W}_{2} - \widetilde{W}_{1} \ge 0$$

The positivity of  $W_L$  follows from the fact that  $\epsilon \cdot W \cdot \epsilon > 0$  for any vector  $\epsilon$  and in particular for the angle  $\theta = 0$  in our frame. We find

$$\frac{\left|\operatorname{Im} \mathbf{F}_{1}^{\lambda}\right|^{2}}{2\pi^{2} \mathbf{e}^{2}} \leq \frac{\rho_{\lambda}}{2M} \frac{W(W+\lambda M)^{2}}{\left[(W+\lambda M)^{2}+q^{2}\right]\left[\nu^{2}+q^{2}\right]} \left\{ \frac{q^{2}\sqrt{\widetilde{W}}_{1}}{W+\lambda M} + \sqrt{q^{2}\widetilde{W}_{L}} \right\}^{2}$$
$$\frac{\left|\operatorname{Im} \mathbf{F}_{2}^{\lambda}\right|^{2}}{2\pi^{2} \mathbf{e}^{2}} \leq \frac{\rho_{\lambda}}{2M} \frac{W}{\left[(W+\lambda M)^{2}+q^{2}\right]\left[\nu^{2}+q^{2}\right]} \left\{ (W+\lambda M)\sqrt{\widetilde{W}_{1}} + \sqrt{q^{2}\widetilde{W}_{L}} \right\}^{2}$$

It is interesting to compare these bounds to the ones given by Cooper and Pagels [5]. That our bounds are an improvement can be seen by fixing  $\widetilde{W}_2$  and choosing  $\widetilde{W}_1$  to maximize the right-hand side of the above inequalities. The two maxima are achieved at different values of the ratio  $(\widetilde{W}_1/\widetilde{W}_2)$ , and these weakened inequalities are

$$\frac{\left|\operatorname{Im} \mathbf{F}_{1}^{\lambda}\right|^{2}}{2\pi^{2} \mathbf{e}^{2}} \leq \frac{\widetilde{\rho}_{\lambda} \mathbf{W}}{2\mathbf{M}} \widetilde{\mathbf{W}}_{2}$$
$$\frac{\left|\operatorname{Im} \mathbf{F}_{2}^{\lambda}\right|^{2}}{2\pi^{2} \mathbf{e}^{2}} \leq \frac{\widetilde{\rho}_{\lambda} \mathbf{W}}{2\mathbf{M} \mathbf{q}^{2}} \widetilde{\mathbf{W}}_{2}$$

Since these bounds are achieved (in the scaling limit) only if  $(\tilde{F}_1/\tilde{F}_2)$  is  $1/2(\omega-1)$  for the former and 1/2 for the latter, they are probably considerably poorer than our previous bound. Finally these can be further weakened to the Cooper-Pagels bounds by using the fact that

$$\widetilde{\rho}_{\lambda} \leq (W - \lambda M)^2 W \rho_1$$

We may also write our expressions in terms of the analogues of the usual deep inelastic structure functions which we write as  $\mathscr{F}_1$  and  $\mathscr{F}_2$  to avoid confusion with the Pauli form factors already introduced. Thus

$$\begin{split} \widetilde{\mathcal{F}}_1 &= \mathbf{M} \widetilde{\mathbf{W}}_1 \\ \widetilde{\mathcal{F}}_2 &= \nu \widetilde{\mathbf{W}}_2 \\ \widetilde{\mathcal{F}}_L &= \mathbf{M} \widetilde{\mathbf{W}}_L = \frac{1}{2} \left\{ (\omega + 2\mathbf{M}/\nu) \, \widetilde{\mathcal{F}}_2 - 2 \, \widetilde{\mathcal{F}}_1 \right\} \ . \end{split}$$

Another useful set of relations is obtained by introducing the so-called longitudinal and transverse cross sections:

$$\widetilde{W}_1 = (\nu - q^2/2M) \,\widetilde{\sigma}_T / 4 \pi^2 \alpha$$

$$\begin{split} \widetilde{W}_{2} &= (\nu - q^{2}/2M) \frac{q^{2}}{(\nu^{2} + q^{2})} (\widetilde{\sigma}_{T} + \widetilde{\sigma}_{L})/4\pi^{2} \alpha \\ \widetilde{W}_{L} &= (\nu - q^{2}/2M) \widetilde{\sigma}_{L}/4\pi^{2} \alpha \end{split}$$

and in terms of  $\widetilde{R} = \widetilde{\sigma}_L^{} / \widetilde{\sigma}_T^{}$  we have

$$\omega \widetilde{\mathscr{F}}_{2}/2 \widetilde{\mathscr{F}}_{1} = (1+\widetilde{R}) \nu^{2}/(\nu^{2}+q^{2})$$
$$\widetilde{\mathscr{F}}_{2} = \widetilde{R} \widetilde{\mathscr{F}}_{1} = \frac{\omega}{2} \frac{\widetilde{R}}{(1+\widetilde{R})} (1+q^{2}/\nu^{2}) \widetilde{\mathscr{F}}_{2}$$

Note that if these "cross sections"  $\tilde{\sigma}_{T}$ ,  $\tilde{\sigma}_{L}$  for space-like photons behaved like conventional partial wave cross sections for fixed  $q^{2}$  as  $W^{2} \rightarrow \infty$ , namely  $\tilde{\sigma} \sim 1/W^{2}$ , recalling  $\nu \sim W^{2}$ , we would find  $\tilde{\mathscr{F}}_{1} \sim \text{constant}$  and  $\tilde{\mathscr{F}}_{2} \sim 1/W^{2}$  as  $W^{2} \rightarrow \infty$ . Finally we remark that it is part of the folklore that for the experimental quantities, i.e., involving summing over states of all angular momenta) R is either zero or very small q as  $q^{2} \rightarrow \infty$ ,  $\omega$  fixed and that  $q^{2}\sigma_{L} \rightarrow 0$  in the same limit and also for fixed  $q^{2}$  as  $W^{2} \rightarrow \infty$  [5]. In terms of  $\tilde{R}$  and  $\tilde{\mathscr{F}}_{1}$ , say, our expressions for the Im  $F_{i}^{\lambda}$ 's become

$$\frac{\left|\operatorname{Im} \operatorname{F}_{1}^{\lambda}\right|^{2}}{\pi^{2} \operatorname{e}^{2}} \leq \frac{\rho_{\lambda} \operatorname{W}(\operatorname{W}+\lambda\operatorname{M})^{2} \widetilde{\mathscr{F}}_{1}}{\left[\left(\operatorname{W}+\lambda\operatorname{M}\right)^{2} + \operatorname{q}^{2}\right] \left[\nu^{2} + \operatorname{q}^{2}\right]} \left\{ \frac{\operatorname{q}^{2}}{\operatorname{M}(\operatorname{W}+\lambda\operatorname{M})} + \sqrt{\frac{\operatorname{q}^{2} \widetilde{\operatorname{R}}}{\operatorname{M}^{2}}} \right\}^{2}}{\frac{\left|\operatorname{Im} \operatorname{F}_{2}^{\lambda}\right|^{2}}{\pi^{2} \operatorname{e}^{2}}} \leq \frac{\rho_{\lambda} \operatorname{W} \widetilde{\mathscr{F}}_{1}}{\left[\left(\operatorname{W}+\lambda\operatorname{M}\right)^{2} + \operatorname{q}^{2}\right] \left[\nu^{2} + \operatorname{q}^{2}\right]} \left\{ \frac{\left(\operatorname{W}+\lambda\operatorname{M}\right)}{\operatorname{M}} + \sqrt{\frac{\operatorname{q}^{2} \widetilde{\operatorname{R}}}{\operatorname{M}^{2}}} \right\}^{2}$$

In the next section we shall use these expressions for the imaginary parts of the form factors in the appropriate dispersion relations and discuss the large  $q^2$  behavior.

VI. LARGE-q<sup>2</sup> BEHAVIOR OF THE NUCLEON FORM FACTORS

Bincer [7] has shown that the Pauli form factors  $F_j^{\lambda} = F_j(q^2, M, \lambda W)$  satisfy dispersion relations in W, and as we have noted, the  $F_j^{\dagger}(q^2, M, M)$  are precisely the usual physical form factors  $F_j(q^2)$ . We shall make the fundamental assumption that the combinations

$$\frac{1}{2} \left[ F_{j}(q^{2}, M, W) + F_{j}(q^{2}, M, -W) \right] ,$$
$$\frac{1}{2W} \left[ F_{j}(q^{2}, M, W) - F_{j}(q^{2}, M, -W) \right]$$

satisfy unsubtracted dispersion relations of the form

$$F(W^{2}) = \frac{1}{\pi} \int_{(M+\mu)^{2}}^{\infty} dW'^{2} \operatorname{Im} F(W'^{2}) \left[ W'^{2} - (W+i\epsilon)^{2} \right]^{-1}$$

from which it follows that

$$F_{j}^{+}(W) \equiv F_{j}(q^{2}, M, W) = \frac{1}{\pi} \int_{(M+\mu)}^{\infty} dW' \left[ \frac{\operatorname{Im} F_{j}^{+}(W')}{W' - W - i\epsilon} + \frac{\operatorname{Im} F_{j}^{-}(W')}{W' + W} \right]$$
$$F_{j}^{-}(W) \equiv F_{j}(q^{2}, M, -W) = \frac{1}{\pi} \int_{(M+\mu)}^{\infty} dW' \left[ \frac{\operatorname{Im} F_{j}^{-}(W')}{W' - W - i\epsilon} + \frac{\operatorname{Im} F_{j}^{+}(W')}{W' + W} \right]$$

We are ultimately interested only in  $F_{i}^{+}(W=+M)$  and it then follows that

$$|\mathbf{F}_{j}(\mathbf{q}^{2})| \leq \frac{1}{\pi} \int_{(\mathbf{M}+\mu)}^{\infty} dW \left[ \frac{|\mathrm{Im} \, \mathbf{F}_{j}^{+}(\mathbf{W})|}{\mathbf{W}-\mathbf{M}} + \frac{|\mathrm{Im} \, \mathbf{F}_{j}^{-}(\mathbf{W})|}{\mathbf{W}+\mathbf{M}} \right]$$

The assumption of an unsubtracted dispersion relation for  $F_2(q^2, M, W)$  probably scares few people but this assumption for  $F_1(q^2, M, W)$  is more unusual. One usually tries to build in the boundary condition that  $F_1(q^2=0, M, M) = e$  and it was assumed by Bincer and by Cooper and Pagels that the more general relation  $F_1(q^2=0, M, W) = e$  holds. It is our feeling that this is not absolutely necessary and that the unsubtracted relation is tenable (especially in composite theories). Dealing as we do with absolute values and inequalities it is sufficient to study the form factors in the abbreviated form

$$|F_{j}(q^{2})| \leq \frac{1}{\pi} \int_{(M+\mu)}^{\infty} dW |Im F_{j}^{+}(W)|(W-M)^{-1}$$

since the two terms in the dispersion relation appear to be of the same order of magnitude.

The large  $q^2$  limit of this inequality will now be considered. Since  $W^2 = M^2 + q^2(\omega - 1)$ , the variable  $\omega$  will be forced to 1 for large  $q^2$  if the W integral is sufficiently convergent. Thus the large  $q^2$  behavior of the form factor will depend upon the threshold behavior of the structure functions  $\widetilde{\mathscr{F}}_i$ . In the threshold limit, it is also reasonable to expect that the higher angular momentum states present in the full structure function, the  $\mathscr{F}$ 's, should become less important. Hence the  $\omega$  near 1 behavior of  $\mathscr{F}_i$  and  $\widetilde{\mathscr{F}}_i$  is expected to be the same.

It is useful to denote the following limiting behaviors:

$$\widetilde{\mathscr{F}}_1 \sim (\omega - 1)^p$$
  
 $\rho_+ (W^2) \sim (W^2)^{-r}$ 

If  $\tilde{R}$  does not grow faster than  $q^2$  in the threshold limit, scaling arguments such as used in Section III on the dispersion integral lead to

$$|F_1(q^2)| \le (q^2)^{-N}$$

where

$$N = \frac{1}{2} \left( r + \frac{1}{2} \right) \quad \text{if } r 
$$N = \frac{1}{2} (p+1) \quad \text{if } r > p + \frac{1}{2} ;$$$$

Thus if r is greater than p+1/2, that is, if  $\rho_+$  is sufficiently convergent, then the DYW relation saturates the inequality and N=1/2 (p+1).

A similar analysis can be carried out for  $F_2$  and one finds (assuming that  $q^2 \widetilde{R} < W^2$ ):

$$|F_2(q^2)| \le (q^2)^{-N-1}$$

where

$$N = \frac{1}{2} \left( r - \frac{1}{2} \right) \qquad \text{if} \qquad r 
$$N = \frac{1}{2} (p+1) \qquad \text{if} \qquad r > p + \frac{3}{2} \quad .$$$$

As before, if r is sufficiently large, the DYW relation is satisfied for the magnetic form factor. The extra power of  $(q^2)^{-1}$  in  $|F_2|$  relative to  $|F_1|$  actually insures that  $G_E$  and  $G_M$  have the same asymptotic behavior in  $q^2$  as required.

## V. CONCLUSIONS

The alert reader will have discerned by now that the inequality used here, while it is quite strong from an abstract point of view, is quite weak when compared to the predictions from simple composite models. The resulting rigorous bounds on the asymptotic behavior of the form factors are not useful unless one can find an interpolating field whose weight functions in the Källén-Lehmann representation fall very rapidly in  $W^2$ . Since the inequalities assume nothing about the composite nature of the particle involved, one might hope that if this information is used, it would be possible to find an interpolating field with the required convergence. Unfortunately, the weight functions for a composite field are a priori expected to behave <u>worse</u> for large  $W^2$  than for an elementary field (e.g., so that  $Z_2$  can vanish). This situation is discussed and clarified in the Appendix for an exactly solvable model. While the above expectations are found to be usually true, interpolating fields with arbitrarily good convergence properties were found. However, we feel that in the realistic situation, it may

not be possible to find a <u>local</u> interpolating field with improved convergence properties. This remains a very interesting open question.

The physical origin of our basic inequality (and perhaps the reason why it is not always very stringent when compared to an explicit composite model) is the following. In any bound state picture, the space-like momentum of the photon is absorbed by a constituent which must then propagate far off shell ( $\sim W^2$ ). The intermediate configuration is one off-shell particle with the remainder of the constituents on shell. The constituents must then interact, thereby sharing the photon momentum, and finally binding into the final bound state as prescribed by the dispersion relations. The inequality arises by replacing the intermediate configuration by <u>all</u> possible momentum partitions that are present in the propagator weight functions at fixed  $W^2$ . This can grossly overcount the number of intermediate states.

There are two features of our general approach that deserve comment. The first feature involves the purely formal question of the full impact of the many relations between S-matrix quantities and purely field theoretic quantities as exemplified by our basic inequality. The second feature is purely phenomeno-logical: what are the full implications on the basic theory if the DYW relation is found to hold experimentally for strongly interacting particles? These questions deserve further study. In any event, one sees that the Drell-Yan-West relation is certainly compatible with the general inequality. The general theoretical problem in tightening this inequality is to find the particular interpolating field that has the most convergent propagator weight functions in any given theory. Acknowledgment

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#### APPENDIX: SOLVABLE MODELS

The purpose of this appendix is to discuss in a simple model the asymptotic behavior of the Källen-Lehmann [8] weight function for a propagator, on its dependence on the particular interpolating field chosen, and all of this on the composite nature of the particle involved. Surely one of the most studied soluble models is that due to T. D. Lee [15], and it will prove very instructive for our purposes. The Hamiltonian for the coupled V-N system is written as

$$\mathbf{H} = (\mathbf{M}_{\mathbf{V}} + \delta \mathbf{M}) \psi_{\mathbf{V}}^{\dagger} \psi_{\mathbf{V}} + \mathbf{M}_{\mathbf{N}} \psi_{\mathbf{N}}^{\dagger} \psi_{\mathbf{N}} + \sum \omega_{\mathbf{k}} \mathbf{a}_{\mathbf{k}}^{\dagger} \mathbf{a}_{\mathbf{k}} + g \left[ \psi_{\mathbf{V}}^{\dagger} \psi_{\mathbf{N}} \mathbf{A} + \mathbf{c. c.} \right] ,$$

where

$$A = \sum_{k} \frac{u(\omega)}{\sqrt{2\omega\Omega}} a_{k}$$

 $\omega^2 = \mu^2 + h^2$ , and  $\Omega$  is the quantization volume.

The solution in the V or N $\theta$  sector is easily obtained. The solution in the V $\theta$  and N $\theta\theta$  sector has been given by Amado [16]. One finds that

$$Z = 1 - g^{2} \mathscr{S}_{2}(\Delta M)$$
$$\delta M = -\frac{g^{2}}{Z} \mathscr{S}_{1}(\Delta M)$$

where  $\Delta M = M_V - M_N$  and

$$\mathscr{S}_{n}(Z) \equiv \sum \frac{u^{2}(\omega)}{2\omega\Omega} (Z-\omega)^{-n}$$

The explicit solutions for the physical V particle and N $\theta$  scattering states are straightforward to determine but we shall not bother to write them down. However, the behavior of the solutions in the limit Z=0 is worth commenting upon. In this limit, with g<sup>2</sup> and M<sub>V</sub> fixed, the bare V-particle mass becomes negatively infinite, and the V-field disappears from the Hamiltonian. The effective interaction term achieves the form of a four field interaction, namely  $\lambda \phi_N^+ \phi_N A^+ A$ , with  $\lambda^{-1} = \mathscr{S}_1(\Delta M)$ . The quantity of major interest here is the spectral function for the V-particle defined by

$$\sigma(\mathbf{E}) = \sum_{k} \delta(\mathbf{E} - \mathbf{M}_{N} - \omega) |<0| \Psi_{V} |N\theta_{k}^{(+)} >|^{2}$$

 $\mathbf{or}$ 

$$\sigma(M_{N}+\omega) = \frac{k}{(2\pi)^{2}} (2\omega\Omega) |<0|\Psi_{V}|N\theta_{k}^{(+)}>|^{2}$$

where  $\Psi_V$  is the chosen interpolating field with the quantum numbers of the Vparticle. The wave function renormalization constant "Z", defined by the spectral representation, is given by

$$("Z")^{-1} = 1 + \int_{M_N^+ \mu}^{\infty} dE \sigma(E),$$

and its value clearly depends upon the choice of the interpolating field. This choice is by no means unique. We hasten to point out that while the propagator is not unique, neither are the vertex functions. All physical on-shell quantities are, of course, independent of the choice of  $\Psi_V$ . A discussion of this situation using dispersion methods has been given by C. Albright [17] for this model.

The simplest choice is  $\Psi_V = \psi_V$ , but an equally reasonable one is  $\Psi_V = \psi_N A$ , or even some linear combination of these two. We shall shortly consider a more general combination of  $\psi_N$  and  $a_k$  as an interpolating field but let us first consider the particular combination

$$\Psi_{\rm V} = {\rm a}\psi_{\rm V} + {\rm b}\psi_{\rm N} {\rm A} \quad ,$$

where a and b must satisfy the normalization constraint

$$<0|\Psi_{\rm V}|V> = 1 = a + bg \mathscr{S}_1(\Delta M)$$

A simple calculation leads to the result

$$"Z" = Z \left[ a^2 + b^2 \mathscr{P}_0 Z \right]^{-1}$$

In the limit b=0, one finds the expected result "Z"=Z, while in the opposite limit of a=0, "Z" =  $(1-Z)\mathscr{S}_1^2/\mathscr{S}_0\mathscr{S}_2 \leq (1-Z)$ . By choosing the value of a and b that maximizes "Z", one finds that

$$\mathbf{Z} \leq \mathbf{Z} = \mathbf{Z} + (1-\mathbf{Z})\mathcal{G}_1^2 / \mathcal{G}_0 \mathcal{G}_2$$

Therefore it is possible to choose an interpolating field so that the wave function renormalization constant "Z" does <u>not</u> vanish even if the basic Z of the theory does vanish (thereby indicating a composite V-particle).

The behavior of the weight function  $\sigma(E)$  for large E depends upon the choice of the interpolating field. For example, the above interpolating field yields the limiting result

$$\sigma(\mathbf{M}_{\mathbf{N}}^{+}\omega) \simeq \frac{\mathrm{ku}^{2}(\omega)}{(2\pi)^{2}} \left[ \mathrm{ag} \left( \mathbf{Z}_{\omega} + \mathrm{g}^{2} \mathscr{S}_{1}^{-}(\Delta \mathbf{M}) \right)^{-1} + \mathrm{b} \right]^{2}$$

If Z=0, both terms in the square bracket produce a constant behavior, and its value is independent of the values of a and b. In the case of an elementary V-particle,  $Z\neq 0$ , and the square bracket again is constant if  $b\neq 0$ , that is if there is an N- $\theta$  component to the interpolating field. If b=0 and  $Z\neq 0$ , the square bracket falls as  $\omega^{-1}$ . Thus we see that with this type of interpolating field, the spectral function is less convergent in the case of a direct product interpolating field, i.e.,  $b\neq 0$ , and also in the case of a composite V-particle, Z=0.

However, this is not the case for more general interpolating X fields. For example, the choice

$$\Psi_{\rm V} = \frac{1}{gW_1(\Delta M)} \psi_{\rm N} \sum_k \frac{u(\omega)}{\sqrt{2\omega\Omega}} w(\omega) a_k$$

where

$$W_{1}(Z) = \sum_{k} \frac{u^{2}(\omega)}{2\omega\Omega} w(\omega) (Z-\omega)^{-1}$$

leads to the weight function

$$\begin{aligned} \sigma(\mathbf{M}_{\mathrm{N}}^{+}\omega) &= \frac{\mathrm{ku}^{2}(\omega)}{(2\pi)^{2}} \frac{1}{\mathrm{g}^{2}\mathrm{W}_{1}^{2}(\Delta \mathrm{M})} \left| \mathbf{w}(\omega) + \mathrm{g}^{2}\mathrm{W}_{1}(\omega) \times \right. \\ & \left. \times \left[ \mathbb{Z}(\omega - \Delta \mathrm{M}) + \mathrm{g}^{2} \left( \mathscr{P}_{1}(\Delta \mathrm{M}) - \mathscr{P}_{1}(\omega) \right) \right]^{-1} \right|^{2} \end{aligned}$$

Whether or not Z=0, it is possible to choose an oscillating weight function  $w(\omega)$ so that the absolute square term above falls with any given power of  $\omega$ . One must choose  $w(\omega)$  so that  $w(\omega)$  and  $W_1(\omega)$  falls sufficiently fast. In fact, for the case Z=0, if  $w(\omega) = (\Delta M - \omega)^{-1}$ , then  $\sigma$  is identically zero for all  $\omega$ ! This is due to the fact that this choice of  $w(\omega)$  produces the exact V-particle state when  $\Psi_V^+$  is applied to the vacuum, and thus is orthogonal to the scattering N $\theta$  state. We realize that the arbitrariness in  $w(\omega)$  that has been utilized here will yield an uncomfortably nonlocal (even in the Lee model sense) interpolating field, but it can be utilized to produce a  $\sigma$  that falls arbitrarily fast in  $\omega$ .

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