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A DYNAMICAL BASIS FOR THE POINCARE STRESSES*

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ABSTRACT

We explore the possibility that the Poincaré stresses needed to stabilize a completely electrodynamical electron arise through a phase transition in the vacuum. By extending our previous work on dynamical symmetry breaking in finite quantum electrodynamics we are able to look for extended structures to see whether a composite electron could be stabilized by critically polarizing its own negative energy sea.

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I. INTRODUCTION

One of the oldest and most intriguing ideas in physics is that the masses of elementary particles arise entirely through dynamics. This has been a recurring theme since classical electron theory up to present day ideas on the role of collective phenomena in theories with an infinite number of degrees of freedom. Though classical electron theory is actually the forerunner of contemporary theories it has not itself been discussed from the viewpoint of broken symmetry. Recently however we have been developing a theory for the electron mass based on the mechanism of dynamical symmetry breaking,[1,2] and it is thus natural to seek the relation between modern work and the original classical electron problem. We shall explore such a relation in detail in this paper. We shall derive a set of mathematical equations which determine whether the electron is an extended object, but must apologize to the reader at the outset as we have not yet succeeded in actually solving these equations. Thus this work must be viewed as a progress report on an ongoing research problem.

Historically the idea that all mass came from self-interaction was suggested by Lorentz who equated the electrostatic energy of a ball of charge with its rest energy, i.e.,

$$\frac{e^2}{r_c} = mc^2 \tag{1}$$

where r_c is the classical electron radius. The two main difficulties with this idea were in getting such a structure to be compatible with Lorentz invariance (given a nonzero r_c) and in securing stability against Coulomb repulsion. Both of these difficulties were resolved by Poincare who introduced an extra attractive force, the so-called Poincare stress, which then allowed an extended classical electron to be a consistent dynamical system (see, e.g., Ref. 3 for a recent discussion). Of course the introduction of the Poincare stresses was an ad hoc addition to the theory and so this resolution was not completely satisfactory.

Since the above discussion ignored quantum mechanics it was hoped that the Poincare stresses could be generated by quantum fluctuations. However, it was soon found that in perturbation theory radiative corrections increase the masses of elementary particles (as is familiar for instance from the fact that perturbatively the proton is heavier than the neutron) rather than produce an attraction which would lower the mass of the system. This specific situation is met in the analysis of the Lamb shift. In perturbation theory the energy of the $2S_{1/2}$ level of the hydrogen atom is decreased by an infinite amount (since kinetic energy behaves as $p^2/2m$ this is equivalent to saying that the mass of the hydrogen atom has increased). Then an ad hoc counterterm is introduced which not merely eliminates this infinity but actually overcompensates so that the $2S_{1/2}$ level finally lies above the $2P_{1/2}$ level in accord with the observed shift.[4] Thus we see that the Poincare stresses cannot be generated in perturbation theory since the normal vacuum is "repulsive", so that if the Poincare stresses are to be electrodynamic at all we must look to possible nonperturbative effects which might be able to produce an "attractive" vacuum. Moreover because of the role that the counterterm plays in Lamb shift calculations a better understanding of the renormalization program might be expected to provide some insight into the origin of Poincare's extra attractive force.

Now during the last few years Johnson, Baker and Willey[5] have made an extensive nonperturbative study of the structure of electrodynamics and found a set of conditions under which the ultraviolet divergences of the theory can organize themselves away nonperturbatively, the theory being known as finite quantum electrodynamics (finite QED). An eigenvalue for the coupling constant of the Gell-Mann Low type is required in order to obtain finite coupling constant renormalization, i.e., finite Z_3. At such an eigenvalue it is possible to find a gauge in which $Z_1=Z_2$ is finite, and more importantly the bare mass is found to satisfy

$$m_0 = m\left(\frac{\Lambda^2}{\mu^2}\right)^{\frac{1}{2}\gamma_\theta(\alpha)} \tag{2}$$

where $\gamma_\theta(\alpha)$ is the anomalous dimension of the composite mass operator $:\bar{\psi}(x)\psi(x):$. [The distinction between m and the subtraction point μ will be made clear later.] Should $\gamma_\theta(\alpha)$ prove to be negative it then follows that m_0 vanishes as the cutoff is sent to infinity leaving $\delta m = m - m_0$ finite with the mass then being entirely dynamical. Moreover the Bethe-Salpeter integral equation for the self-energy (see, e.g., Ref. 2)

$$\left\{\gamma_5, \Sigma(p)\right\}_+ = \int d^4k\ K(p,k)\ S(k+p)\left\{\gamma_5, \Sigma(k+p)\right\}_+ S(k+p)$$

$$+ 2m_0 \int d^4k\ K(p,k)\ S(k+p)\ \gamma_5 S(k+p) \tag{3}$$

becomes homogeneous since m_0 vanishes faster than the integration over the kernel in Eq. (3). Consequently the self-energy equation

$$\left\{\gamma_5, S^{-1}(p)\right\}_+ = \int d^4k\ K(p,k)\left\{\gamma_5, S(k+p)\right\}_+ \tag{4}$$

exists without renormalization (the integration in Eq. (4) being finite because the negative anomalous dimension $\gamma_\theta(\alpha)$ in the solution

$$S^{-1}(p) = \frac{p^2+i\epsilon}{\not{p}} - m_0\left(\frac{-p^2-i\epsilon}{\Lambda^2}\right)^{\frac{1}{2}\gamma_\theta(\alpha)} = \frac{p^2+i\epsilon}{\not{p}} - m\left(\frac{-p^2-i\epsilon}{\mu^2}\right)^{\frac{1}{2}\gamma_\theta(\alpha)} \tag{5}$$

provides the necessary convergence), and the self-energy bootstraps itself. Also the particular form found for m_0 would be infinite order by order in perturbation theory thus suggesting that the counterterms used in calculations such as the Lamb shift have been able in any given order to approximate the effects of the higher order radiative corrections because of their nonperturbative organization.

It is important to add that there is however a hidden infinity in finite QED. From the axial-vector Ward identity it follows that

$$\left\{\gamma_5, \Sigma(p)\right\}_+ = 2m_0\,\Gamma_P(p,p,o) = 2m\,\widetilde{\Gamma}_P(p,p,o) \qquad (6)$$

where Γ_P is the Green's function obtained by inserting $:\bar{\psi}(x)\, i\gamma_5\, \psi(x):$ into the inverse fermion propagator, and where the final equality is obtained by noting that m_0 is a multiplicative renormalizing factor for Γ_P. Equation (6) now shows that $\left\{\gamma_5, \Sigma(p)\right\}_+$ is cutoff independent even though Γ_P needs to be renormalized. It is this renormalization of Γ_P which then allows Eq. (4) to admit of chiral symmetry breaking solutions without the need for an accompanying Goldstone boson, with there being an anomalously nonconserved axial-vector current in the solution.[5] Unfortunately this lack of current conservation has somewhat obscured the symmetry breaking aspects of the theory, since some stability principle is still required in order to guarantee that the theory choose the nontrivial solution to Eq. (4). Moreover since mass is an on-shell concept insight into this problem should not be expected to come from ultraviolet information alone.

In order to answer this stability question we decided to construct the vacuum energy as a function of the self-consistent solution to Eq. (4), and in Refs. 1 and 2 we found that the nontrivial solution becomes energetically favored when $\gamma_\theta(\alpha)=-1$, a new eigenvalue condition for the fine-structure constant. At this critical value the infrared divergences of the massless theory become so severe so as to force the theory into a new nonperturbative chiral degenerate vacuum, $|S\rangle$, in which the order parameter $\langle S|\bar{\psi}(x)\psi(x)|S\rangle$ is nonzero and sets the scale for a self-consistent fermion mass. Thus because of this new eigenvalue condition finite QED becomes a relativistic generalization of the Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity with the mass generation mechanism being of the attractive long range order (infrared) type typical in phase transitions.

Our discussion so far has given us the structure of the self-consistent vacuum whereas the original picture of the Poincare stresses applies to states carrying fermion number. Consequently we must now try to build particle states on the vacuum. The immediate anticipation of course would be that the physical fermion is simply the lowest positive energy plane wave excited out of the translation invariant vacuum itself filled with negative energy plane waves (Fig. 1). However the fermion is able to lower its energy with respect to the plane wave by localizing in space. In order to make such a system stable the order parameter must acquire a space dependence such as that anticipated in Fig. 2, so as to provide a restoring force known as the bag pressure.[6] The new negative energy sea is phase shifted with respect to the original plane waves to define a coherent state, $|C\rangle$, in which $\langle C|\bar{\psi}(x)\psi(x)|C\rangle$ is space dependent (Fig. 3). The observed state with fermion number one is then the lowest positive energy state excited out of these distorted negative energy waves (i.e., out of $|C\rangle$) with the structure of Fig. 2 interpolating between normal symmetry at short distances and broken symmetry at large distances, similar to the structure already found for the propagator of Eq. (5). Since the fermion can lower its energy by getting the negative energy sea to readjust to it as it localizes into an extended structure it is then very natural to identify the Poincare stresses with a dynamically generated bag pressure coming from the polarization of the Dirac sea.

Formally this generation of extended structures is analogous to the existence of vortex solutions in a superconductor since Gorkov was able to derive the macroscopic Ginzburg-Landau theory for the superconducting order parameter by summing the quantum fluctuations of the underlying microscopic BCS theory. Now in a recent paper[7] we have presented a graphical formulation of Gorkov's work suitably generalized to relativistic systems and in this paper we shall therefore apply the techniques of Ref. 7 to finite QED. This in principle (but not yet in practice) allows us to investigate whether the electron is an extended object held together by its own negative energy sea.

In order to avail ourselves of the method of Ref. 7 we shall first (in Section II) reformulate the work of Ref. 2 as a mean field theory. We present the explicit calculation for the extended electron in Section III, and finally in Section IV we examine some novel features of our program (such as the fact that the electron comes accompanied by another lepton) and discuss some possible difficulties that it possesses.

II. FINITE QED AS A MEAN-FIELD THEORY

In this section we briefly review the work of Ref. 2 and recast it as a mean-field theory so that we can proceed to look for extended structures. We consider the conventional Lagrangian of massive QED

$$\mathscr{L}^{(m)}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}i\bar{\psi}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu}\psi - e\bar{\psi}\gamma^{\mu}\psi A_{\mu} - m\bar{\psi}\psi \tag{7}$$

where we presuppose an eigenvalue for the coupling constant. At such an eigenvalue the underlying massless theory (i.e., the one obtained by allowing the parameter m to go to zero in the Green's functions of the above theory) is conformal invariant for all momenta. In particular the Green's function obtained by inserting $:\bar{\psi}(x)\psi(x):$ into the inverse fermion propagator satisfies (up to an irrelevant overall constant)

$$\tilde{\Gamma}^{(o)}_{S,\mu}(p,p,o) = \left(\frac{-p^2-i\epsilon}{\mu^2}\right)^{\frac{1}{2}\gamma_\theta(\alpha)} \tag{8}$$

to describe the quantum fluctuations about the conformal invariant vacuum $\Omega^{(o)}$ with respect to which the fermion is massless. Here μ denotes the off-shell subtraction point used to renormalize the massless theory, and is to be distinguished from m which is defined through the explicit additional mass term in Eq. (7).

The utility of the conformal invariance of the massless theory is that it allows us to calculate the Green's functions of $:\bar{\psi}(x)\psi(x):$[8] in the massive vacuum $\Omega^{(m)}$ of Eq. (7), i.e., in the vacuum where the conformal (and chiral) invariance is explicitly broken. This follows since each Green's function in the presence of the source term $m(x):\bar{\psi}(x)\psi(x):$ is calculable as a sum of an infinite series of Green's functions of the massless theory via the $\bar{\psi}\psi$ generating functional

$$-\int d^4x\, W(m(x)) = \sum \frac{1}{n!} \int d^4x_1 \ldots d^4x_n\, m(x_1) \ldots m(x_n)\, G^{(n)}_{(o)}(x_1 \ldots x_n) \; . \tag{9}$$

Here the $G^{(n)}_{(o)}$ are the connected Green's functions of the $:\bar{\psi}(x)\psi(x):$ composite as given in the massless theory. Noting that m(x) inserts a completely dressed $:\bar{\psi}(x)\psi(x):$ into the vacuum functional we are then able to construct W(m(x)) in terms of $\tilde{\Gamma}^{(o)}_{S,\mu}$ given in Eq. (8). In fact in the explicit case of a space-time independent mass term m we have already noted that the generating functional admits of the loop expansion of Fig. 4, and is explicitly summable as[2]

$$\epsilon(m) = i \int \frac{d^4p}{(2\pi)^4} \,\mathrm{Tr}\, \ln \left\{ \frac{1}{\not p} \left[\not p - m \left(\frac{-p^2}{\mu^2} \right)^{\frac{1}{2}\gamma_\theta(\alpha)} \right] \right\} \quad ; \qquad (10)$$

$\epsilon(m)$ thus has the typical loop expansion form, namely the Fredholm determinant of the full inverse propagator. Three comments are in order. Firstly, unlike the usual loop expansion with bare vertices our loop expansion uses dressed vertices and we discussed in Ref. 2 to what extent Eq. (10) could be exact without approximation (given the eigenvalue of course). Secondly, we have here constructed the propagator of Eq. (5) without ever needing to study the self consistent equation for the massive propagator, Eq. (4), thus showing the consistency of our loop summation of massless graphs with an independent result already known in the massive theory. The third and final remark (which we will discuss in some detail in Section IV since it is clearly puzzling) is that our derivation of $S^{-1}(p)$ requires Eq. (5) to be an exact result for all momenta and not merely for asymptotic momenta where it had originally been derived in Ref. 5. [A consequence of this is that $\tilde{\Gamma}_S^{(m)}(p,p,o)$ of the massive theory is then given by Eq. (8) for all momenta.]

Recognizing $\epsilon(m)$ as the vacuum energy density of massive QED[2] we must now require $\epsilon(m)$ to be negative for a nontrivial value of m in order for the nontrivial solution to Eq. (4) to be energetically favorable, and so in Refs. 1 and 2 we varied $\epsilon(m)$ as a function of $\gamma_\theta(\alpha)$. [Strictly speaking $\epsilon(m)$ is a multisheeted function in the complex m^2 plane and we discuss here only the behavior of its real part. We return to a discussion of its multiple valuedness in Section IV.] Looking only at the behavior of $\epsilon(m)$ near m=0 we note from the first term in the expansion of Fig. 4 that

$$\epsilon''(o) = -\frac{\mu^{-2\gamma}}{8\pi^2}\frac{(p^2)^{(1+\gamma)}}{(1+\gamma)}\Bigg|_{\epsilon^2}^{\Lambda^2} \tag{11}$$

From Eq. (11) we see that the physically interesting piece of $\epsilon''(o)$ for dynamical symmetry breaking, i.e., the infrared divergent piece, changes sign at $\gamma_\theta(\alpha) = -1$ and turns a local minimum into a local maximum at that point, and thus the value of -1 is critical (and should be approached from below). Now in Ref. 2 we have presented a completely independent derivation of this same critical value based on the consistency of the mass bootstrap with the Wilson operator product expansion in the presence of a degenerate vacuum and so we will use this value in the following. The infrared logarithm in Eq. (11) is then removed by summing the rest of the series of Fig. 4 so that at $\gamma_\theta(\alpha) = -1$ (Ref. 2)

$$\epsilon(m) = -\frac{m^2\mu^2}{16\pi^2}\left(\ln\frac{\Lambda^4}{m^2\mu^2} + 1\right) \tag{12}$$

which is negative for any nontrivial value of m, as required.

The origin of the infinity in Eq. (12) is due to the fact that the ultraviolet divergence in $G^{(2)}_{(o)}$ is not removed by any of the conventional counterterms of QED but rather is usually removed by normal ordering with respect to $\Omega^{(m)}$. Since we have already normal ordered once and for all with respect to $\Omega^{(o)}$ we shall thus have to remove this infinity by a counterterm. In order that this new counterterm should define the same mass spectrum as before we shall introduce it through the self-consistent field mechanism presented in Ref. 7. Consider the chiral invariant Lagrangian

$$\begin{aligned}\mathscr{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}i\bar\psi\gamma^\mu\overleftrightarrow{\partial}_\mu\psi - e\bar\psi\gamma^\mu\psi A_\mu - \frac{1}{2}g\left[(\bar\psi\psi)^2 + (\bar\psi i\gamma_5\psi)^2\right] \\ &= \mathscr{L}^{(m)}_{QED} + \frac{m^2}{2g} - \frac{1}{2}g\left[\left(\bar\psi\psi - \frac{m}{g}\right)^2 + (\bar\psi i\gamma_5\psi)^2\right]\end{aligned} \tag{13}$$

written in the mean field form. From $\mathscr{L}^{(m)}_{QED}$ we obtain the tadpole graph of Fig. 5.

$$\epsilon'(m) = \langle\Omega^{(m)}|:\bar{\psi}(x)\psi(x):|\Omega^{(m)}\rangle = -i\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\,\tilde{\Gamma}^{(o)}_{S,\mu}(p,p,o)\,S(p) \quad . \tag{14}$$

In order that the residual 4-Fermi interaction produce no mass corrections to the propagator in the Hartree-Fock approximation we require

$$\langle\Omega^{(m)}|:\bar{\psi}(x)\psi(x):|\Omega^{(m)}\rangle = \frac{m}{g} \tag{15}$$

Denoting by M that particular value of the trial parameter m which satisfies Eq. (15) we obtain

$$M^2 = \frac{\Lambda^4}{\mu^2}\exp\left(\frac{8\pi^2}{\mu^2 g}\right) \quad , \tag{16}$$

strongly reminiscent of the BCS gap equation. [Noting that $g \to 0^-$ as the cutoff is sent to infinity we see that the induced 4-Fermi interaction corresponds to zero-coupling and is thus self-generating in the sense of Brout and Englert.[9]] Returning now to Eq. (13) we can eliminate g to obtain the vacuum energy density of the mean field theory

$$\tilde{\epsilon}(m) = \epsilon(m) - \frac{m^2}{2g} = \frac{m^2\mu^2}{16\pi^2}\left(\ln\frac{m^2}{M^2} - 1\right) \tag{17}$$

which is now completely finite and has the double-well structure of Fig. 6 with nontrivial minima at ±M. Thus by introducing the mean field self-consistently we discover that the imposition of the gap equation automatically renders the vacuum energy to be completely finite.

Thus by dressing the bare fermion loops with photons we are able to generalize to interacting theories the Hartree-Fock method usually used in the 4-Fermi theory. Moreover we may regard Eq. (13) as the Lagrangian of a 4-Fermi interaction to which we add the exchange of a vector field in such a way so as to spread out the 4 fermion vertex into an effective nonlocal interaction. Treating the scattering problem defined by Eq. (13) to lowest nontrivial order in g and simultaneously to all orders in α then yields the graph of Fig. 7. Unlike the quadratically divergent point vertex case we see that this graph is only log divergent (since $\gamma_\theta(\alpha) = -1$), and hence the 4 fermion vertex has been spread out just enough to make it apparently renormalizable as a power series in fully dressed vertices. This method is of course different from the usual procedure of replacing a 4-Fermi interaction by an intermediate boson and proceeding perturbatively. Here we add the boson and treat it nonperturbatively in order to obtain a renormalizable S-matrix. While this remark is of interest in itself it still remains to be studied as to what extent the scattering problems defined by Eqs. (7) and (13) differ from one another and whether the self-consistent way in which the 4-Fermi interaction was introduced prevents any significant new rescattering effects.

Having set up the problem as a mean field theory we can now proceed to look for extended structures.

III. THE ELECTRON AS AN EXTENDED OBJECT

To look for extended structures we merely repeat the calculation of Section II using a space dependent mass term m(x). Introducing $\tilde{\Gamma}_S(x)$ as the Fourier transform of $\tilde{\Gamma}^{(o)}_{S,\mu}(p,p,o)$ we replace Eq. (10) formally by a functional trace,

$$\int d^4x\, W(m(x)) = i\mathrm{Tr}\ \ln\left\{\frac{\left[i\not\partial_x - \int d^4x'\, m(x')\, \tilde{\Gamma}_S(x-x')\right]}{i\not\partial_x}\right\} \tag{18}$$

since the interacting propagator is nonlocal. In order to give a meaning to Eq. (18) we still have to specify the contour for the frequency integration. This will be done below. The self-consistency condition which replaces Eq. (15) (the trace now being the usual one on spin indices only) is

$$\langle C|:\bar{\psi}(x)\psi(x):|C\rangle = -i\,\mathrm{Tr}\int d^4x'\,\tilde{\Gamma}_S(x-x')\;S(x,x')$$

$$= \mathrm{Tr}\int \frac{d\omega}{2\pi i}\,d^3x'\,\tilde{\Gamma}_S(\vec{x}-\vec{x}',\omega)\;S(\vec{x},\vec{x}',\omega) = \frac{m(x)}{g} \tag{19}$$

where S(x, x') is defined via its inverse given in Eq. (18), and where g is defined through Eq. (16). Defining

$$\tilde{W}(m(x)) = W(m(x)) - \frac{m^2(x)}{2g} \tag{20}$$

enables us to rewrite Eq. (19) as a variational condition

$$\frac{\partial}{\partial m(x)}\int d^4x\;\tilde{W}(m(x)) = 0 \tag{21}$$

which serves to determine the trial state |C>.

To evaluate W(m(x)) we try first to expand it as a series of gradients of the order parameter. We introduce

$$\Pi(q^2, m(x)) = -i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\tilde{\Gamma}^{(o)}_{S,\mu}(p+q, p, -q)\,S(p)\,\tilde{\Gamma}^{(o)}_{S,\mu}(p, p+q, q)\,S(p+q)\right]\Bigg|_{m=m(x)}$$

$$= -i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left\{\frac{\left[\left(\frac{-p^2}{\mu^2}\right)\left(\frac{-(p+q)^2}{\mu^2}\right)\right]^{-1/2}}{\left[\not{p} - m\left(\frac{-p^2}{\mu^2}\right)^{-1/2}\right]\left[\not{p}+\not{q} - m\left(\frac{-(p+q)^2}{\mu^2}\right)^{-1/2}\right]}\right\}\Bigg|_{m=m(x)} \tag{22}$$

using the specific form for the Green's functions given in Ref. 2. Then following Ref. 7 we may write an effective Ginzburg-Landau type Lagrangian for the contribution due to the negative energy states of Fig. 3, viz.

$$-\tilde{W}(m(x)) = -\tilde{\epsilon}(m=m(x)) - \frac{1}{2}m(x)\left[\Pi\left(-\partial_\mu\partial^\mu, m(x)\right) - \Pi(o, m(x))\right]m(x) + \ldots$$

$$= -\frac{m^2(x)\mu^2}{16\pi^2}\left[\ln\frac{m^2(x)}{M^2} - 1\right] + \frac{3\mu}{256\pi\, m(x)}\left[\partial_\mu m(x)\right]^2 + \ldots \tag{23}$$

where the dots denote higher gradient terms. We thus see that Eq. (23) has a typical Higgs form since the potential has a double-well structure, with the kinetic energy of the order parameter being the first approximation to an infinite series of higher gradient terms. Equation (23) thus emerges as a completely dynamical Higgs theory in which the fundamental scalar field is replaced by the c-number expectation value of the fermion composite. As we already noted in Section I there is an anomalously nonconserved current in the theory and thus the Higgs field m(x) is not accompanied by any associated scalar bound state, so that there is no observable scalar particle in the theory, even though there is a nontrivial order parameter. Furthermore Eq. (23) is an equation for a c-number Green's function and should not be second quantized. Consequently the existence of higher gradients does not spoil renormalizability since that only has to be discussed at the level of the quantum fluctuations of the fundamental fermion fields, with Eq. (13) already describing a renormalizable theory in the sense we have already discussed. It is important to notice that the coefficients in the expansion of Eq. (23) are all cutoff independent (since $\gamma_\theta(\alpha) = -1$) so that the order parameter itself will be completely finite even at short distances. Furthermore $\widetilde{W}(m(x))$ provides a scalar potential of the type used in the bag models and hence we can anticipate the existence of extended structures such as those found in Ref. 6. For the moment however it does not appear possible to sum the higher gradient terms analytically and so the structure of $\widetilde{W}(m(x))$ is only illustrative.

An alternative way of proceeding is to evaluate W(m(x)) in the basis of its eigenstates (we discuss the basis itself in more detail in Section IV). We set (following Eq. (30) below)

$$S(\vec{x},\vec{x}',\omega) = \frac{1}{2}\sum_n \left\{ \frac{v_n(\vec{x}')\,\bar{v}_n(\vec{x})}{(\omega+\omega_n - i\epsilon)} + \frac{u_n(\vec{x}')\,\bar{u}_n(\vec{x})}{(\omega-\omega_n+i\epsilon)} \right\} \tag{24}$$

where

$$-i\vec{\alpha}\cdot\vec{\partial}\begin{pmatrix} u_n(\vec{x}) \\ v_n(\vec{x}) \end{pmatrix} + \beta\int d^3x'\, m(\vec{x}')\, \tilde{\Gamma}_S(\vec{x}'-\vec{x}, \omega_n)\begin{pmatrix} u_n(\vec{x}') \\ v_n(\vec{x}') \end{pmatrix} = \omega_n \begin{pmatrix} u_n(\vec{x}) \\ -v_n(\vec{x}) \end{pmatrix} \tag{25}$$

Then

$$\int d^3x\, W(m(\vec{x})) = -\sum_n \left(\omega_n - \omega_n(M=0)\right) \tag{26}$$

if we restrict the summation to the negative energy states alone. While such a choice is useful if we want to find sectors of the theory which are topologically distinct from the vacuum our program here is to look for bag states which are in the same Hilbert space as the vacuum. Consequently we must modify the contour so that we also pick up the lowest positive energy state, $u_o(\vec{x})$, of Fig. 3;[7, 10] this adds one more energy, ω_o, to the right-hand side of Eq. (26). Hence we must evaluate $:\bar{\psi}(x)\psi(x):$ in the state $b_o^{(C)+}|C\rangle$ (where $b_o^{(C)}|C\rangle = 0$) rather than in $|C\rangle$ itself so that Eq. (19) becomes

$$\frac{m(\vec{x})}{g} = \frac{1}{2}\sum_n \int d^3x'\, \tilde{\Gamma}_S(\vec{x}-\vec{x}', \omega_n)\, \bar{v}_n(\vec{x})\, v_n(\vec{x}')$$
$$+ \frac{1}{2}\int d^3x'\, \tilde{\Gamma}_S(\vec{x}-\vec{x}', \omega_o)\, \bar{u}_o(\vec{x})\, u_o(\vec{x}') \quad . \tag{27}$$

Equations (25) and (27) now define a self-consistent Hartree-Fock problem in which the potential m(x) (i.e., the negative energy sea) deforms itself and adjusts to the fact that the lowest positive energy level of the coherent basis is occupied. Solutions to this family of coupled equations then give the self-consistent particle states of the theory.

It is important to note that the state $|C\rangle$ itself is not an eigenstate of the theory. The state $|C\rangle$ is constructed from the vacuum $|S\rangle$ (i.e., $\Omega^{(m)}$) by a space-dependent Bogoliubov transform and will have a nontrivial overlap with

$|S\rangle$ and hence not be stable unless of course it carries a topological charge (which would occur only if the phase of the order parameter varies over surfaces at infinity); this we are not considering here. The state $|S\rangle$ found in Section II is the translation invariant vacuum of the theory so there is no breaking of translation invariance. However the state $b_o^{(C)+}|C\rangle$ can still be a better single particle state than $b_o^{(S)+}|S\rangle$ (where $b_o^{(S)}|S\rangle = 0$) even while $|C\rangle$ itself is not merely not a better vacuum than $|S\rangle$ but is not even an eigenstate at all. Moreover since both these particle trial states are in the same Hilbert space and have a nontrivial overlap they cannot simultaneously be eigenstates of the theory. Thus $b_o^{(S)+}|S\rangle$ does not describe some other state with fermion number one, but rather $b_o^{(C)+}|C\rangle$ is the only one of its kind, assuming of course that it has lower energy. Since the localization of Fig. 3 is a property of the particle states and not of the vacuum $|S\rangle$ we remark that the coherent state concept allows matter to localize without any conflict with relativity and translation invariance. Essentially a relativistic particle is not a single degree of freedom but rather it is accompanied by a filled sea, and it is this sea which provides a preferred direction or location and allows extended structures to form in the first place.

While prospects for actually solving the set of coupled Eqs. (25) and (27) seem somewhat remote at the moment (the only apparent method would be to generalize the work of Ref. 10 which would require a solution to the inverse scattering problem for a nonlocal potential with relativistic kinematics in three space dimensions) the equations are at least closed in principle since $\tilde{\Gamma}_S$ is known explicitly and hence define a well-posed mathematical problem. Also they are completely finite.[11] Should these equations admit of an extended structure of the type indicated in Fig. 2 it will then be possible to calculate the radius of bound state as a function of its energy. By using Eq. (1) this would then provide a first principles determination of an order of magnitude estimate for the value of the fine structure constant.

IV. THE TACHYON DIFFICULTY

Throughout this work we have been using the propagator of Eq. (5) both in the asymptotic and nonasymptotic regions. This clearly poses serious problems for the interpretation of the particle spectrum. We shall now discuss the low momentum behavior of the propagator in some detail. This will then clarify the nature of the set of basis states used in Eq. (24).

We return to the case of constant mass and look at the structure of the propagator in the complex p^2 plane when $\gamma_\theta(\alpha) = -1$. We see that the propagator contains two poles, at $p^2 = \pm i m\mu$, and behaves near the poles as

$$S_\mu(p^2) \sim \frac{1}{2}\left\{\frac{\left[\not{p} \mp (im\mu)^{1/2}\right]}{(p^2 - im\mu + i\epsilon)} + \frac{\left[\not{p} \pm (-im\mu)^{1/2}\right]}{(p^2 + im\mu + i\epsilon)}\right\} \quad (28)$$

where the $\pm$ refers to which determination of the square root singularity we make. The reason why the poles are not on the real axis is because of how we located the branch cut in Eq. (5). We find it more convenient instead to consider the propagator

$$S_\nu(p^2) = \left\{\frac{p^2+i\epsilon}{\not{p}} - m\left(\frac{p^2+i\epsilon}{\nu^2}\right)^{-1/2}\right\}^{-1}$$

$$= \frac{1}{2}\left\{\not{p} + m\left(\frac{p^2+i\epsilon}{\nu^2}\right)^{-1/2}\right\}\left\{\frac{1}{\left[p^2 - m\nu + i\epsilon\right]} + \frac{1}{\left[p^2 + m\nu + i\epsilon\right]}\right\} \quad (29)$$

where $\nu^2 > 0$. Then near the poles

$$S_\nu(p^2) \sim \frac{1}{2}\left\{\frac{\not{p} + (m\nu)^{1/2}}{\left[p^2 - m\nu + i\epsilon\right]} + \frac{\not{p} - (-m\nu)^{1/2}}{\left[p^2 + m\nu + i\epsilon\right]}\right\} . \quad (30)$$

Thus the propagator possesses a particle of mass $(m\nu)^{1/2}$ and a (γ_5 rotated) tachyon (positive metric residue and spacelike mass squared) of mass $(-m\nu)^{1/2}$ when $\gamma_\theta(\alpha) = -1$.

In order to make the role of the tachyon more apparent we evaluate the vacuum energy density $\epsilon_\nu(m)$ defined as

$$\epsilon_\nu(m) = i \int \frac{d^4p}{(2\pi)^4} \, \mathrm{Tr}\, \ln \left\{ \frac{\not{p}}{(p^2+i\epsilon)} \left[\frac{p^2+i\epsilon}{\not{p}} - m \left(\frac{p^2+i\epsilon}{\nu^2} \right)^{-1/2} \right] \right\} \tag{31}$$

An integration by parts yields

$$\epsilon_\nu(m) = 4 \int \frac{d^3p}{(2\pi)^3} \frac{dp_o}{2\pi i} \left\{ \frac{p_o^2}{(p^2 - m\nu + i\epsilon)} + \frac{p_o^2}{(p^2 + m\nu + i\epsilon)} - \frac{2p_o^2}{(p^2 + i\epsilon)} \right\} \quad . \tag{32}$$

We now evaluate Eq. (32) as a contour integral, viz.

$$\epsilon_\nu(m) = -2 \int \frac{d^3p}{(2\pi)^3} \left\{ \begin{array}{l} \left(\vec{p}^{\,2} + m\nu\right)^{1/2} + \left(\vec{p}^{\,2} - m\nu\right)^{1/2} \theta\left(\vec{p}^{\,2} - m\nu\right) - 2|\vec{p}| \\ \qquad + i\left(m\nu - \vec{p}^{\,2}\right)^{1/2} \theta\left(m\nu - \vec{p}^{\,2}\right) \end{array} \right\} \tag{33}$$

which is exactly the structure due to filling up the negative energy seas of a free fermion and of a free tachyon fermion. Thus in Eq. (26) we sum over two sets of negative energy seas rather than over one and it is this doubled basis (with m replaced by m(x)) which was used to construct Eq. (24), with one nonlocal potential effectively replaced by two local ones.

It is important to notice that the quadratic divergences of the two seas identically cancel in Eq. (33) leaving $\epsilon_\nu(m)$ only log divergent. It is also of interest to see how this is achieved in the covariant basis. Equation (32) has the structure of the vacuum energy of a system consisting of two free particles. For a free particle of mass κ^2 the vacuum energy is given by summing Fig. 4

with bare vertices[7]

$$\epsilon(\kappa^2) = -\frac{\kappa^4}{32\pi^2}\left[\frac{4\Lambda^2}{\kappa^2} - 2\ln\frac{\Lambda^2}{\kappa^2} - 1\right] \quad . \tag{34}$$

Thus

$$\epsilon(\kappa^2 = m\nu) + \epsilon(\kappa^2 = -m\nu) = \frac{m^2\nu^2}{16\pi^2}\left(\ln\frac{\Lambda^4}{(-m^2\nu^2)} + 1\right) \tag{35}$$

which is a continuation of Eq. (12). Hence we see that the theory softens its short-distance behavior by generating a dynamical fermion tachyon which then cancels off the leading divergence of the input fermion.

We must also specify how the above-mentioned continuation is to be made. Noting that the poles of Eq. (32) are all located in the upper left and lower right quadrants in the p_o plane we may make a Wick rotation to obtain

$$\epsilon_\nu(m) = \frac{m^2\nu^2}{16\pi^2}\left(\ln\frac{\Lambda^4}{(-m^2\nu^2 - i\epsilon)} + 1\right) \quad . \tag{36}$$

The $i\epsilon$ prescription now tells how to determine the value of the logarithm as we continue ν^2 into μ^2. This then gives the purely real form of Eq. (12) on the first sheet with a negative imaginary part $-2n\,\pi i(m^2\mu^2/16\pi^2)$ after continuation through the cuts. Further this also fixes the sign of the imaginary part in Eq. (33) to be negative. Thus the vacuum decays by tachyon emission.

Having discussed the consequences of the existence of the tachyon we must of course also examine its physical implications, since we start with one fermion and finish with two. First of all we believe that we have uncovered a possibly general phenomenon in field theory. Our basic requirement is that the mass be self-generating, which means that $\Sigma(p^2)$ must be a nontrivial function of p^2. Consequently the spectrum of the interacting propagator, i.e., $p^2 = \Sigma^2(p^2)$, would in general be expected to possess more than one solution unless of course there are shielding effects due to branch cuts. Given our calculation in which the cut structure is completely determined by conformal invariance we do in fact find a set of poles. Further if $\gamma_\theta(\alpha)$ were any value other than -1 the poles would be distributed on a circle in the complex p^2 plane, and thus the critical value of -1 emerges as the only value for which all the poles can be put on the real axis. Moreover the theory only needs to generate one new state in order to cancel the quadratic divergence, and thus it may be a general property that leptons appear in pairs and cancel each other's divergences at short distances.

Throughout this work we have been referring to the minimum of the real part of $\epsilon(m)$ as the vacuum since it has the property that it lies lower than the massless vacuum. However we now see that the state $|S\rangle$ in which the propagator is given by Eq. (5) is still not the ultimate ground state of the theory since it can itself decay by tachyon emission. For the moment we have no way of discovering the resulting state to which $|S\rangle$ decays; nor do we know whether such a state even exists. Should such a state exist we may then expect it to be a vacuum in which there is a spontaneous breakdown of the new symmetry obtained by mixing the particle and the tachyon; this would then move the tachyon into the physical region. If this additional symmetry is continuous the tachyon would be related to the muon[12] and if it is discrete the tachyon would be related to the electron neutrino. However a lot more work will have to go into studying this tachyon difficulty to see whether or not it is a desirable feature of the theory.

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11. Equation (27) contains two mutually cancelling infinities, namely the sum over the infinite number of states on the right hand side and g^{-1} which is given by a sum over an infinite number of plane waves in Eq. (15); these two infinities are equal since $m(x)$ is constrained to be equal to its mean value M everywhere except for a finite region where the fermion is localized.

12. While the tachyon might be a candidate for the muon we regard it as unlikely that the muon could emerge as an excited state of the extended structure we have proposed for the electron, an old speculation (see e.g., Ref. 3). This follows since the potential $m(x)$ of Eq. (25) is not given once and for all but rather is determined each time by the motion of whichever states are occupied in Eq. (27). Thus excited states of the potential $m(x)$ for the Dirac equation would not themselves be self-consistent states of the theory.

FIGURE CAPTIONS

1. The translation invariant positive and negative energy plane wave basis used to describe the vacuum $|S\rangle$.
2. The lowest positive energy localized wave bound in the potential due to the order parameter m(x). The structure is spherically symmetric with m(x) tending to M asymptotically in all radial directions.
3. The distorted positive and negative energy wave basis used to describe the coherent state $|C\rangle$. We assume only one localized positive and negative energy level.
4. The loop summation of $\epsilon(m)$ for the interacting theory. The shaded blob vertex represents the complete dressed scalar vertex. The propagators are massless.
5. The tadpole graph for the scalar fermion composite. The vertex is dressed and the propagator is massive.
6. The double-well $\tilde{\epsilon}(m)$ obtained when $\gamma_\theta(\alpha) = -1$.
7. The lowest nontrivial order in g contribution to the 4 fermion scattering amplitude dressed to all orders with electromagnetism.

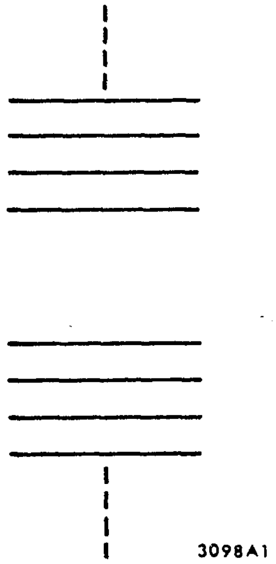

Fig. 1

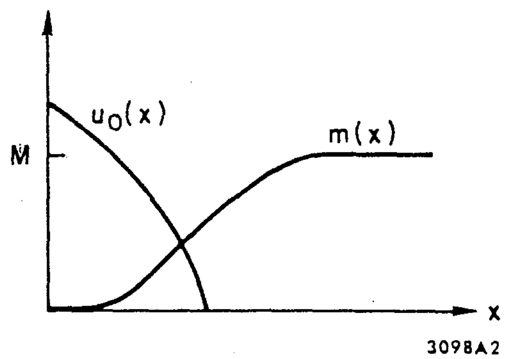

Fig. 2

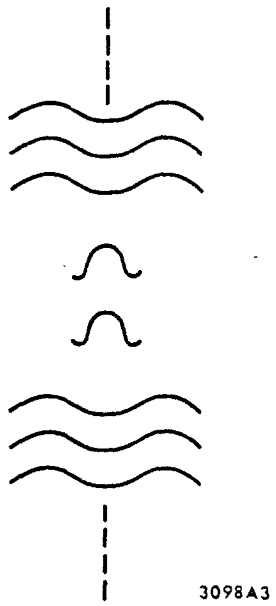

Fig. 3

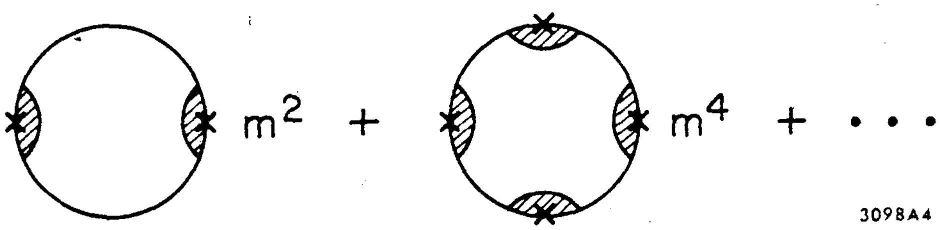

Fig. 4

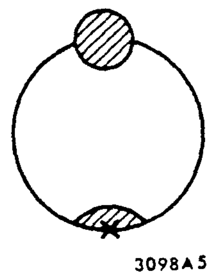

Fig. 5

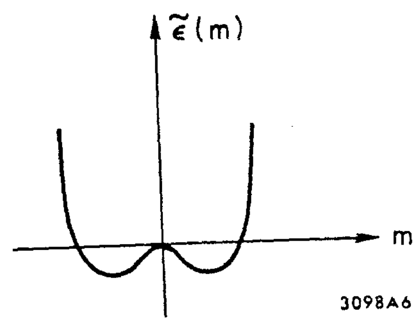

Fig. 6

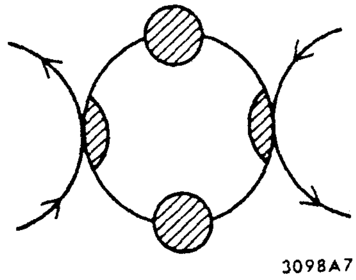

Fig. 7