SLAC-PUB-1884
February 1977

QUANTUM LIFETIME IN ELECTRON STORAGE RINGS*

A. W. Chao

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

Summary

One of the mechanisms which contribute to beam lifetime in electron storage rings is the quantum emission of energetic photons causing particles to be lost from the RF bucket. This quantum lifetime is among other things important in defining the required aperture in a storage ring. An approximate expression of quantum lifetime, predicted by a one-dimensional model which takes into account only the betatron motion, has been used in most storage ring designs. If the beam is aperture-limited at a position with nonzero dispersion, both the betatron and synchrotron motions have to be included and a two-dimensional model must be used. In this paper we offer an exact expression of quantum lifetime for the one-dimensional case and an approximate expression for the two-dimensional case.

Introduction

The stationary particle distribution in an ideal electron storage ring in the presence of radiation damping and quantum fluctuations is well-known to be gaussian with infinitely long beam lifetime.[1,2] This result has the limitation that the oscillation amplitude must not be limited. It is clear that in reality an aperture limit due to, for example, finite vacuum chamber size will truncate the gaussian distribution in the tail and modify the rest of the distribution accordingly. Furthermore, quantum fluctuations will increase the amplitude of particles until they hit the aperture and are lost so that the electron beam will have a finite quantum lifetime. The calculation of beam quantum lifetime is important for storage ring vacuum chamber designs. An approximate expression for the quantum lifetime in a one-dimensional case can be found in Refs. 1 and 2.

If however, the electron beam is horizontally aperture-limited at a location with nonzero energy dispersion function so that the horizontal beam size at such a location contains both horizontal-betatron and synchrotron contributions, the one-dimensional treatment is not applicable. For this case it is necessary to perform a more complicated two-dimensional calculation of quantum lifetime with both the betatron and synchrotron motions taken into consideration.

By using the Fokker-Planck equation[1,3] and an appropriate boundary condition, an exact expression for the quantum lifetime as well as the corresponding particle distribution in the one-dimensional case has been found in Ref. 4. The main results obtained there are summarized in the following section. If the vacuum chamber size is larger than a few times the "natural" beam size, this more accurate expression of quantum lifetime reduces to the usual approximate expression obtained in Refs. 1-2. This approximate result is rederived in the third section by using the Fokker-Planck technique and some simplifying approximation. The method used in this derivation is then generalized to find an approximate quantum lifetime for the two-dimensional case in the last section.

The One-Dimensional Calculation

The Fokker-Planck diffusion equation for the particle distribution function $\psi(a,t)$ can be written as[1,3,4]

$$\frac{\partial\psi}{\partial t} = 2\alpha\psi + \alpha a \frac{\partial\psi}{\partial a} + \frac{D}{2a}\frac{\partial}{\partial a}\left(a\frac{\partial\psi}{\partial a}\right) \tag{1}$$

*Work supported by the Energy Research and Development Administration.

where[1,2] D is the diffusion constant, α is the radiation damping rate and a is the oscillation amplitude. In the presence of an aperture limitation such as $a \leq A$, the particle distribution is described by

$$\psi(a,t) = e^{-t/\tau}\bar{\psi}(a) \tag{2}$$

with τ the quantum lifetime. It is the goal of this section to look for τ and $\bar{\psi}(a)$.

Substituting Eq. (2) into Eq. (1), we find that $\bar{\psi}$ must satisfy

$$\bar{\psi}'' + \left(\frac{1}{a} + \frac{2\alpha}{D}a\right)\bar{\psi}' + \frac{2}{D}\left(2\alpha + \frac{1}{\tau}\right)\bar{\psi} = 0 \quad . \tag{3}$$

The exact solution of this differential equation, in terms of a power series, is

$$\bar{\psi}(a) = \frac{\alpha}{\pi D} e^{-\frac{\alpha}{D}a^2}\left\{1 + \sum_{k=1}^{\infty}\left[(k-1) - \frac{1}{2\alpha\tau}\right]\left[(k-2) - \frac{1}{2\alpha\tau}\right]\cdots\right.$$
$$\left.\cdots\left[-\frac{1}{2\alpha\tau}\right]\left(\frac{\alpha}{D}a^2\right)^k/(k!)^2\right\} \; . \tag{4}$$

The quantum lifetime τ is determined by the boundary condition that at the aperture limit a=A,

$$\bar{\psi}(A) = 0 \quad . \tag{5}$$

Although the problem is in principle solved by Eqs. (4) and (5), these equations are awkward to use in practice. To simplify, we let A be larger than a few times the beam size σ of the "natural" gaussian distribution ψ_0 with infinite quantum lifetime:

$$\psi_0(a) = \frac{1}{2\pi\sigma^2}e^{-a^2/2\sigma^2} \tag{6}$$

with

$$\sigma^2 = D/2\alpha \quad .$$

Under this condition, the quantum lifetime is much longer than the radiation damping time and we can ignore $1/2\alpha\tau$ terms in Eq. (4) as compared to unity. It follows that

$$\tau = \frac{1}{2\alpha}h\left(\frac{A^2}{2\sigma^2}\right) \tag{7}$$

and

$$\bar{\psi}(a) = \psi_0(a)\left[1 - \frac{h(a^2/2\sigma^2)}{h(A^2/2\sigma^2)}\right] \; , \tag{8}$$

where we have defined a function h(x) by

$$h(x) = \sum_{k=1}^{\infty}\frac{x^k}{k(k!)} \quad . \tag{9}$$

Our final expressions (7) and (8) are much easier to use than Eqs. (4) and (5). For $x \gg 1$, h(x) approaches the asymptotic form of e^x/x. Thus Eq. (7) can be further reduced to

$$\tau \approx \frac{1}{\alpha}\frac{\sigma^2}{A^2}\exp\left(A^2/2\sigma^2\right) \tag{10}$$

(Submitted to the 1977 Particle Accelerator Conference, Chicago, Illinois, March 16 - 18, 1977)

This result coincides with that obtained in Refs. 1 and 2. The difference between expressions (7) and (10) can be demonstrated by taking their ratio, yielding $ye^{-y}h(y)$ with $y=A^2/2\sigma^2$. The behavior of this function is shown in Fig. 1.

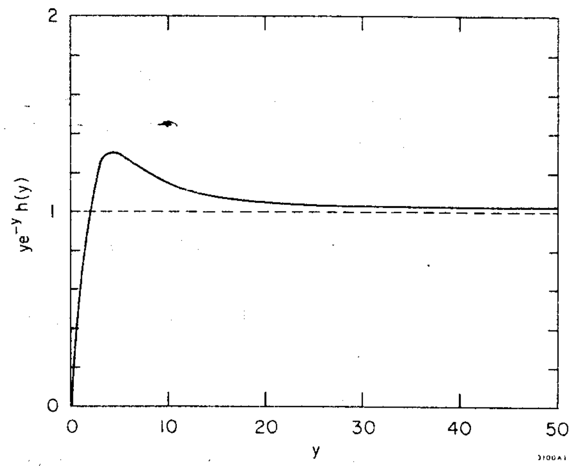

Fig. 1. The function $ye^{-y}h(y)$ vs. y.

It can be seen that the difference is less than 10% if $A>5\sigma$ ($y>12.5$). Plotted in Fig. 2 is the form factor $1-h(a^2/2\sigma^2)/h(A^2/2\sigma^2)$ which appeared in the distribution function $\bar{\psi}(a)$ of Eq. (8). This form factor describes the ratio of $\bar{\psi}(a)$ to the natural gaussian distribution ψ_0, and it can be noticed from Fig. 2 that $\bar{\psi}(a)$ differs from ψ_0 only when a is very close to the aperture boundary A.

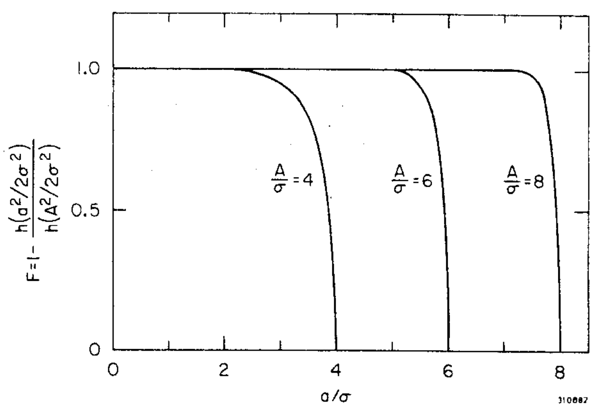

Fig. 2. The form factor $1-h(a^2/2\sigma^2)/h(A^2/2\sigma^2)$ vs. a/σ for different values of A/σ.

Simplified One-Dimensional Model

If we are not interested in the particle distribution $\bar{\psi}(a)$ in the presence of the aperture limitation and only need an approximate expression of the quantum lifetime τ, a simpler derivation of Eq. (10) is possible. To show this we substitute Eq. (2) into Eq. (1) and integrate over $2\pi a da$ from 0 to A. Adopting the normalization convention that $\int_0^A 2\pi a da\, \bar{\psi}(a)=1$, we obtain

$$-\frac{1}{\tau} = 2\alpha + 2\pi\alpha\int_0^A a^2 da\, \bar{\psi}' + \pi D\int_0^A da (a\bar{\psi}')' \quad .$$

An integration by parts, together with the boundary condition $\bar{\psi}(A)=0$, gives

$$-\frac{1}{\tau} = \pi D A\, \bar{\psi}'(A) \quad . \tag{11}$$

In order to use this formula, so far exact, one needs to know $\bar{\psi}$. However, if $A>>\sigma$, Eq. (11) can be approximated by replacing $\bar{\psi}$ on the right-hand side by the natural gaussian distribution ψ_0. Physically this means that the diffusion flux outward across the aperture boundary is approximately unaffected by the boundary. Under this approximation, Eq. (11) becomes Eq. (10). The simplicity of this method makes it possible to generalize to the two-dimensional case, as will be discussed in the following section.

Two-Dimensional Model

If the beam size is limited horizontally at a position with nonzero energy dispersion function, a two-dimensional calculation which includes both the horizontal-betatron and the synchrotron motions has to be considered. For simplicity, we ignore any coupling between the two dimensions. To find the corresponding beam lifetime τ, we follow similar procedure as in the one-dimensional case and let the distribution function be

$$\psi(a_x, a_\delta, t) = e^{-t/\tau}\, \bar{\psi}(a_x, a_\delta) \quad , \tag{12}$$

where a_x and a_δ are the oscillation amplitudes in the horizontal betatron coordinate x and the synchrotron energy deviation $\delta=\Delta E/E$, respectively. If we denote the allowed region in the (a_x, a_δ) space by $\mathscr{R}$ and the boundary of $\mathscr{R}$ by $\mathscr{C}$, we demand

$$\iint_{\mathscr{R}} \bar{\psi}\, 4\pi^2 a_x da_x\, a_\delta da_\delta = 1$$

and

$$\bar{\psi}(\mathscr{C}) = 0 \tag{13}$$

as the normalization and boundary conditions. The Fokker-Planck equation in this case is

$$\frac{\partial\psi}{\partial t} = \sum_{i=x,\delta}\left[2\alpha_i\psi + \alpha_i a_i \frac{\partial\psi}{\partial a_i} + \frac{D_i}{2a_i}\frac{\partial}{\partial a_i}\left(a_i\frac{\partial\psi}{\partial a_i}\right)\right] \quad , \tag{14}$$

where $\alpha_{x,\delta}$ are the radiation damping rates and $D_{x,\delta}$ are the diffusion constants.[1,2] Following a similar method used in the previous section, i.e., substituting (12) into (14) and using (13) to integrate over $\mathscr{R}$, we obtain

$$\frac{1}{\tau} = \frac{D_x}{2} I_x + \frac{D_\delta}{2} I_\delta \tag{15}$$

with

$$I_x = -4\pi^2\int_{\mathscr{R}} a_\delta da_\delta \left[a_x \frac{\partial\bar{\psi}}{\partial a_x}\right]_{\mathscr{C}}$$

$$I_\delta = -4\pi^2\int_{\mathscr{R}} a_x da_x \left[a_\delta \frac{\partial\bar{\psi}}{\partial a_\delta}\right]_{\mathscr{C}} \quad , \tag{16}$$

where $[\ ]_{\mathscr{C}}$ means the quantity is evaluated at the boundary $\mathscr{C}$. We then make the assumption that the exact distribution $\bar{\psi}$ in Eq. (16) can be approximated by the unperturbed gaussian distribution

$$\psi_0(a_x, a_\delta) = \frac{1}{4\pi^2\sigma_x^2\sigma_\delta^2}\exp\left(-\frac{a_x^2}{2\sigma_x^2} - \frac{a_\delta^2}{2\sigma_\delta^2}\right) \tag{17}$$

where

$$\sigma_x^2 = \frac{D_x}{2\alpha_x} \quad \text{and} \quad \sigma_\delta^2 = \frac{D_\delta}{2\alpha_\delta} \quad .$$

Equation (16) then becomes

$$I_x \approx \frac{1}{\sigma_x^4\sigma_\delta^2}\int_{\mathscr{R}} a_\delta da_\delta \left[a_x^2 \exp\left(-\frac{a_x^2}{2\sigma_x^2}-\frac{a_\delta^2}{2\sigma_\delta^2}\right)\right]_{\mathscr{C}}$$

$$I_\delta \approx \frac{1}{\sigma_x^2\sigma_\delta^4}\int_{\mathscr{R}} a_x da_x \left[a_\delta^2 \exp\left(-\frac{a_x^2}{2\sigma_x^2}-\frac{a_\delta^2}{2\sigma_\delta^2}\right)\right]_{\mathscr{C}} \quad . \tag{18}$$

The case we are interested in is $\mathscr{R}$ = (the region $a_x+\eta a_\delta < A$) with η the absolute value of the energy dispersion function at the position where horizontal aperture limitation occurs. A straightforward calculation, using Eqs. (15) and (18), gives

$$\frac{1}{\tau}=n^4 e^{-n^2/2}\int_{-x/y}^{1/xy} du\left(u+\frac{x}{y}\right)\left(\frac{1}{y}-xu\right)\left[\frac{\alpha_x+\alpha_\delta x^2}{y}+(\alpha_\delta-\alpha_x)xu\right]e^{-\frac{n^2u^2y^2}{2}} , \tag{19}$$

where we have defined

$$\begin{aligned} n &= A/\sigma_T \\ \sigma_T &= \sqrt{\sigma_x^2+\eta^2\sigma_\delta^2} \\ x &= \eta\sigma_\delta/\sigma_x \\ y &= \sqrt{1+x^2} \end{aligned} \tag{20}$$

It can be shown[5] that the general expression (19) can be simplified in special cases:

(i) if $\frac{1}{x} >> n >> x$ (valid for small η), Eq. (19) reduces to the one-dimensional result of Eq. (10) with α and σ replaced by α_x and σ_x, respectively.

(ii) Similarly if $x >> n >> \frac{1}{x}$ (valid for very large η), the synchrotron motion dominates and the problem again becomes one-dimensional.

(iii) If $n >> x >> \frac{1}{n}$ (valid when σ_x and $\eta\sigma_\delta$ are comparable), the result is particularly interesting:[6]

$$\frac{1}{\tau_1} \approx \sqrt{2\pi}\, n^3 e^{-n^2/2}\cdot\frac{x}{(1+x^2)^2}\left(\alpha_x+\alpha_\delta x^2\right) \quad . \tag{21}$$

In the following, we will assume $\alpha_\delta = 2\alpha_x$, which is valid for many practical cases. Figure 3 shows the ratio of

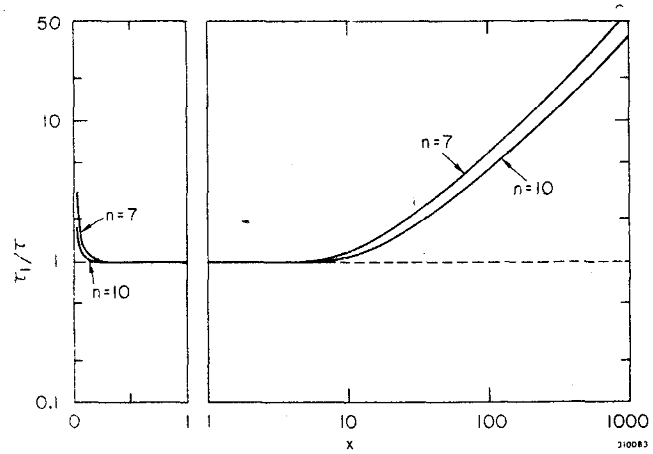

Fig. 3. τ_1/τ vs. x for n=7 and 10, where τ_1 is given by Eq. (21), τ is given by Eq. (19). The horizontal scale is uniform up to x=1 and logarithmic for x>1.

Eq. (19) to Eq. (21) as a function of x for n=7 and 10. It is clear that Eq. (21) is an excellent approximation of Eq. (19) in the entire range $n > x > \frac{1}{n}$.

It is also worthwhile to compare Eq. (19) with the prediction of the one-dimensional model. Unfortunately, Eq. (10) is ambiguous in whether α and σ should be given by the horizontal-betatron or synchrotron parameters. We will choose somewhat arbitrarily

$$\frac{1}{\tau_2} = \alpha_x n^2 e^{-n^2/2} \quad . \tag{22}$$

The ratio τ_2/τ is shown in Fig. 4 for n=5, 7 and 10. At x=0 the beam size contains only the betatron contribution and $\tau_2=\tau$ as expected. At $x=\infty$ the motion is purely synchrotron and we find $\tau_2=2\tau$ as a consequence of choosing α_x instead of α_δ in Eq. (22). Around x=1, the one-dimensional model, Eq. (22), is off by a large factor $\approx 2n$. Experimental verification of this result would be very interesting.

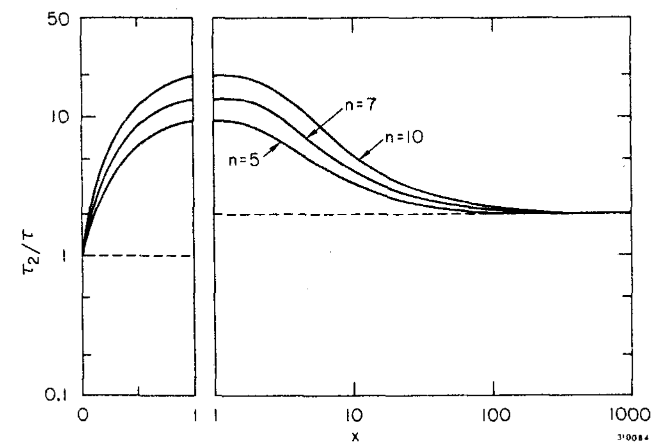

Fig. 4. τ_2/τ vs. x for n=5, 7 and 10, where τ_2 is given by Eq. (22), τ is given by Eq. (19). The horizontal scale is uniform up to x=1 and logarithmic for x>1.

References

1. H. Bruck, Accelerateurs Circulaire de Particules (Press Universitaires de France, Paris, 1966).
2. M. Sands, "The Physics of Electron Storage Rings, An Introduction," Stanford Linear Accelerator Center report SLAC-121 (1970).
3. S. Chandrasekhar, Rev. Mod. Phys. 15, 1 (1943).
4. A. W. Chao, "Particle Distribution and Beam Lifetime with Vacuum Chamber Walls," Stanford Linear Accelerator Center report PEP-136 (1975), unpublished.
5. A. W. Chao, "Electron Beam Lifetime Due to Horizontal Aperture Limitation," Stanford Linear Accelerator Center report PEP-214 (1976), unpublished.
6. The same expression was independently obtained by R. Kohaupt, private communication (1976).