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ALL COUPLING CONSTANT METHODS FOR STUDYING CUTOFF FIELD THEORIES*

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I. INTRODUCTION

Recently we have seen a marked increase of interest in the problem of finding classical and semiclassical solutions to relativistic field theories. The reasons for this renewal of interest in classical problems are quite varied. For some[1] the interest in semiclassical solutions to field theories lies in the hope that they can provide a calculational procedure for discussing relativistic bound state problems. In particular the "holy grail" pursued by these people is the goal of being able to convincingly reconcile the apparent successes of the naive quark model with the fact that one has not yet been able to identify quarks as final states in any physical process. (One example of this sort of thinking was the original SLAC bag model.[2]) For others, the interest in semiclassical solutions to field equations lies in the fact that they indicate one has only begun to catalogue the bestiary of peculiar objects which can exist in a theory by virtue of the existence of topological conservation laws. Whatever one's reasons as a result of this work we have begun to appreciate that more phenomena can (and probably do) occur in a quantum field theory, than are ever dreamed of in perturbation theory.

Because of our desire to understand questions raised by the study of semiclassical solutions to field equations and our desire to calculate the spectrum of states belonging to any given non-Abelian color gauge theory of strong interactions, people have begun to seriously attack the task of developing new—all

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coupling constant methods—for computing the properties of quantum field theory. It is my goal in these lectures to present a simple exposition of the approach to this problem which has been developed here at SLAC by Sid Drell, Shimon Yankielowicz[3] and myself.

Since I would like my discussion to be understandable to anyone familiar with elementary quantum mechanics some coverage of background material will be necessary. Moreover, in order to be able to discuss the ideas I want to develop, in an arena uncluttered by infinite numbers of fields and indices, I will never discuss—except in my "smooth talk" at the end—a field theory which has any hope of having something to do with particle physics; instead my discussion will be limited to a "toy" theory defined in one-space one-time dimension. I will show you, however, that this theory has a rich structure and can serve both as a useful framework on which to hang our discussion of nonperturbative calculational techniques, and as a model which should enable you to understand, in simple terms, what afficianados of the sport mean when they talk about

(i) phase transitions and critical behavior,

(ii) "semiclassical" or "quantum" kinks—or solitons, and

(iii) lattice theories and their connection to continuum field theories.

Hopefully, as an added cultural benefit, our discussion of nonperturbative calculational techniques should leave you with an idea as to one way in which "renormalization group" ideas can be useful in discussing a field theory. Unfortunately, the way in which these concepts, developed by Wilson and Kadanoff[4] within the context of path-integral language (and the ideas developed in the Hamiltonian discussions of Kogut and Susskind[5]) relate to our use of these ideas is not at all clear. For this reason, I don't know how transferable the material developed in my lectures will be to other contexts—but, that is your problem. At least, the general words will look the same, and so you can fool yourself into thinking you understand them when you hear them again.

By way of giving credit where credit is due, I would like to say that the naive treatment to the "toy" model I will discuss in detail, was worked out independently by us, Stoeckly and Scalapino[6] and R. B. Pearson.[7] Moreover, as I will explain later, Bob Pearson made a very important contribution to the development of the more sophisticated calculational techniques which form the backbone of our approach.

II. OUTLINE

Since our discussion will be somewhat lengthy and will be divided into separate topics I will try to follow the dictum of "first telling you what I am about to do, then doing it, and finally telling you what I did! " To begin, let me give you some idea as to how these talks will proceed. To do this I will now present an outline of the material I will cover in the order in which I will cover it. This outline will be of immense benefit to the readers of the written version of this talk since it will enable them to decide what sections they do not wish to read. Unfortunately, it will be somewhat less useful to you, since I am going to discuss all of the material—but you could of course make a deal with your neighbor and have him wake you up for the interesting parts.

The general plan of these lectures will be the following:

A. Heuristics. Since most of the nonperturbative properties of the "toy" model I will discuss are quantum field theory analogues of what happens in more familiar physical systems, I will begin by discussing the ideas to be developed in later sections by talking about the way these ideas relate to a ferromagnet. My purpose in this discussion is to give a semiconcrete and intuitive explanation of what I will mean by the terminology I will use for the rest of my lectures.

B. Classical Field Theory Analogue. Following this brief discussion, I will spend a short time discussing a classical relativistic field theory which has properties analogous to—though not exactly the same as—a ferromagnetic system.

This will be the model we will quantize and reduce to our "toy" model. My goals in this discussion will be to

(i) identify the critical coupling constant in this model,

(ii) discuss the degeneracy properties or long range order of the ground state for different values of the coupling constants, and

(iii) show that the classical theory possesses "soliton" or "kink" states of significant interest.

C. The Connection Between Cutoff Field Theories and Lattice Models. Having presented the classical version of this field theory I will turn to the general question of quantization. The purpose of the discussion in this section is to explain the problems encountered if one proceeds in familiar ways, and to explain why we reformulate our problem as a lattice theory. The discussion presented here will lay down a general framework, but specific detailed examples of what I am talking about will not be given in my talk. A more detailed discussion of some points will appear in appendices to the written version of this talk.

D. A Lattice Analogue of Continuum ϕ^4 Theory. Building upon the remarks of the preceding discussion, I will write down a quantum mechanical lattice version of the classical ϕ^4 theory. My principle purpose at this point is to present a brief, heuristic, discussion of why we can simplify this model—to the "toy" model I referred to—without losing any important physics.

E. Analysis of Toy Model I. In this section we turn to a serious study of the properties of our "toy" theory. We begin with a statement of exact results known for this model. Then we discuss within the framework of perturbation theory how some of these exact results can be understood (the real proof is based upon methods which are very special to this model and devoid of any physical insight).

F. Analysis of Toy Model II: Nonperturbative Methods. This is the key section of my lectures. It is the place I will explain our variational approach and the way in which it leads us to "renormalization group" transformations. This discussion will be developed in two states: first, I will set forth the general idea and give a simple implementation of this idea. Second I will present a more sophisticated and more powerful way of implementing the same idea. The comparison of approximate and exact results for the model in question will be discussed.

G. Smooth Talk. This section summarizes the discussion of the previous section and extends the notion to other more relevant models.

III. HEURISTICS

Before plunging into a discussion of abstract field theories let us try and see how the concepts I will focus on apply to a discussion of a real physical system, namely, a ferromagnet at zero temperature, a system which exhibits spontaneous magnetization. One typically says that since the direction of the average magnetization of this system at any point in space is the same as at any other point, this system has a ground state which exhibits "long range order." The magnitude of the magnetization is referred to as the "order parameter."

One aspect of this system can be appreciated without knowing anything about the detailed dynamics of the system. This is the fact that the basic Hamiltonian of the system is rotationally invariant, and so the energy of the ground state of the magnet cannot depend upon the direction in which the magnetization points. Hence, we say that the ground state of the system is infinitely degenerate. The general idea to be gleaned from this observation is that if there is a state of a system which has the lowest possible energy, and if in this state the expectation value of some operator—like the magnetization—is nonvanishing, then because the magnetization rotates under symmetry transformations of the theory

(i.e., transformations which leave the Hamiltonian unchanged) this state must have degenerate partners. Whenever, during these talks, I refer to a system being in an ordered phase I will mean that it has a degenerate (either infinitely or finitely degenerate) ground state.

To understand what I mean when I discuss critical coupling constants for a theory let us discuss some of the microscopic physics of a ferromagnet. (Please be warned that this is by way of analogy and my discussion of a ferromagnet is by no means complete or even necessarily really correct.) At any rate, the point is that in a ferromagnet there are two competing effects taking place. Presuming that the bulk magnetization is due to an aligning of atomic electron spins, there is the magnetic moment—magnetic moment interaction which tends to align these spins antiparallel to one another—and "exchange forces" which tend to make the atomic electron spins align parallel to one another. Whenever the strength of the exchange terms dominate the magnetic interactions the system is ferromagnetic, i.e., its ground state is degenerate and exhibits long range order. Clearly, if we adjust the strength of the "exchange terms" relative to magnetic terms we eventually reach a point beyond which there is no "spontaneous magnetization" (i.e., the order parameter goes to zero) and so the ground state becomes unique. The value of "coupling constant" or "exchange term strength" at which the ground state changes from being degenerate to unique is what we refer to as the critical coupling constant. As we go through this point we say that the system undergoes a phase transition from an ordered to a disordered phase.

Finally, let us note that the analogue of the ever popular "kink" or "soliton state" in this system is the magnetic domain wall, i.e., that small transition region between two larger regions whose bulk magnetization points in different directions.

IV. A CLASSICAL FIELD THEORY ANALOGUE

To sharpen our definitions and to make more precise statements about "soliton" or "kink" states, let us turn to a discussion of a classical relativistic field theory which has properties quite similar to the ferromagnetic system. The specific features I want to focus your attention upon are:

(i) as one varies a "coupling constant" in this system the ground state goes from being degenerate to nondegenerate, and that at the same time the system goes from an "ordered" to "disordered" state,

(ii) for those values of coupling constant which correspond to having a unique ground state the only elementary excitations of the system are "small vibrations." However, when the system has a doubly degenerate ground state, there are two types of elementary excitations—small vibrations and solitons. Furthermore, near the critical point the soliton becomes the more important excitation.

The theory I will consider is defined either by the Lagrangian density

$$\mathscr{L}(\phi(x), \partial_\mu \phi(x)) = \frac{1}{2} (\partial_\mu \phi(x))^2 - \lambda(\phi^2(x) - f)^2 \tag{4.1}$$

where $\mu=0, 1$, or equivalently by the Hamiltonian

$$H = \int_{-\infty}^{+\infty} dx \left\{ \frac{\Pi^2(x)}{2} + \frac{(\nabla\phi(x))^2}{2} + \lambda(\phi^2(x) - f)^2 \right\}, \tag{4.2}$$

where

$$\Pi(x) \stackrel{\text{def}}{=} \partial_0 \phi(x) \qquad \text{and} \qquad \nabla\phi(x) \stackrel{\text{def}}{=} \partial_1 \phi(x) \quad .$$

If we observe that H is a sum of positive terms we see that the field configuration, ϕ_0, having the lowest possible energy must correspond to $\partial_0\phi_0 = \Pi = 0$ and $\nabla\phi_0 = 0$, i.e., ϕ_0 must be both time independent and independent of 'x'. Inspection of the term $\lambda(\phi^2 - f)^2$ shows that the nature of the groundstate is quite different depending upon whether f is greater or less than zero. If $f \leq 0$ then the lowest possible value $\lambda(\phi^2 + |f|)^2$ can take is $\lambda|f|^2$, and this is achieved for $\phi_0 = 0$.

Hence the lowest state of the system for $f \leq 0$ is unique and is invariant under the transformation $\phi(x) \rightarrow -\phi(x)$, which is an invariance of H. If, however, $f > 0$ then the lowest value $\lambda(\phi^2 - f)^2$ can take is zero, and this occurs for either of two possibilities: namely $\phi_0 = \pm\sqrt{f}$. Hence for $f > 0$ the ground state of this classical system is doubled. This is consistent with the fact that the $+\sqrt{f}$ configuration gets mapped into the $-\sqrt{f}$ configuration by the symmetry $\phi \rightarrow -\phi$. Recalling our ferromagnetic analogy we say that the system undergoes a phase transition at a critical value of the coupling constant f, namely $f_c = 0$.

Note that for the classical theory the value of 'λ' is irrelevant insofar as the degeneracy properties of the ground state are concerned, and so a natural question to ask is "what role, if any does λ play in determining the structure of the theory?" The answer, of course, is that λ is involved in determining the "mass of small vibrations about the ground state." To see this most easily for the case $f \leq 0$ assume that we are examining a small disturbance at t=0, i.e., $\delta\phi(x, 0)$. It is easy to show that this vibration is stable, in that if the initial disturbance at t=0 is small enough it stays small. Thus, one need only keep the linear terms in the equation of motion as a first approximation to the time development of the system. In this way one finds small vibration solutions of frequency squared $\omega(k)^2 = k^2 + m^2$, where the k=0 mass is $m = (\lambda f)^{1/2}$. Similarly, for the case $f > 0$ one can ask what the mass of small vibrations about either one of the states, $\phi_0(f) = \pm\sqrt{f}$, is. As before linearizing the equations of motion tells us that $m = (8\lambda f)^{1/2}$. In either of these cases one can proceed to analyze the problem of finding the true time dependent response to a small displacement by constructing a perturbation series built upon these plane waves. It is in this sense that one refers to the small vibrations of mass, $m = (8\lambda f)^{1/2}$, as the fundamental excitations of the system.

Having said all this let us now note that the $f \leq 0$ case is distinguished from the $f > 0$ case in an important way. That is, for $f > 0$ there are also exact solutions

to the time independent equations of motion different from $\phi_0 = \pm\sqrt{f}$, i.e., there are solutions to

$$-\nabla^2\phi(x) + 4\lambda\phi(x)\,(\phi^2(x) - f) = 0 \quad , \tag{4.3}$$

which are not found in the perturbation expansion, and which have an energy which differs from the ground state energy by a finite amount. These are the so-called "kinks" or "solitons" and are of the general form

$$\phi_{\substack{\text{kink}\\(\text{antikink})}} = \pm\sqrt{f}\,\tanh\left(\sqrt{2\lambda f}\,(x-x_0)\right) \quad . \tag{4.4}$$

Since this function corresponds to the field configuration shown in Fig. 1, we see that it is just a region of "finite width" joining two "ground state configurations." That this is the direct analogue of a domain wall in the ferromagnet is obvious if we think of $\phi = +\sqrt{f}$ as magnetization pointing up and $\phi = -\sqrt{f}$ as magnetization pointing down.

The really important observation which I would like to make about the "kink" state is that direct substitution of (4.4) into (4.2) yields an energy for the kink which is $E_{kink} \cong (\lambda f^3)^{1/2}$. Comparing this to the plane wave of k=0 we see that $E_{kink}/m_{(\text{small vib})} \cong f$; hence, for $f \ll 1$, the kink becomes a much lower energy excitation of the system than the so-called "small vibration." This means, of course, that for $f \ll 1$ the perturbation analysis, in terms of plane wave "small vibrations" about $\phi_0 = \pm\sqrt{f}$, breaks down because the effects of forming "kink-antikink" states, such as that shown in Fig. 2, must be taken into account.

To summarize, we have learned that our classical analogue

(i) exhibits phase transitions from an ordered to disordered phase at a critical coupling $f_c = 0$;

(ii) in the <u>ordered state</u> this theory has both "<u>plane wave</u>" or "<u>small vibration</u>" excitations of mass $m \cong (\lambda f^3)^{1/2}$;

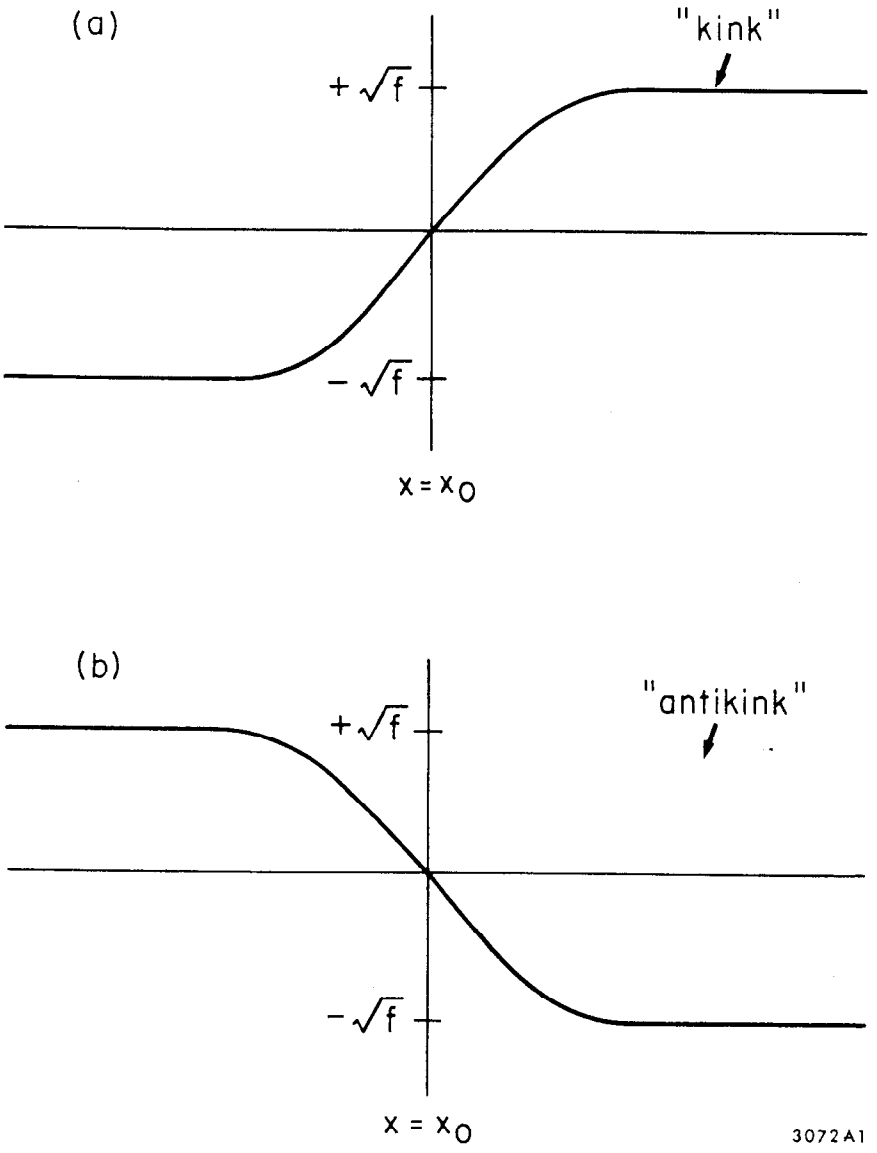


Fig. 1

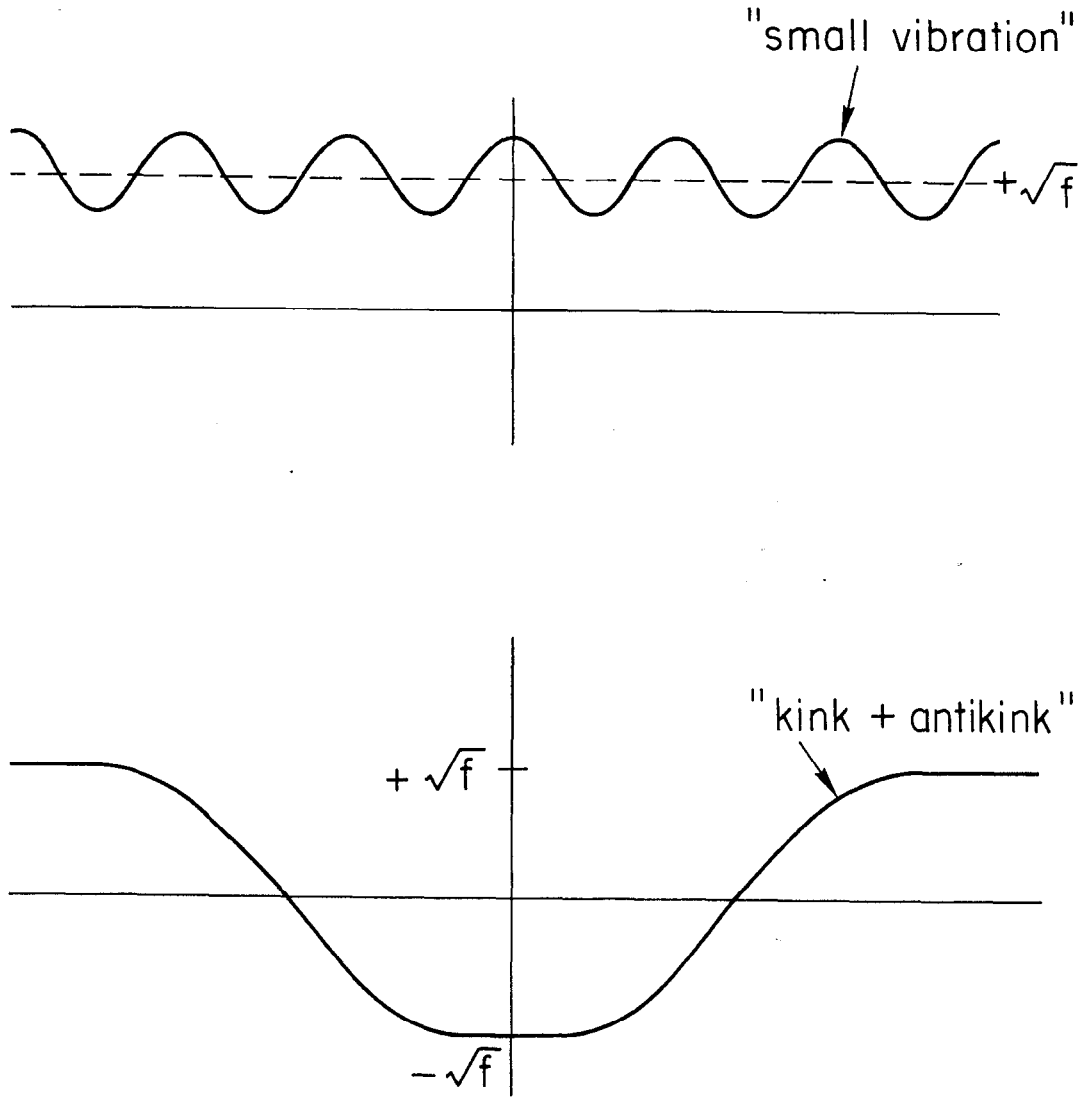


Fig. 2

(iii) the kink state cannot be found from the "perturbation" or "small vibration analysis";

(iv) in the limit $f \ll 1$, when $m_{kink}/m \ll 1$ the usual perturbation analysis breaks down completely because "kinks" and "antikinks" become the lowest lying excitations.

Obviously we could take the Hamiltonian of (4.2) and impose the commutation relations $[\Pi(x), \phi(y)] = -i\delta(x-y)$ in order to quantize the theory. However, our classical discussion already warns us that in the interesting region $f \ll 1$ the usual perturbation methods must fail.

V. ON THE CONNECTION BETWEEN FIELD THEORIES AND LATTICE MODELS

Having argued that we will need to go beyond perturbation theory to study the region $f \ll 1$, we are immediately confronted by the problem of renormalization. We all know that if one adopts the usual perturbation theory approach to dealing with a formal continuum Hamiltonian one does not obtain finite answers unless one resorts to the trick of multiplicative renormalization. Unfortunately, while the infinities encountered in perturbation theory are there whether or not one is dealing with the perturbation expansion, the only well developed method for removing them is tied inextricably to the perturbative approach. Moreover, except for a few special cases no convincing nonperturbative ways for removing these divergences have been put forth. In order to develop a nonperturbative way of computing in a theory free of divergences we find it necessary to abandon—at least temporarily—one or another of our cherished ideas. Our approach will be to abandon manifest (but not practical) Poincare invariance and avoid ultraviolet divergences by establishing a finite momentum space cutoff. Moreover, since we shall be interested in studying phase transitions and other properties which are associated with subtleties of taking infinite volume limits, we will initially define each theory in a box in position space and study the infinite volume limit

afterwards. Actually, what we will really do is study a theory defined on a lattice. Of course, logical completeness requires that I prove to you that studying theories with fixed momentum and spatial cutoffs are equivalent to lattice theories; unfortunately, finiteness of time militates against my doing so. Anyway, as amusing as this correspondence is, my real aim is to tell you what one does with a lattice theory once one has it and not how one goes about finding that lattice theory (or class of lattice theories) which can be shown to be unitarily equivalent to a given cutoff continuum field theory. Hopefully, you will allow me to assure you that such a correspondence can be made and let it go at that for now (in Appendices I and II the way one constructs such an equivalence is described for both free scalar and free fermion field theories, and the way one carries out the same program for interacting theories is clearly indicated). Establishing this correspondence in detail will be irrelevant to this talk anyhow, since for simplicity's sake I will not discuss the theory which is the true unitary equivalent of cutoff ϕ^4 theory at all. Instead I will write down a lattice ϕ^4 theory which at the classical level has properties which are essentially the same as the continuum model we have been discussing, and quantize it. Clearly, even if we make this simplification we still must answer the question, "What do you do with a lattice theory once you have it?", since at first blush strong and intermediate coupling lattice theories are no easier to solve than their continuum counterparts.

VI. A LATTICE ANALOGUE OF CONTINUUM ϕ^4 THEORY

A. The Model. In the sections to follow we shall discuss a simple version of lattice ϕ^4 theory, which will be based upon a nearest neighbor definition of the gradient of a field $\phi(j)$, i.e., $(\nabla\phi)(j) \stackrel{\text{def}}{\equiv} (\phi(j+1) - \phi(j))$. For those of you who are aware of the fact that in general our SLAC collaboration spends its time singing the praises of a different definition of the gradient, let me hasten to assure you that we have not changed our affections. However, since both definitions are

equally simple to work with, and since for scalar fields—as opposed to fermion fields—there is no crucial difference between the two definitions, I adopt the more familiar one so as to minimize the number of new ideas I will ask you to get used to in a short time. I want to emphasize, however, that the methods I will use to study the lattice ϕ^4 theory can be applied to either the theory I will write down, the one using our favorite definition of the gradient or the lattice model truly equivalent to cutoff continuum ϕ^4 theory with roughly equal simplicity, and with roughly the same results.

The theory we will study is defined by

$$H = \Lambda \sum_{j=-N}^{+N} \left[\frac{\Pi(j)^2}{2} + \lambda(\phi(j)^2 - f)^2 + \frac{1}{2}(\phi(j+1) - \phi(j))^2 \right] , \qquad (6.1)$$

where $\Pi(j) = \partial_0 \phi(j)$. We can look upon this as a classical theory of a field defined as a function of a discrete set of points, instead of as a function of a continuum variable 'x'. As in the continuum case we note that H is a sum of positive terms and so the field configurations of lowest energy correspond to $\Pi(j)=0$ and $(\phi(j+1) - \phi(j)) = 0$ (i.e., $\phi(j)$ is a constant). Therefore, the analysis of the structure of the ground state of this theory for $f \leq 0$ and $f > 0$ is the same as the analysis of the continuum theory. In fact the only significant difference between (6.1) and the continuum version of the model is that, for convenience, all fields and coupling constants have been rescaled by dividing by the appropriate powe of inverse lattice spacing $\Lambda \stackrel{\text{def}}{\equiv} 1/a$, so as to make them dimensionless. In this way all of the dimensions of H come from the single explicit factor of mass, Λ, which appears in front of the sum. With this definition one quantizes this model by specifying the equal time commutation relations

$$\left[\Pi(j_1),\ \phi(j_2)\right] = -i\delta_{j_1, j_2} \quad . \qquad (6.2)$$

Assuming, without further ado, that the classical physics of this model is essentially equivalent to the physics of the continuum model, what I will now

attempt to do is convince you that (at least for some range of λ and f) this model can be still further simplified without losing the physics we are interested in. The resulting "toy" model will be the one we will study in detail. In order to formulate this argument it becomes convenient to divide H into two terms,

$$H_1 = \Lambda \sum_j \left[\frac{\Pi^2(j)}{2} + \phi^2(j) + \lambda(\phi^2(j) - f)^2 \right] \quad , \tag{6.3}$$

and

$$H_2 = \Lambda \sum_j \left[-\phi(j)\ \phi(j+1) \right] \quad . \tag{6.4}$$

Having done this we then observe that H_1 is a sum of commuting terms, $H_1(j)$. Thus, the problem of diagonalizing H_1 reduces to the problem of finding the eigenstates of $H_1(j)$.

Since, $[\Pi(j), \phi(j)] = -i$, it is obvious that in the basis in which $\phi(j)$ is diagonal $\Pi(j)$ can be written as $\frac{1}{i}\frac{\partial}{\partial\phi(j)}$ and so, solving the eigenvalue problem specified by each $H_1(j)$ is equivalent to solving the Schrœdinger equation

$$\left\{ -\frac{\partial^2}{\partial\phi^2} + \lambda\left(\phi^4 + \left(\frac{1}{\lambda} - 2f\right)\phi^2 + f^2\right) \right\} \Psi_n(\phi) = E_n \Psi_n(\phi) \tag{6.5}$$

this is just the same as solving the elementary quantum mechanical problem of a particle of mass, m=1, in a double potential well.

I will now try and convince you that so long as we require $(\lambda f^3)^{1/2} >>> (\lambda f)^{1/2} >> 1$, then one can for the purpose of discussing the low lying eigenstates of this theory replace our ϕ^4 theory by a model having only two levels at a site. In order to see why this is so let us consider Fig. 3 which shows that the height of the potential barrier at $\phi=0$ is λf^2 and that the minima of the potential occurs at $\phi_{min} = \pm\sqrt{f - 1/2\lambda}$, where $V(\phi_{min}) = \frac{1}{\lambda}(1 - 1/4\lambda)$. Now the double well problem should be familiar to you from elementary quantum mechanics, where it was studied as a first example of tunneling. For our purposes in this discussion it

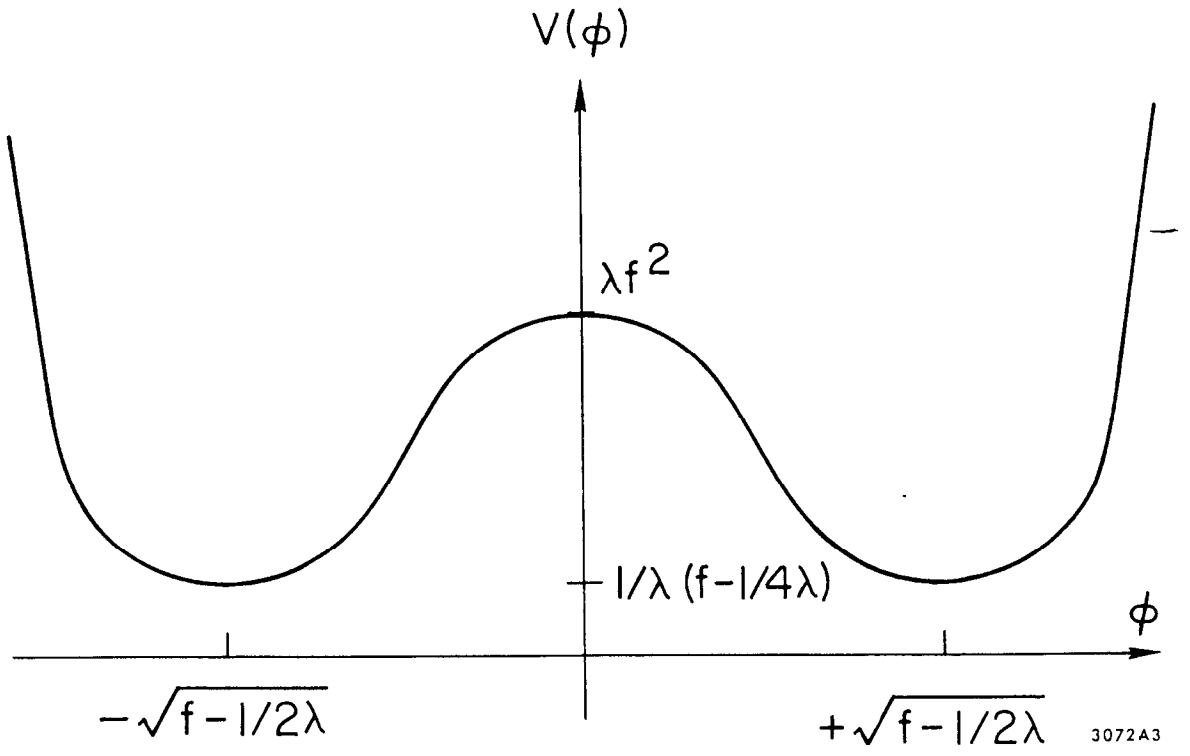


Fig. 3

suffices to recall a few basic facts about this problem, namely:

(i) The eigenstates of the problem can be divided into states which are even or odd under the transformation $\Psi_n(\phi) \rightarrow \Psi'_n(+\phi) = \Psi_n(-\phi)$.

(ii) The first even and odd states are split in energy by an amount proportional to exp $\{-(\text{height of barrier}) \times (\text{width of barrier})\}$ which is on the order of exp $\left\{-\lambda f^3\right\}$. Hence for $(\lambda f^3)^{1/2} >> 1$ the two lowest states of our single-site Schrœdinger problem are nearly degenerate.

(iii) When the first two levels are nearly degenerate one can use naive harmonic oscillator wave functions centered about either minimum to estimate the energy gap to the next level. This argument tells us that the energy of the next level will be essentially $(\lambda f)^{1/2}$ so long as $\lambda f^2 >> (\lambda f)^{1/2}$—or equivalently, so long as $(\lambda f^3)^{1/2} >> 1$. In other words, so long as we satisfy this inequality, the gap between the first two nearly degenerate levels and the next eigenstate of the system can be made as large as desired.

Assuming that we restrict attention to the case $(\lambda f^3)^{1/2} >> 1$ and $(\lambda f)^{1/2} >> 1$, let us try and see what we expect the groundstate of $H=H_1+H_2$ to be like. In this case the groundstate of H_1 is simply given as the product over all j's of the lowest state for each $H_1(j)$. Clearly, by far the most important effect of $H_2 = -\sum_j \phi(j)\,\phi(j+1)$ is to mix this state with states of the same form except having odd parity levels at two adjacent sites "j" and "j+1". Since the lowest even and odd levels are essentially degenerate and since these mixing effects stay finite no matter how large we make $(\lambda f^3)^{1/2}$, for sufficiently small splittings, it behooves us to restrict attention to the two lowest states at each site and to replace $H_2 = -\sum_j \phi(j)\,\phi(j+1)$ by a product of matrices defined for sites "j" and "j+1" which mix these two nearly degenerate levels.

B. A Final Simplification. The previous argument tells us that for large $(\lambda f^3)^{1/2}$ we can replace $H=H_1+H_2$ by an analogue theory of the form

$$H^{(toy)} = \Lambda\sum_j\left[\begin{pmatrix}-\epsilon/2 & 0\\ 0 & +\epsilon/2\end{pmatrix}(j) - \Delta\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}(j)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}(j+1)\right]$$

$$\stackrel{\text{def}}{\equiv} H_1^{(toy)} + H_2^{(toy)} \tag{6.6}$$

In case the passage from what we have said up to now to (6.6) was too rapid let me explain the analogue in greater detail. The term

$$H_1^{(toy)} = \Lambda\sum_j\begin{pmatrix}-\epsilon/2 & 0\\ 0 & +\epsilon/2\end{pmatrix}(j) \tag{6.7}$$

is the replacement for H_1 in (6.3). It merely says that in our analogue system we assume that we have two states or spins, $|\downarrow(j)\rangle$ and $|\uparrow(j)\rangle$, at each site—corresponding to the even and odd parity states of our original theory—and that in the absence of H_2 these states are split in energy by an amount "ϵ" (of course ϵ is some complicated function of λ and f which will not need to be further specified). In other words $H_1^{(toy)}$ is a transcription of the single-site part of H into a theory of spins on a lattice. Obviously,

$$\begin{aligned} H_2^{(toy)} &= -\Lambda\sum_j\left\{\Delta\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}(j)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}(j+1)\right\} \\ &= -\Lambda\sum_j\left\{\Delta\sigma_x(j)\,\sigma_x(j+1)\right\} \end{aligned} \tag{6.8}$$

is just a transcription of H_2 to this spin language. To see why it must have this particular form we need only recall that the operator $\phi(j)$ is odd under the transformation $\phi\rightarrow-\phi$. Hence matrix elements of $\phi(j)$ taken between two even states, or two odd states, must vanish identically. If we restrict attention to the two nearly degenerate eigenstates of $H_1(j)$ then the only matrix element $\phi(j)$ has between these states is $\langle even|\phi(j)|odd\rangle = \langle odd|\phi(j)|even\rangle\,\Delta^{1/2}$. (Note that since only two states are involved one can always choose the relative phase of $|odd\rangle$

and |even> so this matrix element is real.) Therefore, if we think of the two states as $|\downarrow(j)\rangle$ and $|\uparrow(j)\rangle$, $\phi(j)$ can be replaced by $\Delta^{1/2}$ times the matrix which flips spin down to spin up and vice versa, i.e., $\sigma_x(j) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(j)$. Thus we are led to study the theory defined by (6.6), with ϵ and Δ chosen arbitrarily.

VII. ANALYSIS OF THE TOY MODEL: I

A. Exact Results. Having specified the "toy" model we wish to study in detail, the question arises as to whether or not we have oversimplified our original problem and lost all of the interesting nonperturbative phenomena we were out to find. Let me hasten to reassure you that this is not so. This model can be solved exactly and one can show that it has all sorts of interesting properties. Before listing the facts which we know to be true, let me address a question which comes up when one discusses properties of solvable models. "Why study an exactly solvable model in one-space one-time dimension, when we all know the methods for solving these models exactly never work anywhere else?" This is a good question—fortunately (I think) I have a good answer. I couldn't care less about the way in which one goes about solving this model exactly! As I am sure you suspected, the method of obtaining an exact solution to this model involves performing a Pauli-Jordan transformation followed by a Bogoliubov transformation and this method does not generalize to higher dimensions. Even worse, so far as I am concerned, while it allows you to study the properties of this model exactly it leaves you with no "nuts and bolts" feeling for what is going on. However, it is not the model that I care about, but rather the approximate techniques I will use to discuss it. Unlike the techniques used to solve this model exactly, our methods are easily generalizable to any model in any number of space dimensions, and perhaps even more importantly they do give you a feeling for the nuts and bolts of the theory. The only drawback to these methods is that although they seem intuitively reasonable they are after all only approximate

calculations. Therefore, it is enormously important to begin by trying them out on a theory for which an exact solution is known in order to see whether or not they really work. With this comment in mind let us list some interesting properties of the theory defined in (6.6).

(i) The theory defined in (6.6) exhibits a phase transition at the point $\epsilon = 2\Delta$. One can rigorously show that for $\epsilon \leq 2\Delta$ the groundstate—or lowest energy eigenvalue—of H is twofold degenerate, whereas for $\epsilon > 2\Delta$ the groundstate is unique.

(ii) Since the overall scale of H plays no role in the detailed physics only the ratio $\epsilon/\Delta \stackrel{\text{def}}{\equiv} y$ matters, and so $y_c = 2$ is the critical coupling constant of this theory. In effect y_c plays the role of "f" in the classical ϕ^4 model.

(iii) The transformation

$$U = \exp\left(\frac{i\pi}{2}\sum_j \sigma_z(j)\right) = \prod_j (i\sigma_z(j)) \tag{7.1}$$

is the unitary transformation which plays the role, in this quantum theory, of the classical transformation $\phi \to -\phi$. (This is so since we have seen that $\sigma_x(j) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(j)$ is the toy theory equivalent to the operator $\phi(j)$ and it is easy to see that

$$U\sigma_x(j)\,U^{-1} = -\sigma_x(j) \quad .) \tag{7.2}$$

(iv) As in the classical theory we can label states according to their transformation properties with respect to U. In particular the two-dimensional space of states spanned by the degenerate eigenvectors of H, for $\epsilon \leq 2\Delta$, can be generated by eigenvectors of U which we will call $|\text{even}\rangle$ and $|\text{odd}\rangle$. Clearly $\langle \text{even}|\sigma_x(j)|\text{even}\rangle = \langle \text{odd}|\sigma_x(j)|\text{odd}\rangle = 0$ since $U\sigma_x U^{-1} = -\sigma_x$, but in general $\langle \text{even}|\sigma_x(j)|\text{odd}\rangle$ has no reason to

vanish. If it does not vanish it is useful to form the combinations

$$|\pm\tilde{f}> = \frac{1}{\sqrt{2}}\{|\text{even}> \pm |\text{odd}>\} \tag{7.3}$$

for which

$$U|\pm\tilde{f}> = |\mp\tilde{f}> \tag{7.4}$$

and for which

$$<\pm\tilde{f}|\sigma_x(j)|\pm\tilde{f}> = \pm <\text{even}|\sigma_x(j)|\text{odd}> \tag{7.5}$$

is nonvanishing. These states are the quantum analogues of the classical states $\phi_0(f) = \pm\sqrt{f}$, and it can be shown that the "order parameter" or "magnetization" satisfies

$$<\pm\tilde{f}|\sigma_x(j)|\pm\tilde{f}> = \pm\left(1 - \left(\frac{y}{y_c}\right)^2\right)^{1/8} \tag{7.6}$$

for $y \leq y_c$, where $y \overset{\text{def}}{\equiv} \epsilon/\Delta$ and $y_c = 2$. The exponent 1/8 in (7.6) is an important <u>critical exponent</u> of this model.

(v) For $\epsilon > 2\Delta$ the groundstate of the system is unique and develops continuously from the state we have called |even>.

(vi) For $\epsilon \ll 2\Delta$ the lowest lying physical excitations of definite momentum are "kinks", i.e., states which are clearly identifiable as moving "domain walls." The "single kink" spectrum is given by

$$E_{\text{kink}}(k_n) = \Lambda\sqrt{(2\Delta - \epsilon)^2 + 8\Delta\epsilon \sin^2\left(\frac{k_n}{2}\right)} \tag{7.7}$$

where k_n is the <u>dimensionless lattice momentum</u>, $k_n = \frac{2}{(2N+1)}\, n$. In the infinite volume limit, i.e., $N \to \infty$, k_n becomes a continuous variable running over $-\pi \leq k \leq \pi$. (As an aside it is worth noting that (7.7) explains what I mean by the lattice theory abandoning manifest Lorentz invariance but keeping practical Lorentz invariance. It follows from (7.7) that this simple theory, near the

critical point $\epsilon = 2\Delta$, approximates a Lorentz invariant theory arbitrarily closely. To see how this works first define the dimensional momentum $\kappa = \sqrt{2\epsilon\Delta}\ \Lambda k$ so that k runs from $-\pi\Lambda$ to $+\pi\Lambda$, and define a mass $M_0 = \Lambda(2\Delta - \epsilon)$. Next rewrite

$$E_{kink}(k) = \sqrt{M_0^2 + 8\Delta\epsilon\Lambda^2 \sin^2\left(\frac{\kappa}{2\sqrt{2\epsilon\Delta}\,\Lambda}\right)} \tag{7.8}$$

Thus we see that approaching the critical point from below as yields a relativistic theory of "kinks", in that if M_0 is held finite say 1 GeV and Λ is taken arbitrarily large, say 10^{100} GeV, then until we reach values of κ on the order of 10^{100} GeV our spectrum is essentially relativistic.)

(vii) For $\epsilon > 2\Delta$ the lowest lying physical excitations are "particle-like", in that they are identifiable as moving disturbances in a uniform groundstate as opposed to a moving interface between two "groundstate" configurations. The 1-particle spectrum, remarkably enough, is given by the same formula

$$E_{particle}(k_n) = \Lambda\sqrt{(\epsilon - 2\Delta)^2 + 8\Delta\epsilon \sin^2\left(\frac{k_n}{2}\right)}$$

(therefore, approaching the critical point from above defines a relativistic theory of particle excitations).

(viii) The groundstate energy density, $\mathcal{E}_0(y)$ can be calculated exactly and one can prove that $\partial^2 \mathcal{E}_0(y)/\partial y^2$ has a singularity at the critical point $y_c = 2$.

B. <u>A Simple Derivation of Some of These Results</u>. I will not even attempt to explain how these exact results are derived as that would be a seminar in itself. Anyhow, as I have said, even if you know how to solve this theory you don't really get the feeling that you understand what is going on. I would like

to try and give simple arguments to show

(i) for $\epsilon \ll 2\Delta$ the groundstate of H is doubled whereas for $\epsilon \gg 2\Delta$ the groundstate is unique;

(ii) the elementary excitations for $\epsilon \ll 2\Delta$ are "kink-like" in plane wave states, whereas for $\epsilon \gg 2\Delta$ they are "particle-like" plane wave states.

In order to carry out this discussion it will prove convenient to rewrite (6.6) as

$$H = -\frac{\epsilon}{2}\Lambda\left(\sum_j \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(j)\right) + \Lambda\sum_j\left[\begin{pmatrix} 0 & 0 \\ 0 & \epsilon \end{pmatrix}(j) - \Delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(j)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(j+1)\right] \tag{7.8}$$

and to drop the term $-\frac{\epsilon\Lambda}{2}\left[\sum_j \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(j)\right]$, since it is just a multiple of the unit matrix. Having done this let us consider what happens when we look at H for $\epsilon = 0$ (i.e., look at $H(\epsilon = 0)$).

In this case we observe that H can be exactly diagonalized, since all of the terms $\sigma_x(j)\,\sigma_x(j+1)$ commute with one another. It is therefore convenient to go to a new set of single-site states, namely

$$\begin{aligned} |{\rightarrow}\,(j)\rangle &= \frac{1}{\sqrt{2}}\left(|\ (j)\rangle + |\ (j)\rangle\right) \\ |{\leftarrow}\,(j)\rangle &= \frac{1}{\sqrt{2}}\left(|\ (j)\rangle - |\ (j)\rangle\right) \end{aligned} \tag{7.9}$$

so that the complete set of states of the form

$$\ldots\ |{\rightarrow}\,(j)\rangle\ |{\rightarrow}\,(j+1)\rangle \ldots\ |{\rightarrow}\,(j+p)\rangle \ldots$$

or

$$\ldots\ |{\leftarrow}\,(j)\rangle\ |{\leftarrow}\,(j+1)\rangle \ldots\ |{\leftarrow}(j+p)\rangle \ldots$$

or

$$\ldots\ |{\rightarrow}\rangle\ |{\rightarrow}\rangle\ |{\leftarrow}\rangle\ |{\leftarrow}\rangle \ldots$$

will all be eigenstates of $H(\epsilon = 0)$. Observing that

$$\sigma_x(j)\ |{\rightarrow}\,(j)\rangle = +\,|{\rightarrow}\,(j)\rangle$$

and

$$\sigma_x(y)\,|\leftarrow(j)\rangle = -|\leftarrow(j)\rangle \tag{7.10}$$

we can evaluate the eigenenergy of each such configuration, since each term $-\Delta\sigma_x(j)\,\sigma_x(j+1)$ gives a factor of $-\Delta$ whenever the "j" and "j+1" spins point in the same direction and a factor of $+\Delta$ whenever they point in the opposite direction. If, for the sake of definiteness, we restrict ourselves to finite volume, so that our lattice has a finite number of sites (i.e., $\llcorner = 2N+1$) then the two states

$$\ldots|\rightarrow\rangle|\rightarrow\rangle\ldots|\rightarrow\rangle\ldots \stackrel{\text{def}}{\equiv} |\Rightarrow\rangle_0$$

and

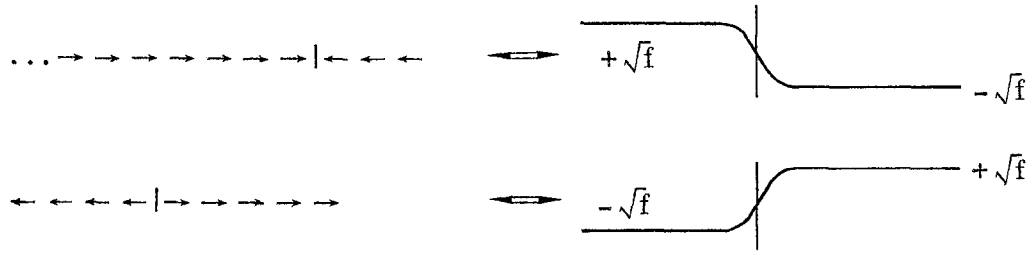
$$\ldots|\leftarrow\rangle|\leftarrow\rangle|\leftarrow\rangle\ldots|\leftarrow\rangle\ldots \stackrel{\text{def}}{\equiv} |\Leftarrow\rangle_0$$

are clearly the eigenstates of $H(\epsilon=0)$ having the lowest possible energy, namely $E_0=-2N\Delta$. Since there are two distinct configurations having this energy we see that for $\epsilon=0$, the groundstate of this theory is degenerate. Moreover, since $\phi(j)$ corresponds to $\Delta^{1/2}\sigma_x(j)$ in this model and the expectation value of $\Delta^{1/2}\sigma_x(j)$ in each of these states is nonvanishing—more precisely

$${}_0\langle\Rightarrow|\sigma_x(j)|\Rightarrow\rangle_0 = -{}_0\langle\Leftarrow|\sigma_x(j)|\Leftarrow\rangle_0 = \Delta^{1/2} \tag{7.11}$$

we see that the order parameter in these states is $\Delta^{1/2}$. Hence, we see that for $\epsilon=0$ our theory has the same sort of groundstate degeneracy as the classical model with $\pm\sqrt{f}$ corresponding to $\pm\Delta^{1/2}$.

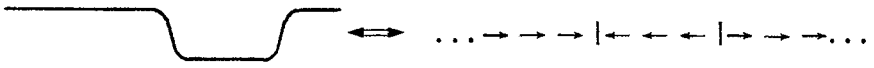
In addition to the groundstates of the $H(\epsilon=0)$ Hamiltonian let us see if we can identify the low lying "kink states" or "magnetic domain configurations." This is trivial since it is clear that configurations like



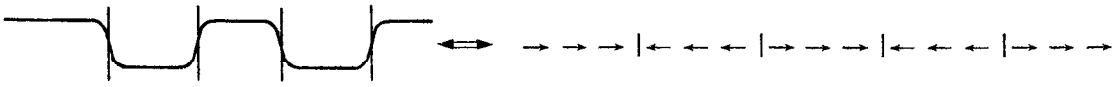
which represent two adjacent blocks of oppositely aligned spins, clearly correspond to "antikink" and "kink" configurations of the classical theory. Since in such a configuration only one pair of adjacent spins are antiparallel the energy of any state like this is

$$E_{1\ \text{kink}} = E_{1\ \text{antikink}} = -2N\Delta + 2\Delta = E_0 + 2\Delta \quad , \tag{7.12}$$

so these states represent "mass=2Δ" configurations. Higher states corresponding to more domains, e.g.,



or



are "multiple kink-antikink" configurations which have higher "mass". In fact, since each mismatched pair of spins contributes $+\Delta$ to the energy whereas a matched pair in the groundstate contributes $-\Delta$ we see that the general rule for calculating the energy of any such configuration is to count the number of interfaces, N_i, and multiply this by $+2\Delta$ to get the difference in energy between multiple kink-antikink configurations and the groundstate. Note further that the energy depends only upon the number of interfaces and not their location.

With these features of the H(ϵ=0) case in mind let us develop a perturbation theory discussion of what happens for very small but nonzero "ϵ". In other words let us treat the term

$$V = \epsilon \sum_j \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(j) \tag{7.13}$$

as a perturbation on the term

$$-\Delta \sum_j \sigma_x(j)\, \sigma_x(j+1) \tag{7.14}$$

Looking at any of the terms in V we see that

$$\epsilon \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}(j)|\rightarrow (j)> = \frac{\epsilon}{2}|\rightarrow (j)> - \frac{\epsilon}{2}|\leftarrow (j)>$$

and

$$\epsilon \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(j)|\leftarrow (j)> = \frac{\epsilon}{2}|\leftarrow (j)> - \frac{\epsilon}{2}|\rightharpoonup (j)> \quad , \tag{7.15}$$

hence in the $|\rightarrow>$, $|\leftarrow>$ basis H can be rewritten as

$$H = \sum_j \left\{ -\Delta \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}(j) \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}(j+1) - \frac{\epsilon}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(j) \right\} + \frac{\epsilon}{2}\,\mathbb{1} \quad . \tag{7.16}$$

It follows that applying V to any eigenstate of $H(\epsilon=0)$ gives us a linear combination of states all of which come from flipping one spin in the original state. Let us analyze what happens to the various eigenstates of the $H(\epsilon=0)$ Hamiltonian in first order perturbation theory. To begin with, consider as our initial state the state $|\Longrightarrow>$ of all spins to the right. Then $V|\Longrightarrow>$ is the state shown in Fig. 4 and is orthogonal to the original state $|\Longrightarrow>$. This means there is no first order shift in the groundstate energy. Clearly the same is true for the state $|\Longleftarrow>$ and so $|\Longrightarrow>$ and $|\Longleftarrow>$ stay degenerate to first order, at an energy $E_0=-2N\Delta$.

Focusing on 1-kink or 1-antikink states yields a different result. If we consider Fig. 5 we see that if we let V operate on a state of this form, the interface gets moved one step to the right or one step to the left or a new pair of interfaces is created. The matrix elements linking the 1-kink or 1-interface state to a 3-interface state does nothing to shift the energy of this state to first order in ϵ. However, the matrix elements which link a 1-interface state to a different 1-interface state do cause a first order shift in the energy since all of these states are degenerate, at energy $E_0+2\Delta$. Recall that the rule for doing degenerate perturbation theory is to first diagonalize V as restricted to the degenerate states and then compute the first order shift for each state $|\psi_n>$, which will be $\delta E_n = <\psi_n|V|\psi_n>$. The diagonalization of V in the 1-kink or

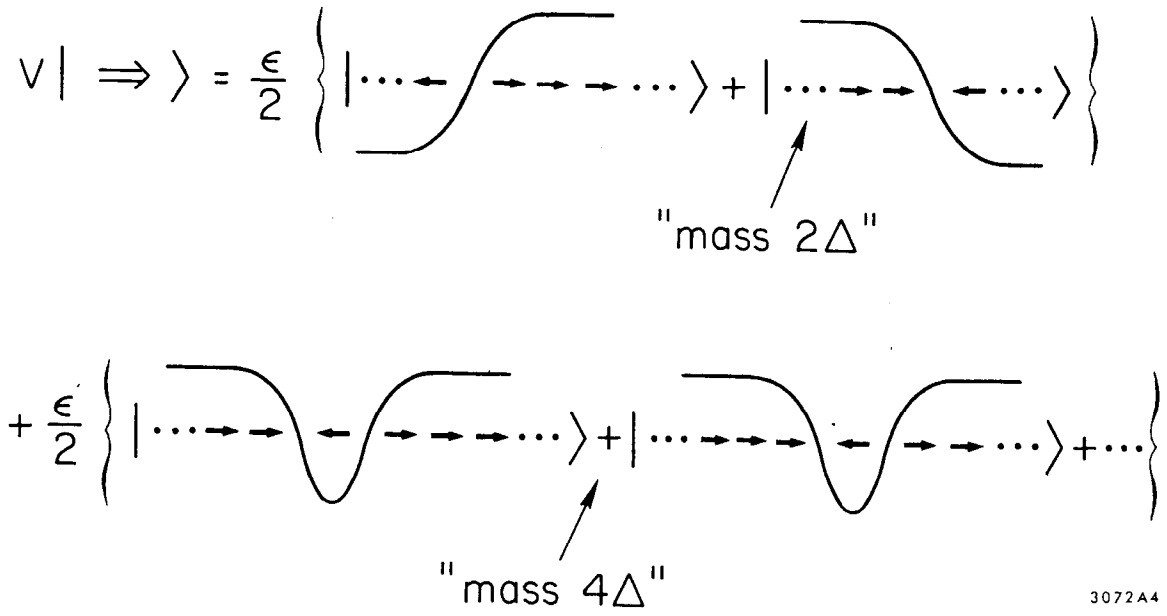


Fig. 4

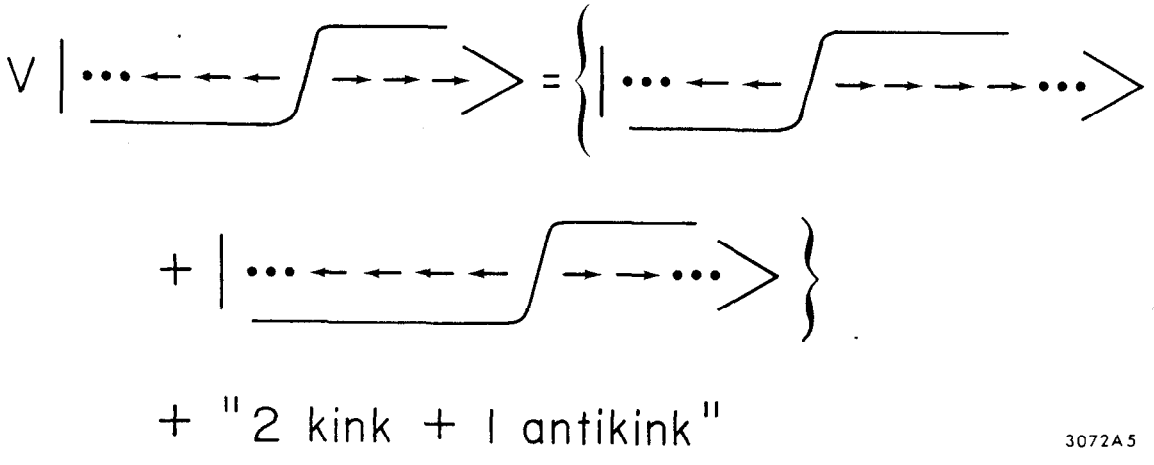


Fig. 5

antikink sector is easily accomplished in the infinite volume limit, $N \to \infty$, because, denoting by $|\psi_j\rangle$ the 1-kink state whose interface lies between 'j' and 'j+1',

$$V|\psi_j\rangle = -\frac{\epsilon}{2}\left(|\psi_{j+1}\rangle + |\psi_{j-1}\rangle\right) \quad . \tag{7.17}$$

Thus, if we form the states

$$|\psi_k\rangle = \sum_j e^{ikj}|\psi_j\rangle \tag{7.18}$$

we see that

$$V|\psi_k\rangle = -\frac{\epsilon}{2}\left(e^{-ik} + e^{+ik}\right)|\psi_k\rangle = -\epsilon \cos(k)|\psi_k\rangle \tag{7.19}$$

It follows that the states $|\psi_k\rangle$, which correspond to "plane wave kink states" or kinks of definite momentum 'k', do get a first order shift in energy; to this order we have

$$H|\psi_k\rangle = E_{1\ \text{kink}}(k) = \left\{E_0 + (2\Delta - \epsilon\cos(k))\right\}|\psi_k\rangle \tag{7.20}$$

Obviously the results for 1-antikink states are identical. (Note that it becomes necessary to measure all energies with respect to the groundstate energy $E_0 = -2N\Delta$ when N goes to infinity in order to obtain finite quantities.)

To summarize, for small enough ϵ the two-fold degeneracy of the groundstate persists, and the lowest lying excitations of the system are single kink or antikink states of momentum 'k'. Moreover (7.20) tells us that the k=0 kink or antikink state has the lowest energy and the "mass" of this state is $(2\Delta - \epsilon)$; this happens to agree with the exact answer.

To conclude our analysis let us consider the limit $\Delta = 0$ or $\epsilon/\Delta \gg 1$. It is clear that in this case the groundstate is the state having all spins down. Having any spin up costs an energy ϵ. In this limit the groundstate is thus unique and the lowest energy particle-like excitations of the system correspond to the (2N+1)-degenerate states having all but 1 spin down. Applying a small Δ/ϵ

perturbation analysis to this case we find that to first order in Δ/ϵ the spectrum of "plane wave" states of "particle" type is $E(k) = E_0 + (\epsilon - 2\Delta \cos(k))$. Once again this agrees with the exact solution in that for $\epsilon >> 2\Delta$ we see that the mass of the k=0 state is $\epsilon - 2\Delta$. Since we have established that this theory changes from a theory with a two-fold degenerate groundstate and plane wave kinks as the lowest lying excitations, to a theory with a unique groundstate and particle-like plane wave excitations, as we go from $\epsilon < 2\Delta$ to $\epsilon > 2\Delta$ it follows that it must undergo a phase transition for some critical value of ϵ. What the perturbation analysis cannot tell us is the value of $\epsilon/2\Delta$ at which this happens, nor—for example—the behavior of $<\sigma_x(j)>$ as we approach this critical point.

VIII. ANALYSIS OF TOY MODEL II: NONPERTURBATIVE METHODS

In this section we turn to an approximate analysis of our "toy" model by variational techniques in an attempt to calculate the properties of the theory for all values of ϵ/Δ. Since guessing a trial wave function for a theory as complicated as this one is a nontrivial feat we shall not do this "at one fell swoop," but shall devise an iterative constructive procedure for finding a suitable infinite parameter trial state. In order to explain the motivation behind the procedure I will set forth allow me to remind you of certain general properties of variational calculations carried out for what I will call a "linear trial wave function."

A. A General Theorem. Assume that we have a Hamiltonian, H, defined on a specific Hilbert space, $\mathscr{H}$. Next, let $|\psi_n>$ stand for any set, in general not a complete set, of orthonormal states in $\mathscr{H}$. We define a "linear trial state with respect to the set $|\psi_n>$" to be a normalized linear combination of the form

$$|\psi^{\text{trial}}> \stackrel{\text{def}}{\equiv} \sum_n \alpha_n |\psi_n> \tag{8.1}$$

where the $\{\alpha_n\}$'s are any set of complex numbers such that

$$<\psi^{\text{trial}}|\psi^{\text{trial}}> = \sum_n |\alpha_n|^2 = 1 \tag{8.2}$$

The question we wish to answer once one has chosen a specific set of $\{\alpha_n\}$'s is "for which choice of $\{\alpha_n\}$'s is the expectation value

$$E(\alpha_1, \ldots) \equiv \langle \psi^{\text{trial}} | H | \psi^{\text{trial}} \rangle = \sum_{n,m} \alpha_n^* \alpha_m \langle \psi_n | H | \psi_m \rangle \qquad (8.3)$$

minimized ?"

Due to the constraint equation, (8.2), we cannot proceed to solve this problem by differentiating $E(\alpha_1, \ldots, \alpha_n, \ldots)$ with respect to the $\{\alpha_n\}$'s and straightforwardly equating the results to zero. Rather we must introduce a Lagrange multiplier λ and minimize the unconstrained expression

$$\langle \psi^{\text{trial}} | H | \psi^{\text{trial}} \rangle - \lambda \langle \psi^{\text{trial}} | \psi^{\text{trial}} \rangle$$

$$= \sum_{n,m} \alpha_n^* \alpha_m \left\{ \langle \psi_n | H | \psi_m \rangle - \lambda \delta_{nm} \right\}$$

$$\stackrel{\text{def}}{\equiv} \sum \alpha_n^* \alpha_m (H^{TR} - \lambda \mathbb{1})_{n,m} \quad . \qquad (8.4)$$

One then solves the resulting equations

$$\sum_m H_{nm}^{TR} \alpha_m = \lambda \alpha_n \quad , \qquad (8.5)$$

$$\sum_n \alpha^*_n H_{nm} = \lambda \alpha_m^* \qquad (8.6)$$

and

$$\sum_n \alpha_n^* \alpha_n = 1 \qquad (8.7)$$

for λ and α_n's. Clearly the solution to this problem is to observe that the α_n's must be the coefficients of the normalized eigenvectors of the matrix H_{nm}^{tr} — which we shall refer to as a truncated or effective Hamiltonian—and the parameter λ must be chosen to be the corresponding eigenvalue. Of course, if the states $|\psi_n\rangle$ are a complete orthonormal set, then (8.27) just says that the linear trial states which extremize the energy are the exact eigenstates of H.

The lesson to be learned from this exercise that if we are faced with the problem of diagonalizing a Hamiltonian which we cannot handle by simple methods, by cleverly choosing a set of orthonormal states (not a complete set) we reduce the problem of finding the groundstate of our system to the problem of finding the groundstate of an effective Hamiltonian. If we are fortunate this effective or truncated Hamiltonian will be amenable to analysis by familiar methods. This notion should be firmly fixed in your minds since it forms the backbone of everything which follows. Our aim will be to develop an algorithm for cleverly choosing a class of states which is intuitively well suited to the problem of finding an accurate upper bound on the groundstate energy of H, and which at the same time leads to an H^{tr} which can be studied by simple means.

B. A Simple Minded Algorithm for Choosing $|\psi_n\rangle$'s. In order to be more concrete let us discuss one example of an algorithm for choosing an orthonormal set of $|\psi_n\rangle$'s. To begin, recall that the space of states corresponding to the Hamiltonian of (6.6) is spanned by the set of wavefunctions formed by taking arbitrary tensor products of states having either spin up, $|\uparrow(j)\rangle$, or spin down, $|\downarrow(j)\rangle$, at each point. Our procedure will be to find a new complete set of states which span the same space and then thin this new set out in some way. We will try to do this in such a way as to keep states which we have good reason to believe are most important in the expansion of the true groundstate of H.

As a guide to making such a choice of $|\psi_n\rangle$'s let us, as indicated in Fig. 6, divide the lattice into disjoint boxes each of which contains two lattice sites. Clearly, one way to get a set of states which span our Hilbert space is to take the four states $|\downarrow\downarrow\rangle$, $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, and $|\downarrow\uparrow\rangle$ associated with each box and then to form from these states four different orthonormal vectors—say for example

$$\frac{(|\uparrow\uparrow\rangle+|\downarrow\downarrow\rangle)}{\sqrt{2}}\ ,\quad \frac{(|\downarrow\downarrow\rangle-|\uparrow\uparrow\rangle)}{\sqrt{2}}\ ,\quad \frac{(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)}{\sqrt{2}}\ ,\ \text{and}\ \frac{(|\downarrow\uparrow\rangle-|\uparrow\downarrow\rangle)}{\sqrt{2}}\ .$$

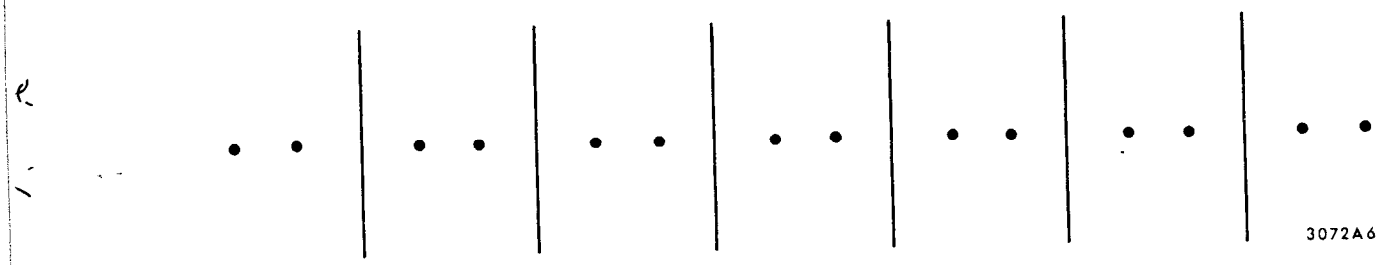


Fig. 6

It is then obvious that we can take a basis for the full Hilbert space, states formed by taking the product over boxes of one of these new vectors for each box. In general the subclass of states, $|\psi_n>$, used to form our linear trial wave functions will be defined by restricting ourselves to two of the four states in each box when forming product states. The key question will be how to decide on which two states in each box to throw away.

One way to throw away states is on the basis of the following overly simple argument. Begin by deleting from the Hamiltonian all terms $-\Delta\sigma_x(j)\,\sigma_x(j+1)$ for which j and j+1 do not lie within the same box. In this event the resulting Hamiltonian can be diagonalized by diagonalizing any one of the single box Hamiltonians

$$H_{(box)} = \epsilon\left\{\begin{pmatrix}0&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$$

$$-\Delta\begin{pmatrix}0&1\\1&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$=\sum_{j=1}^{2}\left\{\begin{pmatrix}0&0\\0&\epsilon\end{pmatrix}(j)\right\}-\Delta\sigma_x(1)\,\sigma_x(2) \qquad (8.8)$$

The Hamiltonian $H_{(box)}$ acts on the four possible states $|\downarrow\downarrow>$, $|\uparrow\uparrow>$, $|\uparrow\downarrow>$ and $|\downarrow\uparrow>$. As an aid to diagonalizing this problem let us observe that $H_{(box)}$ is invariant under a rotation about the z-axis of π. More specifically the unitary transformation

$$U^{(box)} = e^{i\frac{\pi}{2}\sum_{j=1}^{2}\sigma_z(j)} = -\sigma_z(1)\,\sigma_z(2) \qquad (8.9)$$

commutes with $H_{(box)}$; since,

$$U\sigma_z(j)\,U^{-1} = \sigma_z(j); \quad U\sigma_x(j)\,U^{-1} = -\sigma_x(j) \qquad (8.10)$$

Because of this $H_{(box)}$ can only mix states having the same eigenvalue of U, hence the state $|\downarrow\downarrow>$ only mixes with $|\uparrow\uparrow>$ and the state $|\uparrow\downarrow>$ only mixes with $|\downarrow\uparrow>$.

This reduces the problem of diagonalizing $H_{(box)}$ to the problem of diagonalizing two $2\times\pm$ matrices; namely

$$M_1 = \begin{bmatrix} \langle\downarrow\downarrow|H_{box}|\downarrow\downarrow\rangle & \langle\downarrow\downarrow|H_{box}|\uparrow\uparrow\rangle \\ \langle\uparrow\uparrow|H_{box}|\downarrow\downarrow\rangle & \langle\uparrow\uparrow|H_{box}|\uparrow\uparrow\rangle \end{bmatrix} = \begin{bmatrix} 0 & -\Delta \\ -\Delta & Z\epsilon \end{bmatrix} \tag{8.11}$$

and

$$M_2 = \begin{bmatrix} \langle\downarrow\uparrow|H_{box}|\downarrow\uparrow\rangle & \langle\downarrow\uparrow|H_{box}|\uparrow\downarrow\rangle \\ \langle\uparrow\downarrow|H_{box}|\downarrow\uparrow\rangle & \langle\uparrow\downarrow|H_{box}|\uparrow\downarrow\rangle \end{bmatrix} = \begin{bmatrix} \epsilon & -\Delta \\ -\Delta & \epsilon \end{bmatrix} \tag{8.12}$$

Diagonalizing M_1 and M_2 is a trivial exercise and the four eigenfunctions and eigenvalues are given in Table I.

Table I

State	Energy	Energy Relative to Lowest State
$\frac{1}{\sqrt{1+a^2}}(\lvert\downarrow\downarrow\rangle + a\lvert\uparrow\uparrow\rangle)^* \equiv \lvert\downarrow'\rangle$	$\epsilon - \sqrt{\epsilon^2+\Delta^2}$	0
$\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle + \lvert\downarrow\uparrow\rangle) \equiv \lvert\uparrow'\rangle$	$\epsilon - \Delta$	$\sqrt{\epsilon^2+\Delta^2} - \Delta$
$\frac{1}{\sqrt{2}}(\lvert\downarrow\uparrow\rangle - \lvert\uparrow\downarrow\rangle)$	$\epsilon + \Delta$	$\sqrt{\epsilon^2+\Delta^2} + \Delta$
$\frac{1}{\sqrt{1+a^2}}(-a\lvert\downarrow\downarrow\rangle + \lvert\uparrow\uparrow\rangle)$	$\epsilon + \sqrt{\epsilon^2+\Delta^2}$	$2\sqrt{\epsilon^2+\Delta^2}$

$^*a = (\sqrt{\epsilon^2+\Delta^2} - \epsilon)/\Delta$.

Step (i) of our general procedure will be to choose this set of four eigenstates as the new orthonormal system which we will use to construct a basis for . . . Step (ii), the thinning out procedure, is simply accomplished if we argue—with some justification—that when we add back the terms linking different boxes the most important part of the true groundstate will be spanned by states formed by

restricting ourselves to the two lowest energy states in Table I for each box. Having made this statement, we need only construct the truncated or effective Hamiltonian for this choice of linear trial wave functions and see if we can solve it.

Let us begin our computation of $H^{(tr)}$ by labelling each box by an integer 'p' and computing the truncated form of the part, H_1, of the Hamiltonian in (6.6) which does not contain terms $-\Delta\sigma_x(j)\,\sigma_x(j+1)$ where 'j' and 'j+1' are in different boxes. Since our procedure for constructing the $|\psi_n>$'s was to keep for each 'p' the two lowest eigenstates of the $H^{(box)}(p)$ corresponding to that box it is clear that we can write $H_1^{(tr)}$ as

$$H_1^{(tr)} = \sum_p H_{box}^{(tr)}(p) = \sum_p \begin{pmatrix} \epsilon - \sqrt{\epsilon^2+\Delta^2} & 0 \\ 0 & \epsilon-\Delta \end{pmatrix}(p) \tag{8.12}$$

This is perhaps better written as

$$H_1^{(tr)} = \sum_p \left\{ \left(\epsilon - \sqrt{\epsilon^2+\Delta^2}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(p) + \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\epsilon^2+\Delta^2} - \Delta \end{pmatrix}(p) \right\} \tag{8.13}$$

to make it look like the expression in (7.8). The next task which faces us is to compute the truncated version of H_2 which is the sum over all terms $-\Delta\sigma_x(j)\,\sigma_x(j+1)$ which couple different boxes. In order to compute $H_2^{(trunc)}$ all we need to do is see how any one such terms acts on one of the $|\psi_n>$'s we are considering. Since any one of the $|\psi_n>$'s is a state of the form $\prod_{boxes} |\psi_p>$ where the $|\psi_p>$'s can be either of the first two levels in Table I, the way in which a term of the type $-\Delta\sigma_x(j)\,\sigma_x(j+1)$ acts on a $|\psi_n>$ is to flip one spin in each of two adjacent boxes. Therefore, in order to compute the truncated version of this term, we need only compute the overlap of the state in the pth box obtained by flipping one spin in

$$|\psi_0(p)> = \frac{(|\downarrow\downarrow> + a|\uparrow\uparrow>)}{\sqrt{1+a^2}} \tag{8.14}$$

with the state

$$|\psi_1(p)\rangle = \frac{(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)}{\sqrt{2}} \quad . \tag{8.15}$$

Clearly, since $\sigma_x(j)$ is Hermitian and 'a' is real the truncated form of $\sigma_x(j)$ will be symmetric too. Actually since the term we wish to compute is $-\Delta\sigma_x(j)\,\sigma_x(j+1)$ we should note that $\sigma_x(j)$ will flip the right spin in a box whereas $\sigma_x(j+1)$ will flip the left spin, so in principle we must compute the overlap for each of these cases separately. In fact the calculations will give identical results.

The actual computation is quite trivial. Assuming 'j' lies in the pth box we find

$$\sigma_x(j)|\psi_0(p)\rangle = \sigma_x(j)\left[\frac{1}{\sqrt{1+a^2}}(|\downarrow\downarrow\rangle + a|\uparrow\uparrow\rangle)\right]$$

$$= \frac{1}{\sqrt{1+a^2}}\left[|\downarrow\uparrow\rangle + a|\uparrow\downarrow\rangle\right] \tag{8.16}$$

and so

$$\langle\psi_1(p)|\sigma_x(j)|\psi_0(p)\rangle = \frac{(1+a)}{\sqrt{2}\sqrt{1+a^2}} \tag{8.17}$$

Similarly

$$\sigma_x(j+1)|\psi_0(p+1)\rangle = \frac{1}{\sqrt{1+a^2}}\left[|\uparrow\downarrow\rangle + a|\downarrow\uparrow\rangle\right] \tag{8.18}$$

and so

$$\langle\psi_1(p+1)|\sigma_x(j+1)|\psi_0(p+1)\rangle = \frac{(1+a)}{\sqrt{2}\sqrt{1+a^2}} \tag{8.19}$$

It follows from (8.14)-(8.19) that for 'j' in the pth box

$$\sigma_x^{(tr)}(j) = \frac{(1+a)}{\sqrt{2(1+a^2)}}\,\sigma_x(p) \tag{8.20}$$

and

$$\sigma_x^{(tr)}(j+1) = \frac{(1+a)}{\sqrt{2(1+a^2)}}\,\sigma_x(p+1) \quad . \tag{8.21}$$

Thus, we have reduced the problem of finding the best upper bound which one can obtain by choosing trial states from the set of states spanned by forming all possible tensor products of the lowest two states in Table I, to the problem of diagonalizing a new Hamiltonian, $H^{(tr)}$. Interestingly enough $H^{(tr)}$ has exactly the same form as the original Hamiltonian but different coefficients. To be precise

$$H^{(tr)} = \sum_p \left[c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(p) + \epsilon_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(p) - \Delta_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(p+1) \right] \tag{8.22}$$

where

$$c_1 = \epsilon - \sqrt{\epsilon^2 (\Delta^2} \tag{8.23}$$

$$\epsilon_1 = \sqrt{\epsilon^2 + \Delta^2} - \Delta \tag{8.24}$$

and

$$\Delta_1 = \frac{\Delta(1+a)^2}{2(1+a^2)} \tag{8.25}$$

At this point we face one of two possibilities. Either the values of ϵ_1 and Δ_1 are such that we can treat the resulting effective Hamiltonian, H_1, by perturbation theory for $\epsilon_1/\Delta_1 > 1$ or $\epsilon_1/\Delta_1 < 1$; or, we say that we still have too large a class of orthonormal wave functions and we carry out the same procedure as we just went through, but this time for the Hamiltonian H_1. It is a straightforward exercise to convince oneself that each successive restriction of our class of trial wave functions leads us to a new effective Hamiltonian of the same <u>form</u> as the original Hamiltonian, but the coefficients of this effective Hamiltonian are given in terms of the ones obtained from the previous calculations. The general result is that after 'n' successive truncations our variational problem reduces to the

problem of diagonalizing an effective lattice Hamiltonian

$$H_n^{(tr)} = \sum_j \left[d_n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_j + \epsilon_n \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_j - \Delta_n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (j) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (j+1) \right] \tag{8.26}$$

where

$$\begin{aligned} \epsilon_{n+1} &= \sqrt{\epsilon_n^2 + \Delta_n^2} - \Delta_n \quad , \\ \Delta_{n+1} &= \frac{\Delta_n}{2} \frac{(1+a_n)^2}{(1+a_n^2)} \quad , \\ a_n &= \sqrt{\left(\frac{\epsilon_n}{\Delta_n}\right)^2 + 1} - \left(\frac{\epsilon_n}{\Delta_n}\right) \quad , \\ c_{n+1} &= \epsilon_n - \sqrt{\epsilon_n^2 + \Delta_n^2} \quad , \end{aligned} \tag{8.27}$$

and

$$d_{n+1} = c_{n+1} + 2d_n \quad .$$

where, referring back to Eq. (6.6) and Eq. (7.8) we see that $d_0 = -\epsilon/2$. Hopefully, at some state of this process one of the $H_n^{(tr)}$'s will prove to have a ratio of ϵ_n/Δ_n which is solvable or can be handled in perturbation theory. In any event we shall borrow from Wilson and Kadanoff and call the process of generating a new effective Hamiltonian from the one which was obtained in a previous step a "renormalization group transformation." The recursion relations given in (8.26)-(8.27), which define the parameters in $H_n^{(tr)}$ obtained from successive iterations, will be referred to—for want of a better name—as renormalization group equations.

C. Analyzing the Renormalization Group Equations. The preceding discussion reduced the problem of constructing a set of $|\psi_n\rangle$'s by means of a successive thinning out process, to the equivalent problem of computing a series of renormalization group transformations on the coefficients of an effective Hamiltonian. In order to extract all of the information contained in (8.26)-(8.27) the

recursion relations must be studied numerically. However, there are several points which can be understood without putting these equations on a computer. First, we note that both (ϵ=0, Δ arbitrary) and (ϵ arbitrary, Δ=0) are fixed points of the renormalization group transformation since in either case $\epsilon_n=\epsilon$ and $\Delta_n=\Delta$ for all 'n'. In fact, we have already seen that both of these cases can be solved exactly and it is easy to convince oneself that our algorithm for constructing the groundstate wave function constructs the exact eigenstate for these two cases. Second, we observe that a great deal of information can be extracted without completely solving the renormalization group equations if we know whether the ratio ϵ_n/Δ_n increases or decreases with successive iterations. In order to see if this happens let us define

$$y_n \stackrel{\text{def}}{\equiv} \left(\frac{\epsilon_n}{\Delta_n}\right) \tag{8.28}$$

and going back to (8.26)-(8.27) we see that 'y_{n+1}' is a function of 'y_n' alone. To be precise

$$y_{n+1} = \frac{2\left(\sqrt{1+y_n^2}-1\right)\left(1-2y_n\left(\sqrt{1+y_n^2}-y_n\right)\right)}{\left(1+\sqrt{1+y_n^2}-y_n\right)^2} \stackrel{\text{def}}{\equiv} F(y_n) \tag{8.29}$$

If we now wish to see if $y_n=\epsilon_n/\Delta_n$ increases or decreases with each iteration we need only study the function

$$R(y) \stackrel{\text{def}}{\equiv} F(y)-y \tag{8.30}$$

and see what it looks like for all 'y'. R(y) is plotted in Fig. 7, and we obtain useful information from the general shape of the curve alone. For example, since a "fixed point" of the transformation is a value of ϵ and Δ which reproduces itself under the renormalization group transformation, we must have that $F(y)=F(\epsilon/\Delta)=0$, unless of course $\epsilon/\Delta=\infty$ and then $F(\infty)>0$ cannot reduce this value. Hence Fig. 7 shows that there are three fixed points for our transformation;

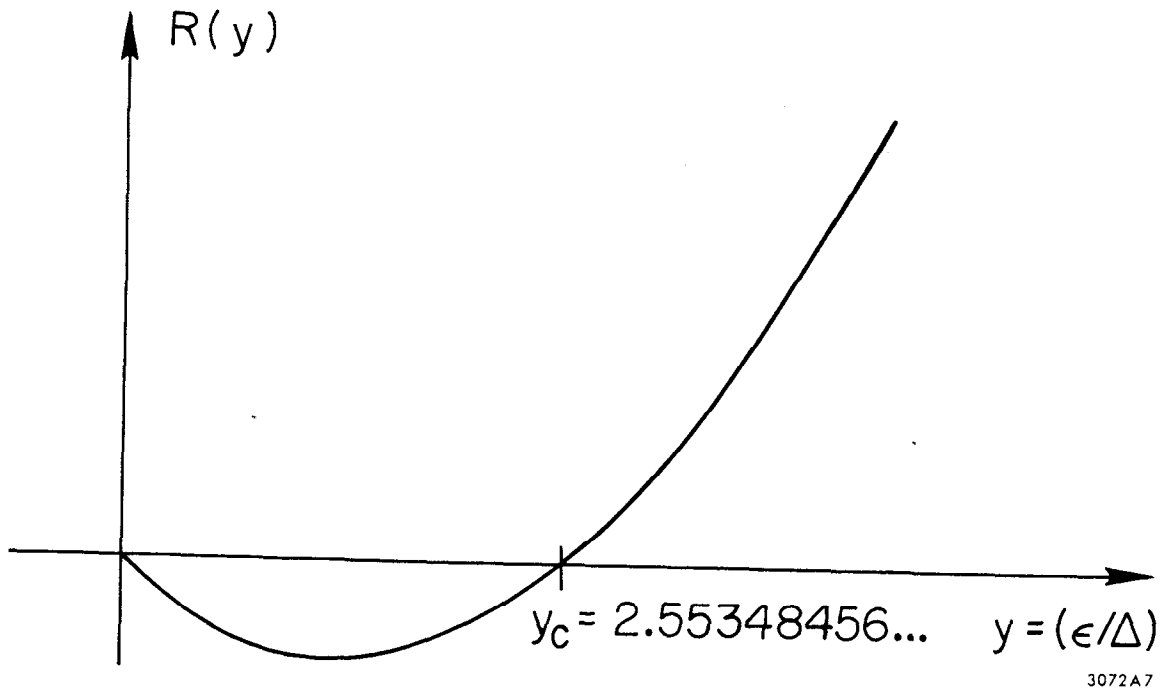


Fig. 7

namely, $\epsilon/\Delta=0$, $\epsilon/\Delta=\infty$ and $\epsilon/\Delta=2.55348456\ldots$. Actually, $F(y)=0$ only requires that the ratio (ϵ/Δ) is unchanged by the iteration and so if $\epsilon_{n+1}=\lambda\epsilon_n$ and $\Delta_{n+1}=\lambda\Delta_n$, i.e., if the Hamiltonian reproduces itself up to an overall scale factor, we will also have a zero of $F(y)$. As we have seen $y=0$ and $y=\infty$ are true fixed points of (8.26)-(8.27), and more careful analysis shows that $y=2.55\ldots$ is a point at which the Hamiltonian is reproduced up to a scale factor $\lambda(y_c)$, which is another critical constant of the theory.

There is additional qualitative information which can be extracted from $F(y)$. In particular, $F(y)=0$ implies that the ratio (ϵ/Δ) decreases for that iteration and so the new (ϵ'/Δ') or y' lies to the left of the y we started with. Since, as shown in Fig. 7, $F(y)$ is negative for $y<y_c$ we see that if we start at any point in this range successive iterations of our truncation procedure will drive us to form for the effective Hamiltonian which we have studied in perturbation theory. On the other hand, since $F(y)>0$ for $y>y_c$ we conclude that successive iterations drive us to $y=\infty$ which implies $\Delta/\epsilon>>1$ which is another form of the Hamiltonian which we have studied in perturbation theory. Hence we conclude that all theories for which $y<y_c$ are theories with a degenerate groundstate, etc.; whereas, all theories for which $y>y_c$ are theories with a unique groundstate. Clearly y_c is the point at which the nature of the groundstate changes, and so y_c is the <u>critical point</u> of this theory. Note, that the result $y_c=2.55348\ldots$ is obtained from our simple procedure is not too bad since $y_c^{(exact)}=2$ is the exact answer. The fixed points $y=0$ and $y=\infty$ are usually referred to the stable fixed points of the <u>renormalization group transformation</u> and the fixed point at $y=y_c$ is called an <u>unstable fixed point</u>. Notice, the fact that at $y=y_c$ the Hamiltonian continues to reproduce itself up to a scale factor says that at a critical point the physics going on at different length scales is essentially the same.

Certainly, there is more information to be gleaned from the recursion relations in (8.26)-(8.27) than one obtains from plotting F(y) alone. In particular, these relations allow us to compute ϵ_n and Δ_n separately. If one examines the result of iterating (8.24)-(8.27) one finds that for initial values, $\epsilon_0/\Delta_0 < y_c$, the successive renormalization group transformations lead to $\lim_{n\to\infty} \epsilon_n = 0$ and $\lim_{n\to\infty} \Delta_n = \Delta_\infty(\epsilon_0/\Delta_0) \neq 0$, whereas for $(\epsilon_0/\Delta_0) > y_c$, $\lim_{n\to\infty} \epsilon_n = \epsilon_\infty(\epsilon_0/\Delta_0) \neq 0$ and $\lim_{n\to\infty} \Delta_n = 0$. The next question is how to use this information to calculate the order parameter $<\sigma_x(j)>$.

The answer to this question is easily divined once one realizes that as we carry out a succession, say N, renormalization group transformations we are effectively finding, by direct construction, those two states within a block of size $(2)^N$-sites which have the lowest energy. Now, for $y < y_c$, in the limit $N\to\infty$ we have shown $\epsilon_N \to 0$ and $\Delta_N \to \Delta$, and so the two states we construct in this limit are degenerate. Since we construct these states by keeping one state which is even under U and one state which is odd, we need only refer to our discussion of exact results, in particular Eq. (7.5), to see that the quantity we want to compute is

$$\lim_{N\to\infty} < \psi^N_{even} | \sigma_x(j) | \psi^N_{odd} > \stackrel{def}{\equiv} <\sigma_x(j)>$$

Going back to the discussion leading to (8.20) we see that because $\left[\sigma_x(j)\right]^{tr}$ is a purely off-diagonal 2×2-matrix calculating $< \psi^{(1)}_{even} | \sigma_x(j) | \psi^{(1)}_{odd} >$ is the same as computing

$$\left[\sigma_x(j)\right]^{1st\text{-}trunc.} = \frac{(1+a_1)}{\sqrt{2(1+a_1^2)}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sqrt{\frac{\Delta_1}{\Delta_0}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (8.31)$$

Hence in sucessive transformations we find that

$$\left[\sigma_x(j)\right]^{Nth\text{-}trunc.} = \prod_{p=1}^{N} \sqrt{\frac{\Delta_p}{\Delta_{p-1}}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sqrt{\frac{\Delta_n}{\Delta_0}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (8.32)$$

This argument therefore tells us that if $\Delta_\infty \equiv \lim_{N\to\infty} \Delta_n$ then the order parameter for a theory such that $y_0 = \epsilon_0/\Delta_0 < y_c$, is given by

$$\tilde{f}\left(\frac{\epsilon_0}{\Delta_0}\right) = \sqrt{\frac{\Delta_\infty}{\Delta_0}} \quad . \tag{8.33}$$

Explicit numerical iteration of (8.26)-(8.27) shows that a very good fit to

$$\tilde{f}\left(\frac{\epsilon_0}{\Delta_0}\right) = \tilde{f}(y_0) = \left(1 - \left(\frac{y}{y_c}\right)^2\right)^{.39} \tag{8.34}$$

with $y_c \cong 2.55348456\ldots$. Considering that we have been so crude in our calculations the agreement of this with the exact answer

$$\tilde{f}_{\text{exact}}(y) = \left(1 - \left(\frac{y}{2}\right)^2\right)^{.125} \tag{8.35}$$

is not too bad.

Nevertheless, the advertised virtue of this method is that one gets a good nuts and bolts feeling for what is going on and so one can see how to do better. In the next section we will show how a simple modification of our approach does produce a significant improvement in results.

D. Nonperturbative Analysis: A More Sophisticated Algorithm. Now that we have seen how a variational procedure based upon successive truncation of a space of "linear trial wave functions" leads one to a renormalization group transformation, let us go on to analyze how one can modify our truncation algorithm to produce better results. In this case "better" will mean a better calculation of the groundstate energy and a better calculation of the general behavior of the order parameter $\langle \sigma_x(j) \rangle$. Actually, we will find that we worsen the prediction of the critical temperature in the process. We are pretty sure that we know why this happens and I will try to explain the reason in more detail at a later point in our discussion. In any event, the improvement in critical exponent by a factor of 2 and the improvement in the qualitative behavior of the groundstate energy

density with respect, y, is even more dramatic. We will see this when we plot $-\partial^2 \mathcal{E}_0/\partial y^2$ and observe that the calculation I am about to describe can see that $\mathcal{E}_0(y)$ possesses a singularity in its second derivative at the critical point—a result which cannot be obtained from a more naive calculation.

As interesting as these results are, however, I do not wish to focus your attention too closely on them. The key point I wish to make in this discussion is that the variational technique based upon the use of linear trial wave functions can be systematically improved upon and that the procedure for implementing this methodically is not much more difficult than the original naive procedure.

To begin, let us note that there are in fact two distinctly different pieces to our algorithm, both susceptible to change and improvement. First we committed ourselves to grouping lattice sites into boxes containing two sites each. We then constructed "box states" and thinned out our complete set by throwing away two out of the four possible states per box. Clearly we could easily generalize this approach both by grouping things into bigger boxes and by keeping more states. However, for now let us assume that this part of our procedure will be left unmodified so that successive truncations of our space of trial wave functions shall always lead to an effective Hamiltonian of the same form as the original one. Having committed ourselves to this procedure, let us turn to the question of improving upon our algorithm for throwing away states. Obviously, although our algorithm for throwing away states based upon a diagonalization of the appropriate two site Hamiltonian is intuitively appealing, it is by no means sacred. Moreover, it certainly neglects important physics. This notion is completely born out if one looks at the argument between the exact answer and the simple method for small ϵ/Δ.

Under these conditions the question becomes "what can we do to correct these defects?" The answer is "adopt a procedure which keeps these small effects." In order to develop such a procedure let us recall that once we noted

the invariance of the theory with respect to the transformation U we were led to classify states according to their U eigenvalues. Hence we divided the "box" states into two classes; e.g., $\{|\uparrow\uparrow>, |\downarrow\downarrow>\}$ and $\{|\downarrow\uparrow>, |\uparrow\downarrow>\}$. We then decided to adopt a procedure for choosing one state from each class and truncated H with respect to this choice. The first question to raise is "Why choose a member from each class? Why not choose two members from the same class?" If we did this we would immediately find that, since $\sigma_x(j)$ cannot link members of the same class, at the first iteration the term Δ_1 would be zero and then H_1 could be exactly diagonalized. It would then be trivial to show this procedure would be constructing a poor class of variational wave functions since the remaining box-box interactions would be rendered inoperative and therefore would not be available for lowering the energy. This argument tells us that we must choose one member of each class, but does not tell us which member to choose.

It is apparent that the most general procedure we could adopt is to decide to form two states

$$|\psi_0> = \cos(\theta)|\downarrow\downarrow> + \sin(\theta)|\uparrow\uparrow>$$

and (8.36)

$$|\psi_1> = \cos(\phi)|\downarrow\uparrow> + \sin(\phi)|\uparrow\downarrow>$$

and compute the new truncated Hamiltonian in terms of the unknown angles θ, ϕ. This would lead to an effective Hamiltonian H_1 of the form

$$H_1 = \sum_p \left[c_1(\epsilon_0, \Delta_0, \theta, \phi) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(p) + \epsilon_1(\epsilon_0, \Delta_0, \theta, \phi) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(p) \right.$$
$$\left. -\Delta_1(\epsilon_0, \Delta_0, \theta, \phi) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(p+1) \right] \quad (8.37)$$

where

$$c_1 = \epsilon_0 - (\epsilon_0 \cos(2\theta) + \Delta_0 \sin(2\theta)) \quad (8.38)$$

$$\epsilon_1 = \epsilon_0 \cos(2\theta) + \Delta_0(\sin(2\theta) - \sin(2\phi)) \quad (8.39)$$

and

$$\Delta_1 = \Delta_0 \cos(\theta - \phi) \sin(\theta + \phi) \tag{8.40}$$

Clearly this procedure could be worked out without deciding upon θ and ϕ in advance. In fact, on general grounds one would expect that a different choice of θ and ϕ to be appropriate for different values of ϵ and Δ. Since each renormalization group transformation produces new values of ϵ_n and Δ_n, we probably should assume that θ and ϕ are arbitrary functions of ϵ and Δ. In this way we make an essentially independent choice of θ and ϕ at each iteration. Once one allows this freedom in the choice of θ and ϕ at each stage of the calculation, the question becomes one of adopting a physical criterion for fixing these two functions. To do this we will make use of a clever observation made by Bob Pearson at FNAL. He suggested we choose $\theta(\epsilon, \Delta)$ and $\phi(\epsilon, \Delta)$ to be functions of ϵ_n/Δ_n) alone, since the overall scale of the Hamiltonian could not matter, and then work out the renormalization group transformations in terms of $\theta(\epsilon/\Delta)$ and $\phi(\epsilon/\Delta)$. If we adopt this prescription we can write the result of the n+1st renormalization group transformation as

$$H_{n+1} = \sum_p \left[d_{n+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(p) + \epsilon_{n+1}(\epsilon_n, \Delta_n) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(p) \right.$$

$$\left. - \Delta_{n+1}(\epsilon_n, \Delta_n) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(p+1) \right] \quad ; \tag{8.41}$$

where

$$c_{n+1} = \epsilon_n - (\epsilon_n \cos(2\theta(\epsilon_n/\Delta_n)) + \Delta_n \sin(2\theta(\epsilon_n/\Delta_n)))$$

and (8.42)

$$d_{n+1} = 2d_n + c_{n+1} \quad ,$$

$$\epsilon_{n+1} = \epsilon_n \cos(2\theta(\epsilon_n/\Delta_n)) + \Delta_n(\sin(2\theta(\epsilon_n/\Delta_n)) - \sin(2\phi(\epsilon_n/\Delta_n))) \tag{8.43}$$

and

$$\Delta_{n+1} = \Delta_n \cos(\theta(\epsilon_n/\Delta_n) - \phi(\epsilon_n/\Delta_n)) \sin(\theta(\epsilon_n/\Delta_n) + \phi(\epsilon_n/\Delta_n)) \quad . \tag{8.44}$$

To fix the unknown functions $\theta(\epsilon/\Delta)$ and $\phi(\epsilon/\Delta)$ we decide at some stage, N, to compute the groundstate energy of H_N, and require that the arbitrary functions θ and ϕ be chosen to minimize this quantity. In order to give a more explicit formulation of this idea let us examine Eqs. (8.42)-(8.44). We can easily convince ourselves that ϵ_n and Δ_n stay bounded in magnitude at least under a wide class of choices of $\theta(\epsilon/\Delta)$ and $\phi(\epsilon/\Delta)$—but that the term proportional to d_{n+1} increases by a power of 2 for each iteration. Hence, for N sufficiently large this term swamps the remainder of the Hamiltonian. This divergence in the coefficient of the unit matrix is just the renormalization group transformation's way of telling us that translation invariance of the groundstate implies that its energy is proportional to the volume of the lattice times a finite number, $\mathscr{E}_0$. This number, $\mathscr{E}_0$, is called the groundstate energy density. Since ϵ_n and Δ_n do not grow with this factor of 2^N, and since each point of the effective Nth-lattice is 2^N-points in the original lattice, it is not hard to convince yourself that the energy density is the limit

$$\mathscr{E}_0 = \lim_{N\to\infty} \left(\frac{1}{2^N} d_N\right) \tag{8.45}$$

or, from (8.42),

$$\mathscr{E}_0 = \lim_{N\to\infty} \sum_{n=0}^{N} \left(\frac{c_n}{2^N}\right) \quad . \tag{8.46}$$

A precise statement of our full procedure is to first parametrize the two unknown functions $\theta(\epsilon/\Delta)$ and $\phi(\epsilon/\Delta)$ in terms of a small number of unknown parameters. Next, carry out the recursion implied by (8.45) and (8.46)—say one hundred times—in order to compute the groundstate energy density, to one-part in $(2)^{100}$. This gives us $\mathscr{E}_0^{(N=100)}$ as a function of the unknown parameters, so we finally vary over these parameters to minimize $\mathscr{E}_0^{(N=100)}$. By this procedure the "renormalization group" algorithm, as well as all quantities of interest, are constructed by a variational procedure. Hence, the only arbitrary

input is the choice of the way in which we group sites and the general choice of the number of states we decide to keep.

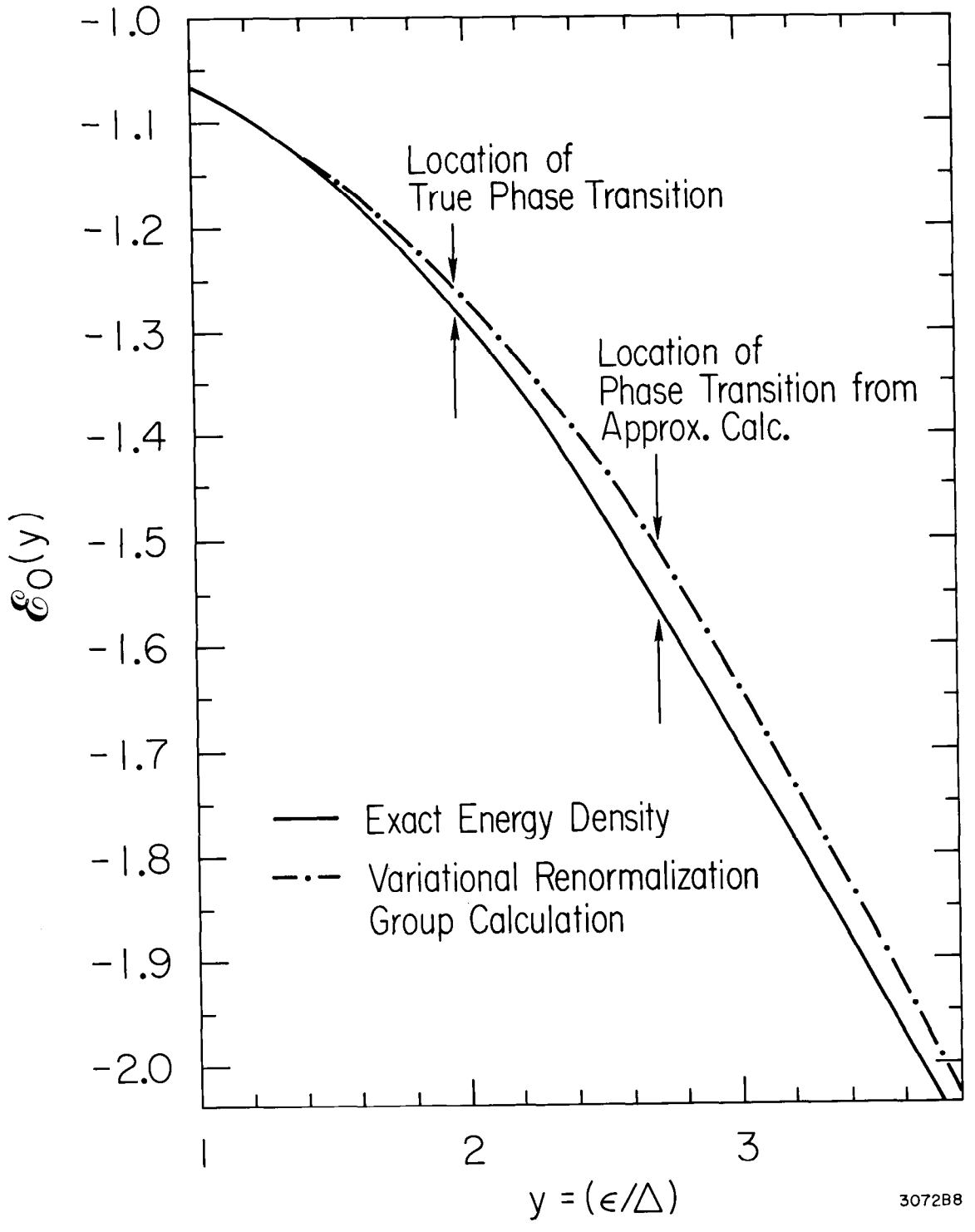
E. A Discussion of Results. It will surprise no one that the procedure just outlined must be carried out numerically, but this is done quite easily and Figs. 8-10 summarize the results of a simple two parameter calculation done for the starting Hamiltonian

$$H_0 = \sum_j \left[-\frac{\epsilon_0}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(j) + \epsilon_0 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(j) - \Delta_0 \sigma_x(j)\, \sigma_x(j+1) \right] \tag{8.47}$$

Because of a symmetry argument based upon invariance of H_0 under any reflection taking the lattice into itself we chose to fix $\phi(\epsilon/\Delta)$ to be a constant equal to $\pi/4$. Therefore, only $\theta(\epsilon/\Delta)$ had to be determined. The choice of $\theta(\epsilon/\Delta)$ was constrained by the fact that for $\epsilon=0$ and any Δ we know that choosing $\theta(\epsilon/\Delta=0)=\pi/4$ leads to a construction of the exact wave function; and the same is true, if for $\Delta=0$, we make the choice $\theta(\epsilon/\Delta=\infty)=0$. These considerations led us to parametrize $\theta(\epsilon/\Delta)$ as:

$$\theta(\epsilon/\Delta) = \frac{\pi}{4}\left[1 - \frac{(\tanh(\rho(\epsilon/\Delta) - u) - \tanh(-u))}{(1 - \tanh(-u))}\right] \tag{8.48}$$

since this function is $\pi/4$ at $\epsilon/\Delta=0$ and vanishes as $\epsilon/\Delta \to \infty$. The parameters ρ and u were the free parameters with respect to which we minimized the groundstate energy density, (8.46). In Fig. 8 we show a comparison of our calculation of the groundstate energy density to the exact answer. Values of ϵ smaller than 1 and greater than 3 are suppressed because for these regions agreement is much better than one-part in 10^3. Examination of these curves shows that our worst disagreement with the exact answer is on the order of 3%. This is a significant improvement over the naive calculation. In Fig. 9 we compare our computation of $f(\epsilon/\Delta)$ with the exact answer. As shown the critical value of ϵ/Δ is now at $(\epsilon/\Delta) \cong 2.75$ which is, as we noted at the outset, somewhat worse than the naive calculation. Interestingly the form of $f(\epsilon/\Delta)$ can be fit to a power law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$\mathscr{E}_0(y)$
Location of
True Phase Transition
Location of
Phase Transition from
Approx. Calc.
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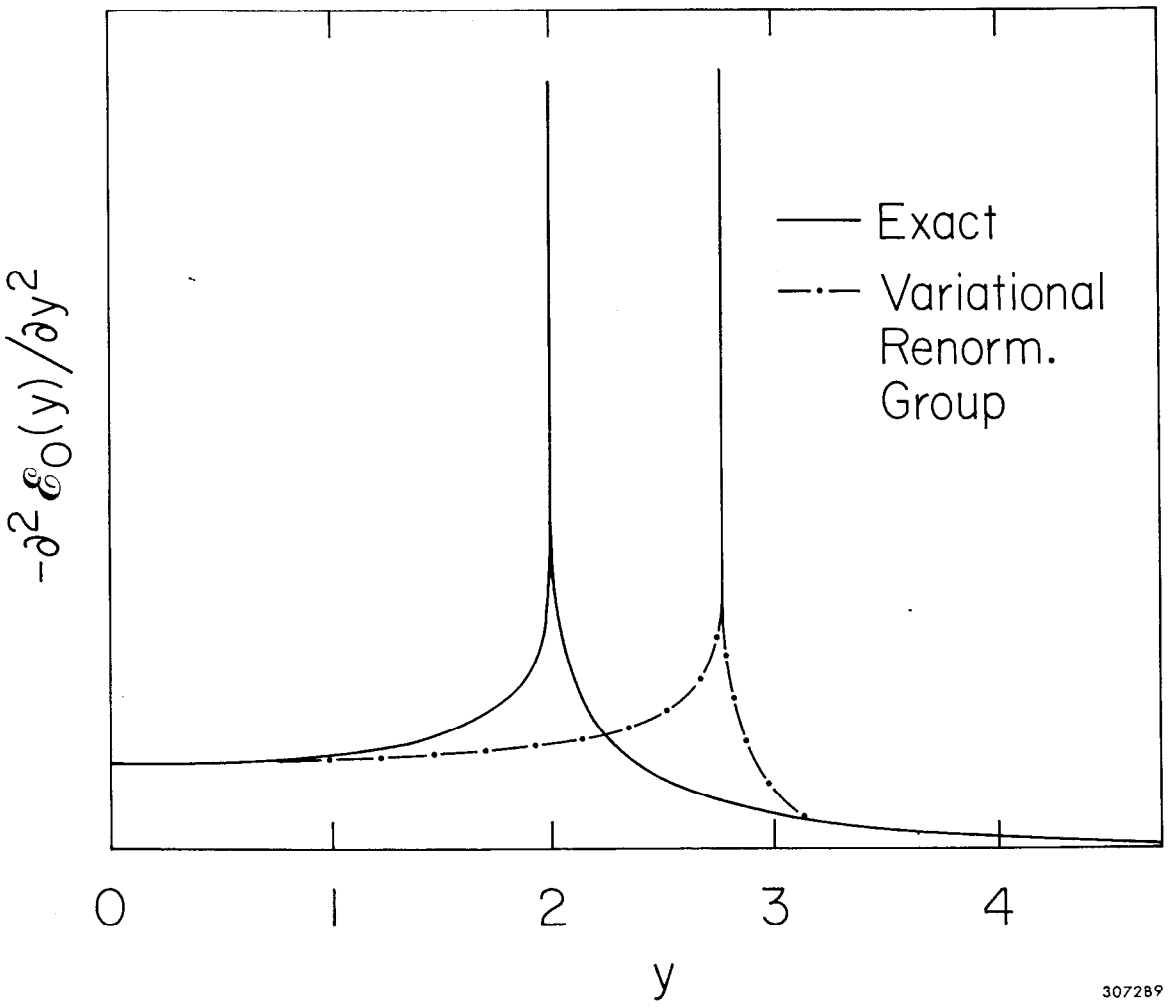
Fig. 8


Exact
Variational
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0
1
2
3
4
y
307289

Fig. 9

of the form

$$\tilde{f}(\epsilon/\Delta) = \left(1 - \left(\frac{y}{y_c}\right)^2\right)^{.21}$$

Note that the critical exponent of 0.21 is a significant improvement over the naive approximation answer of 0.4. I want to emphasize that these calculations are the simplest ones I could think of doing to demonstrate the method and can clearly be improved upon by improving the choice of $\theta(\epsilon/\Delta)$. Surprisingly, even five parameter variation forms for $\theta(\epsilon/\Delta)$ doesn't improve the calculation of the critical point much, although the critical exponent becomes smaller ≈ 0.17. There seems to be a good physical reason for this which is related to the fact that because we work on two sites the function $\phi(\epsilon/\Delta)$ becomes trivial for symmetry reasons, and so the eigenstates of U corresponding to +1 and -1 are not being treated equally. Intuitively, what one would guess is that in order to get the best possible groundstate energy density below the critical point the renormalization group calculation wants to make (ϵ/Δ) small as rapidly as possible, so that Δ approaches its fixed form as quickly as possible. If both $\theta(\epsilon/\Delta)$ and $\phi(\epsilon/\Delta)$ could both vary this could be accomplished while at the same time allowing each individual level to attain a lower energy. Since, however, $\phi(\epsilon/\Delta)$ is forced to be $\pi/4$ as a consequence of the aforementioned symmetry property of the two site truncation process, $\theta(\epsilon/\Delta)$ has to behave somewhat incorrectly in order to produce the best groundstate energy. This artificial situation doesn't arise if one works with blocks of three sites at a time, a procedure which has many other theoretical arguments to recommend it. A computation of this sort based on a 3-site block is quite straightforward, and will be carried out in the near future to see if our understanding of what is really happening is as good as I am arguing it is.

Finally let us discuss the curves shown in Fig. 10. As shown the ground-state energy density has the property that ∂^2 (groundstate energy density/$\partial\epsilon^2$ exhibits a singularity at $y_c=2$, the true critical point. This is really quite a subtle property of the theory and as you can see our variational results also exhibit a singularity at the calculated critical point. Moreover there seems to be, given the crudeness of our procedure, remarkable similarity in the general shape of the curves. It is worth noting that the naive renormalization group procedure discussed first misses this singularity completely.

IX. "SMOOTH-TALK", OR "WHERE DO WE GO FROM HERE?"

The preceding discussion completes our analysis of the "toy model." I would like to spend the remainder of this talk summarizing what I hoped you learned from this discussion and relating it to other already published work which we have done at SLAC.[3] In a sense, these papers on strong coupling lattice theories, should have followed the discussion I have just presented—and so you might think of this talk as paper 0 in our series.

As I indicated at the outset my real interest in the "toy model" I analyzed by the variational—or "renormalization group"—methods was not in the physics of the model but rather in the methods themselves. Hopefully, from this example you have been immediately led to generalize the technique to any field theory; nevertheless, at the risk of overstating the obvious, allow me to list those points which I feel summarize the important features of our discussion.

First we argued that to begin analyzing nonperturbative effects in any field theory one must formulate the theory in a way which does not implicitly make use of the weak coupling expansion. We contended—without giving the proof—that one way to do this was to formulate the theory with a volume and momentum cutoff, a procedure which is completely equivalent to formulating the theory on a lattice.

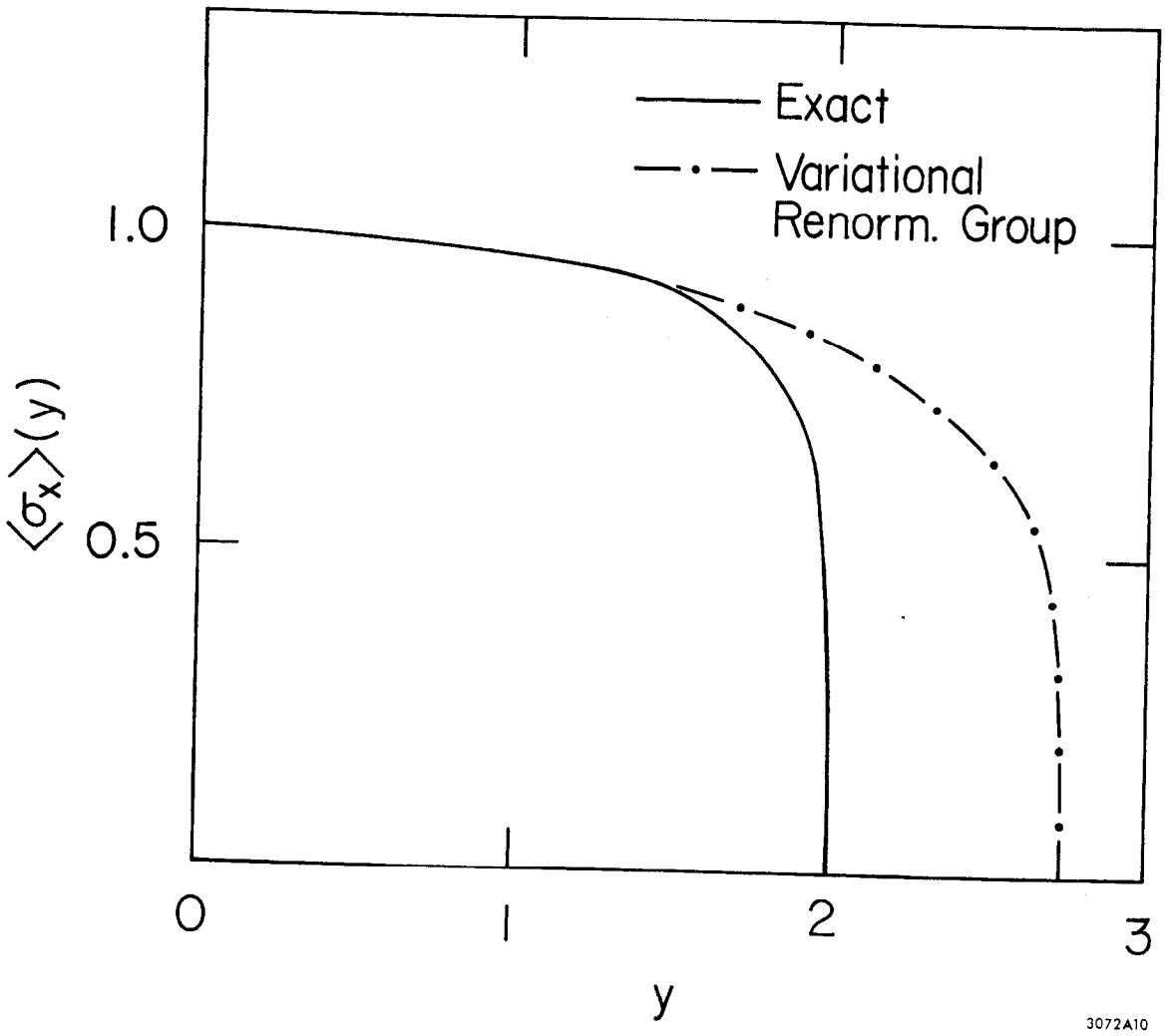
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Fig. 10

Second, we argued one could study the resulting lattice problem by variational methods. We then showed that a variational procedure which could be adopted was one based upon linear trial wave functions, and we showed that the problem of finding a good basis for constructing such trial-wave functions was equivalent to a renormalization group calculation in which the renormalization group itself had to be determined by means of a variational procedure. In effect, the arbitrary input to a calculation was the way in which one grouped single sites into blocks of sites and the assumption of how many states one keeps at each truncation.

Having derived the specific form of the renormalization group transformations for the set of assumptions made in the second step, the next question is to analyze what happens to the form of the truncated or effective Hamiltonian as we thin out our family of linear trial wave functions. As we saw in the specific example, the key first point to understand in these transformations is what happens to the strength of the gradient (site-site recoupling) terms relative to the single-site terms in the Hamiltonian.

Quite generally, it proves useful to study the function which gives the change in the ratio of the single-site couplings to the gradient after a finite number (one or a thousand) interactions, since, as we saw in our "toy model" one can tell a great deal about qualitative features of a theory from this information alone. Suppose, for the sake of argument, we assume that there is only one single-site coupling constant in a theory. Then, defining 'y' to be the strength of the single-site coupling to the gradient term, we can plot the general form of the function R(y) = (change of y in finite number of iterations). A few examples of imaginable R(y)'s are given in Figs. (11a)-(11c) so that we may make a few general remarks about what one would conclude from each picture.

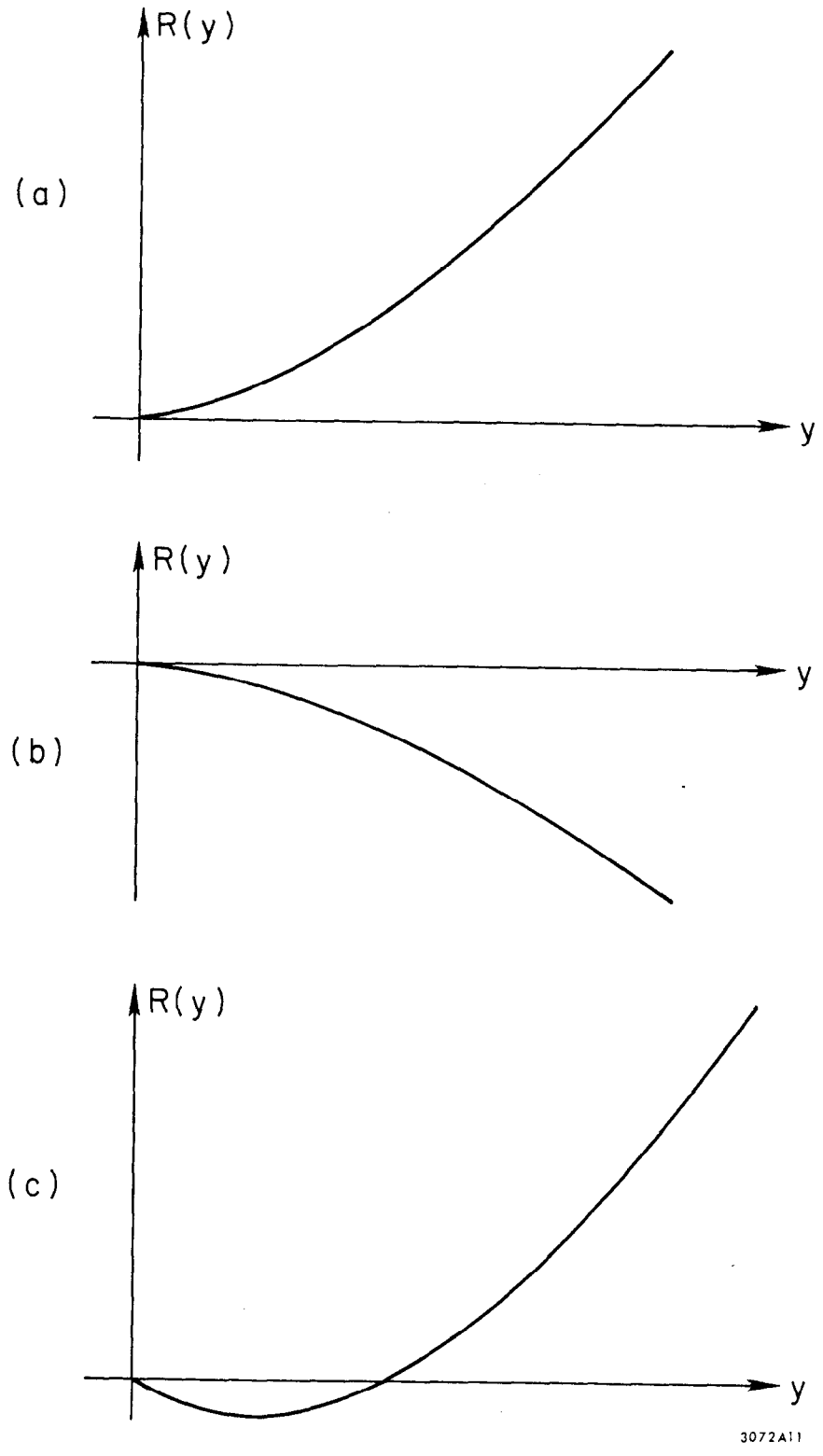

Fig. 11

In Fig. 11a we see that $R(y) \geq 0$ for all values of $0 \leq y \leq \infty$. If a theory has this sort of R (y) we can conclude two things. First, the points y=0 and y=∞ are the only fixed points of the theory. The Hamiltonian at y=0, i.e., zero coupling constant, is a "free field theory," and can be presumably solved exactly, the y=∞ Hamiltonian is solved by solving the single-site Schrœdinger problem. Second, we observe that if we start at some finite value of y successive iterations drive us to larger value of y (i.e., $R(y) > 0$) hence, eventually our problem can be studied by treating the gradient terms as a perturbation on the single-site terms. Hence, in any theory for which $R(y) > 0$ we can conclude that the low energy (or long-wavelength) physics is described by an effectively strong-coupling constant Hamiltonian. One other thing which follows from this discussion is that the mass gap in such a theory will be given by calculating the gap between the first two eigenstates of the effective single-site Schrœdinger problem, and will, therefore, be a function of the effective single-site coupling "g_∞". In general, since the scale of H_0 is set by the cutoff Λ, this means that the lowest mass gap in the theory will be $\approx \Lambda g_\infty$ and so to have practical Lorentz invariance—which we will take to mean that the scale of physical masses should be negligible with respect to the maximum momentum—we are only interested in theories for which $g_\infty \ll 1$, or in other words $g_\infty \Lambda \approx 1$ GeV (for the sake of argument). Since the Hamiltonian at a fixed point reproduces itself up to a scale factor, if one finds a critical point (or fixed point) for which this scale factor, ρ, is less than unity, then by iterating the overall scale of H_N as $N \to \infty$ is $\Lambda \rho^N$. This tells us that the critical theory has no mass gap. This also implies that choosing y close enough to the critical point guarantees that for the first M iterations, H_M, will stay fixed up to a factor of ρ^M and then it will start changing rapidly to its limiting form. Therefore, heuristically we argue that in order to get a practically relativistic theory one needs to start from a value near this fixed point so as to get a scale for $H^M = \rho^M \Lambda \approx 1$ GeV.

It follows from this discussion that for a case like Fig. 11a one need only compute the parameter ρ for the case y=0 to decide whether or not a "practically relativistic" limit exists for the theory in question $\mathcal{H}\rho < 1$ then one can take the cutoff Λ arbitrarily large keeping the mass of the lowest states fixed by taking the original "bare" coupling towards zero as a function of Λ. This would therefore be an example of a theory whose short distance behavior is "free" but whose long wavelength behavior is not.

If we next look at R(y) for Fig. 11b we come up with the opposite conclusion. If $R(y) < 0$ then each successive set of N-iterations will make it smaller. Hence the large wavelength or low energy physics of this theory is given by weak coupling perturbation theory, whereas the single-site or short distance behavior is governed by a strong coupling constant.

Figure 11c tells us that the two different cases can occur depending upon the starting value for y, i.e., whether $y_0 < y_c$ or $y > y_c$. This is of course the form of R(y) calculated for our toy model and so I can refer you back to our discussion of the way in which the exact case leads to an effectively relativistic theory in order to clarify the kind of information you can get from this picture.

Finally, we have reached the point in our discussion where we can understand why the material in Ref. 3 is relevant. As we have seen for cases (11b) and (11c), after a large number of iterations, it becomes necessary to know the solution of the strong coupling lattice theory. The discussion we give (we, meaning Drell, Yankielowicz, and me) for the case of strong coupling fermion theories on a lattice was to show that if a renormalization group calculation of self-coupled fermion, or color-gauge theories leads us to an R(y) of the form shown in Figs. 11b and 11c, then our analysis of these models becomes important. For the case of color gauge theories in particular—where the folklore based upon continuum perturbation theory and asymptotic freedom leads us to conjecture that

Fig. 11b is the relevant form of R(y)—the fact that the infinite or very large coupling analysis of the color gauge theory leads to a "zeroth order spectrum" of physical particles makes the need to study these theories more carefully even greater. To summarize, although much remains to be done along the lines presented in this talk, the analysis of Ref. 3 tells us that if we are lucky the large distance behavior of lattice color gauge theories of strong interactions could be very, very amusing.

APPENDIX A

A. On the Equivalence of Cutoff and Free Field Lattice Theories.

Case I: Boson theory: As an example of the way in which a continuum field theory can be converted to an equivalent lattice-field theory let us study the two simple cases of (i) a free scalar field, and (ii) a free fermion field. In order to maintain a simple notation I shall work things out for the particular case of one space-one time dimensional theories, the generalization of this trick to higher dimensions is discussed in Ref. 3.

To begin let us consider the free massive scalar field theory defined by the Lagrangian density

$$\mathscr{L}(\phi, \partial_\mu \phi) = \frac{1}{2}(\partial_\mu \phi(x))^2 - \frac{\mu^2}{2}\phi^2(x) \tag{A.1}$$

and the canonical equal time commutation relations

$$[\Pi(x,t),\ \phi(y,t)] = -i\delta(x-y)$$

where

$$\Pi(x,t) \stackrel{\text{def}}{\equiv} \frac{\delta \mathscr{L}}{\delta(\partial_0 \phi)} = \partial_0 \phi(x,t) \quad . \tag{A.2}$$

In order to discuss the quantum mechanical theory we proceed canonically to construct the Hamiltonian

$$H = \int_{-\infty}^{+\infty} dx \left\{ \Pi(x)\, \partial_0 \phi(x) - \mathscr{L} \right\}$$
$$= \int_{-\infty}^{+\infty} dx \left\{ \frac{\Pi^2(x)}{2} + \frac{(\nabla\phi(x))^2}{2} + \frac{\mu^2}{2}\phi^2(x) \right\} \qquad (A.3)$$

maintaining the commutation relations specified in Eq. (A.2).

As promised our first step will be to assume that the theory under consideration is defined <u>in a finite volume</u>, i.e., a box of length L. Hence, we think of the functions $\phi(x)$ and $\Pi(x)$ as defined on the interval $-L/2 \leq x \leq L/2$ and so we can expand these functions with respect to the complete set of functions, $e^{ik_n x}$, where

$$k_n \stackrel{\text{def}}{=} \left(\frac{2\pi}{L}\right) n \qquad (A.4)$$

for arbitrary integer n. To be specific we write

$$\phi(x) \equiv \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{+\infty} e^{ik_n x}\, \phi(k_n); \qquad \phi(k_n) = \frac{1}{\sqrt{L}} \int_{-L/2}^{+L/2} dx\, e^{-ik_n x}\, \phi(x) \qquad (A.5)$$

and

$$\Pi(x) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{+\infty} e^{+ik_n x}\, \Pi(k_n); \qquad \Pi(k_n) = \frac{1}{\sqrt{L}} \int_{-L/2}^{+L/2} dx\, e^{-ik_n x}\, \Pi(x) \qquad (A.6)$$

so that

$$H = \sum_{n=-\infty}^{+\infty} \left[\frac{1}{2}\Pi(k_n)\Pi(-k_n) + \frac{\left(k_n^2+\mu^2\right)}{2}\phi(k_n)\phi(-k_n) \right] \qquad (A.7)$$

It follows from Eqs. (A.5) and (A.6) that

$$\left[\Pi(k_n), \phi(-k_n)\right] = -i\delta_{k_n, k_m} \quad . \qquad (A.8)$$

Although we will not use this fact later it is worth noting that we can transform H to the form of a function of 'k' times a number operator by defining $\omega_k = \sqrt{k^2+\mu^2}$ and writing

$$\phi(k) = \frac{1}{\sqrt{2\omega_k}} \left(a_{-k} + a_k^+\right) \tag{A.9}$$

and

$$i\Pi(k) = \sqrt{\frac{\omega_k}{2}} \left(a_{-k} - a_k^+\right) \tag{A.10}$$

so that

$$\left[a_{k_n}^+, a_{k_m}\right] = \delta_{k_n, k_m} \tag{A.11}$$

so that H becomes

$$H = \sum_n \sqrt{k_n^2+\mu^2} \left(a_k^+ a_k + \frac{1}{2}\right) \tag{A.12}$$

Having collected the formulae needed to discuss a theory in finite volume, we now impose a maximum momentum cutoff by assuming $-k_{max} \leq k_n \leq k_{max}$. The easiest way to define k_{max} is to let $L = (2N+1)a$ where "a" has the dimensions of length and then rewrite Eq. (A.4) as

$$k_n = \frac{2\pi}{(2N+1)a} n \qquad \text{for} \quad -N \leq n \leq N \tag{A.13}$$

In this way we can define the cutoff mass $\Lambda = 1/a$ and for $N \to \infty$, $k_{max} \to \pi\Lambda$. Having adopted these definitions it is a trivial matter to show that the cutoff version of the Hamiltonian defined in Eq. (A.7) is unitarily equivalent to a lattice Hamiltonian. To do this one need only define

$$\frac{1}{\Lambda^{1/2}} \phi(j) = \sum_{n=-N}^{+N} \frac{e^{-i\left(\frac{k_n}{\Lambda}\right) j}}{\sqrt{2N+1}} \phi(k_n)$$

and

$$\Lambda^{1/2}\Pi(j) = \sum_{n=-N}^{+N} \frac{e^{-i\left(\frac{k_n}{\Lambda}\right)j}}{\sqrt{2N+1}} \Pi(k_n) \qquad (A.14)$$

where $-N \leq j \leq N$, and $\phi(j)$ and $\Pi(j)$ are dimensionless.

It is a simple exercise to show that

$$[\Pi(j), \phi(j')] = -i\delta_{jj'},$$

so this transformation preserves the canonical commutation relations, and that

$$H = \Lambda\left[\sum_j \frac{1}{2}\left(\Pi(j)^2 + \left(\frac{\mu^2}{\Lambda^2}\right)\phi(j)^2\right) + \sum_{j_1 j_2} \frac{D(j_1-j_2)}{2}\phi(j_1)\phi(j_2)\right] \qquad (A.15)$$

where

$$D(j_1-j_2) = \frac{1}{(2N+1)} \sum_n \left(\frac{k_n}{\Lambda}\right)^2 e^{i\frac{k_n}{\Lambda}(j_1-j_2)} \quad .$$

Since Eq. (A.15) defines a theory of a scalar field defined on (2N+1) lattice sites we have completed our task of establishing the desired correspondence. Note, the amusing feature of this formulation of the lattice theory is that in the limit $(2N+1) \to \infty$ the only way in which this theory fails to be a relativistic theory is that the spectrum $\sqrt{k^2+\mu^2}$ cuts off a maximum momentum, k_{max}. This completes our discussion of the scalar free field.

APPENDIX B

Free Fermion Case. The continuum finite volume fermionic theory starts from a Hamiltonian

$$H = \int_{-L/2}^{+L/2} dx\ \psi^+(x)\left\{\frac{1}{i}\alpha\nabla + \beta m\right\}\psi(x) \qquad (B.1)$$

and canonical equal time anticommutation relations

$$\{\psi^+(x),\ \psi(y)\} = \delta(x-y) \tag{B.2}$$

Proceeding as in the case of the free field we define

$$\psi(x) = \sum_n \frac{e^{ik_n x}}{\sqrt{L}}\,\psi(k_n)\ ; \qquad \psi(k_n) = \frac{1}{\sqrt{L}}\int_{-L/2}^{+L/2} dx\, e^{-ik_n x}\,\psi(x)$$

and

$$\psi^+(x) = \psi(x)^+ = \sum_n \frac{e^{-ik_n x}}{\sqrt{L}}\,\psi^+(k_n) \tag{B.3}$$

where k_n is given by Eq. (A.4). It is then a simple exercise to show that

$$H = \sum_n \psi^+(k)\left\{\alpha\cdot k + \beta m\right\}\psi(k) \tag{B.4}$$

Clearly this can be solved in the usual way by writing

$$\psi(k) = u(k)\, b(k) + v(-k)\, d_k^+ \tag{B.5}$$

where u(k) and v(k) satisfy

$$\begin{aligned}(\alpha k + \beta m)\, u(k) &= E(k)\, u(k)\quad ,\\ (\alpha k + \beta m)\, v(-k) &= -E(k)\, v(-k)\end{aligned} \tag{B.6}$$

with $E(k) = \sqrt{k^2+m^2}$.

Having established the finite volume formulation we now established a k_{max} cutoff as in the boson case and find that the free cutoff continuum model is unitarily equivalent to a lattice Hamiltonian

$$H = \Lambda\left[\sum_{j_1 j_2}\left(-\frac{1}{i}\,\delta'(j_1-j_2)\right)\psi^+(j_1)\,\alpha\psi(j_2)\right.$$

$$\left. + m\sum_j \psi^+(j)\,\beta\psi(j)\right] \tag{B.7}$$

where

$$\psi(j) = \frac{1}{\sqrt{2N+1}} \sum_n e^{-i\left(\frac{k_n}{\Lambda}\right)j} \psi(k)$$

and[8]

$$-\delta'(j_1-j_2) \stackrel{\text{def}}{\equiv} \frac{1}{(2N+1)} \sum \left[i\left(\frac{k_n}{\Lambda}\right) e^{i\left(\frac{k_n}{\Lambda}\right)(j_1-j_2)} \right].$$

This completes our analysis of the two free fields of interest. Obviously the same procedure leads to an equivalent lattice formulation of any interacting field theory (except for those involving gauge fields where some additional questions arise which will not be addressed in these lectures). Deriving the correct form of the Hamiltonian which is truly equivalent to say cutoff continuum ϕ^4 theory is left as an exercise to the student.

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6. Exact solutions to the toy model we discuss appear in recent publications of B. Stoeckly and D. J. Scalapino, Phys. Rev. B 11, 205 (1975); D. J. Scalapino and B. Stoeckly, UCSB preprint (May 1976) and an earlier analysis of this model is P. Pfeuty, Ann. Phys. (N.Y.) 57, 79 (1970).

7. R. B. Pearson (private communication, to be published).

8. For the afficiandoes it is worth observing that with the definition of $\nabla\psi$ given in Eq. (B.6) (i.e., $(\nabla\psi)(j) = -\sum\delta'(j-\ell)\psi(\ell)$) the free field spectrum of a fermionic theory is exactly that of the continuum theory. Hence one encounters no doubling of states and one still preserves the full chiral invariance of the massive theory.