# RENORMALIZATION BY OPERATOR DIFFERENTIATION* 

Martin Wilner<br>Stanford Linear Accelerator Center Stanford University, Stanford, California 94305 and<br>Physics Department<br>Lowell University, Lowell, Massachusetts $01854 \dagger$


#### Abstract

We derive an operator equation with finite coefficients for the renormalized current operator of scalar $A^{3}$ theory, by implementing unitarity, causality, completeness of free fields, and spectral conditions by means of operator derivatives of the scattering operator $S$. The method makes no explicit reference to diagrams, perturbation theory, or the removal of divergences. We prove equivalence with conventional renormalization of this theory.


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## I. INTRODUCTION AND RESULTS

Everybody knows that if you renormalize the neutral scalar field equation

$$
\begin{equation*}
\left(\square_{x}-m_{0}^{2}\right) A_{0}(x)=\frac{1}{2} g_{0}\left(A_{0}(x)^{2}-\left\langle A_{0}(x)^{2}\right\rangle\right) \tag{1.1}
\end{equation*}
$$

according to $\mathrm{A}_{0}=\mathrm{Z}^{1 / 2} \mathrm{~A}, \mathrm{~g}=\mathrm{g}_{0} \mathrm{z}^{3 / 2}$ and $\mathrm{m}^{2}=\mathrm{m}_{0}^{2}+\delta \mathrm{m}^{2}$, you get

$$
\begin{equation*}
K_{x} A(x)=j(x) \tag{1.2}
\end{equation*}
$$

where

$$
\begin{align*}
& K_{x}=\left(D_{x}-m^{2}\right) \\
& j(x)=\frac{1}{2} g\left(A(x)^{2}-\left\langle A(x)^{2}\right\rangle\right)+\int \Lambda_{2}(x-u) A(u) d^{4} u \tag{1.3}
\end{align*}
$$

<> is the vacuum expectation value, and

$$
\begin{equation*}
\Lambda_{2}(\mathrm{x}-\mathrm{u})=\left[-\mathrm{Z} \delta \mathrm{~m}^{2}+(1-\mathrm{Z}) \mathrm{K}_{\mathrm{x}}\right] \delta_{4}(\mathrm{x}-\mathrm{u}) \tag{1.4}
\end{equation*}
$$

If you assume that $A_{0}(x)$ satisfies canonical commutation relations, you get $Z^{-1}=1+I$ and $-Z^{-1} \delta m^{2}=I^{p}$, where

$$
\begin{align*}
& I=\int_{4 \mathrm{~m}^{2}}^{\infty} J(\mathrm{~s}) \mathrm{ds} /\left(\mathrm{s}-\mathrm{m}^{2}\right)^{2}, \\
& I^{\prime}=\int_{4 \mathrm{~m}^{2}}^{\infty} \mathrm{J}(\mathrm{~s}) \mathrm{ds} /\left(\mathrm{s}-\mathrm{m}^{2}\right), \tag{1.5}
\end{align*}
$$

and the spectral weight $J(s)$ is defined by the expectation value with respect to the renormalized free-field vacuum

$$
\begin{equation*}
\langle j(x) j(y)\rangle=-i \int_{4 m^{2}}^{\infty} J(s) d s \Delta_{+}(x-y ; s), \tag{1.6}
\end{equation*}
$$

so that

$$
\begin{equation*}
\Lambda_{2}(\mathrm{x}-\mathrm{u})=\left[\frac{\mathrm{I}}{(1+\mathrm{I})^{2}}+\frac{\mathrm{I}}{1+\mathrm{I}} \mathrm{~K}_{\mathrm{x}}\right] \delta_{4}(\mathrm{x}-\mathrm{u}) \tag{1,7}
\end{equation*}
$$

The integral $I^{\prime}$ diverges in perturbation theory, so the expression (1.3) is no better defined than (1.1), although it does give finite results in perturbation theory.

The final result of this paper is an operator equation with finite coefficients for the renormalized current operator $j(x)$,

$$
\begin{gather*}
\frac{\delta^{2} j(x)}{\delta a(z) \delta a(y)}=g^{\prime} \delta_{4}(x-y)\left[\delta_{4}(x-z)-\int \Delta_{R}(x-u) \frac{\delta j(u)}{\delta a(z)} d^{4} u\right] \\
 \tag{1.8}\\
-i \theta\left(x_{0}-y_{0}\right) \frac{\delta}{\delta a(z)}[j(x), j(y)],
\end{gather*}
$$

where $\mathrm{g}^{\mathrm{I}}=(1+\mathrm{I})^{2} \mathrm{~g}$, and the symbol $\delta / \delta \mathrm{a}(\mathrm{x})$ is the "operator derivative" ${ }^{1-3}$ with respect to the renormalized free field $a(x)$ of mass $m$.

It is not difficult to iterate (1.8) a few times in powers of $\mathrm{g}^{\prime}$ or g , and to verify that to a given order every term on the right side of (1.8) is finite. This program will give the "second operator derivative" as a sum of normal ordered products of $a(x)$, from which it is easy to obtain $j(x)$ to that order, except for the "boundary values", which are given by the vacuum expectation values

$$
\begin{equation*}
\langle\delta \mathrm{j}(\mathrm{x}) / \delta \mathrm{a}(\mathrm{y})\rangle=-\mathrm{K}_{\mathrm{x}} \mathrm{~K} \int_{4 \mathrm{~m}^{2}}^{\infty} \mathrm{ds} J(\mathrm{~s}) \Delta_{\mathrm{R}}(\mathrm{x}-\mathrm{y} ; \mathrm{s}) /\left(\mathrm{s}-\mathrm{m}^{2}\right)^{2} \tag{1.9}
\end{equation*}
$$

for any renormalizable self-interacting neutral scalar field theory, and

$$
\begin{equation*}
\langle\mathrm{j}(\mathrm{x})\rangle=0, \tag{1.10}
\end{equation*}
$$

for the theory considered here. $\Delta_{R}(x ; s)$ is the function $\Delta_{R}(x)$ with mass parameter $s^{1 / 2}$.

The proof that (1.8), (1.9) and (1.10) give $j(x)$ finite to every order and identical with the results of renormalization, which is the burden of this paper, consists of showing that the operator $\mathrm{j}(\mathrm{x})$ which satisfies (1.8) is (formally) identical to (1.3), with $\Lambda_{2}$ given by (1.7). This will be done using a method of renormalization ${ }^{4}$ which obtains the counter-terms by imposing physical
requirements such as unitarity and causality on operator derivatives of the scattering operator $S$, but makes no explicit reference to diagrams, perturbation theory or the removal of infinities. This method was developed in an attempt to treat interactions exactly in Scattering Operator Theory (TSO), ${ }^{5}$ in which they had originally been treated only perturbatively, and not unambiguously in higher orders. ${ }^{6}$ In Ref. 4, unfortunately, equivalence with conventional renormalization was verified only to fourth order: exact equivalence, such as by derivation of (1.3), was not established. Furthermore, the expression for $\Lambda_{2}$ given there is incorrect, although this had no effect up to fourth order. Finally, the treatment there was based on a mix of assumptions about $S$ and $A(x)$. In order to correct these defects we have to paraphrase some of the early treatment, but only briefly. We use only assumptions of TSO. We define $j(x)$ and $A(x)$ in terms of $S$ and its operator derivatives and then we prove (1.3) and (1.7).

The discussion is organized as follows: In Section II we list our assumptions and some direct consequences. We define $j(x)$ and show how the strongest assumptions are summarized in an operator derivative equation for $\mathrm{j}(\mathrm{x})$, from which (1.9) follows. This same equation is later used to derive (1.8). In Section III, starting with an ansatz for $S$, we define $A(x)$ and derive (1.3) and (1.7). In Section IV we derive equal-time commutation relations for $\mathrm{A}(\mathrm{x})$, and use these in Section $V$ to derive (1.8). There is a brief discussion and Appendix.

## II. ASSUMPTIONS AND SOME CONSEQUENCES

A. Completeness of Free Fields

We assume that $S$ has the representation

$$
\begin{equation*}
S=\sum_{n} \frac{1}{n!} \int S_{n}\left(x_{1}, \ldots, x_{n}\right): a\left(x_{1}\right) \ldots a\left(x_{n}\right): d^{4} x_{1} \ldots d^{4} x_{n} \tag{2.1}
\end{equation*}
$$

and that each coefficient $S_{n}$ has a unique off-mass shell extrapolation, which has the same symmetry in the $x_{i}$ off shell as on. This enables us to define the operator derivative

$$
\begin{equation*}
\delta_{y} S=\frac{\delta S}{\delta a(y)}=\sum_{n} \frac{1}{n!} \int S_{n+1}\left(y, x_{1}, \ldots, x_{n}\right): a\left(x_{1}\right) \ldots a\left(x_{n}\right): d^{4} x_{1} \ldots d^{4} x_{n} \tag{2.2}
\end{equation*}
$$

As a special case,

$$
\begin{equation*}
\delta_{y} a(x)=\delta_{4}(x-y) \tag{2.3}
\end{equation*}
$$

The operator derivative is not a variational derivative, but because it is related to commutators by

$$
\begin{equation*}
[a(x), S]=-i \int \Delta(x-u) \delta_{u} S d^{4} u \tag{2.4}
\end{equation*}
$$

one can establish rules for "operator differentiation" that are analogous to those for ordinary differentiation. ${ }^{2}$ These rules are extended off shell by relations analogous to (2.4),

$$
\begin{equation*}
[a(x), s]_{r, a}=-i \int \Delta_{R, A}(x-u) \delta_{u} S d^{4} u \tag{2.5}
\end{equation*}
$$

involving the implicitly retarded or advanced commutator, formed from (2.1) by constructing the explicitly retarded or advanced commutator $\left[\mathrm{a}(\mathrm{x}), \mathrm{a}\left(\mathrm{x}_{\mathrm{i}}\right)\right]_{\mathrm{R}, \mathrm{A}}$. We are going to "take operator derivatives", a procedure made complicated by the fact that it does not commute with ordinary differentiation and integration of some operator functionals, such as those containing factors like $K_{x} \delta_{4}(x-y) .{ }^{3}$ These complications can be avoided, and the same results obtained, by supposing that operator differentiation does commute, and by distinguishing between "strong" operator equations, which remain valid after operator differentiation, and "weak" ones, which don't. ${ }^{7}$ In particular, we will take $\mathrm{K}_{\mathrm{x}} \mathrm{a}(\mathrm{x})=0$ to be weak. That is,

$$
\begin{equation*}
\delta_{y} \mathrm{~K}_{\mathrm{x}} \mathrm{a}(\mathrm{x})=\mathrm{K}_{\mathrm{x}} \delta_{4}(\mathrm{x}-\mathrm{y}) \neq 0 \tag{2.6}
\end{equation*}
$$

In fact, all the other operator equations we will deal with are strong, so no special notation will be used. We will only have to be careful to carry $K_{x} a(x)$ along, until it is obviously safe to drop it.

Finally, we may use operator derivatives to reduce matrix elements. For example, if $|\alpha\rangle$ and $|\beta\rangle$ are single free-particle states, and $|0\rangle$ is the freeparticle vacuum, then

$$
\begin{align*}
\langle\beta| \mathrm{S}|\alpha\rangle & \left.=\delta_{\alpha \beta}+\int \mathrm{d}^{4} \mathrm{x} \mathrm{f}_{\alpha}(\mathrm{x})<\beta\left|\delta_{\mathrm{x}} \mathrm{~S}\right| 0\right\rangle \\
& \left.=\delta_{\alpha \beta}+\int \mathrm{d}^{4} \mathrm{x} \mathrm{f}_{\beta}^{*}(\mathrm{x})<0\left|\delta_{\mathrm{x}} \mathrm{~S}\right| \alpha\right\rangle \tag{2.7}
\end{align*}
$$

where $\mathrm{f}_{\alpha}(\mathrm{x})$ is the wave function of the state $\mid \alpha>$.
B. Strong Unitarity

We assume

$$
\begin{equation*}
s^{+} s=s s^{+}=1 \tag{2.8}
\end{equation*}
$$

is a strong equation. ${ }^{8}$ We define the current operator

$$
\begin{equation*}
j(x)=i S^{+} \delta_{x} S \tag{2.9}
\end{equation*}
$$

Since $a(x)=a^{+}(x)$, one operator derivative of (2.8) gives us $j(x)=j^{+}(x)$ and one more operator differentiation gives

$$
\begin{equation*}
\operatorname{Re}\left(\mathrm{i} \delta_{y} \mathrm{j}(\mathrm{x})\right)=0 \tag{2.10}
\end{equation*}
$$

meaning the self-adjoint part. "Im" will mean the anti-self-adjoint part. Another consequence of (2.8) and (2.9) is ${ }^{7,8}$

$$
\begin{equation*}
i \delta_{y} j(x)-i \delta_{x} j(x)=[j(x), j(y)] \tag{2.11}
\end{equation*}
$$

## C. Strong Bogoliubov Causality and High Energy Behavior

We assume that $\delta_{y} j(x)=0$ strongly outside the forward light cone of $(x-y) .{ }^{9}$ We also assume that if $f(p, q)$ is the Fourier transform of $\delta_{y} j(x)$, that

$$
\begin{equation*}
f(p-\lambda, q+\lambda) / \lambda^{4} \rightarrow 0 \tag{2.12}
\end{equation*}
$$

strongly as $\lambda \rightarrow \infty$, as well as being analytic in the lower half complex $\lambda$ plane. ${ }^{5}$ The notation $p+\lambda=\left(p_{0}+\lambda, \vec{p}\right)$. Equation (2.12) leads to the condition

$$
\begin{equation*}
P_{A}\left(\delta_{y} j(x)\right)=0 \tag{2.13}
\end{equation*}
$$

where the idempotent, mutually orthogonal convolution operators are defined by

$$
\begin{align*}
P_{R, A}(f(x, y))= & K_{x} K_{y} \theta\left( \pm\left(x_{0}-y_{0}\right)\right) \int \Delta_{A, R}(x-u) \\
& \times \Delta_{R, A}(y-v) f(u, v) d^{4} u d^{4} v \tag{2.14}
\end{align*}
$$

$P_{R, A}$ are also orthogonal to the idempotent convolution $B$, defined by

$$
\begin{equation*}
\mathrm{B}=1-\mathrm{P}_{\mathrm{R}}-\mathrm{P}_{\mathrm{A}} \tag{2.15}
\end{equation*}
$$

so that

$$
\begin{align*}
B(f(x, y))= & K_{x} K_{y} \theta\left(x_{0}-y_{0}\right) \int\left[\Delta(x-u) \Delta_{R}(y-v)-\Delta(y-v)\right. \\
& \left.\times \Delta_{R}(x-u)\right] f(u, v) d^{4} u d^{4} v \tag{2.16}
\end{align*}
$$

The entire content of (2.10) and (2.13) may be summarized, with the help of (2.11) and (2.15), in the single equation. ${ }^{10}$

$$
\begin{equation*}
\mathrm{i} \delta_{y} j(x)=P_{R}([j(x), j(y)])+B\left(i \operatorname{Im}\left(i \delta_{y} j(x)\right)\right) \tag{2.17}
\end{equation*}
$$

This is the equation from which (1.8) and $\Lambda_{2}$ will be derived, and is central to our treatment.
D. Stability of the Vacuum and One-Particle States

We assume that

$$
\begin{equation*}
S|0\rangle=|0\rangle \tag{2.18}
\end{equation*}
$$

and

$$
\begin{equation*}
S|\alpha\rangle=|\alpha\rangle \tag{2.19}
\end{equation*}
$$

$10\rangle$ and $|\alpha\rangle$ were defined in connection with (2.7). As a consequence, completely carrying out the reduction of $\langle\beta| S|\alpha\rangle$ leads to the vanishing of $q_{c}(p)$ and $q_{R}(p)$ when $-p^{2}=m^{2}$. ( $q_{c}(p)$ is the Fourier transform of $\left\langle-S^{+} \delta_{x} \delta_{y} S\right\rangle$ and $q_{R}(p)$ is the

Fourier transform of $\left\langle\mathrm{i} \delta_{\mathrm{y}} \mathrm{j}(\mathrm{x})>\right.$.) Then (2.9) and (2.18), together with this vanishing of $q_{R}(p)$ lead to

$$
<0|j(x)| \alpha\rangle=0,
$$

which establishes the lower limit of the integral in (1.6). Finally, by assuming that $q_{c}(p)$ satisfies a subtracted dispersion relation, 9 and by using (2.17) translated into operator derivatives of $S$ to fix the subtraction terms, we can establish that ${ }^{11}$

$$
\begin{equation*}
\left\langle-S^{+} \delta_{x} \delta_{y} S\right\rangle=-i K_{x} K_{y} \int_{4 m^{2}}^{\infty} \frac{J(S)}{\left(s-m^{2}\right)^{2}} \Delta_{c}(x-y ; s) d s \tag{2.20}
\end{equation*}
$$

from which (1.9) follows by (1.6) and the relation

$$
-S^{+} \delta_{x} \delta_{y} S=i \delta_{y} j(x)+j(y) j(x)
$$

III. RENORMALIZATION OF THE MODEL

Suppose that

$$
\begin{equation*}
S=\left(e^{-i H}\right)_{+} \tag{3.1}
\end{equation*}
$$

where

$$
\begin{align*}
H=\frac{1}{6} g \int: a(x) & 3 \cdot d^{4} x+\int \Lambda_{1}(x) a(x) d^{4} x \\
& +\frac{1}{2} \int \Lambda_{2}(x-y): a(x) a(y): d^{4} x d^{4} y \tag{3.2}
\end{align*}
$$

$\Lambda_{2}$ is a distribution of point support, $\Lambda_{1}$ has in fact no $x$-dependence and $S$ is implicitly time-ordered in the $a(x)$. We have omitted an overall phase factor in (3.1) designed to ensure $\langle S\rangle=1$, and instead we will impose (2.18) whenever the question comes up. Since (3.1) is an improper form, manipulations of it must be regarded as purely formal, or as the limit of manipulations of corresponding regularized expressions. It will be sufficient, for example, to cut off integrals
like (1.5). The only purpose of such regularization is to make the following arguments look respectable. It would not be done for the purpose of identifying finite parts.

Define an operator

$$
\begin{equation*}
A(x)=S^{+}(a(x) S)_{+} \tag{3.3}
\end{equation*}
$$

Then

$$
\begin{equation*}
A(x)=a(x)-\int \Delta_{R}(x-u) j(u) d^{4} u \tag{3.4}
\end{equation*}
$$

and

$$
\begin{equation*}
K_{x} A(x)=j(x)+K_{x} a(x) \tag{3.5}
\end{equation*}
$$

which is just (1.2), except for the weakly vanishing $\mathrm{K}_{\mathrm{x}} \mathrm{a}(\mathrm{x})$. According to (2.5), operator differentiation obeys a chain rule, so we may differentiate (3.1) and obtain

$$
\begin{equation*}
j(x)=S^{+}\left(\left(\delta_{x} H\right) S\right)_{+} \tag{3.6}
\end{equation*}
$$

where

$$
\begin{equation*}
\delta_{x} H=\frac{1}{2} g: a(x)^{2}:+\Lambda_{1}+\int \Lambda_{2}(x-u) a(u) d^{4} u \tag{3.7}
\end{equation*}
$$

The contribution of : $a(x)^{2}$ : to (3.6) is evaluated by interpreting it as

$$
: a(x)^{2}:=\lim _{y \rightarrow x}\left[T_{+}(a(x) a(y))-\left\langle T_{+}(a(x) a(y))\right\rangle\right]
$$

where $\mathrm{T}_{+}$signifies explicit time ordering, and by setting

$$
\left(\mathrm{T}_{+}(\mathrm{a}(\mathrm{x}) \mathrm{a}(\mathrm{y})) \mathrm{S}_{+}=(\mathrm{a}(\mathrm{x}) \mathrm{a}(\mathrm{y}) \mathrm{S})_{+},\right.
$$

as is implicit in (3.6). That is, the partial time ordering is subsumed by the overall time ordering. Using ${ }^{12}$

$$
\begin{equation*}
S^{+}(a(x) a(y) S)_{+}=T_{+}(A(x) A(y)) \tag{3.8}
\end{equation*}
$$

and setting $\langle\mathrm{A}(\mathrm{x})>=0$, as is needed in order to satisfy (1.10), we have

$$
\begin{align*}
\Lambda_{1} & =-\frac{1}{2} \mathrm{~g}\left\langle\mathrm{~S}^{+}\left(: \mathrm{a}(\mathrm{x})^{2}: \mathrm{S}\right)_{+}\right\rangle \\
& =-\frac{1}{2} \mathrm{~g}\left\langle\mathrm{~A}(\mathrm{x})^{2}-\mathrm{a}(\mathrm{x})^{2}\right\rangle \tag{3.9}
\end{align*}
$$

$\mathrm{A}(\mathrm{x})^{2}$ is also to be regarded as the limit of a time-ordered product. Inserting (3.2), (3.7), and (3.9) into (3.6), we have (1.3), except for the form of $\Lambda_{2}(x-u)$, which we now determine from (2.13) and (2.20).
$\Lambda_{2}$ has point support, ${ }^{9}$ so suppose that its Fourier transform is a finite polynomial, since we want a local theory. That is,

$$
\begin{equation*}
\lambda_{2}(\mathrm{p})=\sum_{\mathrm{n}=0}^{N} \mathrm{~L}_{\mathrm{n}}\left(\mathrm{p}^{2}+\mathrm{m}^{2}\right)^{\mathrm{n}} \tag{3.10}
\end{equation*}
$$

From (1.3) and (3.4),

$$
\begin{align*}
<\delta_{y} j(x)>=- & \frac{1}{2} g<\left\{A(x), \int \Delta_{R}(x-u) \delta_{y} j(u) d^{4} u\right\}> \\
& +\Lambda_{2}(x-y)-{ }^{r} \Lambda_{2}(x-u) \Delta_{R}(u-v)<\delta_{y} j(v)>d^{4} u d^{4} v \tag{3.11}
\end{align*}
$$

This expression must satisfy (2.13). Now the convolution of $\mathrm{P}_{\mathrm{A}}$ with the first (anticommutator) term on the right side of (3.11) is

$$
\begin{gathered}
K_{x} K_{y} \theta\left(y_{0}-x_{0}\right) \int \Delta_{R}(x-z) d^{4} z<\left\{A(z), \int \Delta_{A}(y-v) \Delta_{R}(z-u)\right. \\
\left.\times \delta_{v} j(u) d^{4} u d^{4} v\right\}>
\end{gathered}
$$

which vanishes, according to (2.12) and (2.13). The convolution of $P_{A}$ with the remaining terms of (3.11), with the use of (1.9), leads to the condition

$$
\begin{equation*}
\int_{-\infty}^{\infty} d \lambda F(p-\lambda) /(\lambda-i \epsilon)=0 \tag{3.12}
\end{equation*}
$$

where

$$
\begin{align*}
& F(p)=\left(\Delta_{R}(p)\right)^{2} \lambda_{2}(p)\left[1+I-\int_{4 m^{2}}^{\infty} \frac{J(s)}{s-m^{2}} \Delta_{R}(p ; s) d s\right]  \tag{3.13}\\
& \Delta_{R}(p ; s)=\left[\vec{p}^{2}+s-\left(p_{0}+i \epsilon\right)^{2}\right]^{-1}
\end{align*}
$$

and

$$
\Delta_{R}(\mathrm{p})=\Delta_{\mathrm{R}}\left(\mathrm{p} ; \mathrm{m}^{2}\right)
$$

Using the expansion (3.10) in (3.13), the terms $\mathrm{n}=0$ and 1 will give no contribution to (3.12). The term $n=2$ contributes $\pi i \mathrm{~L}_{2}(1+\mathrm{I})$, and the terms $\mathrm{n} \geq 3$ diverge in (3.12), and the degree of divergence increases with $n$. In the limit $R \rightarrow \infty$, to leading order in $R$, we obtain from (3.12) the series

$$
\pi i L_{2}(1+\mathrm{I})-4 \mathrm{p}_{0}(1+\mathrm{I}) \sum_{\mathrm{n}=3}^{\mathrm{N}} \frac{\mathrm{n}-2}{2 \mathrm{n}-5} \mathrm{~L}_{\mathrm{n}} \mathrm{R}^{2 \mathrm{n}-5}
$$

The divergent term $\mathrm{R}^{2 \mathrm{~N}-5}$ will dominate, and therefore we must have $\mathrm{L}_{\mathrm{N}}=0$, and in turn each $L_{n}=0, n \geq 2$, in order to satisfy (3.12). Therefore

$$
\begin{equation*}
\lambda_{2}(\mathrm{p})=\mathrm{L}_{0}+\left(\mathrm{p}^{2}+\mathrm{m}^{2}\right) \mathrm{L}_{1} \tag{3.14}
\end{equation*}
$$

and from (1.3) and (3.5),

$$
\begin{equation*}
j(x)=\left(1+L_{1}\right)^{-1}\left[\frac{1}{2} g\left(A(x)^{2}-\left\langle A(x)^{2}>\right)+L_{0} A(x)-L_{1} K_{x} a(x)\right]\right. \tag{3.15}
\end{equation*}
$$

$\mathrm{L}_{0}$ and $\mathrm{L}_{1}$ are determined by (2.20). From (3.6)

$$
-\mathrm{S}^{+} \delta_{x} \delta_{y} \mathrm{~S}=i \mathrm{~S}^{+}\left(\left(\delta_{x} \delta_{y} \mathrm{H}\right) \mathrm{S}\right)_{+}+\mathrm{S}^{+}\left(\left(\delta_{x} \mathrm{H}\right)\left(\delta_{y} \mathrm{H}\right) \mathrm{S}\right)_{+}
$$

where, from (3.7),

$$
\begin{equation*}
\delta_{x} \delta_{y} H=g a(x) \delta_{4}(x-y)+\Lambda_{2}(x-y) \tag{3.16}
\end{equation*}
$$

so that

$$
\begin{equation*}
\left\langle-S^{+} \delta_{x} \delta_{y} S\right\rangle=i \Lambda_{2}(x-y)+\left\langle S^{+}\left(\left(\delta_{x} H\right)\left(\delta_{y} H\right) S\right)_{+}\right\rangle \tag{3.17}
\end{equation*}
$$

The second term on the right side of (3.17) may be expressed in terms of $J(s)$ by using the relation

$$
\begin{equation*}
j(x) j(y)=S^{+}\left(\left(\delta_{x} H\right)\left(\delta_{y} H\right) S\right)_{+}-\left[\delta_{x} H, j(y)\right] a \tag{3.18}
\end{equation*}
$$

which follows from (3.6) and some theorems relating time ordered products to advanced and retarded commutators. ${ }^{13}$ Then

$$
\begin{equation*}
S^{+}\left(\left(\delta_{x} H\right)\left(\delta_{y} H\right) S\right)_{+}=T_{+}(j(x) j(y))+\left\{\theta\left(x_{0}-y_{0}\right)\left[\delta_{x} H, j(y)\right] a+(x \leftrightarrow y)\right\} \tag{3.19}
\end{equation*}
$$

The terms in the curly brackets do not vanish, because the commutators can contain terms proportional to $\delta_{4}(x-y)$, which are not in conflict with $\theta\left(x_{0}-y_{0}\right)$. 14 In particular, when we evaluate (3.17) using (3.19), (3.7), (3.14), (1.9) and relations for $j(x)$ analogous to (2.5), we will encounter the expression

$$
\begin{aligned}
\theta\left(\mathrm{x}_{0}-\mathrm{y}_{0}\right) \mathrm{K}_{\mathrm{y}} \Delta_{\mathrm{A}}(\mathrm{x}-\mathrm{y} ; \mathrm{s}) & \left.=\theta\left(\mathrm{x}_{0}-\mathrm{y}_{0}\right) \Gamma_{-\delta_{4}}(\mathrm{x}-\mathrm{y})+\left(\mathrm{s}-\mathrm{m}^{2}\right) \Delta_{\mathrm{A}}(\mathrm{x}-\mathrm{y} ; \mathrm{s})\right] \\
& =-\theta\left(\mathrm{x}_{0}-\mathrm{y}_{0}\right) \delta_{4}(\mathrm{x}-\mathrm{y})
\end{aligned}
$$

so that

$$
\begin{aligned}
\theta\left(x_{0}-y_{0}\right) & \int \Lambda_{2}(x-u)<[a(u), j(y)]_{a}>d^{4} u \\
& =-i \theta\left(x_{0}-y_{0}\right)\left(L_{0}-L_{1} K_{x}\right) K_{y} \int_{4 m^{2}}^{\infty} \frac{J(s)}{\left(s-m^{2}\right)^{2}} \Delta_{A}(x-y ; s) d s \\
& =i \theta\left(x_{0}-y_{0}\right)\left[I\left(L_{0}-L_{1} K_{x}\right)-L_{1} I^{\prime}\right] \delta_{4}(x-y)
\end{aligned}
$$

Nevertheless,

$$
\begin{equation*}
\theta\left(x_{0}-y_{0}\right)<\left[: a(x)^{2}:, j(y)\right]_{\mathrm{a}}>=0, \tag{3.20}
\end{equation*}
$$

as we will show in the Appendix, so the Fourier transform of $(3,17)$ becomes

$$
\begin{align*}
i(1+I)\left(L_{0}\right. & \left.+\left(p^{2}+m^{2}\right) L_{1}\right)-L_{1} I^{\prime} \\
& =-i \int_{4 m^{2}}^{\infty} J(s) d s \Delta_{c}(p ; s)\left[\frac{\left(p^{2}+m^{2}\right)^{2}}{\left(s-m^{2}\right)^{2}}-1\right] \\
& =I^{\prime}-\left(p^{2}+m^{2}\right) I \tag{3.21}
\end{align*}
$$

This holds for all values of $p^{2}$, so we may solve for $L_{0}$ and $L_{1}$ :

$$
L_{1}=-I /(1+I)
$$

and

$$
\begin{equation*}
\mathrm{L}_{0}=\mathrm{I}^{\mathrm{Y}} /(1+\mathrm{I})^{2} \tag{3.22}
\end{equation*}
$$

which is precisely what is required by (1.7).

## IV. EQUAL-TIME COMMUTATORS

A. $[A(\vec{x}, t), A(\vec{y}, t)]=0$

Proof: Using (3.4), the analog of (2.4) for $j(x)$, and (2.11), we see that

$$
\begin{align*}
{[A(x), A(y)]=} & -i \Delta(x-y)+i \int\left[\Delta_{R}(x-u) \Delta_{A}(y-v) \delta_{v} j(u)\right. \\
& \left.-\Delta_{A}(x-u) \Delta_{R}(y-v) \delta_{u} j(v)\right] d^{4} u d^{4} v \tag{4.2}
\end{align*}
$$

which vanishes when $\mathrm{x}_{0}=\mathrm{y}_{0}$, by virtue of (2.12).
B. $\delta\left(\mathrm{x}_{0}-\mathrm{y}_{0}\right)[\mathrm{A}(\mathrm{x}), \dot{\mathrm{A}}(\mathrm{y})]=\mathrm{i}(1+\mathrm{I}) \delta_{4}(\mathrm{x}-\mathrm{y}) \quad$,
where $\dot{A}(\mathrm{y})=\partial \mathrm{A}(\mathrm{y}) / \partial \mathrm{y}_{0}$. To prove this, we need the following lemma ${ }^{3}$ :

$$
\begin{equation*}
i \delta_{y} A(x)=K_{y}\left\{\theta\left(x_{0}-y_{0}\right)[A(x), A(y)]\right\} \tag{4.4}
\end{equation*}
$$

which follows here from (4.2), using (2.13) and (3.4). Commuting $K_{X}$ with $\theta\left(\mathrm{x}_{0}-\mathrm{y}_{0}\right)$, we have

$$
\begin{equation*}
\delta\left(x_{0}-y_{0}\right)[A(x), \dot{A}(y)]=i \delta_{y} A(x)-\theta\left(x_{0}-y_{0}\right)[A(x), j(y)], \tag{4.5}
\end{equation*}
$$

since $\left[A(x), K_{y} a(y)\right]$ vanishes strongly because it is a commutator, and the $y$-dependence of $a(y)$ is transferred to a c-number. Let us substitute (3.4) for $\mathrm{A}(\mathrm{x})$ on the right side of (4.5), and use (2.4) and (2.11) to change commutators to operator derivatives. Then the right side of (4.5) becomes

$$
\begin{align*}
& \mathrm{i} \delta_{4}(\mathrm{x}-\mathrm{y})-\mathrm{i} \int \mathrm{~d}^{4} \mathrm{u}\left[\theta\left(\mathrm{y}_{0}-\mathrm{x}_{0}\right) \Delta_{\mathrm{R}}(\mathrm{x}-\mathrm{u}) \delta_{y^{j}(\mathrm{u})}\right. \\
&\left.+\theta\left(\mathrm{x}_{0}-\mathrm{y}_{0}\right) \Delta_{A}(\mathrm{x}-\mathrm{u}) \delta_{\mathrm{u}} \mathrm{j}(\mathrm{y})\right] \tag{4.6}
\end{align*}
$$

Now substitute (3.15) and (3.21) in (4.6), and again use (3.4). Then (4.6) becomes

$$
\begin{align*}
& i(1+I) \delta_{4}(x-y)+\frac{i}{2}(1+I) \int d^{4} u d^{4} v\left[\theta\left(y_{0}-x_{0}\right) \Delta_{R}(x-u)\right. \\
& \quad \times \Delta_{R}(u-v)\left\{g A(u)+L_{0}, \delta_{y} j(v)\right\}+\theta\left(x_{0}-y_{0}\right) \Delta_{A}(x-u) \\
& \left.\quad \times \Delta_{R}(y-v)\left\{g A(y)+L_{0}, \delta_{u} j(v)\right\}\right] . \tag{4.7}
\end{align*}
$$

The c-number term is all we need. If we can show that the rest of (4.7) vanishes, then we have established (4.3). Now the term involving $\delta_{u} j(v)$ vanishes at once by (2.13), and the term involving $\delta_{y} j(v)$ can be written as

$$
\begin{gathered}
\int d^{4} p d^{4} q \exp (i p \cdot x+i q \cdot y) \int d^{4} p^{\prime}\left\{g \widetilde{A}\left(p^{\prime}\right)+L_{0} \delta_{4}\left(p^{\prime}\right), \int_{-\infty}^{\infty} \frac{d \lambda}{2 \pi i(\lambda-i \epsilon)}\right. \\
\left.\quad \times f\left(p-p^{\prime}-\lambda, q+\lambda\right) \Delta_{R}(p-\lambda) \Delta_{R}\left(p-p^{\prime}-\lambda\right)\right\},
\end{gathered}
$$

which vanishes according to (2.12). $\widetilde{A}(p)$ is the Fourier transform of $A(x)$, and $f(p, q)$ is defined in connection with (2.12). Thus (4.3) is established.

Two other commutators that will be used later are obtained using (4.1), (4.3), (3.15) and (3.22). They are

$$
\begin{equation*}
[\mathrm{A}(\overrightarrow{\mathrm{x}}, \mathrm{t}), \mathrm{j}(\overrightarrow{\mathrm{y}}, \mathrm{t})]=0, \tag{4.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\delta\left(\mathrm{x}_{0}-\mathrm{y}_{0}\right)[\dot{\mathrm{A}}(\mathrm{x}), \mathrm{j}(\mathrm{y})]=-\mathrm{i}\left[\mathrm{~g}^{\prime} \mathrm{A}(\mathrm{x})+\mathrm{I}^{\prime}\right] \delta_{4}(\mathrm{x}-\mathrm{y}), \tag{4.9}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{g}^{\prime}=(1+\mathrm{I})^{2} \mathrm{~g} \tag{4.10}
\end{equation*}
$$

## V. DERIVATION OF EQUATION (1.8)

We start with (2.17), and in the term $P_{R}([j(x), j(y)])$ commute $K_{x} K_{y}$ with $\theta\left(\mathrm{x}_{0}-\mathrm{y}_{0}\right)$, using (3.4) and

$$
\int \Delta_{A}(x-u) j(u) d^{4} u=a(x)-A(x)-\int \Delta(x-u) j(u) d^{4} u
$$

Then apply the equal time commutation relations (4.1), (4.3), (4.8) and (4.9), and further simplify things by using (2.4) and (2.11). The result is

$$
\begin{align*}
P_{R}([j(x), j(y)])= & \theta\left(x_{0}-y_{0}\right)[j(x), j(y)]+i\left[g^{T} A(x)+I^{\prime}+I K_{x}\right] \\
& \times \delta_{4}(x-y)-\left[2 \delta\left(x_{0}-y_{0}\right) \frac{\partial}{\partial x_{0}}+\frac{\partial \delta\left(x_{0}-y_{0}\right)}{\partial x_{0}}\right] \int \Delta(x-u) i \delta_{y^{\prime}} j(u) d^{4} u \\
& +K_{x}\left\{[ 2 \delta ( x _ { 0 } - y _ { 0 } ) \frac { \partial } { \partial y _ { 0 } } + \frac { \partial \delta ( x _ { 0 } - y _ { 0 } ) } { \partial y _ { 0 } } ] \int d ^ { 4 } u d ^ { 4 } v \left[\Delta_{R}(x-u)\right.\right. \\
& \left.\left.\times \Delta(y-v)-\Delta(x-u) \Delta_{R}(y-v)\right] i \delta_{v} j(u)\right\} \tag{5.1}
\end{align*}
$$

The terms in (5.1) involving $\delta\left(\mathrm{x}_{0}-\mathrm{y}_{0}\right)$ and its first derivatives are identical to $-B\left(i \delta_{y} j(x)\right)$, as can be seen by commuting $K_{x} K_{y}$ with $\theta\left(x_{0}-y_{0}\right)$ in (2.16), with $f(x, y)=i \delta_{y} j(x)$. Remembering that $\operatorname{Re}\left(i \delta_{y} j(x)\right)=0$, Eq. (2.17) then becomes

$$
\begin{equation*}
\delta_{y} j(x)=\left[g^{\prime} A(x)+I^{\prime}+I K_{x}\right] \delta_{4}(x-y)-i \theta\left(x_{0}-y_{0}\right)[j(x), j(y)] . \tag{5.2}
\end{equation*}
$$

The counter-terms appear only as c-numbers, and one more operator differentiation and (3.4) gives (1.8).

If we had started this section with $\mathrm{P}_{\mathrm{R}}([\mathrm{j}(\mathrm{x}), \mathrm{j}(\mathrm{y})]-\langle[\mathrm{j}(\mathrm{x}), \mathrm{j}(\mathrm{y})]>)$, we would have obtained instead of (5.2)

$$
\begin{align*}
\delta_{y} j(x)= & g^{\prime}\left[a(x)-\int \Delta_{R}(x-u) j(u) d^{4} u\right] \delta_{4}(x-y) \\
& -i \theta\left(x_{0}-y_{0}\right)([j(x), j(y)]-<[j(x), j(y)]>)+\left\langle\delta_{y} j(x)>,\right. \tag{5.3}
\end{align*}
$$

which is equivalent to (1.8).
VI. DISCUSSION

Our starting point has been the assumptions of Scattering Operator Theory, but here we have recovered from them the canonical quantized field formalism. Causality and the energy bound (Eq. (2.12)) play the crucial roles in this effort, as is seen in the discussion of (4.7). Thus the scattering operator formalism of

Ref. 5 is seen to be fully equivalent to the quantized field formalism, for the model considered, except for the question of bound states, which we have not attempted to treat. (We have also ignored the question of the existence of the model.)

Our final result (1.8) involves no infinities (and no cancellations) because, in this case, at least, there are at most c-number divergences in $\theta\left(\mathrm{x}_{0}-\mathrm{y}_{0}\right)[\mathrm{j}(\mathrm{x}), \mathrm{j}(\mathrm{y})]$, and one operator derivative takes care of them. In deriving (1.8) from (2.17), the "source term", $\mathrm{g} \delta_{4}(\mathrm{x}-\mathrm{y}) \delta_{4}(\mathrm{x}-\mathrm{z})$ emerges from the commutator $P_{R}([j(x), j(y)])$. The only role of $B(i \delta j j(x))$ is to cancel those terms which do not have the desired off-shell symmetry. This is to be contrasted with the role played by a similar operator in Ref. 6, which must specify the interaction as well as restore the symmetry. (The right side of (1.8) should be symmetrical in $y$ and $z$. This symmetry can be exhibited by carrying out the operator differentiation of the commutator, and then substituting (5.3) in the result.)

The obvious next steps are to investigate nonperturbative solutions of (1.8), and to address the problem of vertex renormalization and overlapping divergences in a harder model. (Coupling constant renormalization was accomplished trivially in (1.8), without any deliberate attempt, by (4.10).)

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## APPENDIX: PROOF OF EQUATION (3.20)

From (3.15) and (2.4) we calculate

$$
\begin{align*}
<\left[: a(x)^{2}:, j(y)\right]_{a}>= & \frac{-i}{2\left(1+L_{1}\right)}<\left\{a(x), \int \Delta_{A}(x-u)\right. \\
& \left.\times\left\{g A(y)+L_{0}, \delta_{u} A(y)\right\}\right\}> \tag{A.1}
\end{align*}
$$

and use (3.4) to evaluate $\delta_{u} A(y)$. The term in (A.1) which comes as a result from $\delta_{u} a(y)$ involves

$$
<\{\mathrm{a}(\mathrm{x}), \mathrm{A}(\mathrm{y})\}>=\Delta_{1}(\mathrm{x}-\mathrm{y})
$$

and

$$
\Delta_{1}(\mathrm{x}-\mathrm{y}) \Delta_{\mathrm{A}}(\mathrm{x}-\mathrm{y})=\frac{\mathrm{i}}{8 \pi^{2}} \int_{4 \mathrm{~m}}\left[1-4 \mathrm{~m}^{2} / \mathrm{s}\right]^{1 / 2} \Delta_{A}(\mathrm{x}-\mathrm{y} ; \mathrm{s}) \mathrm{ds}
$$

which vanishes when $x_{0} \geq y_{0}$. (We cut off $\int_{\text {ds.) What remains of (A. 1) is pro- }}$ portional to

$$
<\left\{a(x),\left\{g A(y)+L_{0}, \int \Delta_{A}(x-u) \Delta_{R}(y-v) \delta_{u} j(v) d^{4} u d^{4} v\right\}\right\}>
$$

which vanishes when $x_{0} \geq y_{0}$, according to (2.12) or (2.13).

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