SLAC-PUB-1824
November 1976
(T)

AN EXACT HEDGEHOG SOLUTION IN CHIRAL DYNAMICS*

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ABSTRACT

We investigate 3+1 dimensional kink solutions of Skyrme's chiral model. These solutions are 'self-dual', and minimize the energy in each homotopy class. We find an exact 1-kink hedgehog by saturation of the lower bound of the energy. It is a stereographic projection, formally akin to the instanton.

(Submitted for publication)

*Work supported in part by the Energy Research and Development Administration.
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The discovery of the instanton [1] in a pure Euclidean Yang-Mills theory shows the paramount importance of an aesthetic geometry underlying field theories admitting exact solutions. Although it lacks explicit renormalizability, nonlinear chiral dynamics does possess a very appealing geometric interpretation [2], and hence may have exact solutions in four dimensions. In this letter, we shall show that this is so in the specific instance of a remarkable chiral model due to Skyrme [2].

We first recall the topological arguments necessary for the existence of the desired solutions of three-dimensional, stable, finite energy, static kinks. Specifically we study the $SU(2) \times SU(2)$ invariant chiral dynamics of Nambu-Goldstone bosons such as pions. The group geometry is as follows.

At each spacetime point, a quaternion field $S(x) = \phi^0 + i\vec{\tau} \cdot \vec{\phi}$ takes value on the nonlinear group manifold M of SU(2). S is a unitary and unimodular 2×2 matrix in a doublet SU(2) representation. τ^i are the usual Pauli matrices. Since $\phi_0^2 + \vec{\phi}^2 = 1$, M is parametrized on a 3-unit sphere S^3 embedded in a four-dimensional Euclidean chiral space. It follows that the four-dimensional rotation group $O(4) \approx SU(2) \times SU(2)$ carries this hypersphere into itself and is the largest group of isometries of S^3.

From the group geometry viewpoint [3] the natural objects of chiral dynamics are the group currents, $J_\mu = S^{-1}\partial_\mu S$, which take values in the Lie algebra. They are antihermitian and traceless matrices: $J_\mu = i\vec{\tau} \cdot \vec{J}_\mu$. They remain invariant under the left-shifts, $S \to U_L S$, and transform simply as $J_\mu \to U_R^{-1} J_\mu U_R$ under right-shifts, $S \to SU_R$, with $U_{L,R}$ being global SU(2) rotations. Chiral Lagrangians can be readily constructed from the left and right invariant combinations of these currents.

Of all possible classical fields $S(x)$, we are only interested in the subset obeying the would-be energy finiteness condition

$$S(x) \xrightarrow[|x| \to \infty]{} I \tag{1}$$

true at all times. Hence at any fixed time t, $S(\vec{x})$ or $J_\mu(x)$ map the physical space R^3 into the group SU(2). Eq. (1) implies that R^3 can be continuously deformed onto S^3, i.e., it is compactified onto S^3:

$$J_i(x) : S^3 \to S^3 \, . \tag{2}$$

In consequence the phase space of the $J_i(x)$ with eq. (1) falls into an infinite number of homotopy classes, the Chern classes of $\pi_3(SU(2)) \approx \pi_3(S^3) = Z_\infty$, where Z_∞ is the additive group of integers [4] which are the degrees of the mappings. Any two mappings of the same class can be continuously deformed into one another while two maps belonging to different classes cannot. Examples of such deformations or homotopies are a global SU(2) transformation or time evolution. So the degree of a mapping is a homotopic invariant, hence conserved irrespective of the dynamics of the system. It depends solely on the <u>periodicity</u> of the field S(x) which arises from the compactness of SU(2), and the condition (1).

We now seek the minima of the energy in each component of the phase space [1]. For that purpose, we express the degree of the mapping, in terms of the currents. From the Minkowskian and chiral geometries, the trivially conserved topological vector current is

$$B_\mu = \frac{i}{4\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}([J_\nu, J_\rho] J_\sigma) \, . \tag{3}$$

$\partial^\mu B_\mu = 0$ follows readily since $\partial^\mu J_\rho = \partial^\mu \partial_\rho \ln S$ is symmetric in μ and ρ, etc. The degree of mapping is then

$$B = \frac{i}{4\pi^2}\int d^3x \; \epsilon_{ijk}\, \mathrm{Tr}([J_i, J_j]J_k) \,. \tag{4}$$

To see the true topological meaning of B we introduce the geodesic parametrization [5] of S^3:

$$S = \cos\frac{\psi}{2} + i\vec{n}\cdot\vec{\tau}\sin\frac{\psi}{2} = e^{\frac{i}{2}\vec{n}\cdot\vec{\tau}\psi} \tag{5}$$

where $n^i = \frac{\phi^i}{\sqrt{\phi^i\phi^i}}$ varies within the sphere

$$0 \leq \psi \equiv \sqrt{\phi^i\phi^i} \leq 2\pi \,.$$

In terms of these geodesic coordinates, we obtain

$$B = \frac{1}{2\pi^2}\int \frac{\sin^2\alpha}{\alpha^2}\, d^3\alpha \tag{6}$$

where $d^3\alpha = (\alpha)^2 \sin\theta \, d\alpha \, d\theta \, d\chi$ is a three-dimensional volume element in the spherical coordinates with the radius α and the angles $0 \leq \theta \leq \pi$ and $0 \leq \chi \leq 2\pi$; for convenience we defined $\alpha = \psi/2$. Notice that this integral is proportional to the surface of S^3. The factor $1/2\pi^2$ in eq. (3) is the proper normalization allowed by the compactness of S^3 to have B take integer values.

From eq. (6) it is clear that B is the degree of the mappings of $S^3 \to S^3$; it measures the number of times S^3 is covered in the course of the mapping [6].

We now introduce our choice of chiral Lagrangian

$$\begin{aligned} \mathscr{L} &= \mathscr{L}_{(2)} + \mathscr{L}_{(4)} \\ \mathscr{L}_{(2)} &= \frac{1}{2c^2}\,\mathrm{Tr}\,(J_\mu J^\mu) \\ \mathscr{L}_{(4)} &= \frac{\epsilon^2}{4}\,\mathrm{Tr}\,[J_\mu, J_\nu]^2 \end{aligned} \tag{7}$$

whose static Hamiltonian is

$$H^{st} = \int d^3x \, \mathrm{Tr}\left(\frac{1}{2c^2} J_i^2 - \frac{\epsilon^2}{4} [J_i, J_j]^2\right) \tag{8}$$

where c is a length, for example the inverse of the pion decay constant, f_π^{-1}, and ϵ is a dimensionless coupling. $\mathscr{L}_{(2)}$ is the standard chiral Lagrangian in the current form [7,3]. The model (7) was first written down and investigated by Skyrme [2] in a series of remarkable papers. Its extension and relevance to chiral gauge invariant theories of realistic hadronic solitons have been recently stressed by Faddeev [8]. In the light of more recent developments in chiral dynamics our own physical motivations for the additional quartic term are the following:

a. A simple application of the Derrick scaling argument [10] shows that $\mathscr{L}_{(2)}$ can only have stable, finite energy static solutions in two space dimensions, such as vortices with finite energy per unit length. To obtain truly three dimensional kinks, the same scaling argument applied to eq. (8) then allows the desired, stable finite energy, static solutions in only 3 spatial dimensions.

b. Secondly, the additional quartic term can be seen as a particular choice of counterterms needed in Slavnov's [11] superpropagator regularization of $\mathscr{L}_{(2)}$. At the one loop level, his procedure leads to a divergence free unitary S matrix when energies are below a certain cutoff. This connection will be important in an eventual quantum soliton expansion.

c. Our most compelling argument for the choice of commutator as the quartic term is rooted in the particular form of the topological current (eq. (3)) [8]. In other words, the dynamics is determined by the geometry in the spirit of the chiral Lagrangian approach [3]. Moreover this selection allows an exact geometrical solution to be presented next.

Our key observation about (8) is that it can be recast in the form

$$H^{st} = \frac{1}{2}\int d^3x\ \mathrm{Tr}\left(\frac{1}{c}J_i - \frac{\epsilon}{2}{*J_i}\right)^2 + \frac{\epsilon}{2c}\int d^3x\ \mathrm{Tr}\,(J_i^* J_i) \geq \frac{2\pi^2\epsilon}{c}\ |B| \tag{9}$$

where we have defined the dual of J_i^a as $*J_i^a = \epsilon_{ijk}\epsilon^{abc}J_j^b J_k^c$. Then eq. (9) gives the lower bound for the kink energy in each homotopy class. We shall see that for $|B| = 1$ this lower bound can be saturated and yields an exact solution. Namely, in place of the highly nonlinear chiral dynamical equations of eq. (7) we can solve instead for the equation

$$*J_i^a = \frac{2}{\epsilon c}\,J_i^a\ . \tag{10}$$

Eq. (10) is the chiral counterpart of the self-dual equation $F^{\alpha}_{\mu\nu} = \pm F^{*\alpha}_{\mu\nu}$ for the Yang-Mills pseudoparticles [1]. Clearly we seek solutions which are invariant under simultaneous spatial and isospatial rotations. The exponential parametrization of S^3 suggests the spherically symmetric hedgehog ansatz [2]

$$\begin{aligned} \vec{\phi} &= \frac{\vec{x}}{r}\sin\frac{\psi(r)}{2}\ , \\ \phi^0 &= \cos\frac{\psi(r)}{2} \end{aligned} \tag{11}$$

for the lowest kink state. The center of the kink is chosen to be at the origin $\vec{x} = 0$. It is easy to calculate J_i^a and $*J_i^a$:

$$J_i^a = \frac{1}{2r}\left[\left(\delta_i^a - \frac{x^a x_i}{r^2}\right)\sin\psi + \frac{x^a x_i}{r^2}\,(r\psi') - \epsilon_{ij}^a\frac{x_j}{r}(1-\cos\psi)\right]$$

and (12)

$$*J_i^a = \frac{1}{2r}\left[\left(\delta_i^a - \frac{x^a x_i}{r^2}\right)\left(\frac{1}{2}\psi'\sin\psi\right) + \frac{x^a x_i}{2r^2}\,(1-\cos\psi) - \epsilon_{ij}^a\frac{x_j}{2r}(1-\cos\psi)\psi'\right]\ .$$

After inserting these explicit forms in eq. (10) we obtain a first order differential equation for the unknown function ψ,

$$r\,\frac{d\psi}{dr} = \mp 2 \sin \frac{\psi}{2} \tag{13}$$

Its solutions are respectively

$$\psi(r) = 4 \tan^{-1}\left(\frac{a}{r}\right) \tag{14a}$$

and

$$\psi(r) = 4 \tan^{-1}\left(\frac{r}{a}\right) \tag{14b}$$

a is a constant of integration, a length which is not arbitrary but is determined below by the lower bound in eq. (9) with B = 1. Eq. (13) gives the solutions

$$\phi^i = \frac{2ax_i}{r^2 + a^2} \,, \qquad \phi^o = \pm \frac{r^2 - a^2}{r^2 + a^2} \,. \tag{15}$$

Here (+) refers to the first solution, and (-) to the second solution. Only the <u>first</u> possibility is the desired physical solution as it satisfies the energy finiteness condition $\psi(\infty) = 0$ resulting from $S \xrightarrow[|x| \to \infty]{} I$. It gives $\psi(0) = 2\pi$ or $S\,(x=0) = -I$. So while the points at spatial infinity are all mapped into the north pole I of S^3, the particle-center (chosen to be at $\vec{x} = 0$) is mapped into the south pole -I. These remarkable points are the focal points of the geodesics and compose the only discrete invariant subgroup of SU(2)[5]. Our exact solution is simply the usual stereographic projection [12] from the hypersphere S^3, the manifold of SU(2), onto R^3, the physical space. It is of degree B = 1 as can be explicitly verified from eqs. (14a) and (6):

$$B = \frac{1}{2\pi} \int_0^{2\pi} d\psi\, (1 - \cos\psi) = 1 \tag{16}$$

For S, the solution takes the Cayley form:

$$S(x) = \frac{I + i\left(\frac{a}{r^2}\right) \vec{\tau} \cdot \vec{x}}{I - i\left(\frac{a}{r^2}\right) \vec{\tau} \cdot \vec{x}} \tag{17}$$

The antiparticle solution with B = -1 is given by S^+. Calculating the energy corresponding to this solution and equating it to the lower bound for B = 1 in eq. (9) we determine a to be $a \approx c/\epsilon$. We have not yet found any solution to eq. (10) with $|B| \neq 1$; many kink solutions may be sought in the same way as the many instanton solutions of ref. [9].

The informed reader must have made two relevant observations. First, there exists a close topological connection between our solution, eq. (17), and the instanton solution [1]. Indeed the group current corresponding to eq. (17) is identical to eq. (4) for the instanton potential A_i^a in ref. [13]

$$J_i^a = -\frac{2a}{(r^2+a^2)^2}\left[(a^2-r^2)\delta_i^a + 2x^a x_i + 2a\epsilon_{ij}^{\ a} x_j\right] . \tag{18}$$

The crucial difference is that, in contrast to the instanton, the solution (17) represents a real extended particle in Minkowski space with a fixed length scale. The connection between chiral dynamics and massive gauge fields is one discovered by Bardakci et al. and Boulware [14]. It is recently emphasized by Polyakov [15].

Secondly, long ago Skyrme [2] obtained the same expression (eq. (18)) for J_i^a by assuming the proportionality between J_i^a and J_i^{a*}. He did so to give a possible definition of the particle singularity and isolates the latter from the remaining nonlinear field structure. This definition is clearly arbitrary in general. The physical nature of our method is completely different. Our eq. (10) is a dynamical consequence of Skyrme's model (7). It has the meaning of a new and simpler equation of motion, one for which an exact 1-kink solution is obtained. Moreover, many kink solutions must also obey eq. (10) in complete analogy to the case of the instantons [9]. Therein lies the essential difference between our work and that of Skyrme, which makes this paper a new development in his scheme [16].

In closing, eq. (17) is one of a few exact kink solutions found in four-dimensional Minkowskian relativistic field theories. Being a pure stereographic map, it can be shown by way of algebraic topology [17] to admit upon canonical quantization a wave functional which is double-valued under a 2π spatial rotation. In the phraseology of Finkelstein [18], the kink theory (7) is said to admit half-integral spins; the quantized kink corresponds to towers of states with both half-integral spins and isospins. If the Lagrangian (7) is a phenomenological model for strong interactions, the kink (17) is seen as a coherent lump of Nambu-Goldstone pions, endowed with a chiral twist, $B = 1$. It has the natural interpretation of a classically degenerate form of a baryon.

The relation of chiral dynamics to a fundamental theory of hadrons may be analogous to that of the Landau-Ginzburg theory to the BCS theory of superconductivity [19]. In this perspective the chiral kink generation mechanism is seen as the obverse of the Nambu-Jona Lasinio theory [20]. In the latter scheme, due to spontaneous γ_5-symmetry breaking, massive fermions arise from massless ones, provided Goldstone pions are created as bound states and restore the symmetry. We recall that nonlinear chiral dynamics was conceived to give a well-defined meaning to the concept of symmetry restoration. Self-consistency then forces the nonlinear chiral invariant group dynamics of Nambu-Goldstone bosons to respond in kind by generating superselection rule sectors which are baryons. This duality property [2] is strongly suggested by a semiclassical analysis, and needs confirmation at the full quantum mechanical level. The detailed investigation of the kink structure of eq. (7), the fermionic spin structure of 1-kink sector, via collective coordinates, the improved Sugawara algebra of currents, and more are the subjects of forthcoming publications.

Acknowledgements

We wish to thank J. D. Bjorken, S. D. Drell, T. D. Lee, and Y. Nambu for helpful discussions, and D. Finkelstein for lectures on kinks.

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