# TWO DIMENSTONAL SU(N) GAUGE THEORY, STRINGS AND WINGS: <br> COMPARATIVE ANALYSIS OF MESON SPECTRA AND COVARIANCE* <br> Andrew J. Hanson ${ }^{\dagger}$ <br> Stanford Linear Accelerator Center Stanford University, Stanford, California 94305 

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## ABSTRACT

't Hooft's two-dimensional $\operatorname{SU}(\mathrm{N})$ gauge theory model for mesons is studied in two different axial gauges. Using numerical techniques employed in aerodynamical wing theory, we compare the bound state spectra in the $A^{+}=0$ and $A_{1}=0$ gauges, finding agreement in the weak coupling limit. Furthermore, Lorentz covariance of the weak-coupling $A_{1}=0$ theory is numerically confirmed. We also investigate the massive-end string model, which is equivalent to 't Hooft's $A^{+}=0$ model when the $x^{+}=\tau$ gauge is chosen. We find that the numerical spectrum of the string model in the $x^{0}=\tau$ gauge differs from the $x^{+}=\tau$ gauge string spectrum as well as from the $A_{1}=0$ gauge theory spectrum. A Bethe-Salpeter equation approach to the spectrum of the gauge theory in the $A_{1}=0$ gauge is developed for any coupling. While the strong coupling theory in this gauge presents severe difficulties, the weak-coupling limit is shown to be completely consistent.
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## INTRODUCTION

't Hooft [1,2] has proposed a model for mesons based on the $1 / \mathrm{N}$ expansion of the Bethe-Salpeter equation for a $U(N)$ Yang-Mills theory of spinor and vector fields in two-dimensional spacetime. In a completely different context, Bardeen, Bars, Hanson and Peccei [3], Bars and Hanson [4] and Bars [5] have investigated models for mesons using point particles attached to the relativistic Nambu string [6]. (For brevity, we will hereafter refer to these models as "the string model.") It is remarkable that 't Hooft's field-theoretic meson spectrum, found by using null-plane quantization and the gauge $A^{+}=0$, agrees with the string model spectrum in the $x^{+}=\tau$ gauge, provided one properly identifies the renormalized spinor mass of the field theory with the pointparticle mass of the string theory.

It is clearly of interest to establish whether this correspondence is an artifact of a judicious gauge choice or is a more general property of these apparently distinct theories. The recent work of Frishman, Sachrajda, Abarbanel and Blankenbecler [7] indicates that there appear to be inconsistencies even between different axial gauge formulations of the 't Hooft model. Thus the gauge invariance of the 't Hooft field theory and the relation of the 't Hooft field theory to the point-particle string model both seem to warrant further investigation.

In this paper we shall study the appropriate bound state equations for the meson spectra of ' $t$ Hooft's model in the $A^{+}=0$ and $A_{1}=0$ axial gauges. We shall also investigate the analogous equations for the string model in the $x^{+}=\tau$ and $x^{0}=\tau$ gauges. For these purposes we shall make use of a number of numerical techniques derived from the theoretical aerodynamics of wings.

The Bethe-Salpeter equation which determines the meson bound state spectrum of the 't Hooft model is considerably more complicated when one adopts
a timelike quantization scheme and the $A_{1}=0$ gauge than when one uses 't Hooft's lightlike quantization and the $A^{+}=0$ gauge. To begin with, the timelike gauge bound state equation, unlike 't Hooft's lightlike case, becomes a $4 \times 4$ equation. Furthermore, before one can attempt to solve this equation one needs to compute the spinor self-energy exactly, which in the $A_{1}=0$ gauge appears to be a prohibitive task. For weak coupling, however, it is possible to reduce the bound state problem in the $A_{1}=0$ gauge to the problem of solving a one-dimensional integral equation similar to the one that is obtained in the $A^{+}=0$ gauge. This weak-coupling equation can be deduced either by taking an appropriate limit of the $A_{1}=0$ Bethe-Salpeter equation or, more simply, by extending Coleman's [8] weak-coupling analysis of the massive Schwinger model to an $S U(N)$ gauge theary. In the weak coupling limit of the $1 / \mathrm{N}$ expansion, we find that the numerical spectra in the timelike and null-plane approaches agree very well. Remarkably enough, the timelike weak-coupling spectrum agrees with the lightlike spectrum to within $10 \%$ far into the strong coupling regime. This is reminiscent of the behavior of the sine-Gordon theory [9].

The fact that our meson spectra calculated in two different gauges agree with one another in the weak coupling limit is not unexpected; the result is consistent with the traditional perturbation theory "proofs" of gauge invariance [10]. The inconsistencies found by Frishman et al. [7] appear in the strong-coupling limit of the theory and hence do not reflect on our weak-coupling results.

Finding solutions to the strong-coupling timelike field theory equations is extremely difficult. In the limit of vanishing bare fermion masses, however, one might hope to be able to solve these equations: in 't Hooft's null-plane analysis, the bound state equation was exactly soluble in this
limit, yielding a zero mass bound state. If the timelike theory is to be equivalent to the null-plane theory, it must then give also a zero mass bound state. While we have been able to solve the self-energy equations for imaginary values of the coupling constant in the zero-mass limit, no solutions have been found for realistic coupling constants. This has prevented us from establishing the equivalence (or inequivalence) of the bound state spectra in different axial gauges for strong-coupling.

To investigate whether the string model and the field theoretical model are equivalent in different gauges, we study the bound state equation for the string model in the timelike gauge, $x^{0}=\tau$. Our numerical results indicate that, save for very weak coupling, the string model in the $x^{0}=\tau$ gauge is not equivalent to the string model in the $x^{+}=\tau$ gauge, and also is not equivalent to 't Hooft's model in either axial gauge.

The gauge dependence of the quantum spectrum of the string is perhaps not totally unexpected. In a recent note, Bardeen, Bars, Hanson and Peccei [11] demonstrated the quantum Poincaré covariance of the string model in two dimensions and noted that the classical Hamiltonians for the model in the $x^{0}=\tau$ and $\mathrm{x}^{+}=\tau$ gauges are related by a canonical transformation. In general, the process of canonical quantization and the process of performing a canonical transformation are not commutative. Our results are an example of this effect. The Nambu string with massive end points possesses a gauge invariant classical action; however, the quantization of this string system does not preserve the gauge invariance and one obtains distinct bound state spectra for different gauge choices.

The plan of this paper is as follows. In Section I we review the 't Hooft model in the lightlike gauge and discuss the numerical solution of the $A^{+}=0$ bound state equation using Multhopp's wing theory techniques [12]. Section II
deals with the 't Hooft model in the $A_{1}=0$ gauge and in the limit of weak coupling. The meson bound state equation is derived by employing a Hamiltonian technique analogous to that used by Coleman [8] for the massive Schwinger model. Numerical solutions for the spectrum are obtained and the results are compared with those of Section $I$. In this section, we also demonstrate the covariance of the mass spectrum by examining the bound state equation in an arbitrary frame. Section III is devoted to an analysis of the bound state spectrum of the string model. We obtain the semiclassical Bohr-Sommerfeld spectra and the quantum spectra of the string model in both the $x^{+}=\tau$ and $x^{0}=\tau$ gauges. The weak-coupling limit of the $x^{0}=\tau$ gauge string bound state spectrum is found to disagree with the $A_{1}=0$ gauge 't Hooft model beginning with terms of order $G^{4}$. Section IV deals with the ' $t$ Hooft model in the $A_{1}=0$ gauge for arbitrary coupling. After deriving the coupled integral equations for the fcrmion self-energy, we discuss the inconsistencies that arise in those equations for vanishing fermion mass. We examine next the Bethe-Salpeter equation and reduce it to a Schrödinger equation for the bound state spectrum. In the weak-coupling limit, we show how the Bethe-Salpeter equation becomes equivalent to the weak coupling equation derived in Section II. Section $V$ contains concluding remarks and a summary of our results. A number of technical matters are relegated to appendices.

## I. 'T HOOF MODEL IN LIGHTLIKE GAUGE

A. Review of Formalism

The 't Hoof model for mesons in two spacetime dimensions is described by a Lagrangian density in which constituent fermions, henceforth called "quarks", interact by means of a $U(N)$ Yang-Mills gauge field. We choose for convenience to work with an $S U(N)$ gauge group [2] instead of a $U(N)$ group since to leading order in $1 / N$ as $N \rightarrow \infty$, we may ignore the distinction between $U(N)$ and $S U(N)$. We take as our Lagrangian density

$$
\begin{gather*}
\mathscr{L}\left(x_{\rho}\right)=-\frac{1}{4} F_{\mu \nu}^{a}\left(x_{\rho}\right) F_{\mu \nu}^{a}\left(x_{\rho}\right)+\bar{\psi}\left(x_{\rho}\right)\left\{i \gamma_{\mu} \partial_{\mu}-m\right\} \psi\left(x_{\rho}\right) \\
-g A_{\mu}^{a}\left(x_{\rho}\right) J_{\mu}^{a}\left(x_{\rho}\right) \tag{1.1}
\end{gather*}
$$

Here $A_{\mu}^{a}$ is the $S U(N)$ Yang-Mills "glue" field, $\psi$ is the quark field and

$$
F_{\mu \nu}^{a}=\partial_{\mu} A_{\nu}^{a}-\partial_{\nu} A_{\mu}^{a}+g f^{a b c} A_{\mu}^{b} A_{\nu}^{c} \text {. } \quad(1.2)
$$

The current $J_{\mu}^{a}$ is given by

$$
\begin{equation*}
J_{\mu}^{a}=\bar{\psi} \gamma_{\mu} \frac{1}{2} \lambda^{a} \psi \tag{1.3}
\end{equation*}
$$

The matrices $\lambda^{a}$ form the fundamental representation of $\operatorname{SU}(N)$. Some of their useful properties are listed in Appendix A along with the gauge transformation properties of $\psi$ and $A_{\mu}^{a}$.

Using null-plane quantization and the lightlike gauge $A_{a}^{+}=0$, 't Hooft [1] has solved, in the $1 / \mathrm{N}$ approximation, the quark self-energy equation and formulated the Bethe-Salpeter equation for two-body meson bound states. ' $t$ Hoof's work is valid to order $1 / \mathrm{N}$ and to all orders in ( $\mathrm{g}^{2} \mathrm{~N}$ ) as $\mathrm{N} \rightarrow \infty$ and $g \rightarrow 0$ with ( $g^{2} N$ ) kept fixed. Because the dimensionless parameter ( $g^{2} N / m^{2}$ ) is
not restricted to be small, 't Hooft's lightlike bound state equation is valid for all values of this parameter. As we shall see, this means that in this case one can obtain the meson spectrum for both weak and strong-coupling from the same equation.

The bound state equation for the meson spectrum obtained by 't toft in the $1 / \mathrm{N}$ approximation reads:

$$
\begin{align*}
M^{2} \phi(K)= & {\left[\frac{m^{2}-G^{2} / \pi}{\frac{1}{2}-K}+\frac{m^{2}-G^{2} / \pi}{\frac{1}{2}+K}\right] \phi(K) } \\
& -\frac{G^{2}}{\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} d K^{\prime} \frac{\phi\left(K^{\prime}\right)}{\left(K^{\prime}-K\right)^{2}} \tag{1.4}
\end{align*}
$$

Here $\mathrm{M}^{2}$ is the invariant mass-squared of the meson states, $m$ is the "bare" quark mass and the notation on the right-hand side indicates a principal value integral as defined in Appendix B. The coupling constant that enters in Eq. (1.4) is an effective coupling defined by

$$
\begin{equation*}
G^{2}=\frac{1}{2} g^{2} N \tag{1.5}
\end{equation*}
$$

We see explicitly that Eq. (1.4) is valid for all values of the dimensionless parameter $\left(g^{2} \mathrm{~N} / \mathrm{m}^{2}\right)$. Hence from it one can obtain the meson spectrum for both weak-coupling $\left[\left(\mathrm{g}^{2} \mathrm{~N} / \mathrm{m}^{2}\right) \ll 1\right]$ and strong-coupling $\left[\left(\mathrm{g}^{2} \mathrm{~N} / \mathrm{m}^{2}\right) \geqslant 1\right]$. The boundary conditions on 't Hooft's equation are such that

$$
\phi(k)=0 \quad \text { for } \quad|k| \geqslant \frac{1}{2} . \quad(1.6)
$$

Using Eq. (1.6) one can integrate the right-hand side of Eq. (1.4) by parts to give:

$$
\begin{gathered}
M^{2} \phi(k)=\left[\frac{m^{2}-G^{2} / \pi}{\frac{1}{2}-k}+\frac{m^{2}-G^{2} / \pi}{\frac{1}{2}+k}\right] \phi(k) \\
-\frac{G^{2}}{\pi} \int_{0}^{\frac{1}{2}} d k^{\prime} \frac{d \phi\left(k^{\prime}\right) / d k^{\prime}}{k^{\prime}-k} . \quad(1.7)
\end{gathered}
$$

This latter equation proves more convenient for numerical investigation.
One can obtain the meson bound state equation, (1.4), directly from a Hamiltonian formalism [13]. In this case the meson state $\left|\mathrm{P}^{+}\right\rangle$is given by

$$
\left.I P^{+}\right\rangle=\int_{-\frac{1}{2}}^{\frac{1}{2}} d k \phi(k) a_{\alpha}^{+}\left[P^{+}\left(\frac{1}{2}+k\right)\right] b_{\alpha}^{\dagger}\left[P^{+}\left(\frac{1}{2}-k\right)\right]|O\rangle .(1.8)
$$

Here $a_{\alpha}^{\dagger}, b_{\alpha}^{\dagger}$ are creation operators for quarks and antiquarks respectively and the bound state wavefunction $\phi(k)$ obeys Eq. (1.4). It is important to remark here that, because of the null-plane quantization used, Eq. (1.4) is manifestly Lorentz covariant. The value of the "center of mass" momentum $\mathrm{P}^{+}$has been scaled out and appears nowhere; no special choice of Lorentz frame has been necessary to find a bound state equation which directly gives the invariant mass-squared.

Using Eq. (1.6), one finds that Eq. (1.7) has a zero mass ground state, $M^{2}=0$, corresponding to the wavefunction

$$
\phi(k)=\theta\left(\frac{1}{2}-k\right) \theta\left(\frac{1}{2}+k\right)
$$

provided the bare quark mass $m$ vanishes. Since the renormalized quark mass $m_{r}$ is given by

$$
m_{r}^{2}=m^{2}-G^{2} / \pi, \quad(1.10)
$$

one concludes that a legitimate zero mass bound state occurs for a particular tachyonic renormalized mass, $m_{r}^{2}=-G^{2} / \pi$.

## B. Numerical Techniques

Equation (1.7) for the mass spectrum of the meson bound states of 't Hoof's model is not to our knowledge soluble in analytic form. Thus we must solve it numerically to determine the spectrum. A numerical solution of Eq. (1.7) has already been obtained by 't Hoof [1]. Here we develop an alternate approach which will be useful also in subsequent sections. Our method is based on Multhopp's elegant techniques for treating the aerodynamics of wings [12].

A description of Multhopp's method is summarized in Appendix C.
We begin by changing variables to the parameter $\theta$,

$$
k=-\frac{1}{2} \cos \theta, \quad\left\{\begin{array}{r}
-\frac{1}{2}<k<\frac{1}{2} \\
0<\theta<\pi
\end{array} \quad(1.11)\right.
$$

so that we can express $\phi(k)$ as a sine series compatible with the boundary conditions (1.6),

$$
\phi(k) \equiv \phi(\theta)=\sum_{j=1}^{\infty} a_{j} \sin j \theta \quad(1.12)
$$

It is convenient to define a dimensionless parameter $R$ to specify the coupling constant regime of the theory. If we take

$$
R=\frac{m^{2} \pi}{G^{2}}, \quad(1.13)
$$

then Eq. (1.10) tells us that the renormalized mass of the quark is given by

$$
m_{r}^{2}=\frac{G^{2}}{\pi}(R-1)
$$

Defining $\mathscr{E}_{\text {LG }}$ (LG - Lightlike Gauge eigenvalue) by

$$
\varepsilon_{L G}=\frac{2 M^{2}}{G^{2}}
$$

and changing variables as in Eq. (1.11), we find that Eq. (1.7) can be rewritten as

$$
\left[\begin{array}{l}
\left.\delta_{L G}-\frac{8}{\pi} \frac{(R-1)}{\sin ^{2} \theta}\right] \sum_{j=1}^{\infty} a \operatorname{aim} j \theta \\
=-\frac{1}{\pi} \sum_{j=1}^{\infty} \sum_{j}^{\infty} a \lim _{j}^{\pi} d \theta \cdot(1.16)
\end{array}\right.
$$

The integral appearing above is among those listed in Appendix B. Using that result, we find

$$
\sum_{j=1}^{\infty} a_{j}\left[\frac{4 j}{\sin \theta}+\frac{8}{\pi} \frac{(R-1)}{\sin ^{2} \theta}-C_{L G}\right] \sin j \theta=0
$$

We have thus been able to convert 't Hooft's lightlike-gauge Bethe-Salpeter equation into a linearized eigenvalue problem.

To solve Eq. (1.17) numerically we use Multhopp's wing theory techniques outlined in Appendix C. It is easily verified that the parity transformation $\kappa \rightarrow-k$ is a symmetry of the bound state integral equation (1.7). (In terms of the $\theta$ variable this parity transformation corresponds to the substitution $\theta \rightarrow \pi-\theta)$. Hence we can search for eigenvalues $\mathscr{E}_{\mathrm{LG}}^{( \pm)}$of definite parity. Using the formulas given in Appendix C, we obtain the even and odd parity eigenvalue problems in standard Multhopp form with the matrix ${ }_{L G} B_{k j}$ defined as follows:
${ }_{L G} B_{k j}=\delta_{k j}\left[\frac{2(n+1)}{\sin \theta_{k}}+\frac{8}{\pi} \frac{(R-1)}{\sin ^{2} \theta_{k}}\right]+\frac{8}{n+1} T_{n}\left(\theta_{k}, \theta_{j}\right)$
where

$$
\begin{aligned}
& T_{n}\left(\theta_{k}, \theta_{j}\right)=\frac{1}{\sin \theta_{k}} \sum_{l=1}^{n} l \sin l \theta_{k} \sin l \theta_{j} \\
& =\left\{\begin{array}{cl}
-\frac{\sin \theta_{j}}{\left(\cos \theta_{j}-\cos \theta_{k}\right)^{2}} & \text { for }|j-k|=1,3,5, \ldots \\
0 & \text { for }|j-k|=2,4,6, \ldots \\
\frac{(n+1)^{2}}{4 \sin \theta_{k}} & \text { for } j=k . \quad(1.19)
\end{array}\right.
\end{aligned}
$$

C. Numerical Results

We already know that for $m=0$, that is $R=0$, the even parity equations have an exact solution for the ground state with mass-squared eigenvalue $M^{2}=0$. This should then, in principle, be an ideal place to test our numerical procedure. Unfortunately, the wavefunction (1.9) appropriate for this case is a step function. Clearly we cannot expect that our truncated series for $\phi(\theta)$ will give a very accurate result.

It is a fact well known to numerical analysts [14] that any finite series representation for a discontinuous function exhibits the Gibbs phenomenon: even in the limit of an infinite number of terms, the series will always overshoot the true value of the function. The standard technique [14] for alleviating this problem is to improve the convergence by inserting Lanczos coefficients in the series; unfortunately the representation of the function is then severely smoothed in the region of the discontinuity.

We now give a simple demonstration that the Multhopp technique approximates discontinuous functions in a satisfactory way differing from both a sine series and a Lanczos-factor sine series. Figure 1 shows a plot of these three different ( 100 term) approximations to a segment of a step function. We see that the Multhopp approximation rises faster than the Lanczos series and overshoots less than the sine series. There is still, however, a small overshoot causing a Gibbs phenomenon which will keep our $R=0$ ground state mass from approaching zero as it should.

In Fig. 2, we illustrate the shape of the numerical ground state mass curve as a function of $R$ for several values of the Multhopp matrix size $n$. We see that for $R>0.2$, the curves agree to better than $1 \%$ for any $n>51$. As we approach $R=0$, we see that the ground state never goes below

$$
\delta_{L G}^{(+)}(R=0)=\left.\frac{2 M^{2}}{G^{2}}\right|_{R=0}=0.3614(1.20)
$$

no matter how large a matrix we use. It is fortunate that we know the exact eigenvalue at $R=0$; otherwise the Gibbs phenomenon could prevent us from ever determining the correct answer, or might even tempt us to declare falsely that Eq. (1.20) gives the true ground state eigenvalue.

Having satisfied ourselves that our numerical procedures check as well as can be expected with the only known solution of the 't Hoof equation, we proceed to examine the numerical spectra for various values of the coupling constant. Figure 3 displays the R-dependence of the even-parity ground state and the first odd-parity excited state. Our calculations were performed for values of $R$ ranging from 0.1 to 5 .

The significance of these curves is best understood by classifying the physical meaning of the various sectors of $R$ :

$$
\begin{array}{ll}
\mathrm{R}=0 & \text { - ultimate strong-coupling } \\
0 \leqq \mathrm{R} \leqq 1 & \text { - super strong-coupling } \\
1 \leqq \mathrm{R} \leqq 5 & \text { - strong-coupling } \\
5 \leqq \mathrm{R}<\infty & \text { - weak-coupling }
\end{array}
$$

The motivation for this classification is that Eq. (1.17) contains only ratios of masses and the coupling constant, namely

$$
R=\frac{m^{2} \pi}{G^{2}}, \quad G_{i G}=\frac{2 M^{2}}{G^{2}} \quad(1.21)
$$

G never appears alone and so does not determine by itself whether the theory is in a strong-coupling or weak-coupling regime. Very small bare quark mass is equally as effective as large $G^{2}$ in producing a strongly-coupled equation. Similarly, either large quark mass or small $\mathrm{G}^{2}$ gives a weak-coupling equation. $R$ is therefore the only dependable indicator of the regime of the equation. $R=0$, with vanishing bare mass (or infinite $G^{2}$ ) and tachyonic renormalized mass, is the ultimate strong-coupling limit. $R=1$ plays a special role because at this point $m_{r}=0$ and the renormalized mass makes the transition between normal and tachyonic properties. The ground state eigenvalue at $R=1$ is

$$
\varepsilon_{L G}^{(+)}(R=1)=4.63119 .(1.22)
$$

As we will see in Section 3 on string models of mesons, this is also the ground state of the massless Nambu string model quantized in the lightlike gauge. The massless string therefore corresponds to the strong coupling sector of the 't Hoof model.

For very large values of $R$, the weak coupling limit and the nonrelativistic limit of the equations are indistinguishable. When the effective quark masses become very large, the motion becomes non-relativistic. We shall see that,
in this limit, the spectra of several different equations related to 't Hooft's equation become indistinguishable.

## II. 'T HOOFT MODEL IN A TIMELINE GAUGE - WEAK COUPLING

A. Generalization of Coleman's weak coupling analysis in the $N \rightarrow \infty$ limit We mentioned in the introduction that 't Hoof's model in the axial gauge $A_{1}^{a}=0$ (which we hereafter refer to as the timelike gauge) is considerably more complicated than the lightlike gauge treatment. In this section we study the model only in the weak coupling limit ( $R \gg 1$ ). We are primarily interested in the meson bound state equation. A physically transparent derivation of this equation may be found by generalizing Coleman's [8] weak-coupling analysis of the massive Schwinger model to the $\operatorname{SU}(\mathrm{N})$ "color" gauge group in the context of the $N \rightarrow \infty$ limit [15].

The Lagrangian for the model has already been given in Eq. (1.1). The theory is super-renormalizable; it requires no infinite renormalization other than standard normal ordering, which amounts to a trivial redefinition of the zero of the energy density. We want to write the theory in Hamiltonian form and retain only the independent degrees of freedom. To do this we must choose a gauge and, as indicated above, we choose the timeline gauge $A_{1}^{a}=0$. The field equation for $A_{0}^{a}$,

$$
\left(\partial_{1}\right)^{2} A_{0}^{a}=-g J_{0}^{a}=-g: \psi^{\dagger} \frac{1}{2} \lambda_{a} \psi: \quad(2.1)
$$

then becomes an equation of constraint. We note that there are no true dynamical degrees of freedom associated with the gauge field because in one space dimension there are no transverse directions. We solve Eq. (2.1) by means of the Coulomb Green's function in one space dimension:

$$
\left\langle x^{\prime}\right|\left(\partial_{1}\right)^{-2}|x\rangle=\frac{1}{2}\left|x-x^{\prime}\right| . \quad 2.2
$$

In (2.2) we have dropped a term proportional to ( $x-x^{\prime}$ ) which corresponds to a constant background colored "electric" field. The general solution to Eq. (2.1) is then

$$
A_{0}^{a}=-g\left(\partial_{1}\right)^{-2} J_{0}^{a}
$$

$$
(2.3)
$$

and the corresponding electric field is

$$
F_{01}^{a}=g\left(\partial_{1}\right)^{-1} J_{0}^{a} \quad(2.4)
$$

We can now find the Hamiltonian density:

$$
f l=: \bar{\psi}\left(+i \gamma_{1} \partial_{1}+m\right) \psi:+\frac{1}{2}: F_{01}^{a} F_{01}^{a}:(2.5)
$$

Restricting ourselves to color neutral states,

$$
Q^{a}=\int_{-\infty}^{+\infty} d x J_{0}^{a}(x)=0
$$

a simple integration by parts yields the Hamiltonian

$$
\begin{aligned}
H= & \int_{-\infty}^{+\infty} d x: \bar{\psi}\left(i \gamma_{1} \partial_{1}+m\right) \psi: \\
& -\frac{g^{2}}{4} \int_{-\infty}^{\infty} d x \int_{-\infty}^{\infty} d y: J_{0}^{a}(x) J_{0}^{a}(y):|x-y|:(2-7)
\end{aligned}
$$

The requirement (2.6) is necessary since the presence of uncompensated color charge in space would lead to a growth of the field at infinity.

In the $N \rightarrow \infty$ limit of the model, the weak coupling regime occurs when the dimensionless ratio $\left(\mathrm{g}^{2} \mathrm{~N} / \mathrm{m}^{2}\right)$ is much less than unity. In this region
we can compute $H$ restricted to the two particle quark-antiquark subspace. Let $q$ be the momentum of the quark and $s$ be the momentum of the antiquark, and let $P=q+s$ be the total center-of-mass momentum of the bound state. Then we can define the two-particle bound state $|P\rangle$ as

$$
|P\rangle=\int_{-\infty}^{+\infty} d q \phi_{p}(q) a_{\alpha}^{\dagger}(q) b_{\alpha}^{+}(p-q)|0\rangle
$$

To obtain the Schrödinger equation for the quark-antiquark bound state, we operate on Eq. (2.8) with the normal ordered Hamiltonian (2.7). A sketch of the calculation is presented in Appendix $D$, which also contains the details of the Dirac field expansions and a list of our conventions. In this calculation we retain only leading order terms in the $1 / \mathrm{N}$ expansion. Hence, for example, quark-antiquark annihilation terms in the Hamiltonian do not contribute. Furthermore, terms in $H$ which, when acting on $|P\rangle$, describe the production of fermion pairs are neglected. These terms are of higher order in ( $g^{2} N / m^{2}$ ) and hence need not be kept in the present weak-coupling approximation. Their effect can be taken into account, if necessary, with the aid of ordinary perturbation theory. However, for strong-coupling, it is more expeditious to deal directly with the Bethe-Salpeter equation. (We shall do this in Section IV.)

Given the above stated approximations, the action of the Hamiltonian (2.7) on the state $|P\rangle$ gives rise to three kinds of terms:
(1) Free particle kinetic energies for the quark and the anti-quark
(2) First order self-energy corrections for the bound state constituents
(3) Coulomb binding potential between the quark and anti-quark We arrive in this way at the following Schrodinger equation for the meson spectra, valid in the weak coupling regime, $G^{2} / \mathrm{m}^{2} \ll 1$ :

$$
\begin{gathered}
H \phi_{P}(q)=W_{P} \phi_{P}(q)=\left(E_{q}+E_{P-q}\right) \phi_{P}(q)-\frac{G^{2}}{2 \pi}\left[\frac{1}{E_{q}}+\frac{1}{E_{P-q}}\right] \phi_{P}(q \\
-\frac{G^{2}}{2 \pi} \int_{-\infty}^{+\infty} d q^{\prime} \frac{\phi_{P}\left(q^{\prime}\right)}{\left(q^{\prime}-q\right)^{2}}\left[\frac{E_{q} E_{q}+q q^{\prime}+m^{2}}{2 E_{q} \cdot E_{q}}\right]^{1 / 2}\left[\frac{E_{P-q^{\prime}} E_{P-q}+(P-q)(P-q)+m^{2}}{2 E_{P-q} E_{P-q}}\right] .
\end{gathered}
$$

Here $W_{p}$ is the eigenvalue of the operator $H$ corresponding to the eigenfunction $\phi_{\mathrm{P}}(\mathrm{q})$ and $\mathrm{E}_{\mathrm{q}}=\left[\mathrm{q}^{2}+\mathrm{m}^{2}\right]^{1 / 2}$. This eigenvalue equation for $H$ assumes an especially simple form in the center-of-mass (C.M.) frame $P=0$ :

$$
\begin{aligned}
& H \phi_{0}(q)=W_{0} \phi_{0}(q)=2 E_{q} \phi_{0}(q)-\frac{G^{2}}{\pi E_{q}} \phi_{0}(q) \\
& -\frac{G^{2}}{2 \pi} \int_{-\infty}^{+\infty} d q^{\prime} \frac{\phi_{0}\left(q^{\prime}\right)}{\left(q^{\prime}-q\right)^{2}}\left[\frac{E_{q} E_{q}+q^{\prime} q+m^{2}}{2 E_{q} E_{q}}\right]
\end{aligned}
$$

The spectrum of Eq. (2.9) would be by definition Lorentz covariant
provided we could show that

$$
M^{2} \equiv W_{0}^{2}=W_{P}^{2}-P^{2}
$$

Because of the two-particle approximation employed in deriving (2.9), we do not expect (2.11) to hold exactly.

Our numerical techniques allow us to evaluate each side of Eq. (2.11) indpendently as a check of covariance. We will see in the course of the next two subsections that Eq. (2.11) holds for all practical purposes for weak coupling, so that in this regime we have demonstrated Lorentz covariance.
B. Numerical Techniques

Our numerical investigation of the weak coupling 't Hoof model in the timelike gauge consists of two distinct parts. First we study the bound state equation (2.10) for the quark-antiquark system in the C.M. frame $P=0$. We then study the bound state equation (2.9) where the total C.M. momentum is nonzero, $P \neq 0$. We now outline the details of the numerical techniques for these two parts separately:
(1) C.M. Frame, $P=0$

The bound state equation that concerns us is (2.10). To solve the aquation numerically we introduce an angle $\theta$ such that:

$$
q=-m \cot \theta,\left\{\begin{array}{r}
-\infty<q<\infty \\
0<\theta<\pi
\end{array} \quad(2.12)\right.
$$

We now express $\phi_{0}(q)$ in a sine series which incorporates the boundary condition that $\phi_{0}(q)$ vanishes at $q= \pm \infty$ :

$$
\phi_{0}(q) \equiv \phi_{0}(\theta)=\sum_{j=1}^{\infty} a_{j} \sin j \theta
$$

Defining $\mathscr{E}_{\text {PG }}(T G=$ Timeline Gauge eigenvalue) by

$$
\sum_{T G}=\frac{\Delta \Delta m \pi}{G^{2}}=\frac{\Delta \Delta R}{m} \quad(2.14)
$$

the bound state equation (2.10) takes the following form in our new parametrization:

$$
\begin{aligned}
& E_{T G} \phi_{0}(\theta)=\left[\frac{2 R}{\sin \theta}-\sin \theta\right] \phi_{0}(\theta) \\
& \left.+\frac{\sin ^{2} \theta}{4}\right]_{0} \int_{0}^{\pi} \frac{d \theta_{0} \theta^{\prime}}{d \theta^{\prime}}\left[\frac{\sin \theta^{\prime}+\sin \theta}{\cos \theta^{\prime}-\cos \theta}\right]
\end{aligned}
$$

To obtain Eq. (2.15), we have integrated the integral on the right-hand side of $\mathrm{Eq} .(2.10)$ by parts. The bound state equation (2.10) (or equivalently (2.15)) is invariant under the parity transformation $q \rightarrow-q$. (For Eq. (2.15), the transformation is $\theta \rightarrow \pi-\theta$ ). Hence we can search for eigenvalues $\mathcal{Q}_{\mathrm{GG}}^{( \pm)}$ which possess definite parity.

Employing again Multhopp's wing theory techniques, the eigenvalue problem for even and odd parity reduces to the standard form given in Appendix C. The Multhopp matrix $\mathrm{TG}^{\mathrm{B}_{\mathrm{kj}}}$ for this case is given by

$$
\begin{aligned}
& T G B_{k j}=\delta_{k j}\left[\frac{2 R}{\sin \theta_{k}}-\sin \theta_{k}\right] \\
& +\frac{\pi T_{n}\left(\theta_{k}, \theta_{j}\right) \sin ^{3} \theta_{k}}{2(n+1)}+\frac{\sin ^{2} \theta_{k}}{2(n+1)} \sum_{l=1}^{n} \sum_{l=1}\left(\theta_{k}\right) \sin j \theta_{l}\left(2 i_{1}\right.
\end{aligned}
$$

where $T_{n}\left(\theta_{k}, \theta_{j}\right)$ is given in Eq. (1.19) and the integral $B_{l}(\theta)$ is:

$$
B_{l}(\theta)=\int_{0}^{\pi} d \theta^{\prime} \frac{\sin \theta^{\prime} \cos \theta^{\prime}}{\cos \theta^{\prime}-\cos \theta} . \quad(2.1)
$$

This integral is evaluated in Appendix B.
(2) Total C.M. Momentum, $P \neq 0$

The bound state equation that concerns us here is (2.9). We still employ the parametrization (2.12) and expand $\phi_{P}(\theta)$ in the sine series:

$$
\begin{equation*}
\theta_{p}(q)=\psi_{p}(\theta)=\sum_{j=1}^{\infty} a_{j}^{P} \sin _{j}^{\infty} \theta \tag{2.18}
\end{equation*}
$$

It is evident that the expression

$$
F\left(q, q^{\prime}, p\right)=\left[\frac{E_{q^{\prime}} E_{q}+q^{\prime} q+m^{2}}{2 E_{q}, E_{q}}\right]^{1 / 2}\left[\frac{E_{p-q} E_{p-q}+\left(P-q^{\prime}\right)(p-q)+m^{2}}{2 E_{P-q}, E_{p-q}}\right]^{1 / 2}
$$

occurring in the integrand of the bound state equations (2.9) makes exact integration fairly hopeless. Therefore, we shall proceed to numerically evaluate the integral on the right-hand side of Eq. (2.9) before employing Multhopp's standard wing theory techniques. However, because the integral in Eq. (2.9) is a principal value integral, we must first isolate the singular pieces of the integral and compute them analytically; then we may evaluate the remaining nonsingular pieces numerically by standard quadrature techniques.

To isolate the singular parts of the integral in (2.9) we make a Taylor expansion of $F\left(q, q^{\prime}, p\right)$ in $\left(q-q^{\prime}\right)$ to give:

$$
F\left(q, q^{\prime} p\right)=1-\frac{m^{2}}{8}\left[\frac{1}{E_{q}}+\frac{1}{E_{q}}\right]\left(q-q^{1}\right)^{2}+\ldots .(2.20)
$$

It is now clear that the integral in (2.9) has only one singular piece with the integrand having a double pole. Isolating this singularity, we rewrite the bound state equation (2.9) as follows:

$$
\begin{aligned}
& W_{p} \phi_{p}(q)=\left(E_{q}+E_{p-q}\right) \phi_{p}(q)-\frac{G^{2}}{\pi}\left[\frac{1}{E_{q}}+\frac{1}{E_{p-q}}\right] \phi_{P}(q) \\
& -\frac{G^{2}}{2 \pi} \iint_{-\infty}^{+\infty} d q^{\prime} \frac{\phi_{p}\left(q^{\prime}\right)}{\left(q^{\prime}-q\right)^{2}}-\frac{G^{2}}{2 \pi} \int_{-\infty}^{+\infty} d q^{\prime} \frac{\phi_{p}\left(q^{\prime}\right)}{\left(q^{\prime}-q\right)^{2}}\left[F\left(q, q^{\prime}, p\right)-1\right] .(2.21)
\end{aligned}
$$

The second integral in the right-hand side of Eq. (2.21) does not contain a principal value prescription since it is perfectly non-singular.

Since the total C.M. momentum $P$ is non-vanishing, it is clear that the bound state equation (2.21) is not invariant under a parity transformation. Thus we cannot classify the eigenvalues by parity. We can, of course, for a given $P$ order the eigenvalues so that the lowest corresponds to the ground state even parity solution with a nonzero C.M. momentum $P$, the next lowest eigenvalue corresponds to the first odd parity excited state solution with a nonzero $C . M$. momentum $P$, etc.

Introducing the notation

$$
\widetilde{P}=P / m, \quad \varepsilon_{P}=\frac{W_{P} m \pi}{G^{2}}, \quad \text { (2.22) }
$$

the eigenvalue problem becomes

$$
\varepsilon_{P} \phi_{P}\left(\theta_{k}\right)=\sum_{j=1}^{n} B_{k j} \phi_{P}\left(\theta_{j}\right), \quad(2.23)
$$

where the standard matrix $P_{k j}$ is defined as follows:

$$
\begin{aligned}
P^{B_{k j}}= & \delta_{k j}\left\{\frac{R}{\sin \theta_{k}}+R\left[\left(\tilde{P}+\cot \theta_{k}\right)^{2}+1\right]^{1 / 2}-\frac{1}{2} \sin \theta_{k}-\frac{1}{2\left[\left(\tilde{P}+\cot \theta_{k}\right)^{2}+1\right]^{1 / 2}}\right\} \\
& -\frac{\sin ^{2} \theta_{k}}{(n+1)} \sum_{l=1}^{n}\left[D_{l}^{P}\left(\theta_{k}\right)-\ell C_{l}\left(\theta_{k}\right)\right] \sin \ell \theta_{j} \quad(2.24)
\end{aligned}
$$

In the above the integral $C_{\ell}\left(\theta_{k}\right)$ is given by

$$
C_{l}(\theta)=\int_{0}^{\pi} d \theta^{\prime} \cos \ell \theta^{\prime} \cot \left(\theta-\theta^{\circ}\right) \quad(2.25)
$$

and is evaluated in Appendix $B$. The quantity $D_{j}^{P}\left(\theta_{k}\right)$ is defined by:

$$
D_{j}^{P}\left(\theta_{k}\right)=\frac{\pi}{\left(n^{2}+1\right)} \sum_{s=1}^{n^{\prime}}\left\{\begin{array}{c}
\frac{\left[F\left(\theta_{k}, \theta_{s}, \tilde{P}\right)-1\right] \sin j \theta_{s}}{\sin ^{2}\left(\theta_{k}-\theta_{s}\right)} \quad \text { for } \theta_{k} \neq \theta_{s} \\
-\frac{1}{8}\left[\sin ^{4} \theta_{k}+\frac{1}{\left.\left[1+\left(\tilde{P}+\cot \theta_{k}\right)^{2}\right]^{2}\right]} \frac{\sin j \theta_{k}}{\sin ^{4} \theta_{k}} \quad \text { for } \theta_{k}=\epsilon\right.
\end{array}\right.
$$

$$
\begin{aligned}
F\left(\theta, \theta^{\prime}, \tilde{P}\right) & =F\left(q=-m \cot \theta, q^{\prime}=-m \cot \theta^{\prime}, P=m \tilde{P}\right) \\
\theta_{s} & =\frac{5 \pi}{n^{\prime}+1}, \quad s=1,2, \ldots, n^{\prime}
\end{aligned}
$$

The expression (2.26) represents the Multhopp quadrature procedure for the nonsingular integral on the right-hand side of Eq. (2.21).

## C. Numerical Results

Equations (2.9) and (2.10) for the $\mathrm{P} \neq 0$ spectrum and the rest frame spectrum of the timeline gauge weak coupling 't Hoof model can be solved numerically using the Multhopp technique with matrices (2.24) and (2.16), respectively. In Fig. 4 we plot the rest frame time like gauge ground state and first excited states eigenvalues $M_{T G}^{2}$ and compare them with the corresponding 't Hoof eigenvalues $M_{L G}^{2}$. For increasing $R$ (the weak coupling limit) the two eigenvalues approach each other very closely and eventually become indistinguishable to five decimal places in our numerical computations.

The timeline ground state eigenvalue $\mathrm{M}_{\mathrm{TG}}^{2}$ continues to remain within $10 \%$ of the corresponding light like gauge eigenvalue $M_{L G}^{2}$ down to $R=0.5$. We find it remarkable that a weak-coupling equation should agree with a strong-coupling equation so far into the strong coupling regime. This behavior is reminiscent
of that noted by Coleman [9] for the sine-Gordon equation. We emphasize, however, that $M_{T G}^{2}$ is to be trusted only for large $R$ and that extrapolating the weak-coupling equation to small R , though amusing, may not be physically relevant.

We now turn, as promised, to checking the covariance of the $\mathrm{P} \neq 0$ time like gauge bound state equation (2.9). For this purpose, we plot the ratio of the two sides of Eq. (2.11) vs. R in Fig. 5 for various values of $P$. In the notation of Eqs. (2.14) and (2.22) this ratio takes the form

$$
\begin{equation*}
z=\frac{\sum_{P}^{2}-R^{2} \tilde{P}^{2}}{\mathcal{E}_{T G}^{2}} \tag{2.28}
\end{equation*}
$$

In particular, the plot of the ratio $Z$ at $P=0$ checks the numerical accuracy of the extra quadrature (2.26) which was necessary in the solution of Eq. (2.23), but which did not occur in Eq. (2.15). We see from Fig. 5 that for values of $R$ and $P$ for which our numerical procedures are expected to be accurate, the ratio $Z$ for the ground state spectrum is typically

$$
\begin{array}{r}
Z(R=1, \widetilde{P}=1)=1.0049, Z(R=100, \tilde{P}=1)=1.000009 \\
Z(R=1, \tilde{P}=10)=1.0584, Z(R=100, \tilde{P}=10)=1.000087 \tag{2.29}
\end{array}
$$

We have therefore checked both gauge invariance and Lorentz covariance of the 't Hoof model in the weak-coupling regime.
III. BOUND STATES OF THE STRING MODEL

In the string model approach to the meson spectra in two dimensions $[3,4,5]$ one assumes that mesons are bound states of two massive particles described by the coordinates $x^{\mu}(1) \equiv x^{\mu}\left(\tau, \sigma=\sigma_{1}\right)$ and $x^{\mu}(2) \equiv x^{\mu}\left(\tau, \sigma=\sigma_{2}\right)$. The Nambu string action [6] generates the required binding potential between the two particles. In this way one arrives at the meson action functional

$$
S=\int_{\tau_{1}}^{\tau_{2}} d \tau\left\{L_{0}\left(\sigma=\sigma_{1}\right)+L_{0}\left(\sigma=\sigma_{2}\right)-\gamma \int_{\sigma_{1}}^{\tau_{2}} d \sigma[-G]^{1 / 2}\right\}
$$

Here $L_{0}(\sigma)$ is the particle Lagrangian corresponding to the desired model for the meson constituents, and

$$
\begin{equation*}
-\gamma[-G]^{1 / 2}=-\gamma\left[\left(x_{\pi} x_{\sigma}\right)^{2}-x_{\pi}^{2} x_{\sigma}^{2}\right]^{1 / 2} \tag{3.2}
\end{equation*}
$$

is the Nambu string Lagrangian density. We define

$$
x_{\sigma}^{\mu}=\frac{\partial x^{\mu}(2, \sigma)}{\partial \sigma}, \quad x^{\mu}=\frac{\partial x^{\mu}(\tau, \sigma)}{\partial \tau} \quad(3,3)
$$

A. I,ightlike Gauge Point Particle

Let us take for $L_{0}(\sigma)$ the standard relativistic point particle Lagrangian

$$
L_{0}\left(\sigma=\sigma_{i}\right)=-\mu_{i}\left|\chi_{\tau}^{2}\left(\tau_{i} \sigma_{i}\right)\right|^{1 / 2} \quad(3.4)
$$

If we then choose the lightlike $\tau$-gauge

$$
x^{+}(\tau, \sigma)=\frac{1}{\sqrt{2}}\left(x^{0}+x^{1}\right)=\tau \quad(3.5)
$$

and the $\sigma$-gauge

$$
\begin{gather*}
x^{-}(\tau, \sigma)=\frac{1}{\sqrt{2}}\left(x^{0}-x^{1}\right) \\
=\frac{1}{2}\left[x^{-}\left(\tau, \sigma_{1}\right)+x^{-}\left(\tau, \sigma_{2}\right)\right] \\
+\frac{1}{2}\left[x^{-}\left(\tau, \sigma_{1}\right)-x^{-}\left(\tau_{1} \sigma_{2}\right)\right] \cos \left[\frac{\pi\left(\sigma-\sigma_{1}\right)}{\sigma_{2}-\sigma_{1}}\right] \tag{3.6}
\end{gather*}
$$

we get a manifestly invariant expression for the mass-squared of the bound state $[3,4,5]$ :

$$
M^{2}=2 P^{+} P^{-}=\frac{\mu_{1}^{2}}{\frac{1}{2}-k}+\frac{\mu_{2}^{2}}{\frac{1}{2}+k}+2 \gamma|\rho| .(3.7)
$$

The Bohr-Sommerfeld approximation to the quantum spectrum was found previously [3, 4] to be given by

$$
\begin{aligned}
& 2 \pi \hbar \gamma(n+\text { constant })=\Delta^{1 / 2}\left(M_{1} \mu_{1}, \mu_{2}\right) \\
& -2 \mu_{1}^{2} \ln \left[\frac{1}{2 M \mu_{1}}\left(M^{2}+\mu_{1}^{2}-\mu_{2}^{2}+\Delta^{1 / 2}\right)\right] \\
& -2 \mu_{2}^{2} \ln \left[\frac{1}{2 M \mu_{2}}\left(M^{2}+\mu_{2}^{2}-\mu_{1}^{2}+\Delta^{1 / 2}\right)\right](3.8
\end{aligned}
$$

where

$$
\Delta=\Delta\left(M, \mu_{1}, \mu_{2}\right)=\left[M^{2}-\left(\mu_{1}+\mu_{2}\right)^{2}\right]\left[M^{2}-\left(\mu_{1}-\mu_{2}\right)^{2}\right](3.9
$$

and $\mathrm{n}=0,1,2 \ldots$
The Schrödinger equation for the quantum mechanical mass-squared spectrum is found by representing $|\rho|$ in Eq. (3.7) as an integral operator with the $M^{2} \phi(k)=\left[\frac{\mu_{1}^{2}}{\frac{1}{2}-k}+\frac{\mu_{2}^{2}}{\frac{1}{2}+k}\right] \phi(k)-\frac{2 \gamma}{\pi} \int_{-1 / 2}^{+1 / 2} d k^{\prime} \frac{\phi\left(k^{\prime}\right)}{\left(k^{\prime}-k\right)^{2}} \cdot(3.10)$

Clearly, Eq. (3.10) is identical to 't Hoof bound state equation (1.4) in the $A^{+}=0$ gauge provided we set $2 \gamma=G^{2}$ and take the string model masses equal to the renormalized field theory mass:

$$
\mu_{1}^{2}=\mu_{2}^{2}=m_{r}^{2}=m^{2}-G^{2} / \pi
$$

With these identifications, the numerical results for the spectrum of the lightlike string are therefore identical to those we found in Section $I$ for 't Hooft's model in the lightlike gauge. In particular, the unfolded mode of the massless Nambu string [3], with

$$
\begin{equation*}
\mu_{1}^{2}=\mu_{2}^{2}=0 \tag{3.12}
\end{equation*}
$$

is identical with 't Hoof's model at $R=1$. The massless Nambu string is thus interpretable as a strong coupling theory.

Recalling that when $m=0$ 't Hoof's lightlike-gauge model has a zero mass bound state, we see that the string model has the same solution when the string model masses become tachyonic:

$$
\begin{equation*}
M_{1}^{2}=\mu_{2}^{2}=-G^{2} / \pi=-2 \gamma / \pi \tag{3.13}
\end{equation*}
$$

It is indeed remarkable that the string model in the lightlike gauge has a perfectly respectable zero mass bound state with tachyonic constituents! The linear potential due to the string is so strong that it overpowers the tachyonic nature of the constituents, which could not appear in a physical theory by themselves.

One can straightforwardly extend this argument to the N-fold lightlikegauge string [3]. The invariant mass-squared is [11]

$$
\begin{aligned}
& M^{2}=\frac{\mu_{1}^{2}}{\frac{1}{N}+k_{1}}+\sum_{v=2}^{N-1} \frac{\mu_{\nu}^{2}}{\frac{1}{N}+k_{v}-k_{v-1}} \\
& \quad+\frac{\mu_{N}^{2}}{\frac{1}{N}-K_{N-1}}+2 \gamma \sum_{v=1}^{N-1}\left|\rho_{v}\right|
\end{aligned}
$$

and the form of the Schrödinger equation analogous to (3.10) follows at once. The wave function

$$
\begin{aligned}
& \phi\left(K_{1}, K_{2}, \ldots, K_{N-1}\right)=\theta\left(\frac{1}{N}+K_{1}\right) \theta\left(\frac{1}{N}-K_{N-1}\right) \prod_{\nu=2}^{N-1} \theta\left(\frac{1}{N}+K_{\nu-1}-K_{\nu}\right) \\
& \text { has the eigenvalue } M^{2}=0 \text { provided that }
\end{aligned}
$$

$$
\begin{aligned}
& \mu_{1}^{2}=\mu_{N}^{2}=-2 \gamma / \pi \\
& \mu_{2}^{2}=\mu_{3}^{2}=\cdots=\mu_{N-1}^{2}=-4 \gamma / \pi
\end{aligned}
$$

Thus we have noticed an entire new class of exact solutions to the $N$-fold massive string in the lightlike gauge, with certain tachyonic constituent masses producing zero-mass ground states.
B. Timelike Gauge Point Particles

The time like gauge point particle system follows from the action (3.1) and the relativistic point particle Lagrangian (3.4) when we choose the timeline $\tau$-gauge

$$
x^{0}(\tau, \sigma)=\tau
$$

$$
(3.17)
$$

and the $\sigma$-gauge

$$
\begin{aligned}
& x^{1}(\tau, \sigma) \equiv x(\tau, \sigma)=\frac{1}{2}\left[x\left(\tau, \sigma_{1}\right)+x\left(\tau, \sigma_{2}\right)\right] \\
& +\frac{1}{2}\left[x\left(\tau, \sigma_{1}\right)-x\left(\tau, \sigma_{2}\right)\right] \cos \left[\frac{\pi\left(\sigma-\sigma_{1}\right)}{\sigma_{2}-\sigma_{1}}\right]
\end{aligned}
$$

When we transform to center-of-mass variables as in Ref. 11, we find the Hamiltonian

$$
H=\left[P^{2}+M^{2}(q, r)\right]^{1 / 2} \quad(3,19)
$$

where

$$
A(q, r)=\left[q^{2}+\mu_{1}^{2}\right]^{1 / 2}+\left[q^{2}+\mu_{2}^{2}\right]^{1 / 2}+6|r| \quad(3.20)
$$

The Bohr-Sommerfeld spectrum of the theory is again of the form given in Eq. (3.8)-(3.9). The Schrödinger equation for the mass spectrum is now found from (3.20) by representing $|r|$ as an integral operator. This yields

$$
\begin{aligned}
& M \phi(q)=\left\{\left[q^{2}+\mu_{1}^{2}\right]^{1 / 2}+\left[q^{2}+\mu_{2}^{2}\right]^{1 / 2}\right\} \phi(q) \\
& -\frac{\gamma}{\pi} \int_{-\infty}^{+\infty} d q^{\prime} \frac{\phi\left(q^{\prime}\right)}{\left(q^{\prime}-q\right)^{2}}
\end{aligned}
$$

Imposing the boundary condition

$$
\phi(q= \pm \infty)=0
$$

$$
(3.22)
$$

we may integrate the right-hand side of Eq. (3.21) by parts obtaining:

$$
\begin{align*}
M \phi(q) & =\left\{\left[q^{2}+\mu_{1}^{2}\right]^{1 / 2}+\left[q^{2}+\mu_{2}^{2}\right]^{1 / 2}\right\} \phi(q) \\
& -\frac{\gamma}{\pi} \int_{-\infty}^{+\infty} d q^{\prime} \frac{d \phi\left(q^{\prime}\right) / d q^{\prime}}{q^{\prime}-q} \tag{3.23}
\end{align*}
$$

We observe that Eq. (3.21) and Eq. (2.10), the bound state equation for the weak coupling ' $t$ Hoof model in the $A_{1}=0$ gauge, are not the same. Making the identifications of parameters as in Eq. (3.11), remembering that $2 \gamma=G^{2}$, and expanding the kinetic energy term does reproduce the first two terms in Eq. (2.10) in the weak coupling limit, $\mathrm{G}^{2} / \mathrm{m}^{2} \ll 1$. However, the kernels remain inequivalent because of the additional kinematic factor

$$
\frac{E_{q^{\prime}} E_{q}+q^{\prime} q+m^{2}}{2 E_{q^{\prime}} E_{q}}
$$

$$
(3.24)
$$

originating from the quark spinors. This factor is unimportant for extremely weak-coupling (the nonrelativistic limit) but it does produce some differences for stronger coupling.

The essential point that needs to be made is that Eq. (3.21) is the bound state equation of the time like string model for any value of $\gamma$ and $\mu$ (ie. for any coupling). As we shall discuss in Sec. IV, the Bethe-Salpeter bound state equation for the strong-coupling timelike 't Hooft model apparently has a structure differing considerably from Eq. (3.21). It therefore seems doubtful that the timelike string model is equivalent to the timeline 't Hoof model, even though the two lightlike-gauge models have identical bound state spectra. Furthermore, as we shall soon discuss, the timelike string and the lightlike string have inequivalent quantum spectra.
C. Numerical Techniques

Equation (3.23) possesses no known exact solutions, so we must again use numerical techniques to analyze its spectrum. We shall consider only the case in which $\mu_{1}=\mu_{2}=\mu$, and transform to a new parameter $\theta$ :

$$
q=-\gamma^{1 / 2} \cot \theta \quad\left\{\begin{array}{r}
-\infty<q<\infty \quad(3.25) \\
0<\theta<\pi
\end{array}\right.
$$

We find it convenient to define the Timelike String eigenvalue

$$
\begin{equation*}
\varepsilon_{T S}=M / \gamma^{1 / 2} \tag{3.26}
\end{equation*}
$$

and to define $R$ by

$$
\frac{\mu^{2}}{\gamma}=\frac{2}{\pi}(R-1)
$$

$$
(3.27)
$$

This last relation is suggested by the identification of $\mu^{2}$ with the quark renormalized mass-squared (1.14). Equation (3.23) now can be written as

$$
\begin{gather*}
\left.\operatorname{Cis}_{1} \prod_{0}+\frac{1}{\pi}+\cot ^{2} \theta+\frac{2}{\pi}(R-1)\right]^{1 / 2} 0(\theta)
\end{gather*}
$$

We note that with $q$ defined as in (3.25), the parity transformation

$$
q \rightarrow-q, \quad \theta \rightarrow \pi-\theta \quad(3.24)
$$

is a symmetry of Eq. (3.28). Hence we will again be able to search for definite parity eigen values $\stackrel{\AA}{\mathrm{G}}_{( \pm)}^{( \pm)}$.

Expanding $\phi(\theta)$ in the by now familiar sine series

$$
\phi(\theta)=\sum_{j=1}^{\infty} a_{j} \sin j \theta,
$$

we can express the eigenvalue problem in the standard Multhopp form of Appendix
C. The matrix ${ }_{T S}{ }^{B_{k j}}$ for the problem at hand is given by

$$
\begin{align*}
& T S B_{k j}=2 \delta_{k j}\left[\cot ^{2} \theta_{k}+\frac{2}{\pi}(R-1)\right]^{1 / 2} \\
& +\frac{2 \sin ^{2} \theta_{k}}{\pi(n+1)} \sum_{l=1}^{n} \ell C_{l}\left(\theta_{k}\right) \sin j \theta_{l} \tag{3.31}
\end{align*}
$$

The function $C_{\ell}\left(\theta_{k}\right)$ appearing in Eq. (3.31) has been defined previously in Eq. (2.25).
D. Numerical Results

Solving the eigenvalue equation (3.28), we find the ground state and the first excited state eigenvalues plotted in Fig. 6 for strong-coupling, $1.0 \leqslant R \leqslant 5.0$. For comparison we have plotted in Fig. 6 the corresponding eigenvalues of the lightlike string. The spectra are clearly different for this R-range. In the timeline gauge, the square root in the kinetic energy prevents us from going below $R=1.0$. At $R=1.0$, where $\mu^{2}=0$, the system corresponds to the no-fold massless Nambu string [3] and has a ground state

$$
\left[\mathcal{E}_{T S}^{(+)}(R=1.0)\right]^{2}=\left.\frac{M^{2}}{\gamma}\right|_{R=1.0}=2.43699 .(3.32)
$$

The timeline gauge ground state is therefore lower than the lightlike gauge ground state

$$
\sum_{L G}^{(+)}(R=1.0)=\left.\frac{2 M^{2}}{G^{2}}\right|_{R=1.0}=4.63119 \quad(3.33)
$$

discussed in Section I and Section III.A. We have here a clear example of the well-known ambiguity in the canonical quantization procedure [11]: If one chooses different canonically equivalent classical variables, the corresponding
quantum systems may be inequivalent.
Quantizing the two-dimensional string model in different gauges produces different spectra, even though the classical systems are canonically equivalent. Furthermore, this discrepancy cannot be resolved by examining Poincaré covariance: it has been shown [11] that both the gauges $x^{0}=\tau$ and $x^{+}=\tau$ can be formulated so as to produce Poincaré-covariant quantum theories. We are faced with the fact that canonical transformations and canonical quantization are operations which do not in general commute. New criteria are needed to choose between inequivalent systems arising in this way.

## E. Comparison of Bohr-Sommerfeld Spectra to Quantum Spectra

The Bohr-Sommerfeld quantum spectrum (3.8) must approximate the true quantum spectra of both the timelike and lightlike string model in the semiclassical limit. We therefore compare the Bohr-Sommerfeld and the quantum spectra in the limit of large principal quantum numbers, which we denote here by $\mathrm{n}=0,1,2 \ldots$

To find the Bohr-Sommerfeld spectrum, we must solve numerically the transcendental equation (3.8) for $M^{2}$ as a function of $R=1+\pi \mu^{2} / 2 \gamma$, the principal quantum number $n$ and the Bohr-Sommerfeld constant. We know of no direct way to compute the constant that enters in the Bohr-Sommerfeld equation for the string model. For most of our work we have adjusted the constant so that the ground state of the Bohr-Sommerfeld spectrum and that of the quantum spectrum agree. Since the ground state mass is different for each value of $R$ and for each gauge, the constant was readjusted for each case considered. Clearly there are other ways in which constants could enter the semiclassical spectrum and other ways to determine the semiclassical ground state; our choice is just a convenient guess.

Figure 7 shows the Bohr-Sommerfeld spectrum versus principal quantum number for various $\mathrm{R}^{\prime}$ s. These curves were computed with vanishing constant in (3.8); if one uses constants that fit the ground state, the curves are nearly identical to those shown. In order to exhibit the differences between the quantum and semiclassical spectrum we plot

$$
\Delta_{T S}(n, R)=\varepsilon_{T S}^{2}-\left.\frac{M^{2}}{\gamma}\right|_{\text {Bohr-Sominerfeld }} \quad(3.34)
$$

and
versus $n$ in Fig. 8. For this latter graph we have forced the ground states to agree separately for each gauge. We observe that the even and odd parity states have very different deviations from the Bohr-Sommerfeld spectrum which approach a common constant for large $n$, which we denote by $\Delta(\infty, R)$. This constant is a function of $R$ and the gauge. We have some evidence that

However, $\Delta_{L G}$ and $\Delta_{T S}$ are quite different at small R.
We have no immediate interpretation for the behavior of $\triangle_{L G}(n, R)$ and $\Delta_{T S}(n, R)$. The shape of the quantum spectra clearly agree with the BohrSommerfeld spectrum for large $n$, as required. To achieve further insight, the Bohr-Sommerfeld approximation must be replaced by a rigorous WKB treatment of the problem, which we shall not attempt here.
IV. 'T HOOFT MODEL IN A TIMELIKE GAUGE - STRONG COUPLING

We now return to treat the timelike-gauge 't Hooft model using methods which, unlike those of Section II, are valid for strong as well as weak coupling. We employ once again 't Hooft's large $N$ expansion, where $G^{2}=\frac{1}{2} g^{2} N$ is kept fixed as $N \rightarrow \infty$, but without the weak-coupling restriction $\mathrm{G}^{2} \ll \mathrm{~m}^{2}$. In the $N \rightarrow \infty$ limit, only "rainbow" graphs contribute to the quark self-energy in any axial gauge. In the timelike gauge, the self-energy problem leads to two coupled nonlinear integral equations which appear very difficult to solve, in contrast to 't Hooft's [1] simple lightlike gauge self-energy problem.

In the superstrong coupling limit, in which the quark bare mass $m$ vanishes, the timelike self-energy equations appear to be inconsistent, as has been noted recently by Frishman et al. [7]. For weak-coupling, on the other hand, one can proceed perturbatively. We shall exhibit the first two terms of this expansion and check that they agree with the weak-coupling results of Section II.

For strong-coupling the appropriate bound state equation is the BetheSalpeter equation. Again, because we are working to leading order in $1 / \mathrm{N}$ and are using an axial gauge, only ladder graphs need to be retained in the bound state equation. Furthermore, since the kernel of the Bethe-Salpeter equation is instantaneous, we may turn the bound state problem into an equivalent Schrödinger equation. Unfortunately, any attempt to solve this problem for strong-coupling requires knowledge of the exact solutions of the quark self-energy equations. For weak-coupling, where these solutions are known, one recovers exactly the bound state Schrödinger problem of Section II.

## A. Quark Self-Energy

The quark self-energy matrix $\Sigma\left(p_{\mu}\right)$ is defined by

$$
\sum\left(p_{\mu}\right)=i\left[S_{0}^{-1}\left(p_{\mu}\right)-S^{-1}\left(p_{\mu}\right)\right],(4.1)
$$

where $S_{0}\left(p_{\mu}\right)$ and $S\left(p_{\mu}\right)$ are the bare and full quark propagators, respectively. In the $A_{1}=0$ gauge, the only diagrams that contribute to the self_energy in the $\mathrm{N} \rightarrow \infty$ limit are planar diagrams with fermions on the boundaries (rainbow graphs). Thus it is easy to deduce that $\Sigma\left(p_{\mu}\right)$ obeys the following integral

$$
\begin{aligned}
& \text { equation: } \\
& \sum\left(p_{\mu}\right)=-\frac{i G^{2}}{4 \pi^{2}} \int_{-\infty}^{+\infty} d q \int_{-\infty}^{+\infty} d E_{q} D\left(p_{\mu}-q_{\mu}\right) \gamma^{0} S_{\mu}^{\infty}\left(q_{\mu}\right) \gamma^{0} \\
& \text { where } q_{\mu}=\left(E_{q}, q\right), p_{\mu}=\left(E_{p}, p\right) \text { and } D\left(p_{\mu}-q_{\mu}\right) \text { is the free "gluon" propagator } \\
& D\left(p_{\mu}-q_{\mu}\right)=\frac{i}{(p-q)^{2}} \quad(4.3)
\end{aligned}
$$

Because $D\left(p_{\mu}-q_{\mu}\right)$ is independent of $E_{p}$ and $E_{q}$ it follows that $\Sigma\left(p_{\mu}\right)$ is independent of $E_{p}$. Then it is easy to see that the most general structure that $\Sigma\left(p_{\mu}\right)$ can have is

$$
\sum\left(p_{\mu}\right) \equiv \sum(p)=A(p)+\gamma, B(p) . \quad(4.4)
$$

Inserting this form for $\Sigma(p)$ into Eq. (4.2) and performing a symmetric intgration in the complex $E_{q}$ plane, we find the following coupled nonlinear integral equations for $A(p)$ and $B(p)$ :

$$
A(p)=\frac{G^{2}}{4 \pi} \int_{-\infty}^{+\infty} \frac{d q}{\omega(q)} \frac{m+A(q)}{(p-q)^{2}}
$$

$$
B(p)=\frac{G^{2}}{4 \pi} f_{-\infty}^{+\infty} \frac{d q}{\omega(q)} \frac{q+B(q)}{(p-q)^{2}}, \quad(4.6)
$$

where

$$
\omega^{2}(q)=[m+A(q)]^{2}+[q+B(q)]^{2} .(4.7)
$$

We have been unable to find exact solutions to the above equations for realistic parameters. However, for zero bare mass and unphysical coupling $\left(G^{2}<0\right)$, we have noted a number of (non-unique) solutions. One such solution is

$$
A(p)=0, \quad B(p)=-\frac{G^{2}}{2 \pi} p \frac{1}{p}, \quad(4.8)
$$

where

$$
\rho_{-\infty}^{1}=\frac{1}{2 i} \int_{-\infty}^{+\infty} d \alpha \in(\alpha) e^{i \alpha p}= \begin{cases}\frac{1}{p} & \text { for } p \neq 0 \\ 0 & \text { for } p=0\end{cases}
$$

One can easily convince oneself that if $G^{2}>0$ the above ceases to be a solution. Indeed, Frishman et al. [7] have shown that Eq. (4.5) and (4.6) are inconsistent for $G^{2}>0$ and $m=0$. It is not clear to us whether this superstrong coupling inconsistency of the self-energy equations has any bearing on the consistency of the bound state equations in the same regime. Obviously, if we can find no solution for $\Sigma(p)$, we shall not be able to construct a solution of the Bethe-Salpeter equation. That is not to say, however, that no bound states exist.

For weak coupling $\left(G^{2} / m^{2} \ll 1\right)$ the self-energy equations are well defined in the usual perturbative sense. We can expand

$$
\begin{aligned}
& A(p)=G^{2} A_{0}(p)+G^{4} A_{1}(p)+\cdots \\
& B(p)=G^{2} B_{0}(p)+G^{4} B_{1}(p)+\cdots,(4.11)
\end{aligned}
$$

This leads to the following expansion for the squared-"energy", $\omega^{2}(p)$, of Eq. (4.7):

$$
\omega^{2}(p)=p^{2}+m^{2}+G^{2} \omega_{0}^{2}(p)+G^{4} \omega_{1}^{2}(p)+\cdots,(4.12)
$$

where

$$
\begin{aligned}
& \omega_{0}^{2}(p)=2 m A_{0}(p)+2 p B_{0}(p) \\
& \omega_{1}^{2}(p)=2 m A_{1}(p)+A_{0}^{2}(p)+B_{0}^{2}(p)+2 p B_{1}(p) .
\end{aligned}
$$

We have calculated $\omega_{0}^{2}(p)$ and $\omega_{1}^{2}(p)$ in Appendix E with the result

$$
\begin{gathered}
\omega_{0}^{2}(p)=-\frac{1}{\pi} \\
\omega_{1}^{2}(p)=\frac{m^{2}}{32 \pi^{2} E_{p}}\{0,(4.14) \\
\left.\omega_{p} \pi^{2}+\ln ^{2}\left(\frac{E_{p}+p}{E_{p}-p}\right)\right\}(4.15
\end{gathered}
$$

It is clear from the above expressions that the timelike-gauge self-energy
is not at all simple. In contrast, in the lightlike gauge the self-energy contribution just gave rise to a mass shift. This difference is a manifestdion of the fact that the quark self-energy is a gauge-variant quantity.

We conclude this section by noting that to order $G^{2}$ the value (4.15)
for the self-energy agrees with the weak-coupling analysis of Section II. Indeed the second term on the right-hand side of Eq. (2.9) corresponds precisely to the expansion:

$$
\begin{equation*}
\omega(p) \approx\left[p^{2}+m^{2}+G^{2} \omega_{0}^{2}(p)\right]^{1 / 2}=E_{p}-\frac{G^{2}}{2 \pi E_{p}}+\cdots ; \tag{4.17}
\end{equation*}
$$

B. Bethe-Salpeter equation for two-body bound states

We now turn to the quark-antiquark bound states of the timeline gauge 't Hoof model in the $1 / \mathrm{N}$ approximation. To leading order in $1 / \mathrm{N}$, the mass spectrum of these bound states is given by the Bethe-Salpeter equation with only ladder diagram contributions [2]. The bound state wave function for "color" singlet bound states is defined as

$$
\left.\Psi_{a, b}^{P}\left(x_{\mu}, y_{\mu}\right)=<0\left|T\left[\psi_{a}^{\alpha}\left(x_{\mu}\right) \psi_{b}^{\alpha}\left(y_{\mu}\right)\right]\right| P\right\rangle(4.18
$$

Here $a, b=1,2$ are spinor indices and $|P\rangle$ is the state vector of the bound state,

$$
P_{o p}^{\mu}|P\rangle=P^{\mu}|P\rangle, \quad(4.19)
$$

with $P^{2}=P_{\mu} P P_{\mu}=M^{2}$. Translation invariance allows us to write

$$
\Psi_{a, b}^{P}\left(x_{\mu,} y_{\mu}\right)=\bigoplus_{a, b}^{P}\left(x_{\mu}-y_{\mu}\right) e^{-\dot{i} P \cdot\left(x+y^{j}\right) / 2}(4.20)
$$

In coordinate space and in the ladder approximation, the bound state wave function obeys the Bethe-Salpeter equation,

$$
\begin{aligned}
& \Phi_{\alpha, b}^{P}\left(x_{\mu}, y_{\mu}\right)=\int d^{2} x^{\prime} \int d^{2} y^{\prime} y^{\prime}\left\{S_{\alpha a^{\prime}}\left(x_{\mu}, x_{\mu}^{\prime}\right) K_{a^{\prime} a^{\prime \prime} ; b^{\prime} b^{\prime \prime}\left(x_{\mu}^{\prime}, y_{\mu}^{\prime}\right.} .\right. \\
& S_{b^{\prime} b}\left(y_{\mu,}^{\prime}, y_{\mu}\right) \\
&\left.\Phi_{a^{\prime \prime}, b^{\prime \prime}}^{P}\left(x_{\mu}^{\prime}, y_{\mu}^{\prime}\right)\right\},
\end{aligned}
$$

where the kernel is given by

$$
\sum_{a b ; c d}\left(x_{\mu} y_{\mu}\right)=\frac{i G^{2} \gamma_{a b}^{0}|x-y| \delta\left(t_{x}-t_{y}\right) \gamma_{d}^{o}}{(H, 2}
$$

This equation is represented pictorially in Fig. 9. Because the kernel is proportional to $\delta\left(t_{x}-t_{y}\right)$, the wave function that enters in the right-hand side of Eq. (4.21) is a onetime wave function. This property allows us to consider equal time wave functions also on the left-hand side of the Bethe-Salpeter equation. Using Eq. (4.20) we see that if $t_{x}=t_{y}=t$ then
so that all dependence on time in the wave function resides in the $e^{-i E t}$
factor, written above in the appropriate relativistic form.
To proceed, it proves convenient to write the Bethe-Salpeter equation in momentum space for the spatial variable $x$, but in ordinary space for the time variable. We define

$$
\phi_{a, b}^{p}(x-y) e^{i P(x+y) / 2}=\int_{-\infty}^{+\infty} \frac{d p_{1}}{2 \pi} \int_{-\infty}^{+\infty} \frac{d p_{2}}{2 \pi} e^{i p_{1} x} e^{i p_{2} p^{p}}\left(4, p_{1}, p_{2}\right)
$$

and

$$
S_{a a^{\prime}}\left(x_{\mu}, x_{\mu}^{\prime}\right) \equiv S_{a a^{\prime}}^{\prime}\left(x_{\mu}-x_{\mu}^{\prime}\right)=\int_{-\infty}^{+\infty} \frac{d q_{0}}{2 \pi} e^{i q\left(x-x^{\prime}\right)} \sum_{a a^{\prime}}\left(q_{i}, t-t^{\prime}\right)
$$

In terms of these definitions the Bethe-Salpeter equation, in the equal time limit, reads

$$
\begin{aligned}
& X_{a, b}^{t, \text { reads }}{ }_{p}\left(p_{1}, p_{2}\right) e^{-i t\left[P^{2}+M^{2}\right]^{1 / 2}}=\int_{-\infty}^{+\infty} d t^{\prime} \int_{-\infty}^{+\infty} d p_{1}^{\prime} d p_{2}^{\prime}\left\{S_{a a^{\prime}}\left(p_{1} ; t-t^{\prime}\right)\right. \\
& \sum_{a^{\prime} a^{\prime \prime} ; b^{\prime} b^{\prime \prime}\left(p_{1} p_{2} ; p_{1}^{\prime} p_{2}^{\prime}\right) S_{b^{\prime} b}\left(-p_{2} ; t^{\prime}-t\right), ~(1)} \\
& \left.e^{-\pi t\left[P^{+}+M^{2}\right]^{\prime 2}} \mathscr{K}_{a ; b^{4}}^{\mathrm{P}}\left(p_{1}^{\prime}, p_{2}^{\prime}\right)\right\}, \quad(4.26)
\end{aligned}
$$

where

$$
\begin{array}{r}
K_{a b ; c d}\left(p_{1} p_{2} ; p_{1}^{\prime} p_{2}^{\prime}\right)=\frac{-i G^{2}}{2 \pi} \gamma_{a b}^{0} \frac{1}{\left(p_{1}-p_{1}\right)^{2}} \gamma_{d c}^{0} \delta\left(p_{1}+p_{2}-p_{1}^{\prime}-p_{2}^{\prime}\right) \\
(4.27)
\end{array}
$$

Because Eq. (4.26) is a one-time equation, it is equivalent to an integral
Schrödinger problem for the meson bound states. Of course since the wave function $X$ carries spinor indices, Eq. (4.26) is a matrix integral problem.

The Green's function that enters in the Bethe-Salpeter equation is the full quark Green's function. Its inverse is related to the self-energy $\Sigma\left(p_{\mu}\right)$ by Eq. (4.1). Because $\Sigma\left(p_{\mu}\right)$ is independent of $E_{p}$ it follows that we can write

$$
\left.S_{a b}{ }^{-1-1}(p ; t)=\left[-i \gamma^{\circ}\left\{\frac{\partial}{\partial t}-H(p)\right)\right\}\right]_{a b} \quad(4.28)
$$

where

$$
H(p)=\gamma^{\circ}\left(m+A(p)+8^{\circ} \gamma^{\prime}(p+B(p)), \quad(4,2 q\right.
$$

Thus the full quark Green's function obeys the differential equation

$$
\left[i \frac{\partial}{\partial t}-H(p)\right]_{a a^{\prime}} S_{a^{\prime} b}\left(p ; t-t^{\prime}\right)=i \gamma_{a b}^{0} \delta\left(t-t^{\prime}\right),(4.30)
$$

Following Schwinger [16], we form the two-body analog of the operator appearing in (4.30) and let it act on the Bethe-Salpeter wave function. This will transform the bound state problem into an ordinary matrix Schrödinger problem. Consider then

$$
\begin{aligned}
& \left\{\delta_{a a^{\prime} b b^{\prime}} \delta^{i} \frac{\partial}{\partial t}-H_{a a^{\prime}}\left(p_{1}\right) \delta_{b b^{\prime}}-\delta_{a a^{\prime}} H_{b b^{\prime}}\left(p_{2}\right)\right\} X_{a^{\prime} b^{\prime}}^{P}\left(p_{1}, p_{2}\right) e^{-i t\left[P^{2}+M^{2}\right]^{1 / 2}}
\end{aligned}
$$

$$
\begin{aligned}
& \text { (4.31) }
\end{aligned}
$$

The right-hand side of (4.31) follows from the Bethe-Salpeter equation (4.26) and Eq. (4.30). The above equation can be simplified further by noting that the full quark Green's function at equal times is given, after a symmetric integration, by

$$
S_{a b}(p ; 0)=\left[-\frac{1}{2} \frac{H(p)}{\omega(p)}\right]_{a a^{\prime}} \gamma_{a^{\prime} b}^{0} . \quad(4.32)
$$

The function $\omega(\mathrm{p})$ in (4.32) is defined in Eq. (4.7). Using the form of the kernel K given in (4.27) one obtains, after some algebra, the following bound state equation:

$$
\begin{aligned}
& \left\{\left[P^{2}+M^{2}\right]^{1 / 2} \delta_{a a^{\prime}} \delta_{b b^{\prime}}-H\left(p_{1}\right) \delta_{b b^{\prime}}-\delta_{a a^{\prime}} H_{b b^{\prime}}\left(p_{2}\right)\right\}\left(p_{1}\right)\left(p_{2}\right) \\
& =-\frac{G^{2}}{4 \pi}\left\{\frac{\left.H_{a a^{\prime}}\left(p_{1}\right) \delta_{b}+\delta_{a} a \frac{H_{b} b^{\prime}\left(p_{2}\right)}{\omega\left(p_{i}\right)}\right\} \quad\left(p_{2}\right)}{Q}\right. \\
& \left.\left.\int_{-\infty}^{+\infty} d_{p_{i}^{\prime}}^{+\infty} p_{2}^{\prime} \frac{p_{1}\left(p_{1}+p_{2}-p_{1}^{\prime}-p_{2}^{\prime}\right)}{\left(p_{1}-p_{1}^{\prime}\right)^{2}}\right)_{a^{\prime} b^{\prime}}^{p_{1}} p_{1}^{\prime}, p_{2}^{\prime}\right):\left(p_{i}:\right.
\end{aligned}
$$

The projection factors $H / \omega$ appearing on the right-hand side of Eq. (4.33) are, essentially, those of the Salpeter equation [17]. However, in Eq. (4.33) these factors contain all orders in $G^{2}$ and are not just given by the ratio of the free Hamiltonian to its eigenvalue.

Equation (4.33) is exact. One cannot, however, even attempt to solve Eq. (4.33) until the full quark self-energy solution is obtained. As we have seen in Section IV A, and as emphasized by Frishman et al. [7], for superstrong coupling there appear to be no solutions to the self-energy equations (at least if one uses the principal value prescription adopted here as an infrared cut off). Because of this difficulty we do not know how to examine Eq. (4.33) in the super strong coupling limit. For weak-coupling, however,
there is a perfectly well-defined expression for the self-energy, and the Bethe-Salpeter equation can be studied in this limit.

To analyze the bound state equation in the weak coupling limit it is useful to diagonalize the kinetic energy terms in Eq. (4.33). This can be done readily by performing a Foldy-Wouthuysen transformation on $H$. It is not hard to check that

$$
\begin{equation*}
u(p)=e^{\frac{1}{2} \gamma_{1} \phi} \tag{4.34}
\end{equation*}
$$

where

$$
\sin \phi=\frac{p+B(p)}{\omega(p)}, \cos \phi=\frac{m+A(p)}{\omega(p)} \quad(4.35)
$$

diagonalizes H :

$$
u(p) H(p) u^{-1}(p)=\gamma^{0} \omega(p)
$$

Defining a new wave function

$$
\left.\bar{X}_{a, b}^{P}\left(p_{1,}, p_{2}\right)=u_{a a^{\prime}}\left(p_{1}\right) u_{b^{\prime} b^{\prime}}\left(p_{2}\right)\right)_{a_{1}^{\prime}, b^{\prime}}^{p}\left(p_{1}, p_{2}\right)
$$

we may partially diagonalize the bound state equation to read:

$$
\begin{aligned}
& \left.\left[p^{2}+M^{2}\right]^{1 / 2} \bar{x}_{a, b}^{P}\left(p_{1}, p_{2}\right)=\left\{\omega\left(p_{1}\right) \gamma_{a a^{\prime}}^{0} \delta_{b b^{\prime}}+\omega\left(p_{2}\right) \delta_{a a^{\prime}}, \gamma_{b^{\prime} b}^{0}\right\}\right)^{\prime} \bar{X}_{a^{\prime}, b^{\prime}}^{P}\left(p_{1}, p_{2}\right) \\
& -\frac{G^{2}}{2 \pi}\left\{\frac{1}{2}\left[\gamma_{a a^{\prime}}^{0} \delta_{b b^{\prime}}+\delta_{a a^{\prime}} \gamma_{b^{\prime} b}^{0}\right]\right\} \\
& \int_{-\infty}^{+\infty} d p_{1}^{\prime} d p_{2}^{\prime} \frac{\delta\left(p_{1}+p_{2}-p_{1}^{\prime}-p_{2}^{\prime}\right)}{\left(p_{1}-p_{1}^{\prime}\right)^{2}} u_{a^{\prime} a^{\prime \prime}}\left(p_{1}\right) u_{a^{\prime \prime}, a^{\prime \prime \prime}}^{-1}\left(p_{1}^{\prime}\right) u_{b^{\prime \prime \prime}, b^{\prime \prime}}^{-1}\left(p_{2}^{\prime}\right) \\
& u_{b^{\prime \prime}, b^{\prime}\left(p_{2}\right)} \bar{x}_{a^{\prime \prime \prime}, b^{\prime \prime \prime \prime}}^{P}\left(p_{1}^{\prime}, p_{2}^{\prime}\right) .
\end{aligned}
$$

Equation (4.38) is still exact. For weak coupling, however, various
approximations can be made:
(1) Only the first two terms in the expansion of $\omega(p)$ need to be kept [Eq. (4.17)]:

$$
\omega(p) \approx E_{p}-\frac{G^{2}}{2 \pi E_{p}}
$$

(2) The matrix $U(p)$ can be calculated with the angle $\phi$ given by the lowest order expression:

$$
\sin \phi \approx \frac{p}{E_{p}}, \cos \phi \approx \frac{m}{E_{p}} . \quad(4.40)
$$

(3) Only the "upper" components of the wave function $\bar{X}{ }_{a}, b$ need be retained (i.e. $a=b=1$ ). This means that effectively

$$
\gamma^{0} \rightarrow 1, u(p) u\left(p^{\prime}\right) \rightarrow\left[\frac{E_{p^{\prime}} E_{p^{\prime}}+p p^{\prime}+m^{2}}{2 E_{p} E_{p^{\prime}}}\right]^{1 / 2} \cdot(4.41)
$$

It is straightforward to check that in this weak-coupling limit, the bound state equation reduces to the following (scalar) form:

$$
\begin{aligned}
& {\left[P^{2}+M^{2}\right]^{1 / 2} X_{11}^{P}\left(p_{1}, p_{2}\right)=\left\{E_{p_{1}}-\frac{G^{2}}{2 \pi E_{p_{1}}}+E_{p_{2}}-\frac{G^{2}}{2 \pi E_{p_{2}}}\left\{\mathcal{S}_{11} p_{1, p}\right.\right.} \\
& -\frac{G^{2}}{2 \pi} \int_{-\infty}^{+\infty} \frac{d p_{1}^{\prime} d p_{2}^{\prime}}{\left(p_{1}-p_{1}^{\prime}\right)^{2}} \delta\left(p_{1}+p_{2}-p_{1}^{\prime}-p_{2}^{\prime}\right)
\end{aligned}
$$

The right-hand side of Eq. (4.42) is identical to that of Eq. (2.9) obtained using the weak-coupling Hamiltonian formalism. However, we demonstrated explicitly in Section II that the eigenvalues have the correct relativistic form indicated on the left-hand side of Eq. (4.42) only for weak-coupling. The deviation of Eq. (4.42) from exactness for strong coupling is exhibited in Fig. 5. In fact, the covariance test plotted in Fig. 5 serves as a sensitive indicator of the value of the coupling parameter at which the techniques we have utilized are expected to fail.

We conclude that in the weak-coupling regime the Bethe-Salpeter equation for the 't Hoof model in the timelike gauge is covariant and yields the same spectrum as the lightlike gauge. Thus, in weak-coupling there are no inconsistencies between different gauge choices. For strong-coupling, however, the situation remains murky and obviously deserves further study.

## V. CONCLUSIONS

Our main concern in this paper has been the investigation of the bound state spectrum of 't Hooft's model in different gauges and its relation to the string model bound state spectrum. To be able to make quantitative statements, we have had to solve numerically a number of singular integral equations. Fortunately, it was possible to adopt for these purposes a technique used extensively in the aerodynamics of wings.

We have obtained three principal results:
(1) We have demonstrated that the spectrum of the 't Hooft model for weak-coupling is the same in the lightlike and timelike gauges.
(2) We have verified that the timelike gauge 't Hooft model is Lorentz covariant for weak coupling.
(3) We have shown that the string model has a quantum spectrum that is gauge variant.

Let us comment briefly on these points.
The gauge invariance and Lorentz covariance of the 't Hooft model, although formally expected, are important results. The recent work of Frishman et al. [7], in strong-coupling, has raised doubts about both the gauge invariance and the covariance of the bound state spectrum. What we have shown is that there exists a range of coupling constants for which no inconsistencies arise in the 't Hooft model.

Our work on the string model, on the other hand, shows that the quantum mechanics of the timelike and lightlike string are canonically inequivalent. Since the quantum system can be formally proven to be Poincaré covariant in either gauge [11], this raises the interesting question of what gauge is the "physical" one. One wonders what criteria, if any, exist to establish a preference for one of the string quantum systems over the other.

The strongly coupled 't Hooft model in the timelike gauge remains an open challenge. If there is to be any chance of retaining gauge invariance in the strong coupling regime, some way must be found to circumvent the problems uncovered by Frishman et al. [7] and to put the bound state equations into a form which can be solved numerically. We remark that it may not be necessary to solve the fermion self-energy equations exactly in order to accomplish this; the quarks are after all unobservable, so that the bound state equation itself could be perfectly well-defined.

We conclude with a comparison of the ground state mass-spectra of the three theories which we have solved numerically in the course of this work. In Figure 10, we plot each of the three possible ground state mass-squared ratios versus the coupling parameter $R$. It is clear that for weak coupling, all these theories are equivalent. Marked differences appear in the strongcoupling regime. The weak-coupling timelike gauge theory spectrum is of course not to be taken seriously in the strong coupling region; nevertheless, it remains remarkably close to the strong-coupling 't Hooft model solution.

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APPENDIX A: $\operatorname{SU}(\mathrm{N})$ Transformation Properties
We display here the transformation properties that the gluon field $A_{\nu}^{a}\left(x_{\mu}\right)$ and the quark field $\psi_{\alpha}\left(x_{\mu}\right)$ have under local $S U(N)$ transformations. Under an infinitesimal gauge transformation generated by $\Lambda^{a}\left(x_{\mu}\right)$, $A_{\nu}^{a}$
transforms as

$$
\delta A_{\nu}^{a}\left(x_{\mu}\right)=-\frac{1}{g} \partial_{v} \Lambda^{a}\left(x_{\mu}\right)+f^{a b c} \Lambda^{b}\left(x_{\mu}\right) A_{v}^{c}\left(x_{\mu}\right),(A \cdot 1)
$$

while $\psi_{\alpha}\left(x_{\mu}\right)$ transforms as

$$
\delta \psi_{\alpha}\left(x_{\mu}\right)=-i \Lambda^{a}\left(x_{\mu}\right) \frac{1}{2} \lambda_{\alpha \beta}^{a} \psi_{\beta}\left(x_{\mu}\right)
$$

Here $f^{a b c}$ are the structure constants of $S U(N)$, and the $\left(N^{2}-1\right)$ matrices $\lambda^{a}$ are the $\mathrm{N} \times \mathrm{N}$ traceless hermitian matrices which define the fundamental
representation of $\mathrm{SU}(\mathrm{N})$. We list below a number of useful properties of these matrices:

$$
\begin{array}{ll}
\text { rices: } \\
{\left[\frac{1}{2} \lambda^{a}, \frac{1}{2} \lambda^{b}\right]=i f^{a b c} \frac{1}{2} \lambda^{c}} & (A .3) \\
\left\{\lambda_{a}, \lambda_{b}\right\}=\frac{4}{N} \delta_{a b}+2 d_{a b c} \lambda_{c} & (A .4) \\
T_{r}\left(\lambda_{a} \lambda_{b}\right)=2 \delta_{a b} & (A .5) \\
\lambda_{a} \lambda_{b}=\frac{2}{N} \delta_{a b}+\left(d_{a b c}+i f_{a b c}\right) \lambda_{c} & \text { (A.6) } \\
\lambda_{\alpha \beta}^{a} \lambda_{\tau \epsilon}^{a}=2\left[\delta_{\alpha \epsilon} \delta_{\beta \tau}-\frac{1}{N} \delta_{\alpha \beta} \delta_{\tau \epsilon}\right] & \text { (A.7) }
\end{array}
$$

## APPENDIX B: Some Useful Principal Value Integrals

The Cauchy principal value integral of a function with a single pole in the interval of integration is defined as:

$$
\int_{a}^{b} \frac{f(y) d y}{y-x}=\lim _{\epsilon \rightarrow 0}\left\{\int_{a+\epsilon}^{x-\epsilon} \frac{f(y) d y}{y-x}+\int_{a}^{b} \frac{f(y) d y}{y-x}\right\}(B .1)
$$

For a function with a double pole in the interval of integration the above definition is modified to read:

In the text we need to know a number of principal value integrals. We list these integrals below and outline their evaluation:
(1) $A_{n}(\theta)=f_{0}^{\pi} \frac{\cos n \theta^{\prime} d \theta^{\prime}}{\cos \theta^{\prime}-\cos \theta}=\frac{\pi \sin (n \theta)}{\sin \theta}$

$$
h=0,1,2, \ldots
$$

$$
0<\theta<\pi .
$$

Proof: One can explicitly check that $A_{0}(\theta)=0$ and $A_{1}(\theta)=\pi$. The following recursion relation for $A_{n}(\theta)$ is easily derived :

$$
\begin{equation*}
A_{n+1}(\theta)-2 \cos \theta A_{n}(\theta)+A_{n-1}(\theta)=0 \tag{B.4}
\end{equation*}
$$

The difference equation ( $B .4$ ) has solutions $A_{n}(\theta) \propto e^{ \pm i n \theta}$ and the boundary conditions provided by $A_{0}$ and $A_{1}$ yield (B. 3 ).

$$
\begin{aligned}
& \text { (2) } B_{n}(\theta)=f_{0}^{\pi} d e^{\prime} \frac{\cos n \theta^{\prime} \sin \theta^{\prime}}{\cos \theta^{\prime}-\cos \theta} \\
& n=0,1,2, \ldots ; \quad 0<\theta<\pi .
\end{aligned}
$$

It is easy to check that the integral (B.5) obeys the following recursion relation:

$$
B_{n+1}(\theta)+B_{n-1}(\theta)=2 \cos \theta B_{n}(\theta)+\frac{2\left[1+(-1)^{n}\right]}{1-n^{2}} \cdot(B \cdot 6)
$$

Using (B.6) and the result of an explicit evaluation of $B_{0}(\theta)$,

$$
B_{0}(\theta)=\ln \left(\frac{1-\cos \theta}{1+\cos \theta}\right), \quad(B \cdot 7)
$$

one can compute (B.5) recursively.

$$
\text { (3) } \quad C_{n}(\theta)=f_{0}^{\pi} \cos n \theta^{\prime} \cot \left(\theta-\theta^{\prime}\right) d \theta^{\prime}
$$

$$
n=0,1,2, \cdots \quad ; \quad 0<\theta<\pi
$$

The integral (B. 8) obeys the recursion relation

$$
C_{n+2}(\theta)+C_{n-2}(\theta)=2 \cos (2 \theta) C_{n}(\theta)+\frac{4\left[1-(-1)^{n}\right]}{4-n^{2}} \text { (B.9) }
$$

For even $n$ this equation is easily solved yielding

$$
C_{2 m}(\theta)=\pi \sin (2 m \theta), \quad m=0,1,2, \ldots \quad(B, 1 c
$$

[Equation (B.10) could have been arrived at more directly by realizing that $C_{2 m}(\theta)$ is a special case of the general Poisson integral formula linking
harmonic conjugate functions [18]]. For odd $n$, one can compute $C_{n}(\theta)$ by using Eq. (B.9) along with the result

$$
C_{1}(\theta)=2+\cos \theta \ln \left(\frac{1-\cos \theta}{1+\cos \theta}\right) . \quad(B .11)
$$

## APPENDIX C: The Multhopp Technique

In this appendix we describe a method for solving the type of singular integral equations that persistently occur in our analysis of strings and two dimensional $\operatorname{SU}(\mathrm{N})$ gauge theories. We adapt an ingenious technique developed almost forty years ago by $H$. Multhopp in connection with the aerodynamic theory of wings [12].

Multhopp's method transforms singular integral equations into (infinite) algebraic equations and then solves these, approximately, by truncation. Consider the singular integral equation

$$
C \psi(x)=\int_{a}^{b} d y \frac{d \psi(y)}{d y} \frac{T^{b}(x, y)}{(x-y)},(C .1)
$$

where $\psi(a)=\psi(b)=0$, which is an equation of the type encountered in the text. Equation (C.1) can be transformed into an algebraic equation by expanding $\psi$ in a complete set of functions and performing the singular integrals on the right hand side. A particularly convenient expansion is provided by a sine series since $\psi$ can then be made to satisfy the boundary conditions trivially. Furthermore, when one uses a sine expansion the integrals on the right hand side of (C.1) can be performed for a variety of simple kernels (see Appendix B).

Mapping the interval ( $a, b$ ) into $(0, \pi)$ and making the expansion

$$
\psi(x) \equiv \psi(\theta)=\sum_{j=1}^{\infty} a_{j} \sin j \theta, \quad 0<\theta<\pi, \quad(c .2)
$$

one transforms Eq. (C.1) into the following transcendental eigenvalue problem:

$$
\sum \psi(\theta)=\sum_{j=1}^{\infty} a_{j}^{\infty} B(j, \theta) \quad(c \cdot 3)
$$

Here $B(j, \theta)$ is the function that is obtained upon performing the singular integral over the kernel $K$ in (C.1).

The eigenvalue problem can be solved approximately by truncating the sums. (C.2) and (C.3). For the problem at hand, it is convenient to truncate at $n$ terms with $n$ being odd. Thus Eqs. (C.2) and (C.3) become, approximately,

$$
\begin{aligned}
& \psi(\theta) \approx \sum_{j=1}^{n} a_{j} \sin j \theta \\
& C(C, 4)
\end{aligned}
$$

Equation (C.5) can be transformed into an $n \times n$ eigenvalue problem by evaluating it at equally spaced angles $\theta_{k}$ (Multhopp's angles),

$$
\theta_{k}=\frac{k \pi}{n+1}, \quad k=1,2, \ldots, n
$$

$$
(C \cdot 6)
$$

The choice of (C.6), made by Multhopp, is especially convenient because one can use the "completeness" relation [12]

$$
\sum_{l=1}^{n} \sin l \theta_{j} \sin l \theta_{k}=\frac{n+1}{2} \delta_{j k}(C, 7)
$$

to solve for $a_{j}$ in terms of $\psi\left(\theta_{k}\right)$. Using (C, 7) one easily sees that

$$
\begin{align*}
\sum_{j=1}^{n} \psi\left(\theta_{k}\right) \sin j \theta_{k} & =\sum_{j=1}^{n} \sum_{l=1}^{n}\left[a_{l} \sin \ell \theta_{k}\right] \sin j \theta_{k} \\
& =\frac{n+1}{2} a_{k} \tag{c.8}
\end{align*}
$$

The result (C.8) allows one to rewrite Eq. (C.5), for $\theta=\theta_{k}$, as

$$
\varepsilon \psi\left(\theta_{k}\right)=\sum_{j=1}^{n} B_{k j} \psi\left(\theta_{j}\right),
$$

where the "Multhopp matrix" $B_{k j}$ is given by

$$
B_{k j}=\frac{2}{n+1} \sum_{\ell=1}^{n} B\left(\ell, \theta_{k}\right) \sin \ell \theta_{j} . \quad(c \cdot 10)
$$

The system of equations (C.9) is simply a linear $n x n$ matrix equation which can be easily solved on a computer to yield the $n$ eigenvalues $\&$ and the corresponding eigenfunction $\psi(\theta)$ evaluated at the $n$ discrete points (C.6). When one has a definite parity for the wave function, as in the text, so that

$$
\begin{array}{lc}
\psi^{(+)}(\theta)=+\psi^{(+)}(\pi-\theta) & \text { [Even parity] } \\
\psi^{(-)}(\theta)=-\psi^{(-)}(\pi-\theta) & \text { [odd parity], }
\end{array}
$$

the system of equations (C.9) can be reduced in an obvious way to systems of $(n+1) / 2$ and $(n-1) / 2$ equations for even $(+)$ and odd ( - ) parity respectively. For the discrete angles $\theta_{k}$ the parity operation (C.11) implies

$$
\psi^{( \pm)}\left(\theta_{k}\right)= \pm \psi^{( \pm)}\left(\theta_{n+1-k}\right) . \quad \text { (c.12) }
$$

Then the eigenvalue problem (C.9) splits into:
(1) Even parity (+):

$$
\varepsilon^{(+)} \psi^{(+)}\left(\theta_{k}\right)=\sum_{j=1}^{(n+1) / 2} B_{k j}^{(+)} \psi^{(+)}\left(\theta_{j}\right) \quad(C .13)
$$

where

$$
B_{k j}^{(+)}=\left\{\begin{array}{cl}
B_{k j}+B_{k, n+1-j} & j \neq \frac{n+1}{2} \\
B_{k, \frac{n+1}{2}} & j=\frac{n+1}{2}
\end{array}\right.
$$

(2) Odd Parity (-):

$$
\sum^{(-)} \psi^{(-)}\left(\theta_{k}\right)=\sum_{j=1}^{(n-1) / 2} B_{k j}^{(-)} \psi^{(-)}\left(\theta_{j}\right) \quad(C .15)
$$

where

$$
B_{k j}^{(-)}=B_{k j}-B_{k, n+1-j} .
$$

$$
(c .16)
$$

These are the equations we use repeatedly in the text.

APPENDIX D: Hamiltonian approach to timelike gauge bound state equation
In this appendix we shall outline the derivation of the bound state equation (2.9) by Hamiltonian methods. The Hamiltonian (2.7) is

$$
H=\int_{-\infty}^{+\infty} d x: \bar{\psi}\left(i \gamma, \partial_{1}+m\right) \psi:-\frac{g^{2}}{4} \int_{-\infty}^{+\infty} d x \int_{-\infty}^{+\infty} d y|x-y|: J_{0}^{a}(x) J_{0}^{a}(y):
$$

The current density entering in the above is given by

$$
\begin{aligned}
J_{0}^{a}(x) & =: \psi^{\dagger}(x) \frac{1}{2} \lambda^{a} \psi(x): \\
& =: \psi_{\alpha}^{\dagger}(x) \psi_{\beta}^{(x)}: \frac{1}{2} \lambda_{\alpha \beta}^{a}
\end{aligned}
$$

For the field $\psi$ we shall use the following zero time field expansion

$$
\psi_{\alpha}(x) \equiv \psi_{\alpha}(x, t=0)=\frac{1}{(2 \pi)^{1 / 2}} \int_{-\infty}^{+\infty} d p\left\{u(p) a(p) e^{i p x}+v(p) b_{\alpha}^{+}(p) e^{-i p x}\right\}
$$

The operators $a_{\alpha}, b_{\alpha}$ obey the usual anticommutation relations

$$
\left\{a_{\alpha}(p), a_{\beta}^{+}\left(p^{\prime}\right)\right\}=\left\{b_{\alpha}(p), b_{\beta}^{\dagger}\left(p^{\prime}\right)\right\}=\delta_{\alpha \beta} \delta\left(p-p^{\prime}\right)
$$

with all other anticommutators vanishing. It is convenient to evaluate the spinors $u(p), v(p)$, in a representation in which

$$
\gamma^{0}=\sigma_{1}=\left(\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right) ; \quad \gamma_{1}=\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right)=i \sigma_{2} ; \quad \gamma_{5}=\gamma_{0} \gamma_{1}=\left(\begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array}\right)=-\sigma_{3}
$$

$(D . S)$

One finds that

$$
\begin{aligned}
& \left.u(p)=[2 E(E-p)]^{-1 / 2}(E-p) ; v(p)=[2 E(E-p)]^{-1 / 2}\binom{p-E}{m}^{m}, 0.6\right)
\end{aligned}
$$

where $E=\sqrt{p^{2}+m^{2}}$. From Eq. (D.6) it follows for example that

$$
u^{+}(p) u(q)=v^{+}(p) v(q)=\left[\frac{E_{p} E_{q}+p q+m^{2}}{2 E_{p} E_{q}}\right]^{1 / 2}
$$

$$
(D, 7)
$$

(In Sec. IV, on the other hand, we use a more conventional representation where $\gamma^{0}$ is diagonal.)

Using Eqs. (D.2) and (D.3) one can write out the Hamiltonian $H$ in terms of the quark and antiquark creation and annihilation operators $a_{\alpha}, a_{\alpha}^{\dagger}, b_{\alpha}$, $b_{\alpha}^{\dagger}$. We are interested in the terms of $H$ which when operating on the state

$$
|p\rangle=\int_{-\infty}^{ \pm \infty} d q \phi_{p}(q) a_{\alpha}^{\dagger}(q) b_{\alpha}^{\dagger}(p-q)|0\rangle
$$

$$
(0.8)
$$

will, to lowest order, reproduce this state. These terms clearly must have pairs of $a_{\alpha}$ and $a_{\alpha}^{\dagger}$, and of $b_{\alpha}$ and $b_{\alpha}^{\dagger}$. The normal ordered Hamiltonian (D.1) contains in general 28 terms. Of these only 10 have the above stated property. We display these terms below:

$$
\begin{aligned}
& H^{e f f}=\int_{-\infty}^{+\infty} d p\left\{u^{+}(p)\left(\gamma_{5} p+\gamma_{0} m\right) u(p) a_{\alpha}^{\dagger}(p) a_{\alpha}(p)\right. \\
& \left.-v^{+}(p)\left(-\gamma_{5} p+\gamma_{0} m\right) v(p) b_{\alpha}^{\dagger}(p) b_{\alpha}(p)\right\} \\
& +\frac{g^{2} N}{8 \pi} \int_{-\infty}^{+\infty} \frac{d p d p^{\prime}}{\left(p-p^{\prime}\right)^{2}}\left\{\left[u^{+}(p) u\left(p^{\prime}\right)\right]_{\alpha}^{2} a_{\alpha}^{+}(p) a_{\alpha}(p)\right. \\
& +\left[v^{+}(p) v\left(p^{\prime}\right)\right]_{\alpha}^{b_{\alpha}^{\prime}}\left(p^{\prime}\right) b_{\alpha}\left(p^{\prime}\right)-\left[v^{\prime}(p) u\left(-p^{\prime}\right) u^{t}\left(-p^{\prime}\right) v(p)\right] a_{\alpha}^{+}\left(-p^{\prime}\right) a_{\alpha}\left(-p^{\prime}\right) \\
& \left.-\left[v^{+}(p) u\left(-p^{\prime}\right) u^{+}\left(-p^{\prime}\right) v(p)\right] b_{\alpha}^{+}(p) b_{\alpha}(p)\right\}
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{g^{2}}{8 \pi} \int_{-\infty}^{+\infty} \frac{d p d p^{\prime} d q d q^{\prime}}{\left(p-p^{\prime}\right)^{2}} \delta\left(p+q^{\prime}-p^{\prime}-q\right) \\
& \left.\begin{array}{l}
\left\{\left[u^{+}(p) u\left(p^{\prime}\right) v^{+}(q) v\left(q^{\prime}\right)\right] a_{\alpha}^{+}(p) b_{\alpha}^{+}\left(q^{\prime}\right) b_{\beta}(q) a_{\beta}\left(p^{\prime}\right)\right. \\
+\left[v^{+}(p) v\left(p^{\prime}\right) u^{+}(q) u\left(q^{\prime}\right)\right] b_{\beta}^{+}\left(p^{\prime}\right) a_{\beta}^{+}(q) a_{\alpha}\left(q^{\prime}\right) b_{\alpha}(p)
\end{array}\right\} \\
& +\frac{g^{2}}{8 \pi} \int_{-\infty}^{+\infty} \frac{d p d p^{\prime} d q d q^{\prime}}{\left(p+p^{\prime}\right)^{2}} \delta\left(p+p^{\prime}-q-q^{\prime}\right) \\
& \\
& \left\{\begin{array}{l}
{\left[v^{+}(p) u\left(p^{\prime}\right) u^{+}(q) v\left(q^{\prime}\right)\right] a_{\beta}^{+}(q) b_{\alpha}^{\dagger}\left(q^{\prime}\right) b_{\alpha}(p) a_{\beta}\left(p^{\prime}\right)} \\
+\left[u^{+}(p) v\left(p^{\prime}\right) v^{+}(q) u\left(q^{\prime}\right)\right] a_{\alpha}^{+}(p) b_{\beta}^{+}\left(p^{\prime}\right) b_{\beta}(q) a_{\alpha}\left(q^{\prime}\right)
\end{array}\right\}
\end{aligned}
$$

In Eq. (D.9) we have gone to momentum space. We have used a principal value prescription to define the Fourier transform of the Coulomb potential $|x-y|$ since it does not introduce any new parameters, unlike 't Hoof's [1] $\lambda$
cutoff. The factor of $N$ occurring in the second group of terms in Eq. (D.9)
comes from the contraction in the internal symmetry space:

$$
\begin{equation*}
\delta_{\alpha \beta} \delta_{\alpha \beta}=N . \tag{D,10}
\end{equation*}
$$

The last two terms in $H^{\text {eff (D.9) when acting on the state (D.8) will not give }}$ rise to any extra factor of $N$. These terms correspond to the annihilation graphs and thus can be neglected in the $\mathrm{N} \rightarrow \infty$ limit.

Armed with (D.9), it is now straightforward to compute the effect of $H^{\text {eff }}$ on $|P\rangle$. The result is displayed in the text in Eq. (2.9). Note that the kinetic energy terms arise from the first two terms in $H$, the self energy


#### Abstract

terms arise from the next four. Finally the 7 th and 8 th term in $H^{\text {eff }}$ give rise to the Coulomb binding term. The effect of these last terms on $|\mathrm{P}\rangle$ yields a factor of $N$ which transforms the effective coupling into $G^{2}=\frac{1}{2} g^{2} N$.


APPENDIX E
In this appendix we give the details of the calculation leading to the results quoted in (4.15) and (4.16). To first order in $G^{2}, A(p)$ and $B(p)$ are determined as follows:

$$
\begin{aligned}
& A_{0}(p)=\frac{1}{4 \pi} f_{-\infty}^{+\infty} \frac{d q}{\left[m^{2}+q^{2}\right]^{1 / 2}} \frac{m}{(p-q)^{2}} . \quad(E, \mid a) \\
& B_{0}(p)=\frac{1}{4 \pi} \int_{-\infty}^{+\infty} \frac{d q}{\left[m^{2}+q^{2}\right]^{1 / 2}} \frac{q}{(p-q)^{2}} \cdot(E, \mid b)
\end{aligned}
$$

The integrals in (E.1) can be straightforwardly calculated giving

$$
\begin{aligned}
& A_{0}(p)=-\frac{1}{4 \pi E_{p}^{2}}\left\{2 m-\frac{m p}{E_{p}} \ln \left(\frac{E_{p}+p}{E_{p}-p}\right)\right\}(E, 2 a) \\
& B_{0}(p)=-\frac{1}{4 \pi E_{p}^{2}}\left\{2 p+\frac{m^{2}}{E_{p}} \ln \left(\frac{E_{p}+p}{E_{p}-p}\right)\right\},(E .2 b)
\end{aligned}
$$

and using (4.13) we verify (4.15). To second order in $G^{2}$ we have:

$$
\begin{aligned}
& A,(p)=\frac{m}{16 \pi^{2}} \int_{-\infty}^{+\infty} \frac{q d q}{E_{q}^{4}(p-q)^{2}} \ln \left(\frac{E q+q}{E q-q}\right) \quad(E .3 a) \\
& B,(p)=-\frac{m^{2}}{16 \pi^{2}} \int_{-\infty}^{+\infty} \frac{d q}{E_{q}^{4}(p-q)^{2}} \ln \left(\frac{E q+q}{E q-q}\right)(E, 3 b) \\
& {\left[\left(\omega_{1}(p)\right]^{2}=\frac{1}{16 \pi^{2} E_{p}^{2}}\left\{4+\frac{m^{2}}{E_{p}^{2}} \ln \left(\frac{E_{p}+p}{E_{p}-p}\right)\right\}+\frac{m^{2}}{8 \pi^{2}} \int_{-\infty}^{+\infty} \frac{d q}{E_{q}(q-p)}\left(\frac{q_{q}}{E_{q}-q}\right) .\right.}
\end{aligned}
$$

We will now give the details of how the integral in Eq. (E.3c) is evaluated. (E.3
The integrals in (E.3a) and (E.3b) can be similarly calculated but are somewhat more tedious.

The integral we wish to concentrate on is

$$
q(p)=\frac{m^{2}}{8 \pi^{2}} \int_{-\infty}^{+\infty} \frac{d q}{E_{q}^{4}(q-p)} \ln \left(\frac{E_{q}+q}{E_{q}-q}\right)
$$

To evaluate $\stackrel{\infty}{0}(p)$ we introduce variables $\phi$ and $\chi$ such that:

$$
\begin{array}{lll}
p=m \sinh \phi & E_{p}=m \cosh \phi & -\infty<\phi<+\infty \\
q=m \sinh x & E_{q}=m \cosh x & -\infty<x<+\infty
\end{array}
$$

In terms of our new variables $\vartheta(\beta)$ takes the following form :

$$
{ }^{0}(p)=d^{0}(\phi)=\frac{1}{4 \pi^{2} m^{2}} \int_{-\infty}^{+\infty} \frac{X d X C}{\cosh ^{3} X(\sinh X-\sinh \phi)} \cdot(E \cdot 6)
$$

The integral in (E.6) is now evaluated by means of contour integration. The contour we choose is shown in Fig. 11. We have assumed without loss of generality that $\phi>0$. The two simple poles at $z=\phi$ and $z=\phi+2 \pi i$ lying on the contour necessitate the Cauchy principal value prescription for evaluating $\rho(\phi)$. The Cauchy principal value is the average of the result obtained with the two simple poles inside and outside the contour $\Gamma$.

We consider first the contour integral

$$
\tilde{\mathscr{O}}(\phi)=\frac{1}{4 \pi^{2} m^{2}} \int_{\Gamma} \frac{z(z-2 \pi i) d z}{\cosh ^{3} z(\sinh z-\sinh \phi)}, \quad(E .7)
$$

where $\Gamma$ is the contour of Fig. 11. As $\rho \rightarrow \infty$, the contributions from the vertical segments vanish and the contributions from the horizontal segments combine to give

$$
\tilde{g}(\phi)=-4 \pi i g(\phi) . \quad(E .8)
$$

On the other hand the residue theorem tells us that

$$
\tilde{\tilde{0}(\phi)=} 2 \pi i\left\{\text { Res. at } \frac{i \pi}{2}+\text { Res. at } \frac{3 \pi i}{2}+\frac{1}{2} \text { Res. at } \dot{\phi}\right\}, \quad(E \cdot 9)
$$

where the poles at $z=\pi i / 2$ and $z=3 \pi i / 2$ are triple poles, while those at $z=\phi$ and $z=\phi+2 \pi i$ are simple poles. A tedious but straightforward evaluation of the
relevant residues gives the following answer:

$$
\partial(\phi)=-\frac{1}{4 \pi^{2} m^{2}}\left\{\frac{1}{2} \frac{\phi^{2}}{\cosh ^{4} \phi}+\frac{1}{\cosh ^{2} \phi}\right\}+\frac{3}{16 m^{2} \cosh ^{4} \phi} \cdot\left(E_{11}\right)
$$

Thus we find

$$
\partial(p)=-\frac{1}{4 \pi^{2} E_{p}^{2}}\left\{1-\frac{3 \pi^{2} m^{2}}{4 E_{p}^{2}}+\frac{m^{2}}{8 E_{p}^{2}} \ln ^{2}\left(\frac{E_{p}+p}{E_{p}-p}\right)\right\}_{1}(E . \|
$$

which combined with (E.3c) verifies Eq. (4.16).

## FIGURE CAPTIONS

Fig. 1: 100 term approximation to a segment of the discontinuous function $\theta(\pi-x) \theta(x)$. The dotted line is the sine series approximation, the dashed line is the Multhopp approximation and the solid line is the sine series with the Lanczos convergence factor approximation.
Fig. 2: Dependence of the ground state eigenvalue $2 M^{2} / G^{2}=\varepsilon_{L G}$ on the dimension $n$ of the Multhopp matrix plotted as a function of $R$. The values of $n$ exhibited are: $\mathrm{n}=11,51,101,201$.

Fig. 3: Dependence of the ground state and first excited state eigenvalue $2 M^{2} / G^{2}=\mathscr{E}_{L G}$ on $R$. The range of $R$ is from 0.1 to 5.0 .
Fig. 4: Comparison of the two lowest eigenvalues $2 \mathrm{M}^{2} / \mathrm{G}^{2}$ of the timelike gauge weak coupling theory with the corresponding eigenvalues of the 1 ightlike gauge 't Hooft model plotted versus R. The range of $R$ is from 0.1 to 5.0. The timelike gauge eigenvalues are shown with a solid line.
Fig. 5: Plot of the ratio $Z=\left(\varrho_{\mathrm{p}}^{2}-\mathrm{R}^{2} \tilde{\mathrm{P}}^{2}\right) / \mathscr{E}_{\mathrm{TG}}{ }^{2}$ versus R for different values of $\tilde{P}=P / m$. The range of $R$ is from 0.1 to 100.0. Curves (a), (b), (c), and (d) correspond to $P=0,1.0,5.0$, and 10.0 , respectively.

Fig. 6: Comparison of the lowest eigenvalues $M^{2} / \gamma=2 M^{2} / G^{2}$ of the timelike gauge string model with the corresponding eigenvalues of the lightlike gauge string model versus $R$. The range of $R$ is from 0.1 to 5.0. The solid curve represents the timelike gauge string model eigenvalues and therefore starts at $R=1.0$.

Fig. 7: Plot of the Bohr-Sommerfeld spectrum $M^{2} / \gamma$ versus principal quantum number $n$ for various values of $R$ : (a) $R=1.0$, (b) $R=10.0$ and (c) $R=100.0$.

Fig. 8: Plot of the difference $\Delta_{T S}(n, R)$ and $\Delta_{L G}(n, R)$ versus principal quantum number $n$ for $R=1.0$. The difference $\Delta_{T S}$ is shown by a $o, \Delta_{L G}$ by a + .

Fig. 9: The Bethe-Salpeter equation for the 't Hooft model in the $1 / \mathrm{N}$ approximation.
Fig. 10: Plot of the three independent ratios of the ground state eigenvalues $M^{2}$ versus $R$. The range of $R$ is from 0.1 to 100.0 . Curve (a) is the ratio $\left(M^{2}\right)_{\mathrm{TS}} /\left(M^{2}\right)_{\mathrm{TG}}$, curve (b) is the ratio $\left(\mathrm{M}^{2}\right)_{\mathrm{TG}} /\left(\mathrm{M}^{2}\right)_{\mathrm{LG}}$ and curve (c) is the ratio $\left(M^{2}\right)_{T S} /\left(M^{2}\right){ }_{L G}$.
Fig. 11: Contour for the integral $0(\Phi)$ in Eq. (E.7).

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Fig. 1


Fig. 2


Fig. 3


Fig. 4


Fig. 5


Fig. 6


Fig. 7


Fig. 8


Fig. 9


Fig. 10


Fig. 11

