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# NONPERTURBATIVE APPROACH TO QUANTUM FIELD THEORIES* <br> Phase Transitions and Confinement 

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[^0]These lectures are devoted to a nonperturbative approach to quantum field theories. We would like to discuss phenomenon for which the usual weak coupling perturbative approach in terms of Feynman diagrams is of no help. We would be interested in studying properties associated with the large distance behavior; i.e., phase transitions, low lying spectra, coherent excitations which are presumably built out of the long wave structure of the theory. These methods will be particularly important for the study of strong coupling field theories and the question of quarks confinement.

Any dynamical theory which incorporate the idea of quarks as fundamental constituents of matter must cope with the fact that quarks behave as if they have lioht mass and are bound together by relatively soft forces although single isoraied quarks have never been observed. Attempts to meet this challenge fall into two categories. There are those schemes which attempt to develop a calculable theory of confined quarks by starting from fundamentally new theoretical concepts like the MIT-bag model, ${ }^{2}$, and there are those schemes which seek to work within the more conservative framework of conventional local quantum field theory, recognizing from the outset that weak coupling perturbation calculations are quite hopeless (at least for the discussions of low lying spectra). ${ }^{3,4}$ Our efforts are along the second line. Questions of the structure of the vacuum, phase transitions, existence of low lying coherent extended states are central to almost all of these approaches.

The methods developed will be applied to calculating the ground state and low lying excitations of various quantum field theories that are rendered finite in terms of a cutoff. The cutoff is expressed by formulating the field theory on a lattice. Since we are interested in studying large distances phenomena, we believe that the lattice theory can teach us (as far as these questions are concerned) about the behavior of the continuum theory. This does not mean that one
can forget about the continuum limit. As a matter of fact as far as the lattice theory is concern there are two limits we should consider

1. The limit of $V$ (the universe volume) $\rightarrow \infty$.
2. The limit of a (the lattice spacing) $\rightarrow 0$.

The limit $\mathrm{V} \rightarrow \infty$ is very important to us since we are interested in phase transitions. As is well known there are no phase transitions in systems of finite volume. The limit $a \rightarrow 0$ is associated with the short distances behavior and is related to the renormalization of the quantum field theory. To take the limit $a \rightarrow 0$ we shall probably have to use renormalization group ideas. This limit is the more difficult one and we shall have very little to say about it. In the large coupling regime we shall be able, however, to exhibit in most of the theories we are going to discuss, low lying states which remain low lying even when the lattice spacing $a \rightarrow 0$ (or alternatively the momentum cutoff $\Lambda \equiv \frac{1}{a} \rightarrow \infty$ ).

1. $\phi^{4}$ THEORY
A. Semiclassical Discussion ${ }^{5,6}$

We shall start our discussion by considering scalar $\phi^{4}$ theory in one spaceone time dimension. The dynamic of this theory is described by the Hamiltonian

$$
\begin{equation*}
\mathrm{H}=\int \mathrm{dx}\left\{\frac{1}{2}\left(\frac{\partial \phi}{\partial \mathrm{t}}\right)^{2}+\frac{1}{2}\left(\frac{\partial \phi}{\partial \mathrm{x}}\right)^{2}+\mathrm{U}(\phi)\right\} \tag{1.1}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathrm{U}(\phi)=\lambda\left(\phi^{2}-\mathrm{f}^{2}\right)^{2}, \quad \mathrm{f}^{2}>0 \tag{1.2}
\end{equation*}
$$

This theory will serve us as an example of a theory which undergos phase transition corresponding to the spontaneous breaking of the discrete symmetry $\phi \rightarrow-\phi$. Moreover, the classical version of this theory possesses a soliton-like "kink" solution which for strong coupling is the low lying state of the theory. In 1x-1t dimensions there is an exactly conserved charge which distinguishes this state from the vacuum state so one might expect that the existence of this extended state
will survive quantization. The scalar $\phi^{4}$ theory is part of the SLAC-bag model and the "kink" is the analogue of the bag in $1 \mathrm{x}-1 \mathrm{t}$ dimensions.

Ignoring quantum aspects of the theory, we drop the momentum $\pi=\frac{\partial \phi}{\partial t}$. To find time independent solutions, Hamiltonian's principle reduces to

$$
\begin{equation*}
\delta \mathrm{V} \equiv \delta \int \mathrm{dx}\left[\frac{1}{2}\left(\frac{\partial \phi}{\partial \mathrm{x}}\right)^{2}+\mathrm{U}(\phi)\right]=0 \tag{1.3}
\end{equation*}
$$

This is equivalent to the mechanical problem of a unit mass particle motion in a potential $-U(x)$

$$
\begin{equation*}
\delta \int d t L=\delta \int d t\left[\frac{1}{2}\left(\frac{d x}{d t}\right)+U(x)\right]=0 \tag{1.4}
\end{equation*}
$$

Figure 1 gives the potential. We are looking for solutions of finite energy. For the energy integral (1.1) to converge, $\phi$ must go to a zero of $u$ as $x$ goes to $\pm$ infinity. 'There are two classes of such solutions. The first class which gives the ground state corresponds to a trivial motion where the particle stays forever at one of the two maxima $\pm f$ (which are also zeroes of $U$ ) of $U$. The other class corresponds to a motion in which the particle starts at time $-\infty$ at one of the maxima and ends at time $+\infty$ at the other maximum. This solution is known as the kink solution. The explicit solutions of the equation of motion gives

$$
\begin{gather*}
\phi^{(0)}= \pm \mathrm{f}  \tag{1.5}\\
\phi_{\mathrm{kink}}= \pm \mathrm{f} \tanh \left[(2 \lambda)^{1 / 2} \mathrm{f}(\mathrm{x}-\mathrm{a})\right] \tag{1.6}
\end{gather*}
$$

Note that for theories (1.1) with $u$ which has only one zero, namely the ground state of the theory is unique, there are no nontrivial time independent solutions of finite energy. Substituting the solutions (1.5) in (1.6) we get the classical energies

$$
\begin{gather*}
\mathrm{E}^{(0)}=0  \tag{1.7}\\
\mathrm{E}_{\text {kink }}=\frac{4}{3} \sqrt{2 \lambda} \mathrm{f}^{3} \tag{1.8}
\end{gather*}
$$

The kink describes a stable configuration even though its energy lies above the ground state energy $\mathrm{E}^{(0)}=0$ for the constant configuration. In 1x-1t dimensions the current

$$
\begin{align*}
& \mathrm{j}_{\mu}(\mathrm{x})=\epsilon_{\mu \nu} \partial_{\nu} \phi \epsilon_{\mu \nu}  \tag{1.9}\\
&=-\epsilon_{\nu \mu} \\
& \epsilon_{01}=1
\end{align*}
$$

is conserved, hence

$$
\begin{equation*}
\mathrm{Q}=\int \mathrm{dx} \mathrm{j}_{0}(\mathrm{x})=\phi(\infty)-\phi(-\infty) \tag{1.10}
\end{equation*}
$$

is time independent. This conserved charge is zero for the vacuum state (1.5); for the kink (1.6), however, $Q^{\text {kink }}= \pm 2 f \neq 0$. Hence the kink is stable.

To discuss quantum fluctuation about a classical solution $g(x)$ we write

$$
\begin{equation*}
\phi(x, t)=g(x)+\phi^{\prime}(x, t) \tag{1.11}
\end{equation*}
$$

Introducing (1.11) into the Hamiltonian (1.1) we obtain

$$
\begin{align*}
H=\int d x\left[\left[\frac{1}{2}\left(\frac{d g}{d x}\right)^{2}+\lambda\left(g^{2}-f^{2}\right)^{2}\right\}\right. & +\left\{\frac{1}{2} \pi^{2}+\frac{1}{2}\left(\frac{\partial \phi^{\prime}}{\partial \mathrm{x}}\right)^{2}+\frac{1}{2} \phi^{2}\left(4 \lambda\left(3 g^{2}-\mathrm{f}^{2}\right)\right)\right\} \\
& \left.+\left\{4 \lambda \mathrm{~g} \phi^{\mathbf{r}^{3}}+\lambda \phi^{\mathbf{t}^{4}}\right\}\right] \tag{1.12}
\end{align*}
$$

where the equation of motion was used to eliminate the linear terms in $\phi^{\prime}$. The usual weak coupling approach correspond to an expansion about the minimum in the classical energy $g= \pm f$. The quadratic terms in (1.12) lead then to normal mode motion for oscillation of mass

$$
\begin{equation*}
\mathrm{m}_{\mathrm{f}}=\sqrt{8 \lambda} \mathrm{f} \tag{1.13}
\end{equation*}
$$

The condition for the kink to be the low lying state is

$$
\begin{equation*}
\mathrm{E}^{\mathrm{kink}} \ll \mathrm{~m}_{\mathrm{f}} \tag{1.14}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathrm{f}^{2} \ll 1 \tag{1.15}
\end{equation*}
$$

This corresponds to strong coupling since for fixed oscillator mass (1.14), it is equivalent to the condition

$$
\begin{equation*}
\lambda \gg \mathrm{m}_{\mathrm{f}}^{2} \text { and } \lambda \mathrm{f} \gg \mathrm{~m}_{\mathrm{f}}^{2} \tag{1.16}
\end{equation*}
$$

In this regime the nonlinear cubic and quartic corrections in (1.12) will be large and their higher order contribution important.

The classical energy difference between the kink solution in the $\mathrm{Q}=2 \mathrm{f}$ sector and the ground state in the $\mathrm{Q}=0$ sector is given by (Eqs. (1.7) and (1.8))

$$
\left(\mathrm{E}^{\mathrm{kink}}-\mathrm{E}^{(0)}\right)_{\text {classical }}=\frac{4}{3} \sqrt{2 \lambda} \mathrm{f}^{3}
$$

The lowest order quantum mechanical energy difference is due to the zero point energies. The one loop correction is calculated by neglecting the cubic and quartic terms in (1.12). The field $\phi^{\prime}(1.11)$ is expanded in normal modes

$$
\begin{gather*}
\phi^{\prime}(x, t)=\sum_{n} \frac{1}{\sqrt{2 E_{n}}}\left(u_{n}(x, t) a_{n}+u_{n}^{*}(x, t) a_{n}^{+}\right)  \tag{1.18}\\
\dot{\phi}(x, t)=\pi(x, t)=-i \sum_{n} \sqrt{\frac{E_{n}}{2}}\left(u_{n} a_{n}-u_{n}^{*} a_{n}^{+}\right)  \tag{1.1.9}\\
{\left[a_{n}, a_{n^{\prime}}^{+}\right]=\delta_{n, n^{i}}} \tag{1.20}
\end{gather*}
$$

For a given classical solution $g(x)$ the $u_{n}{ }^{\prime} s$ and $E_{n}$ 's are determined from the solution to the Schrœdinger equation derived from (1.12)

$$
\begin{equation*}
\left[\frac{d^{2}}{d x^{2}}-4 \lambda\left(3 g^{2}-f^{2}\right)\right] u_{n}=E_{n}^{2} u_{n} \tag{1.21}
\end{equation*}
$$

This equation is easily solved for the $g=f$ case

$$
\begin{equation*}
\frac{1}{2} \sum_{\mathrm{n}} \mathrm{E}_{\mathrm{n}}=\frac{1}{2} \sum_{\mathrm{k}} \sqrt{\mathrm{k}^{2}+8 \lambda \mathrm{f}^{2}} \tag{1.22}
\end{equation*}
$$

It can also be solved for the kink case $g=f \tanh \sqrt{2 \lambda} f x$ and the sift of the zero point energy can be evaluated after performing mass renormalization ${ }^{7}$

$$
\begin{equation*}
\Delta E^{Q M}=\frac{1}{2} \sum_{\mathrm{n}} \mathrm{E}_{\mathrm{n}}(\text { kink })-\mathrm{E}_{\mathrm{n}} \text { (const) }=\left(\frac{\sqrt{3}}{6}-\frac{3}{\pi}\right) \sqrt{2 \lambda} \mathrm{f} \tag{1.23}
\end{equation*}
$$

The kink energy is shifted down as a result of the quantum excitation being drawn into the potential well at the kink boundary. The shift in the 1-loop approximation is of the order of $m_{f}$, much larger than the classical energy for the region of interest where $\mathrm{f}^{2} \ll 1$. An iterative expansion about the classical soliton solution is, therefore, not adequate for the discussion of strong coupling regime.
B. Going to the Lattice

We are going to define the scalar $\phi^{4}$ theory on a spatial lattice. To do it the continuum $\vec{x}$ is replaced by a discrete lattice of linear dimension $L$ and spacing $\mathrm{a}=\frac{1}{\Lambda}$ defined so that there are $2 \mathrm{~N}+1$ points along each direction (see Fig. 2)

$$
\begin{equation*}
L=(2 N+1) / \Lambda \quad V=L^{p} \tag{1.24}
\end{equation*}
$$

The allowed momenta on the lattice are

$$
\begin{align*}
k_{p} & =\frac{2 \pi}{L} n_{p}  \tag{1.25}\\
k_{\max } & =\frac{2 \pi}{L} N
\end{align*} \quad n_{p}=0, \pm 1, \ldots, \pm N
$$

The volume integral becomes

$$
\begin{equation*}
\int d^{p} \mathrm{x}_{\mathrm{x}} \rightarrow \frac{1}{\Lambda^{p}} \sum_{\mathrm{j}} \tag{1.26}
\end{equation*}
$$

The fields at the lattice sites can be expanded in terms of their Fourier components

$$
\begin{align*}
& \pi(\mathrm{x}) \rightarrow \pi_{\mathrm{j}}=\sum_{\mathrm{k}=-\mathrm{k}_{\max }}^{\mathrm{k}_{\max }} \mathrm{e}^{\mathrm{i} \overrightarrow{\mathrm{k}} \cdot \overrightarrow{\mathrm{j}} / \Lambda} \pi(\mathrm{k}) ; \quad \pi(\mathrm{k})=\frac{1}{(2 \mathrm{~N}+1)^{p}} \sum_{\mathrm{j}} \pi_{\mathrm{j}} e^{-\mathrm{i} \overrightarrow{\mathrm{k}} \cdot \overrightarrow{\mathrm{j}} / \Lambda}  \tag{1.27}\\
& \phi(\mathrm{x}) \rightarrow \phi_{\mathrm{j}}=\sum_{\mathrm{k}=-\mathrm{k}_{\max }} \mathrm{e}^{\mathrm{k} \cdot \mathrm{k} \cdot \overrightarrow{\mathrm{j}} / \Lambda} \phi(\mathrm{k}) ; \quad \phi(\mathrm{k})=\frac{1}{(2 \mathrm{~N}+1)^{\mathrm{p}}} \sum_{\mathrm{j}} \phi_{\mathrm{j}} e^{-\mathrm{i} \overrightarrow{\mathrm{k}} \cdot \overrightarrow{\mathrm{j}} / \Lambda} \tag{1.28}
\end{align*}
$$

From the canonical commutation relations on the lattice

$$
\begin{equation*}
\left[\pi_{\vec{j}, \phi_{\vec{j}}}\right]=-\mathrm{i} \Lambda^{\mathrm{p}} \delta_{\vec{j}, \vec{j}} \tag{1.29}
\end{equation*}
$$

it follows

$$
\begin{equation*}
[\pi(\overrightarrow{\mathrm{k}}), \phi(-\vec{l})]=-\mathrm{i} \delta_{\overrightarrow{\mathrm{k}}, \vec{l}} / \mathrm{V} \tag{1.30}
\end{equation*}
$$

$\phi(\mathrm{k})$ and $\pi(\mathrm{k})$ can be written in terms of creation and annihilation operators

$$
\begin{align*}
& \phi(\overrightarrow{\mathrm{k}})=\frac{1}{\sqrt{2 \alpha_{\mathrm{k}}}}\left(\mathrm{a}_{-\overrightarrow{\mathrm{k}}}+\mathrm{a}_{\overrightarrow{\mathrm{k}}}^{+}\right)  \tag{1.31}\\
& \mathrm{i} \pi(\overrightarrow{\mathrm{k}})=\sqrt{\frac{\alpha_{\mathrm{k}}}{2 V}}\left(\mathrm{a}_{-\overrightarrow{\mathrm{k}}}-\mathrm{a}_{\overrightarrow{\mathrm{k}}}^{+}\right)
\end{align*}
$$

Note that the $\alpha_{k}$ can be arbitrary.
To complete our formulation we have to define the gradient operator on the lattice. The most direct way is to define the gradient as a difference operator

$$
\begin{equation*}
\nabla_{\ell} \phi=\Lambda\left(\phi_{\ell+1}-\phi_{\ell}\right) \tag{1.32}
\end{equation*}
$$

However this definition leads to undesirable difficulties with the introduction of fermions as we shall see later on.

Our definition of the gradient starts with the Fourier expansion. For

$$
\begin{equation*}
f_{\vec{j}}=\sum_{\vec{k}} e^{i \vec{k} \cdot \vec{j} / \Lambda} f(\vec{k}) \tag{1.33}
\end{equation*}
$$

we define

$$
\begin{equation*}
\nabla \mathrm{f}_{\overrightarrow{\mathrm{j}}}=\sum_{\overrightarrow{\mathrm{k}}} i \overrightarrow{\mathrm{k}} \mathrm{e}^{\mathrm{i} \overrightarrow{\mathrm{k}} \cdot \overrightarrow{\mathrm{j}} / \Lambda} \mathrm{f}(\overrightarrow{\mathrm{k}})=\sum_{\overrightarrow{\mathrm{j}}} \mathrm{f}_{\overrightarrow{\mathrm{j}}}\left\{\frac{1}{\mathrm{~V}} \sum_{\overrightarrow{\mathrm{k}}} \mathrm{e}^{\mathrm{ik} \cdot\left(\overrightarrow{\mathrm{j}}-\mathrm{j}^{\prime}\right) / \Lambda} \mathrm{ik}\right\} \tag{1.34}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\frac{1}{\Lambda^{\mathrm{p}}} \sum_{\overrightarrow{\mathrm{j}}} \frac{1}{2}(\vec{\nabla} \phi \overrightarrow{\mathrm{j}})^{2}=\frac{V}{2} \sum k^{2} \phi(\overrightarrow{\mathrm{k}}) \phi(\overrightarrow{\mathrm{k}}) \equiv \frac{1}{\Lambda^{p}} \sum_{\overrightarrow{\mathrm{j}}, \vec{j}^{\prime}} \frac{1}{2} \Lambda^{2} \phi \overrightarrow{\mathrm{j}} \phi \overrightarrow{\mathrm{j}^{\prime}} D\left(\mathrm{j}-\mathrm{j}^{\prime}\right) \tag{1.35}
\end{equation*}
$$

where

$$
\begin{equation*}
D\left(j-j^{\prime}\right) \equiv \frac{1}{(2 N+1)^{\mathrm{p}}} \sum_{\overrightarrow{\mathrm{k}}=-\overrightarrow{\mathrm{k}}_{\max }}^{\overrightarrow{\mathrm{k}}_{\max }}\left\{\frac{\mathrm{k}^{2}}{\Lambda^{2}} \mathrm{e}^{\mathrm{i}(\overrightarrow{\mathrm{j}}-\overrightarrow{\mathrm{j}}) \cdot \overrightarrow{\mathrm{k}} / \Lambda}\right\} \tag{1.36}
\end{equation*}
$$

In one space dimension $p=1$

$$
D_{p=1}(j)= \begin{cases}\frac{4 N(N+1)}{(2 N+1)^{2}} \frac{\pi^{2}}{3} \underset{N \rightarrow \infty}{ } \frac{\pi^{2}}{3} & j=0  \tag{1.37}\\ \frac{(2 \pi)^{2}(-)^{j}}{2(2 N+1)} \frac{\cos (\pi j /(2 N+1))}{\sin (\pi j /(2 N+1))^{2}} \underset{N \rightarrow \infty}{\rightarrow \frac{2(-1)^{j}}{j^{2}}} & j \neq 0\end{cases}
$$

For $\mathrm{p}>1$ we get a sum of such correlations in each variable. The important properties of this definition are:
(i) It allows us to define the lattice theory in a way which is essentially isomorphic to momentum cutoff field theory. ${ }^{1 a}$
(ii) For a free field theory the energy momentum dispersion relation has the relativistic form $E(k)=\sqrt{\mathrm{k}^{2}+\mu^{2}}$ for all $\mathrm{k} \leq \mathrm{k}_{\max }$.
(iii) It automatically avoids doubling of the fermion degrees of freedom which result from the fermionic analogue of (1.32). ${ }^{1 \mathrm{~b}}$
(iv) It allows us to write down fermion theories which have local $\gamma_{5}$ invariance. ${ }^{1 b}$

The last two properties will become clear when we discuss fermionic theories. As far as meson theories are concern these properties are not essential and we can use both definitions of the gradient equally well.

The Hamiltonian of the free scalar field on the lattice is given by

$$
\begin{equation*}
\mathrm{H}_{0}=\frac{1}{\Lambda^{\mathrm{p}}} \sum_{\overrightarrow{\mathrm{j}}}\left(\frac{1}{2} \pi_{\vec{j}}^{2}+\frac{1}{2}(\nabla \phi) \frac{2}{\mathrm{j}}+\frac{1}{2} \mu^{2} \phi \stackrel{2}{\mathrm{j}}\right) \tag{1.38}
\end{equation*}
$$

Using the definition (1.34) for the gradient and going to momentum space

$$
\begin{equation*}
\mathrm{H}_{0}=\mathrm{V} \sum_{\overrightarrow{\mathrm{k}}}\left(\frac{1}{2} \pi(\overrightarrow{\mathrm{k}}) \pi(-\overrightarrow{\mathrm{k}})+\frac{1}{2}\left(\mathrm{k}^{2}+\mu^{2}\right) \phi(\overrightarrow{\mathrm{k}}) \phi(-\overrightarrow{\mathrm{k}})\right) \tag{1.39}
\end{equation*}
$$

which leads to the relativistic spectrum $\sqrt{\mathrm{k}^{2}+\mu^{2}}$ for all $\mathrm{k} \leq \mathrm{k}_{\max }$, rather than the form $\sqrt{\mu^{2}+4 \Lambda^{2} \sin ^{2}(k / 2 \Lambda)}$ which would emerge from the nearest neighbors prescription (1.32).

The lattice version of the $\phi^{4}$-theory we choose to work with is

$$
\begin{equation*}
H=\frac{1}{\Lambda^{\mathrm{p}}} \sum_{\vec{j}}\left(\frac{1}{2} \pi_{\vec{j}}^{2}+\frac{1}{2}(\nabla \phi \vec{j})^{2}+\lambda\left(\phi \stackrel{2}{\vec{j}}-f^{2}\right)^{2}\right) \tag{1.40}
\end{equation*}
$$

Rescaling to dimensionsl variables we introduce

$$
\begin{align*}
& x_{\vec{j}}=\Lambda^{1 / 2(1-p)} \phi \vec{j}  \tag{1.41}\\
& p_{\vec{j}}=\Lambda^{-1 / 2(1-p)} \pi_{\vec{j}} \\
& {\left[p_{\vec{j}}, x_{\vec{j}}\right]=-\mathrm{i} \delta \vec{j}, \vec{j}{ }^{\prime}} \tag{1.42}
\end{align*}
$$

Together with

$$
\begin{align*}
& \lambda_{0}=\lambda \Lambda^{p-3}  \tag{1.43}\\
& f_{0}^{2}=f^{2} \Lambda^{(1-p)}
\end{align*}
$$

this gives

$$
\begin{equation*}
\mathrm{H}=\Lambda \sum_{\vec{j}}\left(\frac{1}{2} \mathrm{p} \underset{\mathrm{j}}{2}+\frac{1}{2}(\vec{\nabla} \mathrm{x} \overrightarrow{\mathrm{j}})^{2}+\lambda_{0}\left(\mathrm{x}_{\overrightarrow{\mathrm{j}}}^{2}-\mathrm{f}_{0}^{2}\right)^{2}\right) \tag{1.44}
\end{equation*}
$$

From now on we shall focus on $\phi^{4}$-theory in $1 \mathrm{x}-1 \mathrm{t}$ dimensions. The methods developed will, however, be more general. These methods can be classified as variational and "renormalization group" methods.

## C. Momentum Space Variational Calculation

The momentum space variational calculation is a straightforward example of a variational calculation in quantum field theory. This approach is equivalent to an "average field" or Hartree-Fock approximation ${ }^{8}$ in which the cubic and quartic terms in (1.12) are replaced by

$$
\begin{equation*}
\phi^{3} \rightarrow 3<\phi^{2}>\phi \tag{1.45a}
\end{equation*}
$$

$$
\begin{equation*}
\phi^{4} \rightarrow 6<\phi^{2}>\phi^{2}+3<\phi^{2}> \tag{1.45b}
\end{equation*}
$$

In the fermionic case this method reproduces the effective potential calculation in the one loop approximation.

The trial state we take is a displaced Gaussian packet for each k -mode

$$
\begin{equation*}
\left|\psi\left(\alpha_{\mathrm{k}} ; \mathrm{c}\right)\right\rangle=\mathrm{e}^{-\mathrm{icV} \pi(\mathrm{k}=0)}\left|0_{\alpha_{\mathrm{k}}}\right\rangle \tag{1.46}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{a}_{\alpha_{\mathrm{k}}}{ }^{10} \alpha_{\mathrm{k}}>=0 \quad \text { for all } \mathrm{k} \tag{1.47}
\end{equation*}
$$

The creation and annihilation operators were introduced in (1.31). The $\alpha_{k}{ }^{\prime} s$ which correspond to the width of the packet and $c$ which gives the position of the center of the packet are our variational parameters.

It is easy to calculate the variational ground state energy of the Hamiltonian

$$
\begin{equation*}
\mathrm{E}\left(\alpha_{\mathrm{k}} ; \mathrm{c}\right)=\left\langle\psi\left(\alpha_{\mathrm{k}} ; \mathrm{c}\right)\right| \mathrm{H}\left|\psi\left(\alpha_{\mathrm{k}} ; \mathrm{c}\right)\right\rangle \tag{1.48}
\end{equation*}
$$

by using Eqs. (1.44), (1.27), (1.28) and (1.31).
The variation with respect to $\alpha_{\mathrm{k}}{ }^{\text {'s gives }} \alpha_{\mathrm{k}}^{2}=\mathrm{k}^{2}+\alpha_{0}^{2}$, with $\alpha_{0}$ satisfying the gap equation

$$
\begin{equation*}
\alpha_{0}^{2}=4 \lambda\left(\overline{\mathrm{f}}\left(\alpha_{0}\right)-\mathrm{f}^{2}+3 \mathrm{c}^{2}\right) \tag{1.49}
\end{equation*}
$$

where

$$
\overline{\mathrm{f}}\left(\alpha_{0}\right)=\frac{3}{2 \mathrm{~V}} \sum_{\mathrm{k}} \frac{1}{\sqrt{\mathrm{k}^{2}+\alpha_{0}^{2}}}
$$

To illustrate the behavior for weak $\left(\mathrm{f}_{0}^{2} \gg 1, \lambda_{0}\right.$ fixed $)$, intermediate and strong $\left(\mathrm{f}_{0}^{2} \ll 1, \lambda_{0}\right.$ fixed) coupling regimes, we plot in Fig. 3 the ground state energy $\mathrm{E}_{0}=\mathrm{E}\left(\alpha_{0}(\mathrm{c}), \mathrm{c}\right)$ as a function of the displacement c . The solution of the gap equation (1.49) gives $\alpha_{0}=\alpha_{0}$ (c). There is always a local minimum at $c=0$. As $f_{0}^{2}$ increases, for fixed $\lambda_{0}$, two local minima appear at $\pm c \neq 0$. For $\lambda_{0} \sim 1$ these minima appear when $f_{0}^{2} \sim 1$. For large enough $f_{0}^{2}$ the minima at $c \neq 0$ cross the one
at $c=0$. They then become the degenerate ground states of the theory with $\langle\phi\rangle=+c$ or -c respectively, because the overlap between them vanishes in the $\mathrm{V} \rightarrow \infty$ limit, i.e.,

$$
\begin{equation*}
\left\langle\psi\left(\alpha_{k} ; c\right) \mid \psi\left(\alpha_{k} ;-c\right)\right\rangle=e^{-\alpha_{0} V^{2} c} \tag{1.50}
\end{equation*}
$$

Assuming we choose the one at +c , Fig. 3 shows that as we vary f the lowest eigenstate jumps discontinuously from a state such that $\langle\phi\rangle=\sim \sim \mathrm{f}$ to $\langle\phi\rangle=0$ at some critical value. This first order phase transition is an incorrect prediction coming from the variation choice (1.46) for the ground state; we know this since the behavior implied by it violates a rigorous theorem (specific to this model in 1x-1t dimensions) proved by Simon and Griffiths. ${ }^{9}$. The theorem states that in the presence of an external source $J \phi$ the expectation value of the field $\langle\phi\rangle$ is a monotonic analytic function of $J$ for finite $J \neq 0$.

The ground state energy of the system in the presence of the source is

$$
\begin{equation*}
E_{0}^{\prime}\left(\alpha_{0}(c), c\right)=E_{0}\left(\alpha_{0}, c\right)-J c \tag{1.51}
\end{equation*}
$$

Figure 4 gives schematically the phase diagram. As illustrated in the region $f_{0}^{2}<f_{c r}^{2}$, a ground state developing from the $c=0$ minimum of $E_{0}$ when $J<J{ }_{c r i t}(\lambda)$, jumps discontinuously to the one developing from the $c \neq 0$ minimum of $E_{0}$ when $J>J_{\text {crit }}$. It is this behavior which is forbidden by the Simon-Griffiths theorem. This result suggests that this method is inadequate for studying the strong coupling regime where the tri-critical behavior sets in. It is precisely this region with $0<\mathrm{f}_{0}^{2}<1$ that the semi-classical analysis suggests we must study to find the kink as a low lying state. Note that if this calculation were valid, the "kink" could never exist as a low lying state since by the time the vacuum expectation value $\langle\phi\rangle$ decreases to order of 1 , it jumps discontinuously to zero. For the existence of the kink as a low lying state we need to find a region where $\langle\phi\rangle$ is arbitrarily small but different from zero.

## D. Variational Calculation in the Single Site Basis

To understand the failure of the momentum space calculation let us consider our Hamiltonian (1.44). The single site terms

$$
H_{S S}=\Lambda \sum_{j}\left(\frac{1}{2} p_{j}^{2}+\lambda_{0}\left(x_{j}^{2}-f_{0}^{2}\right)^{2}\right)
$$

describe a Schrödinger problem of a particle moving in an anharmonic potential depicted in Fig. 5. The potential has a bump of height $\lambda f^{4}$ near its center at $x=0$, whereas for small oscillations about the minimum near $x=f$, the zero point energy is $\sim \sqrt{2 \lambda}$ f given by the curvature at the minimum. For weak coupling, i.e., for small field amplitude and harmonic motion near the bottom of the potential well we have

$$
\begin{equation*}
\lambda_{0} f_{0}^{4} \gg \lambda_{0}^{1 / 2} f_{0} \tag{1.52}
\end{equation*}
$$

or

$$
\begin{equation*}
\lambda_{0}^{1 / 2} \mathrm{f}_{0}^{3} \gg 1 \tag{1.53}
\end{equation*}
$$

In this limit the energy splitting between the two lowest energy levels (the symmetric and antisymmetric solutions of the Schrödinger problem) is very small due to the suppressed tunneling of the oscillation amplitude through the center bump. The gradient term in (1.44) mixes the even and odd parity solutions, the amount of mixing is inversely proportional to the energy difference. Hence, in the weak coupling region the mixing is large and it becomes important to treat the gradient term accurately, which of course is precisely what we do by working in momentum space which diagonalizes the kinetic term and by the choice of the ground state. Indeed in the weak coupling region the system is in the $c \neq 0$ phase for its ground state (see Fig. 3). As we approach the strong coupling region the energy difference between the even and odd solutions increascs and the relative strength of the gradient term decreases. In this situation a site basis, i.e., a trial state diagonalizing the terms at individual lattice sites is more natural.

The gradient term in the Hamiltonian is the only term which couples different sites. Neglecting the gradient term leaves us with identical anharmonic Schrödinger problems at each site. In this approximation the eigenstates of our system are a product of eigenstates at each site $j$. The lowest energy state will be

$$
\begin{equation*}
\left|\psi_{0}\right\rangle=\prod_{\mathrm{j}}\left|\psi_{0}\right\rangle \mathrm{j} \tag{1.54}
\end{equation*}
$$

where $\left|\psi_{0}\right\rangle_{j}$ is the ground state at the site $j$. The next level up will correspond to a state for which at each site $j$ the oscillator is in its ground state apart from one site in which it is in the first excited state.

$$
\begin{equation*}
\left|\psi_{1}\right\rangle=\left(\prod_{j<i} \mid \psi_{0}>_{j}\right)\left|\psi_{1}\right\rangle_{i}\left(\prod_{j>i} \mid \psi_{0}>_{j}\right) \tag{1.55}
\end{equation*}
$$

Since the site i can be any of the sites, we have a huge degeneracy. The role of the gradient is to lift this degeneracy. It is when these gradient-induced splittings are small relative to the spacing between the single site excited states that the single site basis is expected to give a good description of the ground state.

The single site variational basis is introduced in terms of creation and annihilation operators at each site j

$$
\begin{align*}
& \mathrm{x}_{\mathrm{j}}=\frac{1}{\sqrt{2 \alpha_{j}}}\left(\mathrm{a}_{\mathrm{j}}+\dot{a}_{\mathrm{j}}^{+}\right) \\
& \mathrm{ip}_{\mathrm{j}}=\sqrt{\frac{\alpha_{j}}{2}}\left(\mathrm{a}_{\mathrm{j}}-\mathrm{a}_{\mathrm{j}}^{+}\right)  \tag{1.56}\\
& {\left[\mathrm{a}_{\mathrm{j}}, \mathrm{a}_{\mathrm{j}^{\prime}}^{+}\right]=\delta_{\mathrm{jj}}}
\end{align*}
$$

The vacuum at site $j$ is defined by

$$
\begin{equation*}
a_{j} \mid 0_{j}>=0 \tag{1.57}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|n_{j}\right\rangle=\frac{1}{\sqrt{n_{j}^{!}}}\left(a_{j}^{+}\right)^{n_{j}}\left|0_{j}\right\rangle \tag{1.58}
\end{equation*}
$$

The trial state will have the form

$$
\begin{equation*}
|\psi\rangle=\prod_{j}\left|\psi_{\mathrm{j}}\right\rangle, \quad\left\langle\psi_{\mathrm{j}} \mid \psi_{\mathrm{j}^{\prime}}\right\rangle=\delta_{\mathrm{j}, \mathrm{j}^{\prime}} \tag{1.59}
\end{equation*}
$$

where

$$
\begin{equation*}
\left|\psi_{j}\right\rangle=\sum_{n_{j}=0}^{\infty} C_{n_{j}}\left|n_{j}\right\rangle \tag{1.60}
\end{equation*}
$$

We assume the ground state to be translationally invariant hence $\left|\psi_{j}\right\rangle$ is the same for each site j. [Note that in this approximation we essentially neglect correlations between sites. Keeping $\left|\psi_{j}\right\rangle$ as a variational wave function we allow, however, for mixing between all the single site levels.] For Hamiltonian of the form

$$
\begin{equation*}
H=\Lambda\left[\sum_{j} H_{s S}(j)+\frac{1}{2} \sum_{j_{1} \neq j_{2}} D\left(j_{1}-j_{2}\right) x_{j_{1}} x_{j_{2}}\right] \tag{1.61}
\end{equation*}
$$

(the diagonal terms in the gradient $\left(j_{1}=j_{2}\right)$ have been taken as a part of the single site Hamiltonian $H_{s s}(\mathrm{j})$ ), the energy in trial state (1.59) is

$$
\begin{equation*}
\langle\psi| \mathrm{H}|\psi\rangle=\Lambda\left[\sum_{\mathrm{j}}\left\langle\psi_{\mathrm{j}}\right| \mathrm{H}_{\mathrm{SS}}(\mathrm{j})\left|\psi_{\mathrm{j}}\right\rangle+\frac{1}{2} \sum_{\mathrm{j}_{1} \neq \mathrm{j}_{2}} \mathrm{D}\left(\mathrm{j}_{1}-\mathrm{j}_{2}\right)\left\langle\psi_{\mathrm{j}_{1}}\right| \mathrm{x}_{\mathrm{j}_{1}}\left|\psi_{\mathrm{j}_{1}}\right\rangle\left\langle\psi_{\mathrm{j}_{2}}\right| \mathrm{x}_{\mathrm{j}_{2}}\left|\psi_{\mathrm{j}_{2}}\right\rangle\right] \tag{1.62}
\end{equation*}
$$

Using the translational invariant of the state and the identity (1.36)

$$
\begin{equation*}
\sum_{j_{1}} \mathrm{D}\left(\mathrm{j}_{1}-\mathrm{j}_{2}\right)=0 \tag{1.63}
\end{equation*}
$$

so that

$$
\begin{equation*}
\sum_{\mathrm{j}_{1} \neq \mathrm{j}_{2}} D\left(\mathrm{j}_{I}-\mathrm{j}_{2}\right)=-\sum_{\mathrm{j}} \mathrm{D}(0)=-\mathrm{L} \mathrm{D}(0) \tag{1.64}
\end{equation*}
$$

we get

$$
\begin{equation*}
\mathrm{E}_{0}(\psi)=\langle\psi| \mathrm{H}|\psi\rangle=\Lambda L\left[\langle\psi| \mathrm{H}_{\mathrm{SS}}(\mathrm{j})|\psi\rangle-\frac{1}{2} \mathrm{D}(0)\langle\psi| \mathrm{x}|\psi\rangle^{2}\right] \tag{1.65}
\end{equation*}
$$

Next we vary the trial state $|\psi\rangle$ so as to minimize $E_{0}(\psi)$. If it were not for the last term, which involves the expectation value square we would just solve the
single site Schrodinger problem $\mathrm{H}_{\mathrm{SS}}(\mathrm{j})$. The actual minimization can be carried in two steps. First the variation is carried with $\langle\psi| x|\psi\rangle$ held fixed by introducing a Lagrange multiplier $J$, then $E_{0}(\psi)$ is varied over all values of $\langle\psi| x|\psi\rangle$. We define

$$
\begin{equation*}
\bar{H}(J)=H_{S S}-J x \tag{1.66}
\end{equation*}
$$

and denote by

$$
\begin{equation*}
\Gamma(J)=\left\langle\psi_{0}\right| \overline{\mathrm{H}}(J)\left|\psi_{0}\right\rangle ; \quad \frac{\partial \Gamma}{\partial J}=-x(J) \tag{1.67}
\end{equation*}
$$

its ground state eigenvalue. The energy density associated with $\mathrm{E}_{0}$ (1.65)

$$
\begin{equation*}
\mathscr{E}(J)=\frac{1}{\Lambda L} E(J)=\Gamma(J)+J x(J)-\frac{1}{2} D(0) x^{2}(J) \tag{1.68}
\end{equation*}
$$

and the problem of minimizing $E_{0}(1.65)$ is reduced to finding the value of $J$ such that

$$
\frac{\partial \mathscr{E}(J)}{\partial J}=0
$$

## E. A Simple Example-Free Field

The free field Hamiltonian

$$
\begin{equation*}
H_{0}=\Lambda\left[\sum_{j}\left\{\frac{p_{j}^{2}}{2}+\frac{1}{2}\left(\frac{\mu^{2}}{\Lambda^{2}}+D(0)\right) x_{j}^{2}\right\}+\frac{1}{2} \cdot \sum_{j_{1} \neq j_{2}} D\left(j_{1}-j_{2}\right) x_{j_{1}} x_{j_{2}}\right] \tag{1.69}
\end{equation*}
$$

has as its single site part

$$
\begin{equation*}
\mathrm{H}_{\mathrm{SS}}=\frac{\mathrm{p}^{2}}{2}+\frac{1}{2}\left(\frac{\mu^{2}}{\Lambda^{2}}+\mathrm{D}(0)\right) \mathrm{x}^{2} \tag{1.70}
\end{equation*}
$$

$\mathrm{H}_{0}$ is completely diagonalized in momentum space and the exact ground state energy is

$$
\begin{equation*}
\mathrm{E}_{0}^{\mathrm{ex}}=\frac{1}{2} \sum_{\mathrm{k}} \sqrt{\mathrm{k}^{2}+\mu^{2}}=\frac{\Lambda^{2} \mathrm{~L}}{2 \pi} \int_{0}^{\pi} \mathrm{dx} \sqrt{\mathrm{x}^{2}+\mu^{2} / \Lambda^{2}} \tag{1.71}
\end{equation*}
$$

with a mass gap $\mu$ to the first single particle excited state, i.e.,

$$
\begin{equation*}
\mathrm{E}_{1}^{\mathrm{ex}}-\mathrm{E}_{0}^{\mathrm{ex}}=\mu \tag{1.72}
\end{equation*}
$$

and the splitting among the excited single particle is

$$
\begin{equation*}
\mathrm{E}_{1}^{\mathrm{ex}}(\mathrm{n})-\mathrm{E}_{0}^{\mathrm{ex}}=\sqrt{\mu^{2}+\left(\frac{2 \pi \mathrm{~m}}{2 \mathrm{~N}+1}\right)^{2} \Lambda^{2}} \quad \mathrm{n}=0, \pm 1, \ldots, \pm \mathrm{N} \tag{1.73}
\end{equation*}
$$

Following the steps described before for the single site variational basis we find that the energy minimizes at $\mathrm{J}=0$ with $\mathrm{x}(\mathrm{J}=0)=0$ and

$$
\begin{equation*}
\mathrm{E}_{0}^{1-\text { site }}=\frac{\Lambda^{2} \mathrm{~L}}{2} \sqrt{\mathrm{D}(0)+\mu^{2} / \Lambda^{2}} \tag{1.74}
\end{equation*}
$$

with a gap to the first excited state

$$
\begin{equation*}
\mathrm{E}_{1}^{1-\text { site }}-\mathrm{E}_{0}^{1-\text { site }}=\sqrt{\mu^{2}+\Lambda^{2} \mathrm{D}(0)} \tag{1.75}
\end{equation*}
$$

The accuracy of the single site calculation is measured by $\Lambda^{2} D(0) / \mu^{2}$. When this ratio is smaller than one the gaps between higher excitations in the site basis are large compared to the splittings among the degenerate one particle levels and hence relatively unimportant.
F. $\phi^{4}$ Theory

In this case

$$
\begin{equation*}
H_{S S}=\frac{p^{2}}{2}+\lambda_{0}\left(x^{2}-f_{0}^{2}\right)^{2}+\frac{1}{2} D(0) x^{2} \tag{1.76}
\end{equation*}
$$

and

$$
\begin{equation*}
\bar{H}(J)=\frac{p^{2}}{2}+\lambda_{0}\left(x^{2}-f_{0}^{2}\right)^{2}+\frac{1}{2} D(0) x^{2}-J x \tag{1.77}
\end{equation*}
$$

For $J=0$ the Schrödinger problem (1.77) has a symmetrical potential and the lowest state is symmetric, hence $x(J=0)=0$. Since the term $-J x$ is an analytic perturbation on $\bar{H}(J=0)$ we have for small $J$

$$
\begin{equation*}
x(J)=C_{1} J\left(1+C_{3} J^{2}+C_{5} J^{4}+\ldots\right) \tag{1.78}
\end{equation*}
$$

Recall that the small $J$ (and therefore small $x$ ) is of interest in exploring the region of small kink mass.

From Eqs. (1.67) and (1.78)

$$
\begin{equation*}
\Gamma(J)=\Gamma(0)-\frac{1}{2} C_{1} J^{2}-\frac{1}{4} C_{1} C_{3} J^{4} \tag{1.79}
\end{equation*}
$$

and (1.68)

$$
\begin{equation*}
\mathscr{E}(J)=\Gamma(0)+\frac{1}{2} C_{1}\left(1-C_{1} D(0)\right) J^{2}+C_{1} C_{3}\left(\frac{3}{4}-C_{1} D(0)\right) J^{4}+\ldots \tag{1.80}
\end{equation*}
$$

Using Eq. (1.78)

$$
\begin{equation*}
\mathscr{E}(\mathrm{x}(J))=\Gamma(0)-\frac{\eta}{2 \mathrm{C}_{1}} \mathrm{x}^{2}-\frac{\mathrm{C}_{3}}{\mathrm{C}_{1}}\left(\frac{1}{1}+\eta\right) \mathrm{x}^{4} \tag{1.81}
\end{equation*}
$$

where

$$
\begin{equation*}
\eta \equiv-1+\mathrm{C}_{1} \mathrm{D}(0) \tag{1.82}
\end{equation*}
$$

Since $C_{1}$ is always positive according to (1.77) and (1.78) this will minimize for small $x$ if ;

$$
\begin{equation*}
0<\eta \ll 1 \quad \text { and } \quad \mathrm{C}_{3}<0 \tag{1.83}
\end{equation*}
$$

with the minimum $x_{c}$ occurring at

$$
\begin{equation*}
\mathrm{x}_{\mathrm{c}}^{2} \cong \frac{\eta \mathrm{C}_{1}^{2}}{\left|\mathrm{C}_{3}\right|}(1-4 \eta) \tag{1.84}
\end{equation*}
$$

Hence $x_{c}^{2} \ll 1$ if

$$
\begin{equation*}
\frac{\eta \mathrm{C}_{1}^{2}}{\left|\mathrm{C}_{3}\right|} \ll 1 \tag{1.85}
\end{equation*}
$$

It remains to show there exists a range of $\lambda_{0}$ and $f_{0}$ for which the conditions on $\mathrm{C}_{1}$ and $\mathrm{C}_{3}$ are satisfied. This can be done analytically by introducing a displaced Gaussian trial state $\mid \psi_{0}>$

$$
\begin{equation*}
\left|\psi_{0}\right\rangle=e^{-\mathrm{iCp}}|0\rangle ; \quad\left\langle\psi_{0}\right| \mathrm{x}\left|\psi_{0}\right\rangle=\mathrm{C} \tag{1.86}
\end{equation*}
$$

The displacement C and mass $\alpha$ (see Eq. (1.56)) are the variational parameters. I'll skip the details of the calculation ${ }^{1 a}$ and just quote the result. For

$$
\begin{equation*}
\mathrm{f}_{0}^{2}>\mathrm{f}_{\mathrm{cr}}^{2}=\frac{3}{2 \sqrt{\mathrm{D}(0)}} \underset{\mathrm{N} \rightarrow \infty}{\longrightarrow} 0.83 \tag{1.87}
\end{equation*}
$$

and

$$
\lambda_{0}<\frac{1}{6}(\mathrm{D}(0))^{3 / 2} \underset{\mathrm{~N} \rightarrow \infty}{\longrightarrow} \frac{3}{18 \sqrt{3}} \sim 1.0
$$

we find a second order phase transition

$$
\begin{equation*}
\mathrm{x}_{\mathrm{c}}=\langle\mathrm{x}\rangle \propto \sqrt{\mathrm{f}_{0}^{2}-\mathrm{f}_{\mathrm{cr}}^{2}} \tag{1.88}
\end{equation*}
$$

F. Upper Bound on the Kink Mass

To calculate the energy of the kink configuration in the single site basis we modify Eq. (1.62) by adding and subtracting the diagonal term $\mathrm{j}_{1}=\mathrm{j}_{2}$ in the double sum,

$$
\begin{align*}
E_{\text {kink }}=\left\langle\psi_{\text {kink }}\right| H\left|\psi_{\text {kink }}\right\rangle= & \Lambda\left[\sum_{j}\left\langle\psi_{j}\right| \frac{p_{j}^{2}}{2}+\lambda_{0}\left(x_{j}^{2}-f_{0}^{2}\right)^{2}+\frac{1}{2} D(0)\left(x_{j}^{2}-x_{j}\left\langle x_{j}\right\rangle\right)\left|\psi_{j}\right\rangle\right] \\
& +\frac{1}{2} \sum_{j_{1} j_{2}} D\left(j_{1}-\mathrm{j}_{2}\right)\left\langle\psi_{j_{1}}\right| x_{j_{1}}\left|\psi_{j_{1}}\right\rangle\left\langle\psi_{j_{2}}\right| x_{j_{2}}\left|\psi_{j_{2}}\right\rangle \tag{1.89}
\end{align*}
$$

The last term is essentially the classical gradient term with the matrix element of the field replacing its classical strength. The form of the kink configuration $\left\langle x_{j}\right\rangle$ is shown in Fig. 6. This state is orthogonal to the vacuum in the limit $L \rightarrow \infty$ since the conserved charge $Q(1.10)$ is different from zero.

The minimization is done by repeating the steps for the vacuum calculation. At each site a Lagrange multiplier $J(j)$ is introduced and we define

$$
\begin{equation*}
\bar{H}(J(j))=\frac{1}{2 N+1} \sum_{j}\left[\frac{p_{j}^{2}}{2}+\frac{D(0) x_{j}^{2}}{2}+\lambda_{0}\left(x_{j}^{2}-f_{0}^{2}\right)^{2}-J(j) x_{j}\right] \tag{1.90}
\end{equation*}
$$

and

$$
\begin{equation*}
\Gamma(J(\mathrm{j}))=\left\langle\psi_{\text {kink }}\right| \overline{\mathrm{H}}\left|\psi_{\text {kink }}\right\rangle=\frac{1}{2 \mathrm{~N}+1} \sum_{\mathrm{j}} \Gamma_{\mathrm{j}}(J(\mathrm{j})) \tag{1.91}
\end{equation*}
$$

where $\Gamma_{j}$ is the same function calculated in (1.73). Then we are instructed to find the local minima with respect to $J(j)$

$$
\begin{align*}
\frac{1}{2 N+1} E_{k i n k}=\frac{1}{2 N+1} \sum_{j}\left\{\Gamma_{j}(J(j))\right. & \left.\left.+J(j)<x_{j}>-\frac{1}{2} D(0)<x_{j}\right\rangle^{2}\right\} \\
& \left.+\frac{1}{2 N+1} \sum_{j_{1} j_{2}} \frac{1}{2} D\left(j_{1}-j_{2}\right)<x_{j_{1}}><x_{j_{2}}\right\rangle \tag{1.92}
\end{align*}
$$

Since apart from a finite length on the latticc $\langle x(j)\rangle$ must be arbitrarily close to the expectation value $\pm x_{c}$ in the vacuum statc (1.88) to assure a finite kink energy, a crude approximation which reveals the dependence of the kink energy on the parameters of the theory is

$$
\begin{align*}
& \left\langle x_{j}\right\rangle-+x_{c} \quad \text { for } j>j_{0} \\
& \left\langle x_{j}\right\rangle=-x_{c} \quad \text { for } j<j_{0}  \tag{1.93}\\
& D=\left(2 j_{0}+1\right) / \Lambda
\end{align*}
$$

The transition width $D$ is treated as a variational parameter. Once again I'll omit the detailed calculation and quote just the result

$$
\begin{equation*}
\mathrm{E}_{\text {kink }}-\mathrm{E}_{0}\left(\mathrm{~J}_{\mathrm{c}}\right)=\Lambda\left[(\mathrm{D} \Lambda) \frac{\eta}{2 \mathrm{C}_{1}} \mathrm{x}_{\mathrm{c}}^{2}+\frac{\mathrm{x}_{\mathrm{c}}^{2}}{(\mathrm{D} \Lambda)}\right] \tag{1.94}
\end{equation*}
$$

minimizing with respect to D we find

$$
\begin{equation*}
D \sim\left(2 \lambda_{0}^{1 / 2} x_{c}\right) \Lambda^{-1}=\left(2 \lambda^{1 / 2} x_{c}\right)^{-1} \tag{1.95}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{E}_{\mathrm{kink}}-\mathrm{E}_{0}\left(\mathrm{~J}_{\mathrm{c}}\right) \approx 4 \lambda^{1 / 2} \mathrm{x}_{\mathrm{c}}^{3}\left(1+\mathrm{O}\left(\mathrm{x}_{\mathrm{c}}\right)\right) \tag{1.96}
\end{equation*}
$$

Assuming that the ground state energy has been evaluated accurately, the kink energy is a rescaled version of the semiclassical result $\sim \lambda^{1 / 2} f^{3}$ (1.17). The
effect of the quantum correction has been simply to rescale the classical field strength f to $\mathrm{X}_{\mathrm{c}}$ for site outside the transition region. The kink mass

$$
\begin{equation*}
\mathrm{m}_{\mathrm{kink}} \sim 4 \lambda_{0}^{1 / 2} \mathrm{x}_{\mathrm{c}}^{3} \Lambda \tag{1.97}
\end{equation*}
$$

can be made finite and small no matter how large the cutoff $\Lambda$ is by going into the region where $x_{c}$ is small enough. This result (1.97) is not an artifact of the lattice since the kink width extends over many lattice sites ((1.93), (1.95))

$$
\left(2 j_{0}+1\right) \sim \mathrm{D} \Lambda \sim \frac{1}{2 \lambda_{0}^{1 / 2} \mathrm{x}_{\mathrm{c}}}
$$

## 2. THE FINITE SPIN APPROXIMATION

A. General Formalism ${ }^{11}$

The $\quad \phi^{4}$ theory with nearest neighbor interactions is described on the lattice by the Hamiltonian

$$
\begin{align*}
H & =\Lambda\left[\sum_{j} \frac{p_{j}^{2}}{2}+\frac{\mu^{2}}{2} x_{j}^{2}+\frac{1}{2}\left(x_{j+1}-x_{j}\right)^{2}+\lambda x_{j}^{4}\right]  \tag{2.1}\\
& =\Lambda\left[\sum_{j} \frac{p_{j}^{2}}{2}+\frac{\mu^{2}+2}{2} x_{j}^{2}+\lambda x_{j}^{4}-x_{j} x_{j+1}\right]
\end{align*}
$$

with periodic boundary conditions.
At each site the single site terms describe the quantum mechanical problem of an anharmonic oscillator. Solving this Schrödinger problem

$$
\begin{equation*}
\left(\frac{p^{2}}{2}+\frac{\mu^{2}+2}{2} x^{2}+\lambda x^{4}\right)|n\rangle=E_{n}|n\rangle \tag{2.2}
\end{equation*}
$$

and choosing as a basis the tensor product of the anharmonic oscillator at each site

$$
\begin{equation*}
\underset{j}{\otimes\left|n_{j}\right\rangle} \quad 0 \leq n_{j} \leq \infty \tag{2.3}
\end{equation*}
$$

the Hamiltonian (2.1) can be written as

$$
\begin{equation*}
H=\sum_{j}\left(E^{j}-X^{j} \otimes X^{j+1}\right) \tag{2.4}
\end{equation*}
$$

where we work with the dimensionless $H(H \equiv H / \Lambda)$. E is a diagonal matrix whose entries are the single site eigenvalues $\left\{E_{n}\right\}$. $X$ is a matrix with nonvanishing elements between even and odd parity states, $X_{n m}=\langle n| x|m\rangle$.

The finite spin approximation is defined by truncating the base to a finite number $S$ of levels at each site

$$
\begin{equation*}
\underset{j}{\otimes}\left|n_{j}\right\rangle \quad 0 \leq n_{j} \leq S-1 \tag{2.5}
\end{equation*}
$$

The truncated Hamiltonian then represent coupled spin s system with $2 \mathrm{~s}+1=\mathrm{S}$.
B. Spin 1/2 Approximation ${ }^{12}$

In this approximation $S=2$ namely only the two low lying states of the single site Schrödinger problem are retained. This approximation is reasonable as long as the coupling is small enough to mix higher states only weakly. In the region $\mu^{2}>0$ the low lying energy levels are dominated by the harmonic term and the levels are equally spaced. As $\mu^{2}$ becomes more negative we reach the situation where the potential has two minima, and tunneling between the two wells splits the energies, of the two lowest eigenstates slightly. The deeper the wells become, the tunneling between them and therefore the splitting of the two lowest levels, decrease exponentially, while the excited states are pushed to higher energies.

The Hamiltonian matrix in this approximation

$$
\left.\begin{array}{rl}
H & =\sum_{j=-N}^{N}\left(\begin{array}{cc}
E_{1} & 0 \\
0 & E_{0}
\end{array}\right)_{j}-\left(\begin{array}{cc}
0 & \langle 1| x|0\rangle \\
\langle 0| x|1\rangle & 0
\end{array}\right)_{j}\left(\begin{array}{cc}
0 & \langle 1| x|0\rangle \\
\langle 0| x|1\rangle & 0
\end{array}\right)_{j+1} \\
& =\sum_{j=-N}^{N}\left[\bar{E} 1+\frac{\epsilon}{2} \sigma_{j}^{Z}-\Delta \sigma_{j}^{\mathrm{x}} \sigma_{j+1}^{\mathrm{x}}\right]=\mathrm{const}+\sum_{j}\left(\frac{\epsilon}{2} \sigma_{j}^{\mathrm{Z}}-\Delta \sigma_{j}^{\mathrm{x}} \sigma_{j+1}^{\mathrm{x}}\right)
\end{array}\right)
$$

where

$$
\begin{align*}
& \overline{\mathrm{E}}=\frac{1}{2}\left(\mathrm{E}_{0}+\mathrm{E}_{1}\right) \\
& \epsilon=\left(\mathrm{E}_{1}-\mathrm{E}_{0}\right) \quad \sigma_{\mathrm{N}+1} \equiv \sigma_{-N}  \tag{2.7}\\
& \Delta=|<0| \mathrm{x}|1>|^{2}
\end{align*}
$$

and the $\sigma^{\prime}$ s are the usual Pauli matrices. Since only two states are considered for each oscillator they can also be represented by the presence or absence of a fermion. The formal transformation to this language is known as the WignerJordan transformation

$$
\sigma_{\mathrm{j}}^{+}=\prod_{\ell=-\mathrm{N}}^{\mathrm{j}-1}(-)^{\mathrm{n}_{\ell}} b_{\mathrm{j}}^{+} \quad \mathrm{n}_{\ell}=b_{\ell}^{+} \mathrm{b}_{\ell}
$$

$$
\begin{align*}
& \sigma_{j}^{-}=\prod_{\ell}^{\mathrm{j}-1}(-)^{\mathrm{n}_{\ell}} \mathrm{b}_{\mathrm{j}} \\
& \sigma_{\mathrm{j}}^{\mathrm{z}}=2 \mathrm{n}_{\mathrm{j}}-1 \tag{2.8}
\end{align*}
$$

where the $b_{j}{ }^{\prime} s$ and $b_{j}{ }^{+}$'s are Fermi operators satisfying anticommutation relations. It is easy to check that for $-N \leq j \leq N-1$

$$
\begin{array}{ll}
\sigma_{j}^{+} \sigma_{j+1}=b_{j}^{+} b_{j+1} & \sigma_{j} \sigma_{j+1}^{+}=-b_{j} b_{j+1}^{+}  \tag{2.9}\\
\sigma_{j}^{+} \sigma_{j+1}^{+}=b_{j}^{+} b_{j+1}^{+} & \sigma_{j} \sigma_{j+1}=-b_{j} b_{j+1}
\end{array}
$$

For the cyclic chain we also have the terms
where

$$
\begin{equation*}
\mathrm{n}=\sum_{\ell}^{\mathrm{N}} \mathrm{n}_{\ell} \tag{2.11}
\end{equation*}
$$

is the total number of particles.
Similarly

$$
\begin{align*}
& \sigma_{N}^{+} \sigma_{-N}^{+}=-\exp (i \pi n) b_{N}^{+} b_{-N}^{+}  \tag{2.12}\\
& \sigma_{N} \sigma_{-N}=\exp (i \pi n) b_{N}{ }^{b}-N \quad \sigma_{N^{\prime}}^{\sigma_{-N}^{+}}=\exp (i \pi n) b_{N^{\prime}} b_{-N}^{+}
\end{align*}
$$

In the fermionic representation the Hamiltonian is

$$
\begin{align*}
H= & L E_{0}+\epsilon \sum_{j=-N}^{N} b_{j}^{+} b_{j}-\Delta \sum_{j=-N}^{N-1}\left(b_{j}^{+}-b_{j}\right)\left(b_{j+1}^{+}+b_{j+1}\right) \\
& +\Delta\left(b_{N}^{+}-b_{N}\right)\left(b_{-N}^{+}+b_{-N}\right) \exp (i \pi n) \\
= & L E_{0}+\epsilon \sum_{j=-N}^{N} b_{j}^{+} b_{j}-\Delta \sum_{j=-N}^{N}\left(b_{j}^{+}-b_{j}\right)\left(b_{j+1}^{+}+b_{j+1}\right) \\
& +\Delta\left(b_{N}^{+}-b_{N}\right)\left(b_{-N^{+}}^{+} b_{-N}\right)(\exp (i \pi n)+1) \tag{2.13}
\end{align*}
$$

$H$ does not conserve particle number but it does preserve the evenness or oddness of the particle number.

For odd number of particles

$$
\begin{equation*}
H^{\text {odd }}=L E_{0}+\epsilon \sum_{j=-N}^{N} b_{j}^{+} b_{j}-\Delta \sum_{j=-N}^{N}\left(b_{j}^{+}-b_{j}\right)\left(b_{j+1}^{+}+b_{j+1}\right) \tag{2.14}
\end{equation*}
$$

which amounts to having periodic boundary conditions $b_{N+1}=b_{-N}$, hence the allowed k's for odd number of particles are

$$
\begin{equation*}
\mathrm{k}=0, \pm \frac{2 \pi}{\mathrm{~L}}, \quad \pm \frac{4 \pi}{\mathrm{~L}}, \ldots \pm \pm \frac{2 \mathrm{~N} \pi}{\mathrm{~L}} \quad \mathrm{~L}=2 \mathrm{~N}+1 \tag{2.15}
\end{equation*}
$$

For even number of particles

$$
\begin{equation*}
H^{\text {even }}=L E_{0}+\epsilon \sum_{j=-N}^{N} b_{j}^{+} b_{j}-\Delta \sum_{j=-N}^{N}\left(b_{j}^{+}-b_{j}\right)\left(b_{j+1}^{+}+b_{j+1}\right) \tag{2.16}
\end{equation*}
$$

with antiperiodic boundary condition $\mathrm{b}_{\mathrm{N}+1}=-\mathrm{b}_{-\mathrm{N}}$, hence the allowed k 's for even number of particles are

$$
\begin{equation*}
\mathrm{k}= \pm \frac{\pi}{\mathrm{L}}, \quad \pm \frac{3 \pi}{\mathrm{~L}}, \ldots, \frac{\mathrm{~L}-2}{\mathrm{~L}} \pi, \quad \pi \tag{2.17}
\end{equation*}
$$

Both $\mathrm{H}^{\text {even }}$ and $\mathrm{H}^{\text {odd }}$ are quadratic in the fermion creation and annihilation operators and therefore can be diagonalized by going into momentum space.

$$
\begin{gather*}
b_{j}=\sum_{k} \frac{e^{-i k j}}{\sqrt{L}} b_{k}  \tag{2,18}\\
H^{\text {odd }}=L E_{0}+(\epsilon-2 \Delta) b_{0}^{+} b_{0}+\sum_{k>0}(\epsilon-2 \Delta \cos k)\left(b_{k}^{+} b_{k}+b_{-k}^{+} b_{-k}\right) \\
-\Delta \sum_{k>0}\left(-2 i \sin k b_{k}^{+} b_{-k}^{+}-2 i \sin k b_{k} b_{-k}\right)  \tag{2.19}\\
H^{\text {even }=}
\end{gather*}
$$

The Hamiltonian is now block diagonalized with $k$ coupled only to $-k$. The resulting quadratic form can be easily diagonalized and the ground state energy of $H^{\text {odd }}$ with odd-particle-number and $\mathrm{H}^{\text {even }}$ with even-particle-number can be computed.

$$
\begin{align*}
& \mathrm{E}^{\mathrm{odd}}=\mathrm{LE} \mathrm{E}_{0}+\epsilon-2 \Delta+\epsilon \frac{\mathrm{L}-1}{2}-2 \Delta \sum_{\mathrm{k}=2 \pi / \mathrm{L}}^{2 \mathrm{~N} \pi / \mathrm{L}} \cos \mathrm{k}-\sum_{\mathrm{k}=2 \pi / \mathrm{L}}^{2 \mathrm{~N} \pi / \mathrm{L}} \sqrt{\epsilon^{2}+4 \Delta^{2}-4 \Delta \epsilon \cos \mathrm{k}} \\
& \mathrm{E}^{\mathrm{even}}=\mathrm{L} \mathrm{E}_{0}+\epsilon \frac{\mathrm{L}-1}{2}-2 \Delta \sum_{\mathrm{k}=\pi / \mathrm{L}}^{\frac{2 N-1}{L} \pi} \cos \mathrm{k}-\sum_{\mathrm{k}=2 \pi / \mathrm{L}}^{\mathrm{L}} \sqrt{\epsilon^{2}+4 \Delta^{2}-4 \Delta \epsilon \cos k} \tag{2.21}
\end{align*}
$$

In the large $L$ limit the sums can be replaced by integrals (however one should remember that $\mathrm{k}_{\mathrm{odd}}=\mathrm{k}_{\text {even }}+\frac{\pi}{\mathrm{L}}$, Eqs. (2.15), (2.17)) with the result

$$
\begin{equation*}
\mathrm{E}^{\mathrm{odd}}=\mathrm{E}^{\text {even }}+(\epsilon-2 \Delta) \theta(\epsilon-2 \Delta) \tag{2.23}
\end{equation*}
$$

$E^{\text {odd }}$ and $E^{\text {even }}$ are the two lowest state of our system (2.13). For $\epsilon>2 \Delta$ the ground state is unique. At $\epsilon=2 \Delta$ the two states coalesce and stay together for $\epsilon<2 \Delta$. $\epsilon=2 \Delta$ defines a phase transition. It has been shown that this calculation produces the exact two dimensional Ising critical index $1 / 8$, i.e.,

$$
\begin{equation*}
\langle x\rangle \sim(-\epsilon+2 \Delta)^{1 / 8} \tag{2.24}
\end{equation*}
$$

(Recall that the $1 \mathrm{x}-1 \mathrm{t}$ scalar $\phi^{4}$ theory is formally related to the two dimensional Ising model. In this respect see Ref. 13.)

The energy of the kink state in this spin $1 / 2$ approximation was computed by Scalapino and Stoeckly. ${ }^{12}$ The solution is obtained by noting that the kink state is generatcd from the original Hamiltonian (2.1) and therefore also (2.6) by requiring antipcriodic boundary condition $\left(\phi_{\mathrm{N}+1}=-\phi_{-\mathrm{N}}\right)$. The kink solutions are, therefore, the cven-particle-number solutions of $H^{\text {odd }}$ (2.19) and the odd-particlenumber solutions of $\mathrm{H}^{\text {even }}(2.20)$. The calculation of the lowest even-particlenumber cigenvalue of $H^{\text {odd }}$ and the lowest odd-particle-number eigenvalue of
$H^{\text {even }}$ is straightforward with the result $(\mathrm{L} \rightarrow \infty)$

$$
E_{\text {kink }}=\left\{\begin{array}{cc}
0 & \epsilon>2 \Delta  \tag{2.25}\\
2 \Delta-\epsilon & \epsilon<2 \Delta
\end{array}\right.
$$

where $E_{\text {kink }}$ is the encrgy of the kink relative to the ground state. Hence, (2. 24)

$$
\begin{equation*}
\mathrm{E}_{\mathrm{kink}} \sim\langle\mathrm{x}\rangle^{8} \tag{2.26}
\end{equation*}
$$

which for small $\langle\mathrm{x}\rangle$ (strong coupling) is lower than the single site result (1.96).
Having the exact result to the truncated problem we can check various variational approximations and compare them to the exact result. This will give us an idea how reliable these variational approximations are. In particular we shall try the mean field (or single site) approximation and a "renormalization group" type of approach.

## C. The Mean Field Approximation

Starting with the spin 1/2 Hamiltonian (2.6) the mean field approximation on this Hamiltonian is obtained by taking as a variational trial wave function

$$
\begin{gather*}
|\psi\rangle=\prod_{j}|\psi\rangle_{j} ; \quad|\psi\rangle_{j}=\binom{\cos \theta}{\sin \theta}_{j}  \tag{2.27}\\
\mathscr{E}(\theta)=\langle\psi| \mathrm{H}|\psi\rangle / \mathrm{L}=\mathrm{const}+\frac{\epsilon}{2} \cos 2 \theta-\Delta \sin ^{2} 2 \theta \tag{2.28}
\end{gather*}
$$

## Hence

$$
\frac{\partial \mathscr{E}}{\partial(2 \theta)}=0=\sin 2 \theta\left(\frac{-\epsilon}{2}-2 \Delta \cos 2 \theta\right)
$$

For $\epsilon>4 \Delta, \mathscr{E}$ minimizes at $\theta=0$, while for $\epsilon<4 \Delta$ the minimum occurs at $\cos 2 \theta=-\epsilon / 4 \Delta$. The order parameter is

$$
\left\langle\sigma_{\mathrm{x}}\right\rangle=\sin 2 \theta=\left\{\begin{array}{cl}
0 & \epsilon>4 \Delta  \tag{2.30}\\
\sqrt{1-\left(\frac{\epsilon-4 \Delta}{4 \Delta}+1\right)^{2}} & \epsilon<4 \Delta
\end{array}\right.
$$

Note that in this approximation we miss on the true transition point $\epsilon=2 \Delta$ by a factor of 2. Near the phase transition $\left\langle\sigma_{x}\right\rangle \sim \sqrt{4 \Delta-\epsilon}$ producing the critical index $1 / 2$ which is characteristic to the mean field approximation.

## D. The Renormalization Group Approach

The idea behind the renormalization group approach is to form blocks of coupled adjacent lattice sites, to diagonalize the Hamiltonian within each block, to couple the original blocks into new big blocks, and then to repeat the process. At each stage as the length scale changes the number of participating degrees of freedom is reduced. The procedure is repeated again and again until the quantities we are interested in computing converge. This will happen when a fixed point of the iteration is found.

The Hamiltonian (2.1) can be rewritten (up to an irrelevant constant) in the form

$$
\begin{equation*}
H=\sum_{j=1}^{N}\left(E^{j}-E_{0}\right)-X^{j} X^{j+1} \tag{2.31}
\end{equation*}
$$

where $E$ is the diagonal matrix whose entries are the single site eigenvalues $E_{n}$

$$
\begin{equation*}
E^{j} \mid n>=E_{n} \ln >\quad \text { for each } j \tag{2.32}
\end{equation*}
$$

We are going to keep only the two lowest states at each site, which can be denoted by $|0\rangle$ and $|1\rangle$. Next we couple each two adjacent sites (i.e., site 1 with 2, site 3 with 4 , etc.). Within each block we have a $4 \times 4$ matrix which we have to diagonalize

$$
\begin{array}{lcccc} 
& |1,1\rangle & |1,0\rangle & |0,1\rangle & |0,0\rangle \\
|1\rangle_{i}|1\rangle_{i+1} \equiv|1,1\rangle & 2 \epsilon_{0} & 0 & 0 & -\Delta_{0} \\
|1\rangle_{i}|0\rangle_{i+1} \equiv|1,0\rangle & 0 & \epsilon_{0} & -\Delta_{0} & 0 \\
|0\rangle_{i}{ }^{|1\rangle_{i+1}} \equiv|0,1\rangle & 0 & -\Delta_{0} & \epsilon_{0} & 0  \tag{2.33}\\
|0\rangle_{i}|0\rangle_{i+1} & \equiv 10,0\rangle & -\Delta_{0} & 0 & 0
\end{array}
$$

where $\epsilon_{0}=E_{1}-E_{0}$ is the original gap and $\left.\Delta_{0}=|\langle 1| x| 0\right\rangle\left.\right|^{2}$. Note that the states $11\rangle|0\rangle$ and 10$\rangle|1\rangle$ just mix between themselves, while the lowest state $10>10>$ mixes only with the state $|1\rangle|1\rangle$. We can, therefore, diagonalize each sub $2 \times 2$
matrix alone. The diagonalization yields the following eigenstates and corresponding eigenvalues

$$
\begin{array}{cl}
\text { state } & \text { eigenvalue } \\
\frac{1}{\sqrt{1+a_{1}^{2}}}\left(\mathrm{a}_{1}|0,0\rangle-|1,1\rangle\right) & \epsilon_{0}+\sqrt{\epsilon_{0}^{2}+\Delta_{0}^{2}} \\
\frac{1}{\sqrt{2}}(|0,1\rangle-|1,0\rangle) & \epsilon_{0}+\Delta_{0} \\
\left.\frac{1}{\sqrt{2}}(10,1\rangle+|1,0\rangle\right) & \epsilon_{0}-\Delta_{0}  \tag{2.34}\\
\left.\frac{1}{\sqrt{1+a_{1}^{2}}}(10,0\rangle+a_{1}|1,1\rangle\right) & \epsilon_{0}-\sqrt{\epsilon_{0}^{2}+\Delta_{0}^{2}}
\end{array}
$$

with

$$
\begin{equation*}
a_{1}=\frac{\sqrt{\epsilon_{0}^{2}+\Delta_{0}^{2}}-\epsilon_{0}}{\Delta_{0}} \tag{2.35}
\end{equation*}
$$

Let's denote the two lowest levels by

$$
\begin{align*}
& \left|\psi_{0}\right\rangle=\frac{1}{\sqrt{1+a_{1}^{2}}}\left(|0,0\rangle+a_{1}|1,1\rangle\right)  \tag{2.36}\\
& \left|\psi_{1}\right\rangle=\frac{1}{\sqrt{2}}(|0,1\rangle+|1,0\rangle)
\end{align*}
$$

The new gap is

$$
\begin{equation*}
\epsilon_{1}=\sqrt{\epsilon_{0}^{2}+\Delta_{0}^{2}}-\Delta_{0} \tag{2.37}
\end{equation*}
$$

In the nextiteration we start with the states $\left|\psi_{0}\right\rangle$ and $\left|\psi_{1}\right\rangle$ within each block and couple each two adjacent blocks (i.e., block $(1,2)$ with $(3,4)$, block $(5,6)$ with $(7,8)$, etc.). The term in the IIamiltonian (2.31) which couples block (i, i+1) to block $(\mathrm{i}+2, \mathrm{i}+3)$ is

$$
\begin{equation*}
H_{i n t}=-\Delta_{0}\left(a_{i+1}+a_{i+1}^{+}\right)\left(a_{i+2}+a_{i+2}^{+}\right) \tag{2.38}
\end{equation*}
$$

It is important to note that the state $\left|\psi_{0}>\right| \psi_{0}>$ mixes only with $\left.\left|\psi_{1}>\right| \psi_{1}\right\rangle$ and the state $\left|\psi_{0}>\right| \psi_{1}>$ mixes only with $\left.\left|\psi_{1}>\right| \psi_{0}\right\rangle$. Consider the block ( $i, i+1$ ) and ( $i+2, i+3$ )

$$
\begin{align*}
& \Delta_{0}\left(a_{i+1}^{+a_{i+1}^{+}}\right)\left(a_{i+2}^{+a_{i+2}^{+}}\right)\left|\psi_{0}>\right| \psi_{0}{ }^{\prime} \\
& \left.=\Delta_{0}\left(a_{i+1}+a_{i+1}^{+}\right)\left(a_{i+2}+a_{i+2}^{+}\right) \frac{1}{\left(1+a_{1}^{2}\right)}\left[\left(10>_{i}|0\rangle_{i+1}+a_{1}\left|1>_{i}\right| 1\right\rangle_{i+1}\right)\left(|0\rangle_{i+2}|0\rangle_{i+3}+a_{1}|1\rangle_{i+2}|1\rangle_{i+3}\right)\right] \\
& =\frac{\Delta_{0}}{1+a_{1}^{2}}\left[|0\rangle_{i}|1\rangle_{i+1}|1\rangle_{i+2}|0\rangle_{i+3}+a_{1}|1\rangle_{i}|0\rangle_{i+1}|1\rangle_{i+2}|0\rangle_{i+3}+a_{1}|0\rangle_{i}|1\rangle_{i+1}|0\rangle_{i+2}|1\rangle_{i+3}\right. \\
& \left.+a_{1}^{2}|1\rangle_{i}{ }^{10\rangle_{i+1}}{ }_{i+2}^{|0\rangle_{i}}{ }_{i+3}\right]  \tag{2.39}\\
& 7
\end{align*}
$$

The new overlap parameter is determined by taking the scalar product of (2.39) with $\left.\left|\psi_{1}>\right| \psi_{1}\right\rangle$

$$
\begin{equation*}
\Delta_{1}=\frac{\Delta_{0}}{2} \frac{\left(1+a_{1}\right)^{2}}{\left(1+a_{1}^{2}\right)} \tag{2.40}
\end{equation*}
$$

$\Delta_{1}$ is also the overlap between $H_{\text {int }}\left|\psi_{0}>\right| \psi_{1}>$ and $\left|\psi_{1}>\right| \psi_{0}>$. Hence the new $4 \times 4$ matrix we have to diagonalize is the same as (2.33) with the replacement

$$
\begin{align*}
& \epsilon_{0} \rightarrow \epsilon_{1} \\
& \Delta_{0} \rightarrow \Delta_{1} \tag{2.41}
\end{align*}
$$

We obtain, therefore, the following renormalization group equations

$$
\begin{align*}
& \epsilon_{p}=\sqrt{\epsilon_{p-1}^{2}+\Delta_{p-1}^{2}}-\Delta_{p-1} \\
& a_{p}=\frac{1}{\Delta_{p-1}}\left(\sqrt{\epsilon_{p-1}^{2}+\Delta_{p-1}^{2}}-\epsilon_{p-1}\right)  \tag{2.42}\\
& \Delta_{p}=\frac{\Delta_{p-1}}{2} \frac{\left(1+a_{p}\right)^{2}}{1+a_{p}^{2}}
\end{align*}
$$

For $\epsilon_{0} \ll, \Delta_{0}$ the equations give

$$
\begin{equation*}
\epsilon_{p}=x^{p} \epsilon_{0}, \quad x=\frac{\epsilon_{0}}{2 \Delta_{0}}<1 \tag{2.43}
\end{equation*}
$$

hence the gap goes to zero which is an indication of a phase transition.
For $\epsilon_{0} \gg \Delta_{0}$

$$
\begin{align*}
& \epsilon_{p}=\epsilon_{0}-\Delta_{p-1} \\
& a_{p}=1  \tag{2.44}\\
& \Delta_{p}=\frac{\Delta_{p-1}}{2}
\end{align*}
$$

hence

$$
\begin{equation*}
\epsilon=\lim _{\mathrm{p} \rightarrow \infty} \epsilon_{\mathrm{p}}=\epsilon_{0}-\Delta_{0}\left(1+\frac{1}{2}+\frac{1}{4}+\ldots\right)=\epsilon_{0}-2 \Delta_{0} \tag{2.45}
\end{equation*}
$$

The order parameter for our system is $\xi=\left\langle\mathrm{a}+\mathrm{a}^{+}\right\rangle$which is closely related to the vacuum expectation value of the field (1.56). The expectation value of $\xi$ in either $\left|\psi_{0}\right\rangle$ or $\left|\psi_{1}\right\rangle$ is zero. It is only when the phase transition occurs and $\left|\psi_{0}\right\rangle$ and $\left|\psi_{1}\right\rangle$ become degenerate that the expectation value of $\xi$ in the ground state (which is then an arbitrary linear combination of $\left|\psi_{0}\right\rangle$ and $\left|\psi_{1}\right\rangle$ ) can be different from zero. In the first iteration

$$
\begin{equation*}
\langle\xi\rangle_{1}=\left\langle\psi_{1}\right| \xi\left|\psi_{0}\right\rangle=\frac{1+\mathrm{a}_{1}}{\sqrt{2\left(1+\mathrm{a}_{1}^{2}\right)}} \tag{2.46}
\end{equation*}
$$

The next iteration the two lowest states are

$$
\begin{align*}
& \left|\psi_{0}^{(2)}\right\rangle=\frac{1}{1+\mathrm{a}_{2}^{2}}\left(\left|\psi_{0}>\left|\psi_{0}>+\mathrm{a}_{2}\right| \psi_{1}>\right| \psi_{1}>\right) \\
& \left.\left.\left|\psi_{1}^{(2)}\right\rangle=\frac{1}{\sqrt{2}}\left(\left|\psi_{0}>\right| \psi_{1}\right\rangle+\left|\psi_{1}>\right| \psi_{0}\right\rangle\right) \tag{2.47}
\end{align*}
$$

and

$$
\begin{equation*}
\langle\xi\rangle_{2}=\left\langle\psi_{1}^{(2)}\right| \xi\left|\psi_{0}^{(2)}\right\rangle=\frac{1+\mathrm{a}_{2}}{\sqrt{2\left(1+\mathrm{a}_{2}\right)^{2}}}\left\langle\psi_{1}\right| \xi\left|\psi_{0}\right\rangle=\frac{1+\mathrm{a}_{2}}{\sqrt{2\left(1+\mathrm{a}_{2}^{2}\right)}} \frac{1+\mathrm{a}_{1}}{\sqrt{2\left(1+\mathrm{a}_{1}^{2}\right)}} \tag{2.48}
\end{equation*}
$$

Recalling Eq. (2.40)

$$
\begin{equation*}
\langle\xi\rangle_{2}=\left(\frac{\Delta_{2}}{\Delta_{1}}\right)^{1 / 2} \times\left(\frac{\Delta_{1}}{\Delta_{0}}\right)^{1 / 2}=\left(\frac{\Delta_{2}}{\Delta_{0}}\right)^{1 / 2} \tag{2.49}
\end{equation*}
$$

In the p-th iteration the order parameter is

$$
\begin{equation*}
\langle\xi\rangle_{p}=\left(\frac{\Delta_{p}}{\Delta_{0}}\right)^{1 / 2} \tag{2.50}
\end{equation*}
$$

The numerical solution of the renormalization group equations (2.42) yields $\epsilon_{0}=2.55 \Delta_{0}$ as the phase transition point. Above the phase transition $\left(\epsilon_{0}>2.55 \Delta_{0}\right)$ $\langle\xi\rangle=0$. Below the phase transition point $\left.\left(\epsilon_{0}<2.55 \Delta_{0}\right)<\xi\right\rangle \neq 0$. The critical index which determine the approach of $\langle\xi\rangle$ to zero near the phase transition point turns out to be $\sim 0.5$.

To conclude let us see for what range of the original parameters of the $\phi^{4}$ theory (Eq. (1.14) or Eq. (2.1)) the spin $1 / 2$ approximation is reasonable. The single site potential for the $\phi^{4}$ theory with negative mass $\mu^{2}=-2 \lambda f^{2}<0$ is depicted in Fig. 10. The two lowest states are the symmetric and antisymmetric states whose energy difference is governed by the amount of tunneling through the $\lambda f^{4}$ barrier. Since we want the gap to be small we shall demand

$$
\lambda f^{4} \gg \lambda^{1 / 2} f
$$

or

$$
\begin{equation*}
\mathrm{z} \equiv \lambda^{1 / 2} \mathrm{f}^{3} \gg 1 \tag{2.51}
\end{equation*}
$$

where $\omega_{0} \approx \lambda^{1 / 2} \mathrm{f}$ is the curvature at the minimum of the well. Under this condition the first energy gap is given by

$$
\begin{equation*}
\epsilon_{0} \approx \mathrm{e}^{-\left(\lambda^{1 / 2} \mathrm{f}\right) \mathrm{f}}\left|\lambda^{1 / 2} \mathrm{f}\right| \approx \mathrm{e}^{-\mathrm{z}}|\mu| \tag{2.52}
\end{equation*}
$$

The next state up is at energy $i \mu \mid$ relative to these two states (Fig. 10). The $\operatorname{spin} 1 / 2$ calculation demands

$$
\begin{equation*}
\epsilon_{0} \ll|\mu| \tag{2.53}
\end{equation*}
$$

which is clearly satisfied ((2.52), (2.52)).
The parameter $\Delta_{0}(2.7)$ is the matrix element square of $x$ between the symmetric and antisymmetric states

$$
\begin{equation*}
\Delta_{0} \approx \mathrm{f}^{2} \tag{2.54}
\end{equation*}
$$

At the phase transition (2.23)

$$
\begin{equation*}
\epsilon_{0}=2 \Delta_{0} \tag{2.55}
\end{equation*}
$$

hence,

$$
\begin{equation*}
\mathrm{e}^{-\mathrm{z}}|\mu|=\mathrm{f}^{2}=\frac{\mathrm{z}}{|\mu|} \tag{2.56}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathrm{e}^{-\mathrm{z}}=\frac{\mathrm{z}}{\left|\mu^{2}\right|} \tag{2.57}
\end{equation*}
$$

For large $z,|\dot{\mu}|$ should also be large. Condition (2.57) determine the regime of $\lambda$ and $f$ for which the spin $1 / 2$ calculation of the phase transition is reliable.
3. COMPLEX $\phi^{4}$ THEORY AND THE ABSENCE OF GOLDSTONE BOSON IN 1x-1t DIMENSIONS
A. General Survey

The Lagrangian for the complex $\phi^{4}$ theory is given by

$$
\begin{equation*}
\mathscr{L}=\frac{1}{2}\left(\partial_{\mu} \phi^{*}\right)\left(\partial_{\mu} \phi\right)-\lambda\left(\phi^{*} \phi-\mathrm{f}^{2}\right)^{2} \tag{3.1}
\end{equation*}
$$

This Lagrangian can be written in terms of two scalar fields $\sigma$ and $\pi$

$$
\begin{equation*}
\mathscr{L}=\frac{1}{2}\left(\partial_{\mu} \pi\right)^{2}+\frac{1}{2}\left(\partial_{\mu} \sigma\right)^{2}-\lambda\left(\pi^{2}+\sigma^{2}-\mathrm{f}^{2}\right)^{2} \tag{3.2}
\end{equation*}
$$

The theory has a continuous rotation symmetry in the ( $\sigma, \pi$ ) plane. Classically the theory undergoes spontaneous symmetry breaking in which $\sigma$ and/or $\pi$ acquire nonvanishing vacuum expectation value. To be precise if we first restrict the system to finite volume $V$ and add a source term Jo which singles out the direction of $\sigma$ in the $(\sigma, \pi)$ plane, then

$$
\begin{equation*}
\langle\sigma\rangle=\lim _{\mathrm{J} \rightarrow 0} \lim _{\mathrm{V} \rightarrow \infty}\langle 0| \sigma(\mathrm{x})|0\rangle \neq 0 \tag{3.3}
\end{equation*}
$$

The Hamiltonian obtained from (3.2) is a sum of positive terms. Classically it is clear that a constant vector $[\sigma, \pi]$ of length $f^{2}$ minimizes the energy. The added source term $J \sigma$ tend to fix the direction of this vector to be along $\sigma$. The classical lowest energy state is therefore $\sigma=f$ and $\pi=0$. The spontaneous symmetry breaking is known also as the Goldstone phenomenon and Goldstone's theorem states that this phenomenon is accompanied by the appearance of massless scalar boson (the Goldstone boson). ${ }^{14}$

When the full quantum mechanical aspect of the theory is taken into account there are quantum fluctuations which should be considered. If the quantum fluctuations are large enough all memory of the classical preferred direction may be lost and the ground state expectation value of $\sigma$ and $\pi$ will be then $\langle\sigma\rangle=\langle\pi\rangle=0 .{ }^{15}$ In this case no long range order associated with spontaneous symmetry breaking will occur. This is what happens in 1x-1t dimensions and is the basis for

Coleman's theorem which states that there are no Goldstone boson in $1 \mathrm{x}-1 \mathrm{t}$ dimensions. ${ }^{16}$

At this point it is convenient to introduce polar coordinates

$$
\begin{align*}
& \sigma=\mathrm{r} \sin \theta \\
& \pi=\mathrm{r} \cos \theta \tag{3.4}
\end{align*} \quad 0 \leq \mu \leq \infty ; \quad 0 \leq \theta \leq 2 \pi,
$$

It is the $\theta$ field which is the candidate to be the Goldstone boson. It is, therefore, the fluctuation in $\theta$ which we want to consider. For this we can replace $r$ by its average c-number value f. The r-potential is shown in Fig. 7. It is clear that for $\lambda f^{4}>\lambda^{1 / 2} \mathrm{f}$ the r part of the ground state wave function is well localized at $\mathrm{r}=\mathrm{f}$. Once $r$ is frozen to be equal to $f$ the $\theta$ part of the Lagrangian is

$$
\begin{equation*}
\mathscr{X}_{\theta}=\frac{1}{2} \mathrm{f}^{2}\left(\partial_{\mu} \theta\right)^{2} \tag{3.6}
\end{equation*}
$$

which looks like a free massless field Lagrangian apart from the fact that $0 \leq \theta \leq 2 \pi$.

## B. Going to the Lattice

In our lattice formulation the Hamiltonian obtained from the Lagrangian (3.2) is

$$
\begin{equation*}
H=\sum_{j} \frac{p_{\sigma}^{2}(j)}{2}+\frac{p_{\pi}^{2}(j)}{2}+\lambda\left(\sigma^{2}+\pi^{2}-f^{2}\right)+\frac{1}{2} \sum_{j_{1} j_{2}} D\left(j_{1}-j_{2}\right)\left[(\sigma+i \pi) j_{1}(\sigma-i \pi)_{j_{2}}+h . c .\right] \tag{3.7}
\end{equation*}
$$

where

$$
\begin{align*}
& \phi_{j}=r_{j} e^{i 0_{j}}=r_{j} \cos \theta_{j}+i r_{j} \sin \theta_{j}=\sigma_{j}+i \pi_{j}  \tag{3.8}\\
& p_{\sigma}=\frac{1}{i} \frac{\partial}{\partial \sigma}, \quad p_{\pi}=\frac{1}{i} \frac{\partial}{\partial \pi}
\end{align*}
$$

In polar coordinates

$$
\begin{equation*}
H=\sum_{j}-\frac{1}{2}\left(\frac{1}{r_{j}} \frac{\partial}{\partial r_{j}} r_{j}\right)-\frac{1}{2 r_{j}^{2}} \frac{\partial^{2}}{\partial \theta_{j}^{2}}+\lambda\left(r_{j}^{2}-f^{2}\right)^{2}+\frac{D(0)}{2} r_{j}^{2}+\frac{1}{2} \sum_{j_{1} \neq j_{2}} D\left(j_{1}-j_{2}\right) r_{j_{1}} r_{j_{2}}\left(e^{i \theta_{j_{1}}} e^{-i \theta j_{2}}+h . c .\right) \tag{3.9}
\end{equation*}
$$

If we freeze the radial coordinate as argued before we are left with the $\theta$ Hamiltonian

$$
\begin{equation*}
H=\sum_{j}-\frac{1}{2 f^{2}} \frac{\partial^{2}}{\partial \theta_{j}^{2}}+\frac{1}{2} f^{2} \sum_{j_{1} \neq j_{2}} D\left(j_{1}-j_{2}\right)\left(e^{i \theta} j^{-i \theta_{j}} e+\text { h.c. }\right) \tag{3.10}
\end{equation*}
$$

which can be rewritton as

$$
\begin{equation*}
H=\sum_{j}-\frac{1}{2 f^{2}} J_{z}^{2}(j)+\frac{1}{2} f^{2} \sum_{j_{1} \neq j_{2}} D\left(j_{1}-j_{2}\right)\left(J_{j_{1}}^{+} J_{j_{2}}^{-}+h . c .\right) \tag{3.11}
\end{equation*}
$$

Now we can see in what sense we can think of $\theta$ as a free massless field. Note that for $f^{2} \rightarrow 0$ the $J_{z}$ term is the most important term and we get a theory with an energy gap. On the other hand we can consider the theory for very large $f$ starting with the Hamiltonian (3.10) with nearest neighbors interactions and adding a small source term $\epsilon \mathrm{f}^{2} \sin ^{2} \theta$ in the $\sigma$ direction

$$
\begin{equation*}
H=\sum_{j}\left[\frac{-1}{2 f^{2}} \frac{\partial^{2}}{\partial \theta_{j}^{2}}-2 f^{2} \cos \left(\theta_{j+1}-\theta_{j}\right)+\epsilon f^{2} \sin ^{2} \theta_{j}\right] \tag{3.12}
\end{equation*}
$$

Note that this source term does not break completely the rotation symmetry; there is still a symmetry under rotation by $\pi$.

Choosing $\theta$ to lie between $-\pi / 2$ and $3 \pi / 2$ and rescaling the $\theta$ variable

$$
\begin{equation*}
\tilde{\theta}=\mathrm{f} \theta \quad-\frac{\pi}{2} \mathrm{f} \leq \tilde{\theta} \leq \frac{3 \pi}{2} \mathrm{f} \tag{3.13}
\end{equation*}
$$

we have

$$
\begin{equation*}
H=\sum_{j}\left(-\frac{1}{2} \frac{\partial^{2}}{\partial \widetilde{\theta}_{j}^{2}}-2 f^{2} \cos \left(\frac{\widetilde{\theta}_{j+1}-\widetilde{\theta}_{j}}{f}\right)+\epsilon f^{2} \sin ^{2}\left(\frac{\widetilde{\theta}_{j}}{f}\right)\right) \tag{3,14}
\end{equation*}
$$

The potential term in (3.14) has the form depicted in Fig. 8. The height of the middle bump is $\sim \epsilon \mathrm{f}^{2}$ while the tunneling distance (the distance between the two minima) is $\sim \pi f$. The curvature at the bottom of the well $\epsilon$ determine the energies of the low lying levels in the well. For large $f$ the amount of tunneling is small, the energy of the symmetric state is $\frac{1}{2} \sqrt{\epsilon}$ and of the antisymmetric state $\frac{1}{2} \sqrt{\epsilon}+\mathrm{e}^{-\mathrm{f}^{2}}$ hence these states are almost degenerate. The next level up is at energy $\sim \frac{3}{2} \sqrt{\epsilon}$. The number of levels which can be accurately described by harmonic states within one well is of the order of the height at the middle ( $\mathrm{Ef}^{2}$ ) divided by the energy gap $\sqrt{\epsilon}$. We want this number to be large

$$
\begin{equation*}
\mathscr{N}=\frac{\epsilon \mathrm{f}^{2}}{\sqrt{\epsilon}}=\sqrt{\epsilon} \mathrm{f}^{2} \gg 1 \tag{3.15}
\end{equation*}
$$

Since the wave functions are of the form $e^{-\sqrt{\epsilon} \widetilde{\theta}^{2}}$ the condition that all of these states will be well localized is that their width will be small relative to the tunneling distance $f$

$$
\tilde{\theta} \sim \frac{1}{\epsilon^{1 / 4}}<\mathrm{f}
$$

or

$$
\begin{equation*}
\epsilon^{1 / 4} f \gg 1 \tag{3.16}
\end{equation*}
$$

This last condition (3.16) allows us to expand both the $\cos$ and $\sin$ in the expression (3.14) of the Hamiltonian. Note also that the condition (3.15) $\mathscr{N} \gg 1$ is compatible with the condition

$$
\begin{equation*}
\epsilon \mathrm{f}^{2} \ll 1 \tag{3.17}
\end{equation*}
$$

which states that the source term is an arbitrarily small perturbation. The Hamiltonian is now (up to irrelevant constant)

$$
\begin{equation*}
\mathrm{H}=\sum_{\mathrm{j}}\left[-\frac{1}{2} \frac{\partial^{2}}{\partial \widetilde{\theta}_{\mathrm{j}}^{2}}+\frac{1}{2} \epsilon \widetilde{\theta}_{\mathrm{j}}^{2}+\frac{1}{2}\left(\widetilde{\theta}_{\mathrm{j}+1}-\widetilde{\theta}_{\mathrm{j}}\right)^{2}\right] \tag{3.18}
\end{equation*}
$$

which is a free field Hamiltonian $\left(-^{\infty} \leq \tilde{\theta} \leq \infty\right.$ for $\left.f \rightarrow \infty\right)$. Once we established the fact that for large $f, \tilde{\theta}$ is essentially a free field of mass $\sqrt{\epsilon}$ it is easy to understand Coleman theorem (in the region for which the free field approximation is acceptable). ${ }^{15}$

The bigger f is more level $\mathscr{N}(3.15)$ are correctly described by a free field expansion of $\tilde{\theta}$

$$
\begin{equation*}
\tilde{\theta}_{j}=\sum_{k} \frac{1}{\left(2 \omega_{k} \mathrm{~V}\right)^{1 / 2}}\left(a_{k} e^{-i k j}+a_{k}^{+} e^{i k j}\right) \tag{3.19}
\end{equation*}
$$

The ground state is 10$\rangle=\prod_{\mathrm{k}} \mid 0_{\mathrm{k}}>$ and the correlation function is

$$
\begin{equation*}
\Delta(\mathrm{j})=\lim _{\mathrm{V} \rightarrow \infty}\langle 0| \theta(\mathrm{j}) \theta(0)|0\rangle=\frac{1}{\mathrm{f}^{2}} \lim _{\mathrm{V} \rightarrow \infty}\langle 0| \tilde{\theta}(\mathrm{j}) \tilde{\theta}(0)|0\rangle=\frac{1}{2 \mathrm{f}^{2}} \int_{0}^{\pi} \frac{\mathrm{d}^{\mathrm{D}} \mathrm{e}^{-\mathrm{ikj}}}{2\left(\mathrm{k}^{2}+\epsilon^{2}\right)^{1 / 2}} \tag{3.20}
\end{equation*}
$$

It is clear that in $1 \mathrm{x}-1 \mathrm{~d}$ dimensions due to the infra-red singularity as $\epsilon \rightarrow 0$ the fluctuations in $\theta$ are infinitely large. In fact

$$
\begin{equation*}
\langle 0| \mathrm{e}^{\mathrm{i} \theta(\mathrm{j})}|0\rangle=\mathrm{e}^{-\Delta(0) / 2} \underset{\epsilon \rightarrow 0}{\longrightarrow} 0 \tag{3.21}
\end{equation*}
$$

and therefore

$$
\begin{align*}
& \langle\sigma\rangle \propto\langle\sin \theta\rangle=0  \tag{3.22}\\
& \langle\pi\rangle \alpha\langle\cos \theta\rangle=0
\end{align*}
$$

## C. Renormalization Group Approach to the Goldstone Boson

The argument presented before is heuristic and nonrigorous. We would like now to present a simple renormalization group computation from which the physics of the Goldstone boson becomes more clear.

Starting with the Hamiltonian (3.11) with nearest neighbors interactions

$$
\begin{equation*}
H=\sum_{j}\left[\epsilon_{0} J_{z}^{2}(j)-\Delta_{0}\left(J_{+}(j) J_{-}(j+1)+h . c .\right)\right] \tag{3.23}
\end{equation*}
$$

where

$$
\begin{equation*}
\epsilon_{0}=\frac{1}{2 \mathrm{f}^{2}} \quad \Delta_{0}=\mathrm{f}^{2} \tag{3.24}
\end{equation*}
$$

The calculation is similar to the one done for the spin $1 / 2$ case. At each sile we take the three lowest levels of $\epsilon J_{z}^{2}$ which we shall denote by $|0\rangle,|+1\rangle,|-1\rangle$. Note that we cannot truncate at two levels since the states $|+1\rangle$ and $|-1\rangle$ are degenerate. Coupling into blocks of two adjacent sites there are 9 -states we have to consider. The states are shown in Fig. 9. The states $|1,1\rangle$ and $|-1,-1\rangle$ are eigenstates of the Hamiltonian. The state 10,0$\rangle$ can be connected only to the state $\frac{1}{\sqrt{2}}(|1,-1\rangle+|-1,1\rangle)$ while the state $|1,0\rangle$ mixes only with 10,1$\rangle$, and the state $|-1,0\rangle$ mixes only with $|0,-1\rangle$. The matrices we are instructed to diagonalize are

$$
\underset{10,0\rangle}{\frac{1}{\sqrt{2}}(|1,-1\rangle+|-1,1\rangle)}\left(\begin{array}{cc}
|0,0\rangle & \frac{1}{\sqrt{2}}(|1,-1\rangle+|-1,1\rangle) \\
0 & -\sqrt{2} \Delta_{0}  \tag{3.25}\\
-\sqrt{2} \Delta_{0} & 2 \epsilon
\end{array}\right)
$$

with eigenvalues $\epsilon_{0} \pm \sqrt{\epsilon_{0}^{2}+2 \Delta_{0}^{2}}$ and

$$
\begin{align*}
& 10,1> \\
& |1,0\rangle \\
& 10,-1> \\
& |-1,0\rangle \\
& 10,1\rangle\left(\begin{array}{cc}
\epsilon & -\Delta_{0} \\
-\Delta_{0} & \epsilon
\end{array}\right) ; \begin{array}{cc}
10,-1\rangle \\
-1-1,0\rangle
\end{array}\left(\begin{array}{cc}
\epsilon & -\Delta_{0} \\
-\Delta_{0} & \epsilon
\end{array}\right) \tag{3.26}
\end{align*}
$$

with eigenvalues $\epsilon_{0} \pm \Delta_{0}$.

The level*structure of the 3 lowest states in each block is
state

$$
\begin{array}{ll}
\left.\left|\psi_{0}\right\rangle=\frac{1}{\sqrt{1+a_{1}^{2}}}(10,0\rangle+\frac{a_{1}}{\sqrt{2}}(|1,-1\rangle+|-1,1\rangle)\right) & \epsilon_{0}-\sqrt{\epsilon} \\
\left|\psi_{+}\right\rangle=\frac{1}{\sqrt{2}}(|0,1\rangle+|1,0\rangle) & \epsilon_{0}-\Delta_{0} \\
\left|\psi_{>}\right\rangle=\frac{1}{\sqrt{2}}(|0,-1\rangle+|-1,0\rangle) & \epsilon_{0}-\Delta_{0}
\end{array}
$$

with

$$
\begin{equation*}
a_{1}=\frac{\sqrt{\epsilon_{0}^{2}+2 \Delta_{0}^{2}}-\epsilon_{0}}{\sqrt{2} \Delta_{0}} \tag{3.28}
\end{equation*}
$$

Note that as in the zeroth itcration we have two degenerate levels $\left|\psi_{+}\right\rangle$and $\left|\psi_{\ldots}\right\rangle$. The new gap is

$$
\begin{equation*}
\epsilon_{1}=\sqrt{\epsilon_{0}^{2}+2 \Delta_{0}^{2}}-\Delta_{0} \tag{3.29}
\end{equation*}
$$

In the next iteration we start with these three states within each block of two and couple each two adjacent blocks (i.e., block $(1,2)$ to block $(3,4)$, block $(5,6)$ to block $(7,8)$, etc.). The term in the Hamiltonian which couple block ( $\mathbf{i}, \mathrm{i}+1$ ) to block ( $\mathbf{i}+2, i+3$ ) is

$$
\begin{equation*}
H_{\text {int }}=-\Delta_{0}\left(J_{+}(i+1) J_{-}(i+2)+J_{-}(i+1) J_{+}(i+2)\right) \tag{3.30}
\end{equation*}
$$

Once again we have the same pattern of levels depicted in Fig. 9. Note that the state $\left.\left|\psi_{0}>\right| \psi_{0}\right\rangle$ mixes only with the state $\frac{1}{\sqrt{2}}\left(\left|\psi_{+}\right\rangle\left|\psi_{-}\right\rangle+\left|\psi_{-}\right\rangle\left|\psi_{+}\right\rangle\right)$, and that the state $\left.\left.\left|\psi_{0}>\right| \psi_{+}\right\rangle\left(\left|\psi_{0}>\right| \psi_{-}\right\rangle\right)$mixes only with $\left.\left.\left|\psi_{+}>\right| \psi_{0}\right\rangle\left(\left|\psi_{-}>\right| \psi_{0}\right\rangle\right)$. The new $2 \times 2$ matrices to be diagonalized turn out to be identical to the matrices in the zeroth iteration (3.25, 3.26) with the replacement

$$
\epsilon_{0} \rightarrow \epsilon_{1}
$$

$$
\begin{equation*}
\Delta_{0} \rightarrow \Delta_{1}=\frac{\Delta_{0}}{2} \frac{\left(1+\frac{a_{1}}{\sqrt{2}}\right)^{2}}{1+a_{1}^{2}} \tag{3.31}
\end{equation*}
$$

(The calculation of $\Delta_{1}$ is straightforward and proceeds in the same way as for the $\operatorname{spin} 1 / 2$ case.)

The renormalization group equations obtained in the p-th iteration are, therefore

$$
\begin{gather*}
\epsilon_{p}=\sqrt{\epsilon_{p-1}^{2}+2 \Delta_{p-1}^{2}}-\Delta_{p-1} \\
a_{p}=\frac{1}{\sqrt{2} \Delta_{p-1}}\left(\sqrt{\epsilon_{p-1}^{2}+2 \Delta_{p-1}^{2}}-\epsilon_{p-1}\right)=\frac{1}{\sqrt{2} \Delta_{p-1}}\left(\epsilon_{p}+\Delta_{p-1}-\epsilon_{p-1}\right) \\
\Delta_{p}=\frac{\Delta_{p-1}}{2} \frac{\left(1+\frac{a_{p}}{\sqrt{2}}\right)^{2}}{1+a_{p}^{2}} \tag{3.32}
\end{gather*}
$$

In this case it is clear that there can be no fixed point for which $\epsilon$ iterates to zero while $\Delta$ stays finite. It is also easy to see that for a solution with $\epsilon \neq 0, a \rightarrow \frac{1}{\sqrt{2}}$ and therefore $\Delta_{p}=\frac{3 \Delta_{p}-1}{4}$ hence $\Delta \rightarrow 0$. To summarize, the only fixed points are those with $\Delta=0$. As for the $\operatorname{spin} 1 / 2$ case the order parameter is proportional to $\left(\Delta / \Delta_{0}\right)^{1 / 2}$. The actual numerical solution gives $\epsilon_{0}=1.7 \Delta_{0}$ as the transition point. However, even when the phase transition occurs and the gap goes to zero, $\Delta=0$ which means that there is no long range order in the system.

The obvious question now is what happens when we go to higher dimensions. Repeating the renormalization group calculation we have preliminary indications that in higher dimensions the system undergoes a phase transition with long range order. The difference from the 1 x -1t dimensions is due to the fact that now we have more interactions (the number of nearest neighbors at each stage of the iteration grows), more degenerate levels appear and the mixing pattern is more complicated. The difference between 1x and higher dimensions, which allows Goldstone boson and long range order to appear in higher dimensions is, in short, phase space.

## 4. FERMIONS ON A LATTICE

Following our study of scalar field theories formulated on a spatial lattice we turn to fermionic field theories. For fermion fields our definition of the gradient will turn out to be of great importance. It allows us to over come the problem of doubling of states encountered in the nearest neighbor prescription and also allows us to write down lattice theories which are locally $\gamma_{5}$ invariant. We will concentrate on the behavior of these theories in the large coupling regime. We will show that in this case the usual fermionic degrees of freedom become very massive, but new massless degrees of freedom (fermion-antifermion bound states) become low lying (even when the lattice spacing $a=\frac{1}{\Lambda} \rightarrow 0$ ). Since we consider theories with $\gamma_{5}$ invariance, the question of the $\gamma_{5}$ symmetry properties of the ground state (the realization of the $\gamma_{5}$-symmetry) will be of great interest. At least for large coupling constant we shall once again (as for the $\phi^{4}$ case) see that diagonalizing first the single site terms in the Hamiltonian leads to the correct physical picture (e.g., the $\gamma_{5}$ symmetry of the ground state). The role of the gradient term is to split the huge degeneracy which occurs in the single site diagonalization by allowing states to move.
A. Back to the Gradient Definition-The Free Field Example

The simplest fermion theory is of course the free field theory

$$
\begin{equation*}
\mathrm{H}=\int \mathrm{d}^{3} \mathrm{x}\left\{\psi+\left(\frac{\vec{\alpha} \cdot \vec{\nabla}}{\mathrm{i}}+\beta \mathrm{m}\right) \psi\right\} \tag{4.1}
\end{equation*}
$$

which yields the Dirac equation for the fermion field

$$
\begin{equation*}
\mathrm{E}_{\psi}=\left(\frac{\vec{\alpha} \cdot \vec{\nabla}}{\mathrm{i}}+\beta \mathrm{m}\right) \psi \tag{4.2}
\end{equation*}
$$

For simplicity let us consider the lattice version of the Dirac equation in $1 \mathrm{x}-1 \mathrm{t}$ dimensions. If we try to adopt the usual transcription of the gradient as a difference operator

$$
\begin{equation*}
(\nabla \psi)_{\mathrm{j}}=\Lambda(\psi(\mathrm{j}+1)-\psi(\mathrm{j})) \tag{4.3}
\end{equation*}
$$

we have .

$$
\begin{equation*}
\mathrm{E}_{\mathrm{j}}=-\frac{\mathrm{i}}{2} \alpha\left(\psi_{\mathrm{j}}-\psi_{\mathrm{j}+1}\right) \Lambda-\frac{i}{2} \alpha\left(\psi_{\mathrm{j}-1}-\psi_{\mathrm{j}}\right) \Lambda+\beta \mathrm{m} \psi_{\mathrm{j}} \tag{4.4}
\end{equation*}
$$

Using the representation

$$
\alpha=\left(\begin{array}{cc}
1 & 0  \tag{4.5}\\
0 & -1
\end{array}\right) \quad \beta=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)
$$

and the Fourier expansion

$$
\begin{equation*}
\psi_{j}=\sum_{k} \frac{e^{i k j / \Lambda}}{\sqrt{V}}\binom{b-k}{d_{k}^{+}} \tag{4.6}
\end{equation*}
$$

we find, for a free Dirac particle, the energy momentum dispersion relation

$$
\begin{equation*}
\mathrm{E}(\mathrm{k})=\sqrt{\mathrm{m}^{2}+\Lambda^{2} \sin ^{2} \frac{\mathrm{k}}{\Lambda}} \tag{4.7}
\end{equation*}
$$

where $-\pi \leq \mathrm{k} \Lambda \leq \pi$. As illustrated in Fig. 11 this formula shows that to each eigenvalue E there correspond two distinct states of $\mathrm{k}>0$ and two of $\mathrm{k}<0$; hence the spectrum of states possesses a doubling of levels not encountered in the continuum theory. Kogut and Susskind have proposed one technique for avoiding this problem. ${ }^{18}$ In $1 \mathrm{x}+1 \mathrm{t}$ dimensions they simply put the upper (lower) components of the Dirac spinors on even (odd) lattice sites. The disadvantage of this procedure is that it makes it impossible to write down locally chiral invariant interactions since one does not have both particles and antiparticles at the same point. In higher dimensions this procedure will demand to split spin components in $2 x+1$ dimensions and to double the number of fermions in $3 x+1 t$ dimensions. An alternative projection operator technique which was introduced by Wilson in his action formulation also destroys local $\gamma_{5}$ invariance. Our way of defining the gradient operator on the lattice avoid this difficulty (i.e., there is no doubling of states and we can write down theories which are locally $\gamma_{5}$-invariant on the lattice). To see why this is so recall that for

$$
\begin{equation*}
f(\mathrm{j})=\sum_{\mathrm{k}} \mathrm{f}(\mathrm{k}) \mathrm{e}^{\mathrm{ikj} / \Lambda} \tag{4.8}
\end{equation*}
$$

we define

$$
\begin{align*}
(\nabla f)_{j} & =\sum_{k} i k f(k) e^{i k j / \Lambda} \\
& =\sum_{j^{\prime}} f\left(j^{\prime}\right)\left\{\sum_{k} \frac{i k}{2 N+1} e^{i k\left(j-j^{\prime}\right) / \Lambda}\right\} \equiv \sum_{j^{\prime}} f\left(j^{\prime}\right)\left(-\delta^{\prime}\left(j-j^{\prime}\right)\right) \tag{4.9}
\end{align*}
$$

where $(2 N+1)$ is the number of sites in the lattice. In $3 x$ dimensions (4.9) becomes

$$
\begin{equation*}
\left(\nabla_{x} f^{f} j_{x} j_{y} j_{z}=\sum_{j_{x}^{\prime}} f\left(j_{x}^{\prime}, j_{y}, j_{z}\right)\left(-\delta^{t}\left(j_{x}-j_{x}^{\prime}\right)\right)\right. \tag{4.10}
\end{equation*}
$$

It is easy to check that this prescription gives the exact relativistic energy momentum relation for free fermion of mass $m$

$$
\begin{equation*}
\mathrm{E}(\mathrm{k})=\sqrt{\mathrm{k}^{2}+\mathrm{m}^{2}} \tag{4.11}
\end{equation*}
$$

The only difference from the continuum free fermion theory is that the lattice version has a maximum allowable momentum $\left|k_{\max }\right|=\pi \Lambda$. Since there is no doubling of states and no need to split field components onto different lattice sites, we can easily incorporate exact $\gamma_{5}$ (chiral) invariance into theories with this formalism.
B. Lattice Thirring Models

We first study the lattice version of the pure fermionic theory based upon the chirally invariant Hamiltonian

$$
\begin{equation*}
\mathrm{H}=\int \mathrm{d} \mathrm{p}\left\{\psi+\frac{\vec{\alpha} \cdot \vec{\nabla}}{\mathrm{i}} \psi-\frac{\mathrm{g}_{0}}{2}\left[(\bar{\psi} \psi)^{2}-\left(\bar{\psi} \gamma_{5} \psi\right)^{2}\right]\right\} \tag{4.12}
\end{equation*}
$$

which canbe Fierz transformed into a current-current interaction

$$
\begin{equation*}
\mathrm{H}=\int \mathrm{d} \mathrm{p}_{\mathrm{x}}\left\{\psi+\frac{\vec{\alpha} \cdot \vec{\nabla}}{\mathrm{i}} \psi+\frac{1}{4} \mathrm{~g}_{0}\left[\left(\bar{\psi} \gamma_{\mu} \psi\right)^{2}-\left(\bar{\psi} \gamma_{\mu} \gamma_{5} \psi\right)^{2}\right]\right\} \tag{4.13}
\end{equation*}
$$

The theory is invariant under the transformations

$$
\begin{array}{ll}
\psi \rightarrow \mathrm{e}^{\mathrm{i} \theta} \psi & \text { and } \\
\bar{\psi} \rightarrow \bar{\psi} \mathrm{e}^{-\mathrm{i} \theta} & \psi \mathrm{e}^{\mathrm{i} \gamma_{5} \theta} \psi \\
& \bar{\psi} \rightarrow \bar{\psi} \mathrm{e}^{-\mathrm{i} \gamma_{5} \theta} \tag{4.14}
\end{array}
$$

for constant $\theta$. Hence the generators

$$
\begin{align*}
\mathrm{Q} & =\int \psi^{+} \psi \mathrm{d}^{\mathrm{p}} \mathrm{x} \\
\mathrm{Q}_{5} & =\int \psi^{+} \gamma_{5} \psi \mathrm{~d}^{\mathrm{p}} \mathrm{x} \tag{4.15}
\end{align*}
$$

commute with H .
In constructing the lattice version of (4.12) we introduce the dimensionless variables $\bar{\chi}(\mathrm{j}), \chi(\mathrm{j})$ and g via

$$
\begin{align*}
\psi(\vec{j}) & =\Lambda^{p / 2} \chi(\vec{j}) \\
\bar{\psi}(\vec{j}) & =\Lambda^{p / 2} \vec{\chi}(\vec{j})  \tag{4.16}\\
g_{0} & =g \Lambda^{1-p}
\end{align*}
$$

The canonical anticommutation relations are

$$
\begin{equation*}
\left\{\chi^{+}\left(\overrightarrow{\mathrm{j}_{1}}\right), \chi\left(\overrightarrow{\mathrm{j}_{2}}\right)\right\}=\mathbb{u} \delta_{\overrightarrow{\mathrm{j}_{1}}}, \overrightarrow{\mathrm{j}_{2}} \tag{4.17}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{H}=\Lambda\left[\sum_{\dot{j}_{1} \mathrm{j}_{2}} \chi^{+}\left(\overrightarrow{\mathrm{j}_{1}}\right) \mathbf{i} \vec{\alpha} \cdot \vec{\delta}^{\prime}\left(\overrightarrow{\mathrm{j}_{1}}-\overrightarrow{\mathrm{j}_{2}}\right) \chi\left(\overrightarrow{\mathrm{j}_{2}}\right)-\frac{\mathrm{g}}{2} \sum_{\overrightarrow{\mathrm{j}}}(\bar{\chi}(\overrightarrow{\mathrm{j}}) \chi(\overrightarrow{\mathrm{j}}))^{2}-\left(\vec{\chi}(\overrightarrow{\mathrm{j}}) \gamma_{5} \chi(\vec{j})\right)^{2}\right] \tag{4.18}
\end{equation*}
$$

The conserved charges are

$$
\begin{align*}
Q & =\sum_{\vec{j}} \chi^{+}(\vec{j}) \chi(j)  \tag{4.19}\\
Q_{5} & =\sum_{\vec{j}} \chi^{+}(\vec{j}) \gamma_{5} \chi(j)
\end{align*}
$$

We can also construct conserved but nonlocal currents on the lattice $j_{\mu}(\vec{j})$ and $\mathrm{j}_{5 \mu}(\vec{j})$ which have the form

$$
j_{0}(\vec{j})=\chi^{+}(\vec{j}) \chi(\vec{j}), \quad j_{50}(\vec{j})=\chi^{+}(\vec{j}) \gamma_{5} \chi(\vec{j})
$$

$$
\begin{align*}
& \overrightarrow{\mathrm{j}}(\overrightarrow{\mathrm{j}})=\chi^{+}(\overrightarrow{\mathrm{j}}) \cdot \vec{\alpha} \chi(\mathrm{j})+\sum_{\overrightarrow{j_{1}}, \overrightarrow{\mathrm{j}_{2}}} S\left(\overrightarrow{\mathrm{j}^{\prime}} ; \overrightarrow{\mathrm{j}_{1}}, \overrightarrow{\mathrm{j}_{2}}\right) \chi^{+}\left(\overrightarrow{\mathrm{j}_{1}}\right) \vec{\alpha} \chi\left(\overrightarrow{\mathrm{j}_{2}}\right) \\
& \overrightarrow{\mathrm{j}_{5}}(\overrightarrow{\mathrm{j}})=\chi^{+}(\overrightarrow{\mathrm{j}}) \cdot \vec{\alpha} \gamma_{5} \chi(\overrightarrow{\mathrm{j}})+\sum_{\overrightarrow{\mathrm{j}_{1}}, \overrightarrow{j_{2}}} S\left(\vec{j} ; \overrightarrow{\mathrm{j}_{1}}, \overrightarrow{\mathrm{j}_{2}}\right) \chi^{+}\left(\overrightarrow{\mathrm{j}_{1}}\right) \vec{\alpha} \gamma_{5} \chi\left(\overrightarrow{\mathrm{j}_{2}}\right) \tag{4.20}
\end{align*}
$$

Where $\mathrm{S}\left(\overrightarrow{\mathrm{j}} ; \overrightarrow{\mathrm{j}_{1}}, \overrightarrow{\mathrm{j}_{2}}\right)$ is uniquely defined. ${ }^{17}$ If we evaluate commutators involving time and space components of these currents we obtain a nonlocal term which, in the continuum limit, becomes the familiar Schwinger term.

## C. Momentum Space Variational Approach

The momentum space variational approach is equivalent to the effective potential calculation in the one loop approximation first carried out by Nambu and Jona-Lasinio. ${ }^{20}$ To obtain their equation for the mass gap we take the expectation value of the Hamiltonian (4.18) in the trial ground state $|\psi(\mathrm{m})\rangle$ defined by

$$
\begin{align*}
& \mathrm{b}_{\mathrm{m}}(\mathrm{k}) \mid \psi(\mathrm{m})>=0  \tag{4.21}\\
& \mathrm{~d}_{\mathrm{m}}(\mathrm{k}) \mid \psi(\mathrm{m})>=0
\end{align*}
$$

where the $b_{m}(\bar{k})$ and $d_{m}(k)$ are the fermion (antifermion) annihilation operators defined by a plane wave expansion with an arbitrary mass $m$

$$
\begin{align*}
& \psi(\overrightarrow{\mathrm{j}})=\frac{1}{\sqrt{\mathrm{~V}}} \sum_{\mathrm{k}} \mathrm{e}^{\mathrm{i} \overrightarrow{\mathrm{k}} \cdot \overrightarrow{\mathrm{j}} / \Lambda^{\prime}} \psi(\overrightarrow{\mathrm{k}})  \tag{4.22}\\
& \psi(\overrightarrow{\mathrm{k}})=\sum_{\alpha}\left[\mathrm{u}_{\mathrm{m}}^{\alpha}(\mathrm{k}) b_{\mathrm{m}}^{\alpha}(\mathrm{k})+\mathrm{v}_{\mathrm{m}}^{\alpha}(-\mathrm{k}) \mathrm{d}_{\mathrm{m}}^{+\alpha}(-\mathrm{k})\right]
\end{align*}
$$

The $u_{m}^{\alpha}(\vec{k})\left(v_{m}^{\alpha}(\vec{k})\right)$ are the positive (negative) energy solutions to the Dirac equation. The expectation value of the Hamiltonian (4.18) in this state is

$$
\begin{align*}
\mathscr{E}(m) & =\frac{E(m)}{V}=\frac{1}{V}\left\langle 0_{m}\right| H\left|0_{m}\right\rangle \\
& =-\frac{1}{V} \sum_{k}\left(k^{2}+m^{2}\right)^{1 / 2}+m^{2} \frac{1}{V} \sum_{k} \frac{1}{\left(k^{2}+m^{2}\right)^{1 / 2}}-2 g_{0} m^{2}\left[\frac{1}{V} \sum_{k} \frac{1}{\left(k^{2}+m^{2}\right)}\right]^{2 / 2} \tag{4.23}
\end{align*}
$$

Minimizing $\mathscr{E}(\mathrm{m})$ with respect to m leads to the gap equation

$$
\begin{equation*}
m\left[1-4 g \frac{\Lambda^{1-p}}{L^{\mathrm{p}}} \sum_{\mathrm{k}} \frac{1}{\sqrt{\mathrm{k}^{2}+\mathrm{m}^{2}}}\right]=0 \tag{4.24}
\end{equation*}
$$

Whenever $g$ has values such that there exist a solution besides $m=0$, this solution $(\mathrm{m} \neq 0)$ corresponds to a local minimum. Due to the continuous $\gamma_{5}$ symmetry when $\mathrm{m} \neq 0$, corresponding to the existence of massive fermion states, the ground state is infinitely degenerate implying the existence of massless Goldstone boson.

This result is in conflict with Coleman's theorem for $\mathrm{p}=1$ which states that no Goldstone boson occurs in 1 x -1t dimensions. ${ }^{16}$ Moreover for $\mathrm{p}=3$ one can show ${ }^{17}$ that both in the strong and weak coupling regimes the ground state is unique. The lesson to be drawn is that the one-loop effective potential method which is equivalent to the Hartree-Fock approximation

$$
\begin{equation*}
\left\langle(\bar{\psi} \psi)^{2}\right\rangle \rightarrow 2\langle\bar{\psi} \psi\rangle^{2} \tag{4.25}
\end{equation*}
$$

is misleading for determining chiral properties of the ground state. As we argued in our study of the scalar $\phi^{4}$ theory, the momentum space variational approach which diagonalizes the gradient term exactly might be reliable for weak coupling, but it can be very misleading when applied in the strong coupling regime. We turn therefore to a configuration space approach to discuss the large coupling $g>1$. It will turn out that a site basis gives a lower ground state energy and furthermore the $\gamma_{5}$ symmetry of the theory is realized in a normal way.
D. The Thirring Model in Two Dimensions

The Thirring model in $1 \mathrm{x}-1 \mathrm{t}$ dimensions will serve us as an example of strong coupling calculation in a fermionic theory. ${ }^{21}$ This theory has important features in common with gauge theories so that this analysis will prove useful to our subsequent discussions. As we shall see the strong coupling limit of this model describes a system of massless fermion-antifermion bound states in addition to super-massive charged fermions of mass $\sim \operatorname{g} \Lambda \gg \Lambda$. Hence this model provides
a concrete example of a theory for which, as $\mathrm{g} \rightarrow \infty$, the original fermionic degrees of freedom become "frozen out", but new massless degrees of freedom are left behind.

For $g \gg 1$ we first diagonalize the quartic part of the Hamiltonian (4.18)
exactly. A convenient representation is

$$
\begin{gather*}
\gamma_{5}=\alpha=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) \quad \gamma_{0}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right) \\
\chi(\mathrm{j})=\binom{b(\mathrm{j})}{\mathrm{d}^{+}(\mathrm{j})} \tag{4.26}
\end{gather*}
$$

$b(j)$ and $d(j)$ satisfy the anticommutation relations

$$
\begin{align*}
& \left\{\mathrm{b}(\mathrm{j}), \mathrm{b}^{+}\left(\mathrm{j}^{\prime}\right)\right\}=\left\{\mathrm{d}(\mathrm{j}), \mathrm{d}^{+}\left(\mathrm{j}^{\prime}\right)\right\}=\delta_{\mathrm{jj}} \\
& \left\{\mathrm{~b}(\mathrm{j}), \mathrm{d}\left(\mathrm{j}^{\prime}\right)\right\}=0, \text { etc. } \tag{4.27}
\end{align*}
$$

In this representation the Eamiltonian (4.18) has the form

$$
\begin{align*}
H & =\Lambda\left[\sum_{j_{1} j_{2}} i \delta^{\prime}\left(j_{1}-j_{2}\right)\left\{b_{j_{1}}^{+} b_{j_{2}}-d_{j_{1}}^{+} d_{j_{2}}\right\}-g \sum_{j}\left\{n_{b}(j)+n_{d}(j)-I\right\}^{2}\right] \\
& \equiv K+V \tag{4.28}
\end{align*}
$$

where we have used $\delta^{\prime}\left(\mathrm{j}_{1}-\mathrm{j}_{2}\right)=-\delta^{\prime}\left(\mathrm{j}_{2}-\mathrm{j}_{1}\right)$ and

$$
\begin{align*}
& n_{b}(j)=b_{j}^{+} b_{j} \\
& n_{d}(j)=d_{j}^{+} d_{j} \tag{4.29}
\end{align*}
$$

Since the potential V is a sum of commuting single site terms we can diagonalize each term separately and form a product basis over all sites. If we define 10 (j)> by

$$
\begin{equation*}
b(\mathrm{j})|0(\mathrm{j})\rangle=\mathrm{d}(\mathrm{j})|0(\mathrm{j})\rangle=0 \tag{4.30}
\end{equation*}
$$

we have the following four eigenstates at each site

At the single site level the ground state is two-fold degenerate with

$$
\begin{equation*}
E_{0}(\mathrm{j})=\mathrm{E}_{ \pm}(\mathrm{j})=-\mathrm{g} \Lambda \tag{4.32}
\end{equation*}
$$

The electric and $\gamma_{5}$ charges are given in accordance with (4.15)

$$
\begin{array}{rl}
Q & Q=\sum_{j}\left\{n_{b}(j)-n_{d}(j)\right\} \equiv \sum_{j} Q(j)  \tag{4.33}\\
Q_{5} & =\sum_{j}\left\{n_{b}(j)+n_{d}(j)-1\right\} \equiv \sum_{j} Q_{5}(j)
\end{array}
$$

As we shall see later on the ground state of the full Hamiltonian in the large coupling regime is also two-fold degenerate. Note that the single site charged states with $Q(j)= \pm 1, Q_{5}(j)=0$ lic high above the ground state from which they are separated by a gap $\sim g \Lambda$.

The two-fold degeneracy at the single site level means that the ground state of $V(4.28)$ is $2^{2 N+1}$-fold degenerate. The total electric charge of these degenerate states is zero and their $\gamma_{5}$ charge $Q_{5}$ can take any odd integer value from $-(2 N+1)$ to ( $2 \mathrm{~N}+1$ ) depending on the number of sites occupied by pairs. The role of K , the kinetic term in (4.28) is to split the degeneracies among the low lying $\mathrm{Q}=0$ states. Since $K$ commutes with $Q$ and $Q_{5}$ it connects only state within each $Q$ and $Q_{5}$. Thus, we will treat it as a perturbation for $g \gg 1$ and work within the $Q=0$ sector to construct the low lying energy spectrum. It is clear from the form (4.28) that $K$ which moves a single fermion or antifermion from one lattice site to another, gives no first order energy shift to the low lying states. We are forced, therefore,
to do second order degenerate perturbation theory in the ground state sector which is characterized by having $n_{b}(j)=n_{d}(j)$ for each site $j$. Since all energy denominators between the ground state and an excited state with one unbound pair are the same, $\mathrm{E}_{\mathrm{x}}-\mathrm{E}_{0}=2 \mathrm{~g} \Lambda$, the intermediate state sum can be performed to obtain an effective second order Hamiltonian for the ground state sector

$$
\begin{equation*}
H_{e f f}=-\frac{\Lambda}{2 g} \sum_{j_{1} j_{2}}\left\{\delta^{\prime}\left(j_{1}-j_{2}\right)\right\}^{2}\left[2 n_{b}\left(j_{1}\right)\left(1-n_{b}\left(j_{2}\right)\right)+2 d^{+}\left(j_{1}\right) b^{+}\left(j_{1}\right) b\left(j_{2}\right) d\left(j_{2}\right)\right] \tag{4.34}
\end{equation*}
$$

A spin formalism is suggestive at this stage. At each site only two eigenstates, $10\rangle$ and $| \pm\rangle$, which correspond to "spin down" and "spin up", respectively, occur in the $\mathrm{Q}=0$ sector. Hence we define "spin" raising and lowering operators

$$
\begin{align*}
& S_{+}(j) \equiv d^{+}(j) b^{+}(j)  \tag{4.35}\\
& S_{-}(j) \equiv b(j) d(j)=\left\{S_{+}(j)\right\}^{+}
\end{align*}
$$

such that

$$
\begin{equation*}
\left| \pm(\mathrm{j})>=\mathrm{S}_{+}(\mathrm{j})\right| 0(\mathrm{j})> \tag{4.36}
\end{equation*}
$$

and introduce -

$$
\begin{equation*}
n_{b}(j)=n_{d}(j)=S_{3}(j)+\frac{1}{2} \tag{4.37}
\end{equation*}
$$

$\mathrm{H}_{\text {eff }}$ can be rewritten

$$
\begin{equation*}
H_{e f f}=-\frac{\Lambda}{g} \sum_{j_{1} j_{2}}\left(\delta^{\prime}\left(j_{1}-j_{2}\right)\right)^{2}\left\{\frac{1}{4}+S_{+}\left(j_{1}\right) S_{-}\left(j_{2}\right)-S_{3}\left(j_{1}\right) S_{3}\left(j_{2}\right)\right\} \tag{4.38}
\end{equation*}
$$

Except for the relative minus sign between the spin-spin terms $H_{\text {eff }}$ describes the Heisenberg antiferromagnetic chain. This analogy is even clearer if we return for a moment to the definition of the gradient as a difference, in which case (4.38) contains only nearest neighbor interactions. Now we can make a unitary transformation changing the representation ( $(4.35),(4.37))$ by rotating through angle $\pi$ about the three axis at every other lattice site; i.e.,

$$
\begin{equation*}
\mathrm{S}_{ \pm}(\mathrm{j}) \rightarrow(-)^{\mathrm{j}} \mathrm{~S}_{ \pm}(\mathrm{j}) ; \quad \mathrm{S}_{3}(\mathrm{j}) \rightarrow \mathrm{S}_{3}(\mathrm{j}) \tag{4.39}
\end{equation*}
$$

The effective Hamiltonian becomes

$$
\begin{equation*}
H_{e f f}=\frac{\Lambda}{g} \sum_{j}\left(\vec{S}(j) \cdot \vec{S}(j+1)-\frac{1}{4}\right) \tag{4.40}
\end{equation*}
$$

The eigenstates of $\mathrm{H}_{\text {eff }}$ can be classified into degenerate multiplets of the total spin as well as of its third component, $\frac{1}{2} Q_{5}=\sum_{j} S_{3}(j)$. If we further assume that the lattice has an even number of sites and impose cyclic boundary conditions we 22 can refer to two exact theorems.

Theorem 1 When $\frac{\Lambda}{\mathrm{g}}>0$ corresponding to an antiferromagnetic interaction, the ground state of (4.40) has total sp in $\mathrm{S}=0$ and is unique.

Theorem 2 The theory has no mass gap in the limit as the length of the linear chain becomes infinite.

A corollary to theorem 1 which is independent of having nearest neighbors interactions in the antiferromagnetic chain (and therefore applies directly to our case) states that within each $S_{3}$ sector the ground state is unique. This, however, still does not tell us that the real ground state of the system is unique since a priori the various lowest states within each $S_{3}$ sector could have been degenerate. For nearest neighbors interaction we have theorem 1 which assures us that the ground state is indeed unique.

For a lattice with an odd number of sites it is impossible to form a state of $S_{3}=0$. In this case $S_{3}= \pm \frac{1}{2}$ is the lowest possible value and the ground state is twofold degenerate corresponding to the invariance $Q_{5} \rightarrow-Q_{5}$. The original solution for the ground state and excitation spectrum of (4.40) is due to Bethe. ${ }^{23}$ His method also shows that the excitation spectrum starts off linearly in $k$, corresponding to a massless particle spectrum.

If these results carry over to the solution of (4.38), which we constructed using (4.9) to avoid the doubling of the free fermion states on lattice, we see from the theorems that to leading order in $1 / g$ there exists a low lying spectrum of massless excitations of the Thirring model in addition to the arbitrarily massive
( $\sim \mathrm{g} \Lambda$ ) normal fermion excitations. This low lying excitation spectrum corresponds to bounded fermion-antifermion pairs but, like a fermion, obeys the exclusion principle of no more than one pair per lattice site. This spectrum is built upon a unique or doubly degenerate vacuum (depending on whether we use an even or odd number of lattice sites), and there is no spontaneous breaking of the $\gamma_{5}$-invariance.

On the basis of the analysis of (4.40) we expect the ground state solution of (4.38) to lie in the $Q_{5}= \pm 1=2 \mathrm{~S}_{3}$ sector. This can be easily verified for a lattice of three or five sites. For the general case we must, however, rely on variational calculations to construct upper bounds on the ground state energy in each $Q_{5}$ sector. Before describing the variational calculation note that the sectors with $S_{3}=\mp \frac{1}{2}(2 \mathrm{~N}+1)$, corresponding to all sites empty, or all occupied by a pair, are eigenstates of $H_{\text {eff }}$ and are the nondegenerate ground states in their respective sectors of $Q_{5}=\mp(2 N+1)$. The energy of these states is $-\mathrm{g} \Lambda(2 N+1)$. A less trivial case is the exact solution of (4.38) in the sector $Q_{5}=\mp(2 N-1)$ in which a single bound pair is present (or absent). The ground state in this sector lies below the one in the $Q_{5}= \pm(2 N+1)$ sector and the excitation spectrum is found to start off linearly in $k,{ }^{17}$ i.e.,

$$
\underset{Q_{5}= \pm(2 N-1)}{E(k)}=\Lambda\left\{-g(2 N+1)-\frac{2 \pi^{2}}{3 g}\right\}+\frac{\pi}{g}|k|-\frac{|k|^{2}}{2 g \Lambda}
$$

For the variational calculation in the sector, $Q_{5}=-(2 N+1)+2 p$, with pairs at $0 \leq \mathrm{p} \leq 2 \mathrm{~N}+1$ sites, we use a fully symmetrized trial state

$$
\begin{align*}
\mid \psi(p)> & =\frac{1}{p!}\binom{2 N+1}{p}^{1} \sum_{i_{1}} \ldots i_{p} s_{i_{1}}^{+} S_{i_{2}}^{+} \ldots S_{i_{p}}^{+}|0\rangle \\
& \left.=\lim _{\alpha \rightarrow 0} \frac{1}{p!\sqrt{\binom{2 N+1}{p}}}\left(\frac{d}{d \alpha}\right)^{p} e^{\alpha \sum_{i=-N}^{N} S_{i}^{+}} \right\rvert\, 0> \tag{4.42}
\end{align*}
$$

The upper bound on the total energy obtained is

$$
\begin{equation*}
E_{\operatorname{var}}(p)=-(2 N+1) \Lambda g-\frac{2 \pi^{2}}{3 g} \frac{p(2 N+1-p)}{2 N} \Lambda \quad p=0,1, \ldots, 2 N+1 \tag{4.43}
\end{equation*}
$$

This bound is also the exact result for $p=0$ and $p=1$. Equation (4.43) describes a parabola as a function of $p$ with a doubly degenerate minimum at $p=N$ and $p=N+1$ corresponding to $S_{3}=\mp \frac{1}{2}$. This suggests that (4.38) has the same general structure as the theory defined by (4.40); namely, the ground state is a $\gamma_{5}$ doublet, and the spectrum has no mass gap.

As a check on our choice of the variational wave function we can use it to calculate the ground state ( $p=N$ ) for the nearest neighbor case

$$
\begin{equation*}
\mathrm{E}_{\operatorname{var}}(\mathrm{N})=-(2 \mathrm{~N}+1) \Lambda \mathrm{g}-\frac{\Lambda}{\mathrm{g}}(\mathrm{~N}+1) \tag{4.44}
\end{equation*}
$$

This should be compared with Bethe's exact result

$$
\begin{equation*}
E_{\text {Bethe }}(N)=-(2 N+1) A g-\frac{\Lambda}{g}(N+1)(2 \log 2) \tag{4.45}
\end{equation*}
$$

This comparison suggests that the trial state (4.40) of a symmetric spin function without correlations is a reasonable guess.

## 5. GAUGE THEORIES ON THE LATTICE

Since the current opinion holds that non-Abelian gauge theories of quarks coupled to color gauge gluons comprise the class from which "the theory" will emerge, we would like next to discuss such theories. Being familiar by now with the description of a fermion field on the lattice, the new ingredient here is the gauge field. We shall adopt the prescription of Wilson ${ }^{3}$ and Kogut-Susskind ${ }^{3}$ in which we associate a gauge field with the links between lattice points. Hence, each link corresponds to an independent degree of freedom. The simplest model of interacting fermions plus gauge fields is the Schwinger model, or QED in 1x-1t dimensions. ${ }^{24}$ This model is soluble in the continuum; we shall show that it is soluble also on the lattice for strong coupling since then we can reduce it to a variant of the linear Heisenberg antiferromagnetic chain. We shall encounter once again the phenomenon that while the usual degrees of freedom become very massive there are low lying gauge invariant states corresponding to bound states of fermion-antifermion left behind. This result is intimately connected to the fact that our formalism for handling the fermion field allows for gauge invariant states with fermions and antifermions at the same lattice site and no flux links.

The same phenomenon occurs also for the non-Abelian gauge theory in $3 \mathrm{x}-1$ t dimensions. The most interesting results obtained in the large coupling limit are:
(i) 'The only "gauge invariant states" which remain at low mass have the quantum number of physical hadrons.
(ii) The resulting "effective strong coupling" theory preserves the full chiral symmetry of the exact theory $(\mathrm{SU}(3) \times \mathrm{SU}(3)$ for three colors and three flavors) and describes a theory of "massless bare hadrons" interacting with one another through a quark interchange mechanism of finite strength.

In obtaining these results for the non-Abelian gauge theory we were using KogutSusskind formalism for the gauge field. In this formalism the Hamiltonian is written in terms of angle variables (rotors). An important question is whether by adopting the prescription to work in terms of angle variable one does not build in implicitly confinement. In this respect the Abelian case (QED), which is known experimentally not to confine, should serve as a test case.

## A. The Schwinger Model

The Hamiltonian of the Schwinger model on a lattice in $1 \mathrm{x}-1 \mathrm{t}$ dimension in the gauge

$$
\begin{equation*}
A_{0}=0 \tag{5.1}
\end{equation*}
$$

is written as

$$
\begin{equation*}
\mathrm{H}=\Lambda\left\{\sum_{\mathrm{j}_{1} \mathrm{j}_{2}} \chi_{\mathrm{j}_{1}}^{+} \alpha\left(\mathrm{i} \delta^{\prime}\left(\mathrm{j}_{1}-\mathrm{j}_{2}\right) \mathrm{U}\left(\mathrm{j}_{1}-\mathrm{j}_{2}\right) \chi_{\mathrm{j}_{2}}+\frac{1}{2} \sum_{\ell} \mathrm{E}^{2}(\ell)\right\}\right. \tag{5.2}
\end{equation*}
$$

in terms of the fermion field $\chi_{j}$ and the gauge field $A(\ell)$. The electric field $E(\ell)$ is given by

$$
\begin{equation*}
E(\ell)=-\dot{A}(\ell) \tag{5.3}
\end{equation*}
$$

The index j runs over the lattice points, while $\ell$ runs over all the links.

$$
\begin{equation*}
U\left(j_{1}-j_{2}\right) \equiv \prod_{j_{1} \leq \ell \leq j_{2}} U(\ell)=U\left(\ell_{j_{1}}, j_{1}+1\right) U\left(\ell_{j_{1}+1, j_{1}+2}\right) \ldots U\left(\ell_{j_{2}}-1, j_{2}\right) \tag{5.4}
\end{equation*}
$$

where the product goes over all links, $\ell$, between the lattice points $j_{1}$ and $j_{2}$.

$$
\begin{equation*}
\mathrm{U}(\ell)=e^{\operatorname{iga~A(\ell )}}=e^{\operatorname{ig}_{0} A(\ell)} \quad \mathrm{a} \equiv 1 / \Lambda, \quad g_{0}=\mathrm{ga} \tag{5.5}
\end{equation*}
$$

with the convention

$$
\begin{equation*}
A\left(\ell_{1,2}\right)=-A\left(\ell_{2,1}\right)=-A\left(-\ell_{1,2}\right) \tag{5.6}
\end{equation*}
$$

so that

$$
\begin{equation*}
\mathrm{U}\left(\ell_{1,2}\right)=\mathrm{U}^{+}\left(\ell_{2,1}\right)=\mathrm{U}^{+}\left(-\ell_{1,2}\right) \tag{5.7}
\end{equation*}
$$

Note that the electric flux has a direction associated with it. We have scaled all degrees of freedom by the appropriate powers of $\Lambda$ so as to work with dimensionless fields. The dimensionless coupling constant is $\mathrm{g}_{0} \equiv$ ga.

Formally the Hamiltonian (5.2) reduces to the usual continuum Schwinger model in the limit $a \rightarrow 0$. The canonical commutation relations of the fermion field is given by (4.17), while those for the gauge field are

$$
\begin{align*}
& {[\mathrm{A}(\mathrm{n}), \mathrm{A}(\mathrm{~m})]=[\mathrm{E}(\mathrm{n}), \mathrm{E}(\mathrm{~m})]=0} \\
& {[\mathrm{~A}(\mathrm{n}), \mathrm{E}(\mathrm{~m})]=\mathrm{i} \delta_{\mathrm{n}, \mathrm{~m}}} \tag{5.8}
\end{align*}
$$

The condition (5.1) does not fix the gauge completely. The Hamiltonian (5.2) is still invariant under time independent gauge transformation

$$
\begin{align*}
\chi_{j} & \rightarrow e^{i g_{0} \Lambda_{j}} \chi_{j} \\
\chi_{j}^{+} & \rightarrow \chi_{j}^{+} e^{-i g_{0} \Lambda_{j}}  \tag{5.9}\\
A\left(\ell_{j, j+1}\right) & \rightarrow A\left(\ell_{j, j+1}\right)+\left(\Lambda_{j+1}-\Lambda_{j}\right)
\end{align*}
$$

We can talk about gauge transformation at each given point $j_{0}$ on the lattice. Under such a transformation only $\Lambda_{j_{0}}$ in (5.9) is different from zero. The generator of this transformation is

$$
\begin{equation*}
G_{j_{0}}=E\left(\ell_{j_{0}}, j_{0}+1\right)-E\left(\ell_{j_{0}}-1, j_{0}\right)-g_{0} x_{j_{0}}^{+} \chi_{j_{0}} \tag{5.10}
\end{equation*}
$$

The gauge invariance condition demands that $G_{j_{0}}$ annihilate every physical state

$$
\begin{equation*}
\mathrm{G}_{\mathrm{j}_{0}}\left|\psi_{\text {physical }}\right\rangle=0 \tag{5.11}
\end{equation*}
$$

which gives us the lattice version of Gauss law ( $\nabla \mathrm{E}=\rho$ ). Using the canonical commutation relations (5.8) it is easy to compute

$$
\begin{equation*}
[E(n), U(m)]=\left[E(n), e^{i g_{0} A(m)}\right]=g_{0} e^{\operatorname{ig}_{0} A(n)} \delta_{n, m}=g_{0} U(n) \delta_{n, m} \tag{5.12}
\end{equation*}
$$

Hence, the operator $U$ acts as a ladder operator on the eigenstates of $E$ shifting them by the value $\mathrm{g}_{0} . \mathrm{U}\left(\ell_{1,2}\right)$ is, therefore, an operator which create a unit flux (of magnitude $\frac{1}{2} g_{0}^{2}$ oriented from site 1 to site 2 ). We are interested in studying the strong coupling behavior of (5.2), $\mathrm{g}_{0} \rightarrow \infty$. Note that this limit means that $\mathrm{g}=\mathrm{g}_{0} / \mathrm{a} \gg 1 / \mathrm{a}$ as $\mathrm{a} \rightarrow 0$. It is convenient to rescale the gauge field by means of a canonical transformation

$$
\begin{align*}
\mathscr{A}(\ell) & =\mathrm{g}_{0} \mathrm{~A}(\ell)  \tag{5,13}\\
\mathscr{E}(\ell) & =\frac{1}{\mathrm{~g}_{0}} \mathrm{E}(\ell)
\end{align*}
$$

so that

$$
\begin{equation*}
[\mathscr{E}(\mathrm{n}), \mathrm{U}(\mathrm{~m})]=\mathrm{U}(\mathrm{n}) \delta_{\mathrm{n}, \mathrm{~m}} \tag{5.14}
\end{equation*}
$$

When $U(n)$ operates on an eigenstate of $\mathscr{E}$ containing $S$ unit of gauge flux on link $n$, i.e.,

$$
\begin{equation*}
\mathscr{E}(n)|S(n)\rangle=S|S(n)\rangle \tag{5.15}
\end{equation*}
$$

it increases the flux by one unit

$$
\begin{equation*}
\mathrm{U}(\mathrm{n})|\mathrm{S}(\mathrm{n})\rangle=|(\mathrm{S}+1)(\mathrm{n})\rangle \tag{5.16}
\end{equation*}
$$

The rescaled version of the Hamiltonian is

$$
\begin{align*}
H & =\Lambda\left\{\frac{1}{2} g_{0}^{2} \sum_{\ell} \mathscr{E}^{2}(\ell)+\sum_{\mathrm{j}_{1} \mathrm{j}_{2}} \chi_{\mathrm{j}_{1}}^{+} \alpha\left(\mathrm{i} \delta^{\prime}\left(\mathrm{j}_{1}-\mathrm{j}_{2}\right)\right) \prod_{\mathrm{j}_{1} \leq \ell \leq \mathrm{j}_{2}} \mathrm{e}^{\mathrm{i} \mathscr{A}(\ell)} \chi_{\mathrm{j}_{2}}\right\} \\
& =\mathrm{H}_{0}+\mathrm{K} \tag{5.17}
\end{align*}
$$

From this expression it is clear that the free gauge field $H_{0}$ is dominant for large $\mathrm{g}_{0} \gg 1$. The ground state of $\mathrm{H}_{0}$

$$
\begin{equation*}
\mathrm{H}_{0}=\frac{\Lambda}{2} g_{0}^{2} \sum_{\ell} \mathscr{E}^{2}(\ell) \tag{5.18}
\end{equation*}
$$

is the state of zero flux at each link. All other states with one or more nonzero flux links lie higher by at least $\frac{1}{2} g_{0}^{2} \Lambda \gg \Lambda$. Since $H_{0}$ contains no reference to the fermionic configuration, all zero flux states $|\psi, 0\rangle$, where $\psi$ is any fermionic
configuration, are degenerate. It is up to the kinetic term K to lift this degeneracy. We are going to do perturbation theory in K. It is clear that, since $U\left(j_{1}-\mathrm{j}_{2}\right)$ connects $|\psi, 0\rangle$ to state with nonzero flux link whose energy is $\frac{1}{2}\left|\mathrm{j}_{1}-\mathrm{j}_{2}\right| g_{0}^{2} \Lambda$, there is no energy shift in first order.

The first contribution is obtained in second order. Using the representation (4.26) we find the effective second order Hamiltonian

$$
\begin{gather*}
H_{\text {eff }}=-\Lambda \sum_{j_{1} j_{2} j_{3} j_{4}}\left(-i \delta^{\prime}\left(j_{1}-j_{2}\right)\right)\left(-i \delta^{\prime}\left(j_{3}-j_{4}\right)\right) \sum_{n}\left\{\left(b_{j_{1}}^{+} b_{j_{2}}-d_{j_{1}} d_{j_{2}}^{+}\right) U\left(j_{1}-j_{2}\right) \ln >\right. \\
\left.\left.\frac{1}{E_{n}}<n \right\rvert\,\left(b_{j_{3}}^{+} b_{j_{4}}-d_{j_{3}} d^{+}\right) U\left(j_{3}-j_{4}\right)\right\} \tag{5.19}
\end{gather*}
$$

The only intermediate states which contribute are those for which the electric flux created by $\mathrm{U}\left(\mathrm{j}_{3}-\mathrm{j}_{4}\right)$ is annihilated by $\mathrm{U}\left(\mathrm{j}_{1}-\mathrm{j}_{2}\right)$. Hence, the sums are restricted to $\mathrm{j}_{2}=\mathrm{j}_{3}, \mathrm{j}_{1}=\mathrm{j}_{4}$. The energy denominator $\mathrm{E}_{\mathrm{n}}$ is

$$
\begin{equation*}
\mathrm{H}_{0}\left\{\mathrm{U}\left(\mathrm{j}_{1}-\mathrm{j}_{2}\right)|0\rangle\right\}=\mathrm{H}_{0} \prod_{\mathrm{j}_{1} \leq \ell \leq \mathrm{j}_{2}} \mathrm{U}(\ell)|0\rangle=\frac{\Lambda}{2} \mathrm{~g}_{0}^{2}\left|\mathrm{j}_{1}-\mathrm{j}_{2}\right|\left\{\mathrm{U}\left(\mathrm{j}_{1}-\mathrm{j}_{2}\right)|0\rangle\right\} \tag{5.20}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
H_{e f f}=-\frac{2 A}{g_{0}^{2}} \sum_{j_{1} j_{2}} \frac{\left(\delta^{\prime}\left(j_{1}-j_{2}\right)\right)^{2}}{\left|j_{1}-j_{2}\right|}\left\{n_{b}\left(j_{1}\right)\left(1-n_{b}\left(j_{2}\right)\right)+n_{d}\left(j_{1}\right)\left(1-n_{d}\left(j_{2}\right)\right)+2 C_{j_{1}}^{+} C_{j_{2}}\right\} \tag{5.21}
\end{equation*}
$$

where

$$
C_{j} \equiv d_{j} b_{j}
$$

The gauge invariant condition (5.11) demands that any nonzero charge should be accompanied by the appropriate electric field. Since we limit ourselves to zero flux states there must be either a fermion-antifermion pair or nothing at each lattice site. Note that the huge degeneracy encountered has to do with the fact that our definition of the gradient allows particle and antiparticle to be at the same
site. As for the Thirring model we can define conserved fermion charges $Q$ and $Q_{5}$ (4.33). All the zero flux gauge invariant states have $\mathrm{Q}=0$ but differ in their $Q_{5}$ eigenvalues depending on the number of sites, $p$, occupied by pairs, i.e., $Q_{5}=-(2 N+1)+2 p$. Within this subspace $n_{b}(j)=n_{d}(j)$

$$
\begin{align*}
H_{e f f} & =-\frac{4 \Lambda}{2} \sum_{0} \frac{\left(\delta^{\prime}\left(j_{1}-j_{2}\right)\right)^{2}}{\left|j_{1}-j_{2}\right|}\left(n_{j_{1}}\left(1-n_{j_{2}}\right)+C_{j_{1}}^{+} C_{j_{2}}\right) \\
& =-\frac{4 \Lambda}{g_{0}^{2}} \sum_{j_{1} j_{2}} \frac{\left(\delta^{\prime}\left(j_{1}-j_{2}\right)\right)^{2}}{\left|j_{1}-j_{2}\right|}\left(\frac{1}{4}+s_{j_{1}}^{+} S_{j_{2}}^{-}-S_{j_{1}}^{3} S_{j_{2}}^{3}\right) \tag{5.22}
\end{align*}
$$

where the spin operators have been introduced in the same way as for the Thirring model. For the low lying gauge invariant states in the strong coupling limit the Schwinger model is equivalent to a linear Heisenberg antiferromagnetic chain, and the analysis applied to the Thirring model is relevant for this case too. Moreover due to the extra convergence factor in the correlation function of the Schwinger model

$$
\begin{equation*}
X\left(j_{1}-j_{2}\right)=\frac{\left(\delta^{\prime}\left(j_{1}-j_{2}\right)\right)^{2}}{\left|j_{1}-j_{2}\right|} \tag{5.23}
\end{equation*}
$$

relative to that of the Thirring model (4.38) we can prove, provided we neglect boundary effects at the end points of the lattice so that $\mathrm{H}_{\text {eff }}$ describe a translationally invariant system, that there is no mass gap in the excitation spectrum. We encounter here once again a situation in which for large coupling the normal excitations become extremely heavy. However there is a zero mass spectrum left behind (even when $\mathrm{a} \rightarrow 0$ ) which after a wave function renormalization looks perfectly relativistic.

The continuum Schwinger model has been solved exactly. It is known that in the sector of gauge invariant states only fermion-antifermion bound states exist which cannot be pulled apart. This is due to the fact that in 1 x dimensions the Coulomb energy is proportional to the distance between the fermion and the
antifermion. The same thing happens also for the lattice theory. All gauge invariant states must have zero total charge, $Q=0$, and a fermion at site $j_{1}$ must be joined to an antifermion at site $\mathrm{j}_{2}$ by a flux line which costs an energy $\left.\frac{1}{2} \mathrm{~g}_{0}^{2} \mathrm{lj}_{1}-\mathrm{j}_{2} \right\rvert\, \Lambda$. The Schwinger model is exactly soluble in the continuum. ${ }^{24}$ It is known ${ }^{25}$ that the solution is equivalent to the solution of the Thirring model plus extra massive photon whose mass is proportional to the coupling constant. Clearly we find the same result in the strong coupling limit of the lattice theory. The Hamiltonian (5.22) is equivalent to the strong coupling Thirring model (4.38) aside from the extra factor $1 /\left|j_{1}-j_{2}\right|$ in $H_{\text {eff }}$. The massive "photon" state will comprise of a quark and an antiquark joined together via a flux line (to ensure gauge invariance). Since the "mass" of a flux link is proportional to the coupling constant, the mass of this state is proportional too to the coupling constant. In particular Banks, Kogut, and Susskind show ${ }^{18}$ that the gauge invariant state

$$
\begin{equation*}
|\gamma\rangle=\sum_{j} b_{j}^{+} U\left(\ell_{j, j+1}\right) d_{j+1}^{+}|0\rangle \tag{5.24}
\end{equation*}
$$

represents a massive "photon" with zero momentum.
In comparing the lattice Schwinger model spectrum to the continuum spectrum the following comments should be remembered. ${ }^{25}$ The diagonalization of the Hamiltonian in the continuum leads to a spectrum consisting of a massive photon and a massless scalar boson (corresponding to a solution of the Thirring model). The full theory is therefore equivalent to a direct sum of two free noninteracting theories and in this respect is in a way uninteresting. If we restrict ourselves to the states created by the algebra of gauge invariant operators and choose to work in one of the noncovariant gauges we can select out the massive photon part of the theory. This is the reason why in the usual continuum treatment one talks only about the massive photon state. The algebra of gauge invariant operators contains, besides the electric field,bilocal quantities defined by a split point method
which in the lattice language corresponds to having particles and antiparticles at different lattice sites with gauge field links between them. States created by these operators will be very massive in the large coupling limit. In particular the operator $\bar{\psi}(\mathrm{x}) \psi(\mathrm{x})$ whose discrete analogue on the lattice does create the zero mass state is singular and must be defined in an appropriate way (by splitting the points and subtracting away the singularity in the limit). The usual normal ordering definition using the split point method decouples the zero mass state from the algebra of gauge invariant operators. On the lattice, however, the zero mass states are created by the well defined gauge invariant operator $\bar{\psi}_{j} \psi_{j}$. Hence the algebra of gauge invariant operators on the lattice is larger and does not select the massive photon. The question of the continuum limit is certainly nontrivial and deserved much more study.

## B. The Non-Abelian Color Gauge Theories in 3x-1t Dimensions

The results obtained in the preceding discussion can be directly generalized to non-Abelian gauge models in higher dimensions. As in the preceding discussion, because our formalism includes gauge invariant states with fermions and antifermions at the same lattice site and no flux links, there are low mass states in the strong coupling limit. We follow the same prescription of Wilson ${ }^{3}$ and Kogut and Susskind ${ }^{3}$ for the gauge field in $3 x-1 t$ dimensions. For the fermion field the gradient operator as defined in (1.6) leads to the Hamiltonian

$$
\begin{align*}
H & =H_{0}\binom{\text { electric part of }}{\text { gauge field }}+\Lambda \sum_{\overrightarrow{\mathrm{j}_{1}} \overrightarrow{\mathrm{j}}_{2}} \chi^{+}\left(\overrightarrow{\mathrm{j}_{1}}\right)\{\overrightarrow{\mathrm{c}}\} \\
& \equiv \mathrm{H}_{0}+\underset{\mathrm{j}}{ }+\mathrm{K}+\mathrm{O}\left(\frac{1}{\mathrm{~g}^{2}}\right) \tag{5.25}
\end{align*}
$$

where

$$
\begin{equation*}
\delta_{1}^{\prime}(\vec{j}-\vec{l}) \equiv \delta^{\prime}\left(j_{1}-l_{1}\right) \delta_{j_{2}}, \ell_{2} \delta_{j_{3}}, l_{3}, \text { etc. } \tag{5.26}
\end{equation*}
$$

and $U\left(\overrightarrow{j_{1}}-\overrightarrow{\mathrm{j}_{2}}\right)$ is a product of terms of the form

$$
\begin{equation*}
U(\vec{\ell})=e^{i g \lambda \cdot \vec{A}(\vec{\ell}) \cdot \vec{\ell}} \tag{5.27}
\end{equation*}
$$

where $A(l)$ are the canonical link fields, and $\lambda$ are $c$-number matrices belonging to a specific ( $N, \bar{N}$ ) representation of the gauge group as determined by the choice of representation for the fermion fields. Equation (5.2) defines the obvious straight line path on the lattice for the flux links joining $\vec{j}_{1}$ to $\vec{j}_{2}$.

In the strong coupling region of large $g_{0}$ the important properties of (5.25) are:

1) The low mass states are those with zero flux links and an arbitrary configuration of fermions. All others are pushed up in energy above $\sim g_{0}^{2} \Lambda$.
2) When $U(\ell)$ hits an unoccupied link-i.e., one for which no gauge field has been excited-it excites the link and increases the energy of the state by $\sim \mathrm{g}_{0}^{2} \Lambda$.

At this point we proceed in close parallel to the discussion of the Schwinger model with strong coupling. Focusing our attention on the sector of gauge invariant states, we study the way in which the fermionic part of $H$ mixes all the zero energy eigenstates of $\mathrm{H}_{0}$ (gauge) that have no flux links to split their degeneracy. In a theory with the $\operatorname{SU}(3) \times[\operatorname{SU}(3)]$ color symmetry of the quark model all states with ( $q \bar{q}$ ) or ( $q q q$ ) at a lattice site in color singlet states are included in the low lying sector of gauge invariant states. These are the states having the quantum numbers of ordinary hadrons.

If we choose the spinor representation

$$
\begin{gathered}
x_{j}=\binom{B_{j}}{D_{j}^{+}} \quad B=\binom{b_{j}^{+}}{b_{j}^{-}} \quad D_{j}=\binom{d_{j}^{+}}{d_{j}^{-}} \\
\gamma^{\ell}=\left(\begin{array}{cc}
0 & \sigma^{\ell} \\
-\sigma^{\ell} & 0
\end{array}\right)
\end{gathered}
$$

we can rewrite the fermionic part of H as

$$
\begin{equation*}
K=\sum_{\overrightarrow{\mathrm{j}}_{1} \overrightarrow{\mathrm{j}}_{2}} \overrightarrow{\mathrm{i} \vec{\delta}^{\prime}}\left(\overrightarrow{\mathrm{j}_{1}}-\overrightarrow{\mathrm{j}_{2}}\right) U\left(\overrightarrow{\mathrm{j}}_{1}-\overrightarrow{\mathrm{j}_{2}}\right)\left[B^{+}\left(\mathrm{j}_{1}\right) \vec{\sigma} B\left(\mathrm{j}_{2}\right)-D\left(\mathrm{j}_{1}\right) \vec{\sigma} D^{+}\left(\mathrm{j}_{2}\right)\right] \tag{5.28}
\end{equation*}
$$

As before $K$ moves a quark in a straight line from $\overrightarrow{\mathrm{j}}_{2}$ to $\overrightarrow{\mathrm{j}_{1}} \neq \overrightarrow{\mathrm{j}}_{2}$ (or an antiquark from $\overrightarrow{\mathrm{j}_{1}}$ to $\overrightarrow{\mathrm{j}_{2}}$ ) and at the same time excites a unit of gauge flux on each intervening link. Therefore we must go to second or higher order in $K$ in order to mix the degenerate color singlet fluxless states.

Furthermore since $\delta^{\prime}(0)=0$, the action of $K$ in second order allows scattering and interaction among these states but it introduces no self mass term involving only quarks all at the same lattice site. Hence our effective Hamiltonian for the low lying gauge invariant states of "bare colorless hadrons" corresponds to a theory of bare massless strongly interacting particles. Our starting point is a
strong coupling theory with the full chiral $\operatorname{SU}(3) \times \operatorname{SU}(3)$ symmetry if we choose a fundamental quark triplet. Instead of having to drive the pion mass down to zero to insure PCAC we have a zero mass starting point and must solve the problem of generating the hadronic masses either by a dynamical breakdown mechanism yet to be explored or by explicitly introducing chiral breaking interactions into H ab initio.

The real work of solving for the hadronic spectra and interactions still remains to be done. What we have formulated here is a starting point in terms of a chirally invariant gauge theory (of color) which reduces in the strong coupling region to a system of interacting "bare" particles with hadronic quantum numbers. In the gauge invariant sector the quark and gluon degrees of freedom are frozen out since such states with excited flux links are pushed up to very high energy above $\approx \mathrm{g}_{0}^{2} \Lambda>\Lambda$. This is a very different starting point from earlier formulations that destroy local chiral invariance by splitting fermion field components onto different lattice sites.

In conclusion we make some general observations:

1) According to $(5.28) \mathrm{K}$ acting on a fluxless gauge invariant state moves a quark or an antiquark, creating the associated flux link. To second order it can either move a quark (or antiquark) from an initial site to an intermediate one, and then move it back again to where it started, thereby cancelling the flux link, since $U\left(j_{1}-j_{2}\right) U\left(j_{2}-j_{1}\right)=1$; or it can move both a quark and an antiquark from site $j_{1}$ to $\mathrm{j}_{2}$ without creating flux links in the final state. This is illustrated in Figs. 12a and b . This amounts to a kinetic energy term as we saw in the analysis of the Thirring and Schwinger models.
2) If there are two hadrons present on different sites, the second order application of $K$ can lead to a quark inter change interaction between them, as illustrated in Fig. 13. Starting from color singlet states the hadrons will also
end up as individual color singlets if no flux links are created in the final state. However, $\operatorname{SU}(3)$ quantum numbers can be changed.
3) A single three-quark baryon can move from one site to another on the lattice only as a result of 3 rd and higher order applications of K . This is because each order of $K$ can move but one quark at a time. This means that baryonic masses are displaced relative to the zero order degenerate eigenvalue of $\mathrm{H}_{0}$ by factors of order $1 / \mathrm{g}_{0}^{4}$ in contrast to the $1 / \mathrm{g}_{0}^{2}$ shift from second order application of K to the meson states. The significance of this for hadronic mass spectra and for the choice of coupling strengths $\mathrm{g}_{0}^{2}$ remains to be studied. So does the entire question of how our bare massless mesonic states become dressed to form the true physical states containing ( $q \bar{q}$ ) clouds with which they can interact via the quark interchange mechanism.
4) In the gauge invariant sector, all exotic states of nonzero triality contain flux links and are therefore pushed very high up in energy above $g_{0}^{2} \Lambda$. Exotic states of the second kind-namely, states such as $\left\{(q \bar{q})_{\text {octet }}(q \bar{q})_{\text {octet }}\right\}_{\text {singlet }}$ with quarks and antiquarks finally coupled to color singlet configurations, but not contained in the normal quark model-do occur. However, whereas the vacuum and ordinary $q \bar{q}$ mesonic states will have their degeneracy split and can be pushed down in energy with second order application of $K$, these exotics of the second kind are shifted only in higher order since it takes fourth order application of K to move them on the lattice. Hence, if they were stable, we would expect to find them lying higher in the energy spectrum. In fact, it is easy to see that such states can decay, in second order, to ordinary separated $q \bar{q}$ states.
5) Glue balls-that is states of pure gluon, or flux link, configurations-lie very high in energy above our low mass gauge invariant sector since they will have the energy of at least four flux links, $2 \mathrm{~g}_{0}^{2} \Lambda$.

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## REFERENCES

1. All the new material covered in these lectures has been worked out together with Prof. Sidney Drell and Dr. Marvin Weinstein at SLAC.
a. S. D. Drell, M. Weinstein and S. Yankielowicz, "Strong Coupling Field Theories-I. Variational Approach to $\phi^{4}$ Theory, "Stanford Linear Accelerator Center preprint SLAC-PUB-1719, Phys. Rev. D (in press).
b. S. D. Drell, M. Weinstein and S. Yankielowicz, "Strong Goupling Field Theoriess-II. Fermions and Gauge Fields on a Lattice," Stanford Linear Accelerator Center preprint SLAC-PUB-1752.
2. A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn and V. F. Weisskopf, Phys. Rev. D 9, 3471 (1974).
A. Chodos, R. L. Jaffe, K. Johnson and C. B. Thorn, Phys. Rev. D 10, 2599 (1974).
T. DeGrand, R. L. Jaffe, K. Johnson and J. Kiskis, Massachusetts Institute of Technology preprint MLT-CTP-475.
3. K. G. Wilson, Phys. Rev. D 10, 2445 (1974).
K. G. Wilson, Erice School of Physics, Cornell preprint CLNS-321 (1975).
J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975).

1
W. A. Bardeen and R. B. Pearson, "Local Gauge Invariance and Bound State

Nature of Hadrons, " FermiLab-PUB-76/24-THY.
R. Balian, J. Drouffe and C. Itzykson, Phys. Rev. D 10, 3376 (1974).
4. W. A. Bardeen, M. S. Chanowitz, S. D. Drell, M. Weinstein and T. M. Yan, Phys. Rev. D 11, 1094 (1975).
M. Creutz and K. S. Soh, Phys. Rev. D 12, 443 (1975).
5. See Ref. 1a.
6. Reviews of the semi-classical approach can be found also in Sidney Coleman, "Classical Lumps and Their Quantum Descendants," Erice School of Physics (1975), Harvard preprint.
R. Rajaraman, "Some Nonperturbative Semi-Classical Methods in-Quantum Field Theory," Institute for Advance Study, Princeton preprint COO-2220-47 (1975).
R. F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D 10, 4130 (1974);

Phys. Rev. D 10, 4114 (1974); Phys. Rev. D 10, 4138 (1974).
7. R. F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D 10, 4130 (1974).
8. S. J. Chang, Phys. Rev. D 12, 1071 (1975).
S. J. Chang and J. A. Wright, Phys. Rev. D 12, 1595 (1975).
9. B. Simon and R. B. Griffith, Commun. Math. Phys. 33, 145 (1973).
10. See Ref. 1a.
11. A finite spin approximation for the $\phi^{4}$ theory has been worked out also by R. B. Pearson who carried out also renormalization group calculations.
12. The exact solution for the spin $1 / 2$ approximation described in this section is based on the following works: B. Stoeckly and D. J. Scalapino, Phys. Rev. B 11, 205 (1975).
D. J. Scalapino and B. Stoeckly, "A Quantum Mechanical Kink Solution," University of California, Santa Barbara preprint (May 1976).
13. See, for example, K. G. Wilson and J. Kogut, "The Renormalization Group and the $\epsilon$-Expansion, " Physics Reports Vol. 12, 75 (1974).
14. J. Goldstone, A. Salem and S. Weinberg, Phys. Rev. 127, 965 (1962).
15. S. K. Ma and R. Rajaraman, Phys. Rev. D 11, 1701 (1975).
16. S. Coleman, Commun. Math. Phys. 31, 259 (1973).
17. See Ref. 1b.
18. J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975).
L. Susskind, Lectures at Bonn Summer School (1974).
T. Banks, J. Kogut and L. Susskind, Phys. Rev. D 13, 1043 (1976).
19. K. G. Wilson, Erice School of Physics, Cornell University preprint CLNS--321 (1975).
20. Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 395 (1961); Phys. Rev. 124, 246 (1961).
21. A discussion of the continuum Thirring model can be found in the following works: W. Thirring, Ann. Phys. (N.Y.) 3, 91 (1958).
K. Johnson, Nuovo Cimento 20, 773 (1.961).
G. F. Dell'Antonio, Y. Frishman and D. Zwanziger, Phys. Rev. D 6, 988 (1972).;
22. E. Lieb, T. Schultz and D. Mattis, Ann. Phys. 16, 407 (1961).
E. Lieb and D. Mattis, Mathematical Physics in One Dimension (Academic Press, New York, 1966), p. 466.
23. H. A. Bethe, Z. Phys. 71, 205 (1931).
24. A discussion of the continuum Schwinger model can be found in the following works: J. Schwinger, Phys. Rev. 128, 2425 (1962).
J. Lowenstein and J. Swieca, Ann. Phys. (N. Y.) 68, 172 (1971).
25. J. Lowenstein and J. Swieca, Ann. Phys. (N. Y.) 68, 172 (1971).

## FIGURE CAPTIONS

1. The equivalent potential of the $\phi^{4}$ theory.
2. The lattice notation.
3. The ground state energy as a function of the vacuum expectation value $c$ for fixed $\lambda$ and different $f$, for the momentum space variational calculation.
4. Phase diagram for the momentum space calculation in the presence of an external source J.
5. The single site anharmonic potential.
6. The "kink" configuration.
7. The radial potential.
8. The $\theta$-potential.
9. The level structure for the renormalization group calculation of the spin 1 problem.
10. The levels in the anharmonic potential of the $\phi^{4}$ theory.
11. Energy momentum dispersion relation for a free Dirac particle.
12. Motion of a $q \bar{q}$ state on a lattice to second order $1 / \mathrm{g}^{2}$. (a) $q$ (or $\bar{q}$ ) moves to a different latice site exciting the intervening flux links and then returns;
(b) $q$ moves to a new site and is then followed by the $\bar{q}$.
13. Quark interchange interaction between mesons.


Fig. 1


$$
a \equiv \frac{1}{\Lambda}
$$

Fig. 2
(a)
 $f^{2}>f_{c r} \quad$ "weak coupling".
(b)

$f^{2}=f_{c r} \quad$ "intermediate coupling"
(c)


$$
f^{2}<f_{c r}
$$

"strong coupling"
(d)


$$
f^{2}<f_{c r}
$$

"very strong coupling"

Fig. 3


Fig. 4


Fig. 5


Fig. 6


Fig. 7


Fig. 8


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Fig. 9


Fig 10


Fig. 11
(a)

$$
\begin{aligned}
& \times \stackrel{j_{1}}{\times} \times \stackrel{j_{2}}{\times} \times \xrightarrow{\text { firstorder }} \times \stackrel{j_{1}}{j_{2}} \times \\
& \pm 0 \quad-\cdots++ \\
& \xrightarrow{\text { second order }} \times \stackrel{j_{1}}{\times} \times \stackrel{j_{2}}{\times} \times \\
& \pm-\longleftarrow-0
\end{aligned}
$$

(b)

$$
\begin{aligned}
& \times \underset{ }{\substack{j_{1} \\
\\
\pm}} \times \underset{\sim}{j_{2}} \times \xrightarrow{\text { first order }} \times \underset{-\rightarrow-+}{j_{1}} \times \\
& \xrightarrow{\text { second order }} \times \stackrel{\mathrm{j}_{1}}{\times} \times \stackrel{\mathrm{j}_{2}}{\times} \times \\
& 0--7- \pm_{2803 \mathrm{AB}}
\end{aligned}
$$

Fig. 12

Fig. 13


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