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GENERALIZATION OF WEYL'S UNIFIED THEORY TO ENCOMPASS A NON-ABELIAN INTERNAL SYMMETRY GROUP*

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ABSTRACT

Weyl's unified theory of the gravitational and the electromagnetic field is directly generalized to a unified theory of the gravitational field and a non-Abelian, internal symmetry gauge field. We show that a second gauge field naturally arises which transforms homogeneously under transformations of the gauge group. Certain formal similarities with a quark confining model of the strong interactions are pointed out.

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In a series of papers [1] beginning in 1918, Weyl developed a unified theory of gravitation and electromagnetism which was based on a generalization of Riemannian geometry. While the theory failed as a model of the universe, it has had a seminal impact on theoretical physics because it contained the essence of the gauge group concept. Indeed, the name "gauge group" is indicative of its origin in Weyl's theory.

In this paper we shall generalize Weyl's theory in a straightforward way to obtain a unified theory of the gravitational field, and a gauge field transforming according to a non-Abelian internal symmetry group. We shall see that this theory necessarily contains two Yang-Mills fields—one which is introduced by hand through the Weyl ansatz and which transforms inhomogeneously under gauge group transformations; and another which becomes prominent in certain contractions of the Riemann tensor and which transforms homogeneously under gauge group transformations. Some formal similarities to a gauge theory of the strong interactions will also become apparent.

We shall first outline Weyl's theory [2]. In Riemannian geometry the transference of the direction of a vector from a point P to a point P' depends in general on the path taken. Weyl goes beyond this by permitting a path dependence for the transference of length. As a result it is only possible to compare lengths measured at one and the same world point. Suppose we assume the existence of a metric, $g_{\mu\nu}$, and define a length

$$\ell^2 = ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \tag{1}$$

where dx^μ are the coordinate differences of the endpoints of the length in question. Suppose further that the length is displaced along a certain curve, $x^\mu = x^\mu(t)$, from the point P(t) to the point P'(t+dt). Weyl makes the ansatz that the length changes by a definite fraction of ℓ

$$\frac{d\ell^2}{dt} = -\ell^2 \frac{d\phi}{dt} \tag{2}$$

where ϕ is a function of t but independent of ℓ. Weyl then assumes that the components of a vector remain unchanged (in a certain coordinate system) under an infinitesimal parallel displacement, so that

$$\frac{dV^\mu}{dt} = -\Gamma^\mu_{\lambda\sigma} \frac{dx^\sigma}{dt} V^\lambda \quad . \tag{3}$$

Consistency between eqs. (2) and (3) requires $d\phi/dt$ to be linear in dx^ν/dt:

$$\frac{d\phi}{dt} = \phi_\nu \frac{dx^\nu}{dt} \tag{4}$$

and leads to the relation

$$2\Gamma_{\mu,\lambda\sigma} = \left(\frac{\partial}{\partial x^\sigma} + \phi_\sigma\right) g_{\mu\lambda} + \left(\frac{\partial}{\partial x^\lambda} + \phi_\lambda\right) g_{\mu\sigma} - \left(\frac{\partial}{\partial x^\mu} + \phi_\mu\right) g_{\lambda\sigma} \quad . \tag{5}$$

Under a gauge transformation

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu} = \Lambda(x)\, g_{\mu\nu}(x) \tag{6}$$

we find that

$$\phi_\nu \rightarrow \phi'_\nu = \phi_\nu - \frac{\partial}{\partial x^\nu} \ell n\, \Lambda \tag{7}$$

from eq. (2) and $\Gamma_{\mu,\lambda\sigma}$ remains unchanged. The gauge transformation of eq. (6) corresponds to a change of scale. However there is a marked similarity [3] to the gauge transformation of electrodynamics which becomes apparent when we make the substitution

$$\phi_\mu \rightarrow -ie\, A_\mu \tag{8}$$

and let $\Lambda \rightarrow \exp[i\theta(x)]$. We shall exploit this similarity in generalizing the analog of the electromagnetic field to a non-Abelian gauge field.

Let us assume the existence of a "metric" tensor which, besides being a second rank covariant tensor under general coordinate transformations, is a matrix in the fundamental representation of SU(N)

$$\tilde{g}_{\mu\nu} = g^a_{\mu\nu}\,\tilde{T}^a = \tilde{g}_{\nu\mu} \qquad (9)$$

where $\tilde{T}^a$ is a matrix representing a generator of SU(N) for $a=1,2,\ldots,N^2-1$ and $\tilde{T}^a$ is proportional to the identity matrix, $\tilde{I}$, for $a=N^2$. We define an inverse by

$$\tilde{g}_{\mu\alpha}\tilde{g}^{\alpha\nu} = \tilde{I}g_\mu{}^\nu \qquad (10)$$

with tildes indicating matrical quantities while $g_\mu{}^\nu = \delta_\mu{}^\nu$. Consider a "length",

$$\tilde{\ell}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu \quad , \qquad (11)$$

which, in analogy to eq. (2), changes by a definite fraction upon displacement along a curve

$$\frac{d\tilde{\ell}^2}{dt} = -i\,\frac{d\tilde{A}}{dt} \times \tilde{\ell}^2 \qquad (12)$$

where $\tilde{B}\times\tilde{C} = \tilde{B}\tilde{C} - \tilde{C}\tilde{B}$. We provisionally let

$$\frac{d\tilde{A}}{dt} = \tilde{A}_\mu\,\frac{dx^\mu}{dt} \qquad (13)$$

and rewrite eq. (12) in the form

$$D_t\tilde{\ell}^2 \equiv \frac{dx^\mu}{dt}\,D_\mu\tilde{\ell}^2 = 0 \qquad (14)$$

where

$$D_\mu\tilde{\ell}^2 = \left(\frac{\partial}{\partial x^\mu} + i\tilde{A}_\mu\times\right)\tilde{\ell}^2 \quad . \qquad (15)$$

Note that the part of $\tilde{\ell}^2$ proportional to the identity matrix, $\tilde{I}$, is unchanged upon displacement along a curve.

We now generalize eq. (3), which describes the effect of an infinitesimal parallel displacement on the direction of a vector, to

$$D_t \tilde{V}_\mu = \tilde{\Gamma}^\alpha_{\mu\sigma} \circ \tilde{V}_\alpha \frac{dx^\sigma}{dt} \tag{16}$$

where $\tilde{B} \circ \tilde{C} = \tilde{B}\tilde{C} + \tilde{C}\tilde{B}$ and $\tilde{\Gamma}^\alpha_{\mu\sigma} = \tilde{\Gamma}^\alpha_{\sigma\mu}$. If we let

$$\tilde{V}_\mu = \tilde{g}_{\mu\alpha} V^\alpha \tag{17}$$

then eqs. (14) and (16) imply

$$\tilde{\Gamma}_{\nu,\mu\sigma} + \tilde{\Gamma}_{\mu,\nu\sigma} - \left(\partial_\sigma + i\tilde{A}_\sigma \times\right) \tilde{g}_{\mu\nu} = 0 \tag{18}$$

where

$$\tilde{\Gamma}_{\mu,\nu\sigma} = \tilde{\Gamma}^\alpha_{\nu\sigma} \circ \tilde{g}_{\alpha\mu} \ . \tag{19}$$

Algebraic manipulation of eq. (18) gives

$$2\tilde{\Gamma}_{\mu,\nu\sigma} = \left(\partial_\sigma + i\tilde{A}_\sigma \times\right)\tilde{g}_{\mu\nu} + \left(\partial_\nu + i\tilde{A}_\nu \times\right)\tilde{g}_{\mu\sigma} - \left(\partial_\mu + i\tilde{A}_\mu \times\right)\tilde{g}_{\sigma\nu} \tag{20}$$

in close analogy to eq. (5). Now if we consider a local gauge transformation, $S=S(x)$, of $\tilde{g}_{\mu\nu}$

$$\tilde{g}_{\mu\nu} \rightarrow S^{-1}\tilde{g}_{\mu\nu} S \tag{21}$$

then we find that gauge covariance of eq. (15) requires

$$\tilde{A}_\mu \rightarrow S^{-1}\tilde{A}_\mu S - iS^{-1}\partial_\mu S \tag{22}$$

so that $\tilde{A}_\mu$ is a Yang-Mills gauge field [4]. In addition, eq. (20) implies that $\tilde{\Gamma}_{\mu,\nu\sigma}$ (and thus $\tilde{\Gamma}^\alpha_{\nu\sigma}$) transforms homogeneously

$$\tilde{\Gamma}_{\mu,\nu\sigma} \rightarrow S^{-1}\tilde{\Gamma}_{\mu,\nu\sigma} S \ . \tag{23}$$

Equation (20) also reveals an interesting facet of the general coordinate transformation properties of $\tilde{\Gamma}_{\mu,\nu\sigma}$. Under a general coordinate transformation we find

$$\tilde{\Gamma}_{\mu,\nu\sigma} \rightarrow \tilde{\Gamma}'_{\mu,\nu\sigma} = \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial x^\gamma}{\partial x'^\nu} \frac{\partial x^\delta}{\partial x'^\sigma} \tilde{\Gamma}_{\beta,\gamma\delta} + \frac{\partial^2 x^\gamma}{\partial x'^\nu \partial x'^\sigma} \frac{\partial x^\beta}{\partial x'^\mu} \tilde{g}_{\beta\gamma} \tag{24}$$

so that $\tilde{\Gamma}^\mu_{\nu\sigma} \equiv \Gamma^{\mu a}_{\nu\sigma}\tilde{T}^a$ is a tensor with respect to general coordinate transformations <u>except</u> for the coefficient of $\tilde{I}$, namely $\Gamma^{\mu a}_{\nu\sigma}$ for $a=N^2$, which is not a tensor.

Derivatives which are covariant both under general coordinate transformations and gauge transformations are defined by

$$D_\sigma \tilde{V} = \left(\partial_\sigma + i\tilde{A}_\sigma \times\right) \tilde{V} \tag{25}$$

$$D_\sigma \tilde{V}_\mu = \left(\partial_\sigma + i\tilde{A}_\sigma \times\right) \tilde{V}_\mu - \tilde{\Gamma}^\alpha_{\mu\sigma} \circ \tilde{V}_\alpha \tag{26}$$

$$D_\sigma \tilde{V}_{\mu\nu} = \left(\partial_\sigma + i\tilde{A}_\sigma \times\right) \tilde{V}_{\mu\nu} - \tilde{\Gamma}^\alpha_{\mu\sigma} \circ \tilde{V}_{\alpha\nu} - \tilde{\Gamma}^\alpha_{\nu\sigma} \circ \tilde{V}_{\mu\alpha} \tag{27}$$

for a scalar, covariant vector and second rank covariant tensor respectively. We note that

$$D_\sigma \tilde{g}_{\mu\nu} = 0 \tag{28}$$

by eqs. (27) and (20).

We shall now turn to an examination of the covariant derivative of "determinants" to obtain the generalization of the usual form of the covariant derivative of a density. Let $\tilde{V}_{\mu\nu}$ be a tensor,

$$\tilde{V} = \epsilon^{\mu\nu\lambda\sigma}\, T\left[\tilde{V}_{0\mu}\tilde{V}_{1\nu}\tilde{V}_{2\lambda}\tilde{V}_{3\sigma}\right] \tag{29}$$

and

$$\tilde{V}_{\alpha\beta\gamma\delta} = \epsilon^{\mu\nu\lambda\sigma}\, T\left[\tilde{V}_{\alpha\mu}\tilde{V}_{\beta\nu}\tilde{V}_{\gamma\lambda}\tilde{V}_{\delta\sigma}\right] \tag{30}$$

where $T[\ldots]$ means a totally symmetric sum of products of the matrices. Then following our definition of covariant derivative, eqs. (25), (26), and (27), we see

$$D_\tau \tilde{V} = \left(\partial_\tau + i\tilde{A}_\tau \times\right)\tilde{V} - \tilde{\Gamma}^\beta_{0\tau} \circ \tilde{V}_{\beta 123} - \tilde{\Gamma}^\beta_{1\tau} \circ \tilde{V}_{0\beta 23} - \tilde{\Gamma}^\beta_{2\tau} \circ \tilde{V}_{01\beta 3} - \tilde{\Gamma}^\beta_{3\tau} \circ \tilde{V}_{012\beta} \tag{31}$$

which, taking account of the total antisymmetry of $\widetilde{V}_{\alpha\beta\gamma\delta}$, leads to

$$D_\tau \widetilde{V} = \left(\partial_\tau + i\widetilde{A}_\tau \times - \widetilde{\Gamma}_\tau \circ\right) \widetilde{V} \tag{32}$$

with $\widetilde{\Gamma}_\tau = \widetilde{\Gamma}^{\alpha}_{\alpha\tau}$.

The curvature tensor can be obtained from the commutator of covariant derivatives of a vector, $\widetilde{V}_\mu = V^a_\mu \widetilde{T}^a$,

$$\left(D_\sigma D_\lambda - D_\lambda D_\sigma\right) \widetilde{V}_\nu = -R^{\mu a}_{\nu\lambda\sigma} V^a_\mu \quad . \tag{33}$$

After some algebra we find

$$R^{\mu a}_{\nu\lambda\sigma} = -ig^\mu_\nu \widetilde{F}_{\sigma\lambda} \times \widetilde{T}^a + \widetilde{H}^\mu_{\nu\sigma\lambda} \circ \widetilde{T}^a - \widetilde{\Gamma}^\beta_{\nu\sigma} \circ \left(\widetilde{\Gamma}^\mu_{\beta\lambda} \circ \widetilde{T}^a\right) + \widetilde{\Gamma}^\beta_{\nu\lambda} \circ \left(\widetilde{\Gamma}^\mu_{\beta\sigma} \circ \widetilde{T}^a\right) \tag{34}$$

with

$$\widetilde{F}_{\sigma\lambda} = \partial_\sigma \widetilde{A}_\lambda - \partial_\lambda \widetilde{A}_\sigma + i\widetilde{A}_\sigma \times \widetilde{A}_\lambda \tag{35}$$

and

$$\widetilde{H}^\mu_{\nu\sigma\lambda} = \left(\partial_\sigma + i\widetilde{A}_\sigma \times\right) \widetilde{\Gamma}^\mu_{\nu\lambda} - \left(\partial_\lambda + i\widetilde{A}_\lambda \times\right) \widetilde{\Gamma}^\mu_{\nu\sigma} \quad . \tag{36}$$

The Ricci tensor is

$$R^a_{\nu\sigma} = R^{\beta a}_{\nu\beta\sigma} = -i\widetilde{F}_{\sigma\nu} \times \widetilde{T}^a + \widetilde{H}^\beta_{\nu\sigma\beta} \circ \widetilde{T}^a - \widetilde{\Gamma}^\beta_{\nu\sigma} \circ \left(\widetilde{\Gamma}_\beta \circ \widetilde{T}^a\right) + \widetilde{\Gamma}^\beta_{\nu\gamma} \circ \left(\widetilde{\Gamma}^\gamma_{\beta\sigma} \circ \widetilde{T}^a\right) . \tag{37}$$

Its antisymmetric part is

$$2A^a_{\nu\sigma} = R^a_{\nu\sigma} - R^a_{\sigma\nu} = -2i\widetilde{F}_{\sigma\nu} \times \widetilde{T}^a + \widetilde{H}_{\sigma\nu} \circ \widetilde{T}^a + \left(\widetilde{\Gamma}^\beta_{\nu\gamma} \times \widetilde{\Gamma}^\gamma_{\beta\sigma}\right) \times \widetilde{T}^a \tag{38}$$

with

$$\widetilde{H}_{\sigma\nu} = \left(\partial_\sigma + i\widetilde{A}_\sigma \times\right) \widetilde{\Gamma}_\nu - \left(\partial_\nu + i\widetilde{A}_\nu \times\right) \widetilde{\Gamma}_\sigma \quad . \tag{39}$$

Another nontrivial contraction of the curvature tensor is

$$S^a_{\lambda\sigma} = R^{\mu a}_{\mu\lambda\sigma} = -4i\widetilde{F}_{\sigma\lambda} \times \widetilde{T}^a + \widetilde{H}_{\sigma\lambda} \circ \widetilde{T}^a + \left(\widetilde{\Gamma}^\beta_{\alpha\lambda} \times \widetilde{\Gamma}^\alpha_{\beta\sigma}\right) \times \widetilde{T}^a \quad . \tag{40}$$

$\widetilde{\Gamma}_\mu$ appears in the antisymmetric contractions of the curvature tensor (eqs. (38) and (40)) through the second rank tensors $\widetilde{H}_{\sigma\nu}$. If we eliminate the nonvector part of $\widetilde{\Gamma}_\mu$ by defining

$$\widetilde{\Gamma}'_\mu = \widetilde{\Gamma}_\mu - \frac{1}{N}\widetilde{I}\,\mathrm{Tr}\,\widetilde{\Gamma}_\mu \tag{41}$$

then we find that this generalization of Weyl's model contains a second Yang-Mills field which transforms homogeneously under gauge transformations

$$\widetilde{\Gamma}'_\mu \rightarrow S^{-1}\widetilde{\Gamma}'_\mu S \quad , \tag{42}$$

and as a covariant vector under general coordinate transformations. Furthermore, one finds that its gauge covariant curl,

$$\widetilde{H}_{\sigma\lambda} = \left(\partial_\sigma + i\widetilde{A}_\sigma \times\right)\widetilde{\Gamma}'_\lambda - \left(\partial_\lambda + i\widetilde{A}_\lambda \times\right)\widetilde{\Gamma}'_\sigma \quad , \tag{43}$$

naturally arises in contractions of the curvature tensor. $\widetilde{H}_{\sigma\lambda}$ would appear to play the same role with respect to $\widetilde{\Gamma}'_\mu$ as $\widetilde{F}_{\mu\nu}$ plays with respect to $\widetilde{A}_\mu$. The set of quantities, $\widetilde{A}_\mu$, $\widetilde{\Gamma}'_\nu$, $\widetilde{F}_{\mu\nu}$, and $\widetilde{H}_{\mu\nu}$, with their attendant gauge group transformation properties are in complete analogy with non-Abelian gluon fields which have been used to construct a field theory of the strong interactions with quark confinement [5]. The geometrical origin of $\widetilde{\Gamma}'_\mu$ together with other considerations have led us to propose a unified theory of gravitation and the strong interaction which is described elsewhere [6].

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