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A NEW LEVINSON'S THEOREM*

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ABSTRACT

We investigate the form Levinson's theorem takes when the two-body scattering amplitude is not decomposed into partial waves. It is found that the theorem changes its structure in this case and is not merely the sum over angular momentum of the well-known partial wave results. The energy dependent quantity that replaces the partial wave phase shift turns out to be the trace of the two-body time delay operator. This new version of the theorem remains valid for scattering by nonspherically symmetric potentials.

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I. INTRODUCTION

We study an extension of Levinson's theorem for two particle scattering. This extension states the theorem as a moment property of the trace of the two-body time delay operator. In this form obtained here the theorem is valid for the entire—nonpartial wave decomposed—amplitude. The resultant form of the theorem found here is <u>not</u> what one would surmise on the basis of simply summing the well-known partial wave statements in terms of phase shifts. Our derivation will be rigorously carried out for the class of local potentials that belong to $L^1 \cap L^2$.

To begin with we list the known features of time delay in two-particle potential scattering which we must employ in this analysis. This outline is too brief to be a balanced introduction to the theory of time delay concepts in scattering. Such a general discussion is found in Ref. 1, which also gives a survey of the recent literature on this topic.

The scattering system studied here is characterized by an interacting Hamiltonian h and an asymptotic Hamiltonian h_0. In these two Hamiltonians the center-of-mass motion has been removed. If $\vec{x}$ is the vector separation of the two particles, then h and h_0 act on a Hilbert space $\mathscr{H}$ composed of square integrable functions of $\vec{x}$. On $\mathscr{H}$ one defines the Møller wave operator by the strong limit,

$$\Omega^{(\pm)} = s - \lim_{t \to \mp\infty} e^{iht} e^{-ih_0 t} \tag{I. 1}$$

where t is the real parameter denoting time. Each f in $\mathscr{H}$ may correspond to a possible incident wave packet. The symbol $\phi(t)$ will always represent the time dependent noninteracting wave packet associated with f. Likewise $\psi(t)$ will be the function that is the fully interacting wave packet evolving in time according to h.

These two functions are given by

$$\phi(t) = e^{-ih_0 t} f \ , \qquad \text{(I. 2)}$$

$$\psi(t) = e^{-iht}\, \Omega^{(\pm)} f \ . \qquad \text{(I. 3)}$$

The time delay of a scattering process is defined by the following construction. Let us describe a family of projection operators that is specified by the equations

$$\begin{aligned} P(R)\, g(\vec{x}) &= g(\vec{x}) \qquad && x \leq R \\ &= 0 && x > R \end{aligned} \qquad \text{(I. 4)}$$

where g is any function belonging to $\mathscr{H}$. Thus P(R) projects any function onto a sphere of radius R measured from the collision center at $\vec{x} = 0$. Given an incident wave packet f and a specific value of R the time delay is determined by the expression,

$$T(R, f) = \int_{-\infty}^{\infty} dt \left[(\psi(t), P(R)\, \psi(t)) - (\phi(t), P(R)\, \phi(t))\right] \qquad \text{(I. 5)}$$

The inner product is that of $\mathscr{H}$. The second member of the integrand gives the probability that at time t the wave packet $\phi(t)$ is inside the sphere P(R). The integral over t of this real quantity is just the total time $\phi(t)$ spends inside the sphere P(R). The same interpretation applies to the first inner product involving $\psi(t)$. Consequently T(R, f) is the difference of time the two waves reside in the sphere.

Consider now the description of the scattering problem in momentum space. The relative two-particle momentum will be the vector $\vec{p}$. The corresponding kinetic energy of relative motion will be $E = \vec{p}^{\,2}/2\mu$, where μ is the reduced mass of the two particles. The symbol $\hat{p}$ will denote the unit vector direction of $\vec{p}$. We introduce a Hilbert space $\mathscr{H}_{\wedge}$ of L^2 functions of $\hat{p}$—namely that space determined by the inner product,

$$(g,g')_{\wedge} = \int g(\hat{p})^* \, g'(\hat{p}) \, d\hat{p} \quad . \tag{I.6}$$

The theory of time delay allows one to construct a family of operators q(E,R) acting on $\mathscr{H}_{\wedge}$. This family has the property[2,3]

$$T(R,f) = \int_0^{\infty} dE \, \mu p \iint d\hat{p} \, d\hat{p}' \, f^*(p\hat{p}) \langle \hat{p} | q(E,R) | \hat{p}' \rangle f(p\hat{p}') \tag{I.7}$$

where $p = \sqrt{2\mu E}$. In expression (I.7) $\langle \hat{p} | q(E,R) | \hat{p}' \rangle$ is the kernel representation of the operator q(E,R). Furthermore, for well behaved potentials, the $R \to \infty$ limit of T(R,f) exists and is associated with an operator q(R), viz.

$$\lim_{R \to \infty} T(R,f) = \int_0^{\infty} dE \, \mu p \iint d\hat{p} \, d\hat{p}' \, f^*(p\hat{p}) \langle \hat{p} | q(E) | \hat{p}' \rangle \, f(p\hat{p}') \tag{I.8}$$

The operator q(E) is known[2] to be simply determined by the S-matrix. The full S-matrix that acts on $\mathscr{H}$ is defined by the product of wave operators:

$$S = \Omega^{(-)\dagger} \, \Omega^{(+)} \quad . \tag{I.9}$$

If one takes the momentum space matrix elements of Eq. (I.9) one is led to a natural definition of a reduced, energy dependent S-matrix, s(E), that acts on $\mathscr{H}_{\wedge}$. The operator s(E) is specified by its kernel $\langle \hat{p} | s(E) | \hat{p}' \rangle$, which is determined from S by the expression,

$$\langle \vec{p} \, | S | \vec{p}' \rangle = \frac{\delta(E-E')}{\mu p} \langle \hat{p} | s(E) | \hat{p}' \rangle \quad , \tag{I.10}$$

for $E = \vec{p}^{\,2}/2\mu$. The energy dependent delta function, of course, indicates the physical conservation of energy in the scattering process. In terms of s(E) the operator q(E) may be expressed[2,4] as,

$$q(E) = -i \, s^{\dagger}(E) \frac{d}{dE} s(E) \quad . \tag{I.11}$$

It is interesting to note here that structure of Eq. (I.11) is such that the unitarity of s(E) implies that q(E) must be hermitian.

The feature of time delay that is vital for our analysis is known as the spectral property. Let $r(z) = (h-z)^{-1}$ and $r_0(z) = (h_0-z)^{-1}$ be the resolvents of h and h_0 defined for complex energy z. Then the spectral property is the relation,

$$2\, \mathscr{I}m\ \mathrm{Tr}\left[r(E+i0) - r_0(E+i0)\right] = \mathrm{tr}\ q(E) \tag{I. 12}$$

In this equation Tr is the trace on $\mathscr{H}$ and tr is the trace on $\mathscr{H}_\wedge$. This relation has a simple physical interpretation. The right hand side is just the trace of the time delay operator $q(E)$ and is proportional to the total time delay experienced by an incident plane wave of energy E. The left hand side is the change of state density produced by the interaction v. In fact we shall require a somewhat more general version of Eq. (I. 12), specifically

$$\mathscr{I}m\ \mathrm{Tr}\left[r(E+i\eta) - r_0(E+i\eta)\right] = \sum_{i=1}^{N} \frac{\eta}{|E_i+E+i\eta|^2} + \frac{1}{2\pi}\int_0^\infty dE' \frac{\eta}{(E-E')^2+\eta^2}\ \mathrm{tr}\ q(E') \tag{I. 13}$$

The E_i appearing here are the negative of the eigenvalues of h. This equation is given explicitly in Ref. 5. The spectral property is readily obtained from Eq. (I. 13) by letting the imaginary parameter η go to zero. The advantage inherent in this version of the spectral property is that it allows one to estimate how rapidly $\mathscr{I}m\ \mathrm{Tr}\left[r(E+i\eta) - r_0(E+i\eta)\right]$ approaches its $\eta=0$ value.

Throughout this study we will consistently assume that the potential belongs to $L^1 \cap L^2$. This means that $v(\vec{x})$ is such that

$$\int d\vec{x}\ |v(\vec{x})| = B_1 < \infty\ , \qquad \left(\int d\vec{x}\ |v(\vec{x})|^2\right)^{1/2} = B_2 < \infty\ . \tag{A}$$

This class of potentials is broad enough to include most cases of physical interest. However both hard core potentials and Coulomb potentials are excluded by (A). We note that the L^1 restriction of (A) dictates that behavior of v for $|\vec{x}|$ very large must be like $\sim |\vec{x}|^{-3-\delta}$, where δ is an arbitrarily small positive number. The L^2 requirement of (A) implies that most severe local singularities can be $\sim |\vec{x}|^{-3/2+\delta}$.

The time delay formalism has been rigorously studied under assumption (A). In particular Kato[6] has proved that the wave operators $\Omega^{(\pm)}$ in Eq. (I. 1) exist when (A) holds. Jauch, Sinha and Misra[3] prove the existence of the limit given in Eq. (I.8). Equation (I. 13) is found in Ref. 5. This equation, which is central to discussion, may also be easily inferred from the results of Jauch, Sinha and Misra.[7] Another rigorous analysis of the time delay formalism above has been recently given by Martin[4] for slightly different assumptions on the potential.

One may question whether or not it is necessary to treat this problem in a rigorous fashion. For example, is not Levinson's theorem valid so long as the potential falls off more rapidly than the Coulomb force? Two observations indicate why a careful and detailed analysis is necessary. First, as indicated at the beginning of this section a simple sum of known partial wave results does not lead to the correct answer for the entire scattering problem. The form of the answer is sensitive to the order of integration and limiting processes, thus each change of order must be justified. A second observation emphasizes the need to specify precisely the behavior of the potential. Suppose one considers the following central potential

$$v_1(r) = \frac{\lambda_1}{r^2} \int_0^r dr' \, g(r')^2 + \frac{\lambda_2}{r^2} \;, \qquad \lambda_1 > 0 \tag{I. 14}$$

where λ_1 and λ_2 are real parameters and $g(r)$ is an arbitrary real function. In this case one can prove[8] that the momentum derivative of all partial wave phase shifts is positive for all k, so that

$$\int_0^\infty dk \, \frac{d}{dk} \, \delta_\ell(k) > 0 \;, \qquad \text{all } \ell \;. \tag{I. 15}$$

By way of contrast the Levinson's theorem for partial wave phase shifts states

$$\int_0^\infty dk \, \frac{d}{dk} \delta_\ell(k) = \delta_\ell(\infty) - \delta_\ell(0) = -\pi N_\ell \tag{I. 16}$$

where N_ℓ is the number of two-body bound states with angular momentum ℓ. Thus potentials of the form v_1 violate the theorem for every partial wave. Sufficient conditions for the existence of the partial wave form of Levinson's theorem are that the moments,

$$M_i = \int_0^\infty dr \, r^i \, |v(r)| \;, \qquad i=1,2 \tag{I. 17}$$

be finite.[9] By this criteria we see that potentials like v_1 fall off too slowly in r to lead to a reasonable phase shift behavior. Also in the extended case, the potential v_1 would be excluded by condition (A).

The proof we shall give of our extended Levinson's theorem is based on two elements. One is the spectral property Eq. (I. 13). The second is the analytic behavior of Tr $[r(z) - r_0(z)]$ in the complex z plane. Section II of this paper gives a rigorous proof of the various aspects of the analytic behavior we need. Section III combines this analytic behavior with Eq. (I. 13) to complete the proof. In Section IV we give a general discussion of these results and also describe a second approach to the problem that is based on the asymptotic completeness of the wave operators. A quick, albeit nonrigorous, understanding of our results may be obtained by just reading Section IV.

II. ANALYTIC PROPERTIES OF Tr $[r(z) - r_0(z)]$

This section is devoted to the study of Tr $[r(z) - r_0(z)]$. We always assume condition (A) is obeyed by the potential v. One very useful consequence of (A) is that it implies that our potential v is in the Rollnik class that Simon[10] has extensively studied, viz.

$$\iint d\vec{x}\, d\vec{y}\, \frac{|v(\vec{x})\| v(\vec{y})|}{|\vec{x}-\vec{y}|^2} = B_r < \infty \quad . \tag{II. 1}$$

Our analysis will make extensive use of the well-known operators $r_0(z)$, $V r_0(z)$ and $A(z)$. These operators all act on $\mathscr{H}$ and depend parametrically on z. They are conveniently defined by their kernel representations in coordinate space:

$$<\vec{x}\,|r_0(z)|\,\vec{y}> = \frac{\mu}{2\pi}\, \frac{e^{ik|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} \quad , \tag{II. 2}$$

$$<\vec{x}\,|V r_0(z)|\,\vec{y}> = \frac{\mu}{2\pi}\, v(\vec{x})\, \frac{e^{ik|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} \quad , \tag{II. 3}$$

$$<\vec{x}\,|A(z)|\,\vec{y}> = \frac{\mu}{2\pi}\, v^{1/2}(\vec{x})\, \frac{e^{ik|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}\, |v(\vec{x})|^{1/2} \quad . \tag{II. 4}$$

Where $k=\sqrt{2\mu z}$, and $v^{1/2}(\vec{x}) = |v(\vec{x})|^{1/2}$ sgn $v(\vec{x})$, and $A(z) = V^{1/2} r_0(z)\, |V|^{1/2}$. The set of points in the z plane a distance δ or greater from positive real axis we will denote by Π_δ. The symbol Π_0 will denote the cut z-plane obtained by letting $\delta \to 0$. For $z \in \Pi_\delta$ or Π_0 then k clearly belongs to the upper-half complex k plane.

We shall use three different norms to describe operators on $\mathscr{H}$. First, the usual operator norm will be represented by the symbol $\|\cdot\|$. Second, we define the Schmidt norm of an operator A on $\mathscr{H}$ by

$$\|A\|_2 = \left(\iint d\vec{x}\, d\vec{y}\, |A(\vec{x},\vec{y})|^2\right)^{1/2}$$

where $A(\vec{x},\vec{y})$ is the kernel generated by A. The class of all operators on $\mathscr{H}$ with finite Schmidt norm is called the Schmidt class. This class is denoted by $\mathscr{B}_2$. Our third norm is the trace norm defined by

$$\|A\|_1 = \sum_i^{\infty} (\phi_i, |A| \phi_i) \quad , \tag{II.6}$$

where $A = (A^{\dagger}A)^{1/2}$. When A has finite $\|A\|_1$ it belongs to the trace class of operators on $\mathscr{H}$. The trace class is labeled $\mathscr{B}_1$. When $A \in \mathscr{B}_1$ then operator has a well-defined trace given by the sum,

$$\mathrm{Tr}\, A = \sum_i^{\infty} (\phi_i, A\, \phi_i) \quad . \tag{II.7}$$

Of course this sum is independent of the basis set $\{\phi_i\}$. Our analysis will frequently use the following general properties of the trace and the Schmidt operator.

(i) $A \in \mathscr{B}_1$ if and only if it can be written as the product $A=BC$ where $B \in \mathscr{B}_2$ and $C \in \mathscr{B}_2$. Furthermore $\|A\|_1 \leq \|B\|_2 \|C\|_2$.

(ii) If $B \in \mathscr{B}_2$ then $B^{\dagger} \in \mathscr{B}_2$.

(iii) If $B \in \mathscr{B}_2$, and $C \in \mathscr{B}_2$ then $\mathrm{Tr}\, BC = \mathrm{Tr}\, CB$.

(iv) If $B \in \mathscr{B}_1$ and A has finite norm, $\|A\|$, then $BA \in \mathscr{B}_1$ and $AB \in \mathscr{B}_1$ and $\|BA\|_1 \leq \|A\| \; \|B\|_1$.

(v) If $A \in \mathscr{B}_1$ then $|\mathrm{Tr}\, A| \leq \sum_i^{\infty} |(\phi_i, A\, \phi_i)| \leq \|A\|_1$.

(vi) If B and C are Schmidt class then Tr BC has the representation

$$\mathrm{Tr}\, BC = \iint d\vec{x}\; d\vec{y}\; B(\vec{y},\vec{x})\; C(\vec{x},\vec{y})$$

where $B(\vec{y},\vec{x})$ and $C(\vec{x},\vec{y})$ are the L^2 kernels generated by the operators B and C.

Shatten[11] gives proofs of all of these statements.

For later convenience let us collect here some well known estimates for norms of the operators occurring in this problem. We shall show that the operators $V^{1/2}r_0(z)$, $|V|^{1/2}r_0(z)$, and, $r_0(z)V$ are Schmidt class for all $z \in \Pi_\delta$. Consider the first operator in the list above. If we employ the integral form of the Schmidt norm to compute $\|V^{1/2}r_0(z)\|_2$ we have,

$$\|V^{1/2}r_0(z)\|_2 = \frac{\mu}{2\pi}\left(\iint d\vec{x}\, d\vec{y}\; \frac{|v(\vec{x})|e^{-2\mathscr{I}m\, k|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|^2}\right)^2$$

$$= \mu\left(\frac{B_1}{2\pi\mathscr{I}m\, k}\right)^{1/2} \tag{II.8}$$

The same expression holds for the norm $\|\,|V|^{1/2}r_0(z)\|_2$. Similar considerations show that

$$\|V r_0(z)\|_2 = \mu B_2\left(\frac{1}{2\pi\mathscr{I}m\, k}\right) \tag{II.9}$$

Now let us examine the operator A(z) given in Eq. (II.4). This operator is Schmidt class in the entire z-plane Π_0,

$$\|A(z)\|_2 = \frac{\mu}{2\pi}\left(\iint d\vec{x}\, d\vec{y}\; \frac{|v(\vec{x})\|v(\vec{y})|}{|\vec{x}-\vec{y}|^2}\, e^{-2\mathscr{I}m\, k|\vec{x}-\vec{y}|}\right)^{1/2}$$

$$\leq \frac{\mu}{2\pi}\, B_r^{1/2} \tag{II.10}$$

where B_r is the constant entering the Rollnik bound on v. Another useful bound pertains to $A(z)^2$. One may show, using the Riemann-Lebesque lemma that

$$\lim_{|\mathrm{Re}\, k|\to\infty} \|A(z)^2\|_2 = 0 \quad . \tag{II.11}$$

The convergence is uniform in $\mathscr{I}m\, k \geq 0$. Equation (II.11) means that there exists a finite k_r such that for all $|\mathrm{Re}\, k| > k_r$ then $\|A(z)^2\|_2 < \frac{1}{2}$. The number k_r depends only on v. We will not write out the proof of Eq. (II.11) and estimate for $\|A(z)^2\|_2$. Theorem I.23 of Simon's book is very nearly result (II.11). The difference is that Simon requires k to be real. It is a simple modification of Simon's proof to extend it to complex k in the upper half plane and to show the convergence is uniform in $\mathscr{Im}\, k$.

Lemma 1. Let the potential v satisfy (A). For all positive integers n, the operators $r_0(z)[V r_0(z)]^n$ are trace class for $z \in \Pi_\delta$. The function $\mathrm{Tr}\, r_0(z)[V r_0(z)]^n$ is an analytic function of z in the Π_δ domain. Furthermore the order of the trace operation and $\frac{d}{dz}$ may be freely interchanged in Π_δ.

Proof: We first establish $r_0[V r_0]^n$ is trace class. Consider n=1. This operator may be written as the product of $r_0 |V|^{1/2}$ and $V^{1/2} r_0$, each of which in Schmidt class in Π_δ. Thus employing property (i) of the trace gives,

$$\| r_0 V r_0 \|_1 \leq \| r_0 |V|^{1/2} \|_2^2 \quad . \qquad (\text{II}.12)$$

Estimate (II.8) tells us the right hand side is finite. For n>1 we may write

$$\| r_0 [V r_0]^n \|_1 \leq \| r_0 V r_0 \|_1 \| V r_0 \|_2^{n-1} \quad . \qquad (\text{II}.13)$$

In obtaining (II.13) we have used trace property (iv) together with the general inequality $\|A\| \leq \|A\|_2$. Estimates (II.8) and (II.9) then imply that the right hand side of (II.13) is infinite.

Next consider the analyticity of $\mathrm{Tr}\left(r_0(z)[V r_0(z)]^n\right)$. Set n=1. As noted above the operator is the product of two Schmidt operators $r_0 |V|^{1/2}$ and $V^{1/2} r_0$ in domain Π_δ. Invoking property (vi) of the trace for the operator $r_0 V r_0$ gives us

$$\mathrm{Tr}\,(r_0 V r_0) = \left(\frac{\mu}{2\pi}\right)^2 \iint d\vec{x}\, d\vec{y}\, \frac{v(\vec{x})}{|\vec{y}-\vec{x}|^2}\, e^{2ik|\vec{y}-\vec{x}|} \qquad (\text{II}.14)$$

For values of k restricted by the condition $\mathscr{Im}\, k \geq \delta' \geq 0$ the expression

$$\frac{|v(\vec{x})|\, e^{-2\delta' |\vec{y}-\vec{x}|}}{|\vec{y}-\vec{x}|^2}$$

bounds the integrand above uniformly in k. This bound is absolutely integrable with respect to $(\vec{x}, \vec{y})$. Thus the integral in Eq. (II. 14) defines an analytic function of k and thus z. This argument may be extended to show $\mathrm{Tr}\, r_0[V r_0]^n$ is analytic for all n and $z \in \Pi_\delta$.

Finally let us examine the differential properties of $\mathrm{Tr}[V r_0(z)]^n$. The trace diverges for n=1, but is well defined for $n \geq 2$. Consider the case n=2. If we examine the integral representation of $\mathrm{Tr}[V r_0(z)]^2$ the Rollnik condition (II. 1) guarantees that the integral is uniformly convergent in Π_δ. Thus we can differentiate under the integral to obtain

$$\frac{d}{dz} \mathrm{Tr}\,[V r_0(z)]^2 = \left(\frac{\mu}{2\pi}\right)^2 \iint d\vec{x}\, d\vec{y}\, \frac{2i\mu\, v(\vec{x}) v(\vec{y})}{k\, |\vec{y}-\vec{x}|}\, e^{2ik\, |\vec{y}-\vec{x}|} \quad . \tag{II. 15}$$

One then observes that the Hilbert identity for $r_0(z)$,

$$r_0(z_1) - r_0(z_2) = (z_1 - z_2)\, r_0(z_1)\, r_0(z_2) \tag{II. 16}$$

implies the operator relation,

$$\frac{dr_0(z)}{dz} = r_0(z)^2 \quad . \tag{II. 17}$$

The kernel form of this last identity is

$$\frac{i\mu}{k}\, e^{ik\, |\vec{x}-\vec{y}|} = \frac{\mu}{2\pi} \int d\vec{s}\, \frac{e^{ik(\,|\vec{x}-\vec{s}| + |\vec{s}-\vec{y}|)}}{|\vec{x}-\vec{s}|\; |\vec{s}-\vec{y}|} \tag{II. 18}$$

Inserting Eq. (II. 18) into the right hand side to (II. 15) gives

$$\begin{aligned} \frac{d}{dz} \mathrm{Tr}[V r_0(z)]^2 &= 2\, \mathrm{Tr}\left(r_0(z) [V r_0(z)]^2\right) \\ &= \mathrm{Tr}\left(\frac{d}{dz} [V r_0(z)]^2\right) \end{aligned} \tag{II. 19}$$

These arguments extend to the n>2 cases. There Eq. (II. 19) becomes

$$\frac{d}{dz} \mathrm{Tr}\,[V r_0(z)]^n = n\, \mathrm{Tr}\left(r_0(z) [V r_0(z)]^n\right) \tag{II. 20}$$

This completes the demonstration of lemma 1.

Lemma 2. There exist finite k_r and k_i such that for all $|z| > \left(k_r^2 + k_i^2\right)/2\mu$ the Born series expansion

$$\mathrm{Tr}\left[r(z) - r_0(z)\right] = \sum_{n=1}^{\infty} (-1)^n \,\mathrm{Tr}\left(r_0(z)\left[V r_0(z)\right]^n\right) \tag{II.21}$$

is valid. The series is uniformly convergent in z.

Proof: As usual set $z = k^2/2\mu$ and choose z so that it has values $\mathscr{Im}\, k > k_i = \mu B_2/2\pi$. For z so restricted operator Born expansion

$$r(z) - r_0(z) = \sum_{n=1}^{\infty} (-1)^n \, r_0(z)\left[V r_0(z)\right]^n \tag{II.22}$$

is valid. It is easy to see that series (II.22) is convergent in operator norm for $\mathscr{Im}\, k > k_i$. A given term in this series has norm,

$$\| r_0(z)\left[V r_0(z)\right]^n \| \le \| r_0(z) \| \, \| V r_0(z) \|^n \le \frac{2\mu}{k_i^2} \| V r_0(z) \|_2^n \tag{II.23}$$

For the restricted values of k, $\| V r_0(z) \|_2$ is less than one. The sum of terms (II.23) with respect to n then converges absolutely.

Since we know $r(z) - r_0(z)$ is trace class, we can take the trace of Eq. (II.22) to obtain

$$\mathrm{Tr}\left[r(z) - r_0(z)\right] = \mathrm{Tr}\left\{\sum_{n=1}^{\infty} (-1)^n r_0(z)\left[V r_0(z)\right]^n\right\} \tag{II.24}$$

for $\mathscr{Im}\, k > k_i$. The series on the right of Eq. (II.24) suggests we consider the related series

$$\sum_{n=1}^{\infty} (-1)^n \,\mathrm{Tr}\left(r_0(z)\left[V r_0(z)\right]^n\right) . \tag{II.25}$$

Introducing the definition of the trace into this expression gives the double sum

$$\sum_{n}^{\infty} \sum_{i}^{\infty} (-1)^n \left(\phi_i, r_0(z)\left[V r_0(z)\right]^n \phi_i\right) \tag{II.26}$$

It is easy to demonstrate that this double series is absolutely convergent. Employing the general trace identity (v) we have

$$\sum_{i}^{\infty}\left|\left(\phi_i, r_0(z)[V r_0(z)]^n \phi_i\right)\right| \leq \| r_0(z)[V r_0(z)]^n \|_1$$

$$\leq \| r_0(z) \| \; \| V r_0(z) \|_2^n \qquad \text{(II. 27)}$$

As in Eq. (II. 25), when $\mathscr{Im}\, k > k_i$ the sum over n of the terms on the right of Eq. (II. 27) converge uniformly in k. This shows that the double series in expression (II. 26) is absolutely convergent. Thus the order of summation may be changed. And so, Eq. (II. 24) may be written in the form given by Eq. (II. 21).

Let us consider the validity of (II. 21) in a different region of z. Suppose $|\mathrm{Re}\, k| > k_r$. The trace norm appearing in Eq. (II. 27) may be estimated by,

$$\| r_0(z)[V r_0(z)]^n \|_1 = \| r_0(z)\, |V|^{1/2} A(z)^{n-1} V^{1/2} r_0(z) \|_1$$

$$\leq \| r_0(z)\, |V|^{1/2} \|_2^2 \; \| A(z)^2 \|_2^{(n-1)/2} \qquad n = \text{odd}\, , \qquad \text{(II. 28)}$$

$$\leq \| r_0(z)\, |V|^{1/2} \|_2^2 \; \| A(z) \|_2 \; \| A(z)^2 \|_2^{(n-2)/2} \qquad n = \text{even}\, .$$

Bounds given in Eqs. (II. 9), (II. 11) and (II. 12) show that the sum over n of $\| r_0(z)[V r_0(z)]^n \|_1$ converge. Again, the double sum in Eq. (II. 26) is absolutely convergent and formula (II. 21) is valid for all $|\mathrm{Re}\, k| > k_r$. In fact the domain specified by $|z| \geq \left(k_i^2 + k_2^2\right)/2\mu$ lies in the union of $|\mathrm{Re}\, k| > k_r$ and $|\mathscr{Im}\, k| > k_i$. So lemma 2 is proved.

<u>Lemma 3</u>. For all integers $n \geq 2$, $\mathrm{Tr}[V r_0(z)]^n$ satisfies,

$$\lim_{|\mathrm{Re}\, k| \to \infty} \mathrm{Tr}[V r_0(z)]^n = 0 \qquad \text{(II. 29)}$$

for all $\mathscr{Im}\, k \geq 0$.

Proof: We note first that for $n \geq 2$

$$\mathrm{Tr}[V\, r_0(z)]^n = \mathrm{Tr}\, |V|^{1/2}\left(V^{1/2} r_0(z) |V|^{1/2}\right)^{n-1} V^{1/2} r_0(z) \tag{II. 30}$$

For $z \in \Pi_\delta$ the operators $A(z)$, $V^{1/2} r_0(z)$ are Schmidt class and $|V|^{1/2}$ is bounded, thus we may use (iii) to obtain

$$\mathrm{Tr}[V\, r_0(z)]^n = \mathrm{Tr}\; A(z)^n \tag{II. 31}$$

For $n \geq 4$ we have the bounds,

$$\begin{aligned} |\mathrm{Tr}[V\, r_0(z)]^n| &\leq \| A^2(z) \|_2^{n/2}\,, \qquad && n = \text{even} \\ &\leq \| A(z) \|_2 \, \| A^2(z) \|_2^{(n-1)/2}\,, && n = \text{odd} \end{aligned} \tag{II. 32}$$

Applying result Eq. (II. 11) gives the statement (II. 29) in the lemma.

There remains only the cases n=2, 3 to prove. Consider n=2. Using the integral representation of the trace,

$$\mathrm{Tr}[V\, r_0(z)]^2 = \left(\frac{\mu}{2\pi}\right)^2 \int d\vec{x}\, d\vec{y}\; \frac{v(\vec{x})\, v(\vec{y})\; e^{2ik|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|^2} \tag{II. 33}$$

Because of the condition (II. 1) the non-oscillatory part of the integrand

$$v(\vec{x})\, v(\vec{y})\, |\vec{x}-\vec{y}|^{-1}\, e^{-2\mathscr{Im}\, k\, |\vec{x}-\vec{y}|}$$

is L^1 over $(\vec{x}, \vec{y})$. Thus we can apply the Riemann-Lebesque lemma to conclude that $\mathrm{Tr}[V\, r_0(z)]^2$ vanishes as $|\mathrm{Re}\; k| \to \infty$. A similar argument works for $\mathrm{Tr}[V\, r_0(z)]^3$.

Lemma 4. For all $z \in \Pi_0$ the value of $\mathrm{Tr}\,(r_0(z)\, V\, r_0(z))$ is given by

$$\mathrm{Tr}\,(r_0(z)\, V\, r_0(z)) = \frac{i}{\pi}\left(\frac{\mu}{2}\right)^{3/2} \frac{1}{\sqrt{z}} \int d\vec{x}\; v(\vec{x}) \tag{II. 34}$$

<u>Proof:</u> For $z \in \Pi_\delta$, $r_0(z)\, V\, r_0(z)$ is the product of two Schmidt operators, so by trace property (vi)

$$\mathrm{Tr}\,(r_0(z)\, V\, r_0(z)) = \left(\frac{\mu}{2\pi}\right)^2 \iint d\vec{y}\, d\vec{x}\, \frac{v(\vec{x})\, e^{2ik|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}\,|^2} \tag{II. 35}$$

for all $\mathscr{Im}\, k > 0$ the double integral

$$\iint d\vec{y}\, d\vec{x}\, \frac{|v(\vec{x})|\, e^{-2\mathscr{Im}\, k|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|^2}$$

exists since $v \in L^1$. Thus, employing the Fubini theorem on interchange of integration, we can write (II. 35) in the form of an iterated integral

$$\mathrm{Tr}\,(r_0(z)\, V\, r_0(z)) = \left(\frac{\mu}{2\pi}\right)^2 \int d\vec{\eta}\, \frac{e^{2ik|\vec{\eta}|}}{|\vec{\eta}|^2} \int d\vec{x}\, v(\vec{x}) \quad , \tag{II. 36}$$

where we have set $\vec{\eta} = \vec{x} - \vec{y}$. The integral is trivial and gives Eq. (II. 34). So far the equality (II. 34) is established for $z \in \Pi_\delta$. However, the right hand side of (II. 34) has Π_0 as its natural domain of analyticity. Thus (II. 34) represents the analytic extension of $\mathrm{Tr}\,(r_0(z)\, V\, r_0(z))$ to the domain Π_0.

III. LEVINSON'S THEOREM

In this section we combine the analytic properties of $\mathrm{Tr}\left[r(z) - r_0(z)\right]$ established in the previous section with known features of time delay outlined in the introduction to complete our derivation of Levinson's theorem. Our proof will require one additional technical assumption about time delay. We assume the existence of the following integral

$$\int_0^\infty dE \left|\mathrm{tr}\; q(E) + \frac{2}{\pi}\left(\frac{\mu}{2}\right)^{3/2} \frac{\tilde{v}}{\sqrt{E}}\right| \tag{B}$$

where $\tilde{v}$ is defined by

$$\tilde{v} = \int d\vec{x}\; v(\vec{x}) \;, \quad |\tilde{v}| < B_1 \quad . \tag{III. 1}$$

Ideally assumption (B) should be verified directly from the potential property (A). But it would take us far afield to establish (B) in this manner. There are strong physical arguments for believing (B), that will be discussed in the next section. To our knowledge (B) is not established anywhere in the literature on time delay.

Consider the function $Q(z)$ defined by

$$Q(z) = \mathrm{Tr}\left[r(z) - r_0(z) + r_0(z)\, V\, r_0(z)\right] \tag{III. 2}$$

We have established that this function is analytic in Π_δ. Bound states of the Hamiltonian H appear as simple poles of the resolvent $r(z)$, with residues that are projection operators onto the bound state eigenfunction space. Physically interesting potentials will always have negative bound state energies. So our formalism will always imply this situation. Only a small change in the notation is required if in fact positive energy eigenfunctions exist.

Suppose z_0 is some point in Π_δ and C_0 some small circular contour about z_0. Then the Cauchy-Coursat theorem tells one that the integral,

$$\oint_{C_0} dz\; Q(z) = 0 \quad . \tag{III. 3}$$

Our version of Levinson's theorem is based on this identity. We open up the contour C_0 as indicated in Fig. 1. Now C_0 may be replaced by the contour segments C_Γ, C_δ and C_i. The contours C_i which are P in number encircle the P distinct eigenvalues of the Hamiltonian h. Path C_δ is a symmetric about the real axis, always a distance δ away from the positive real axis and ending where the real value of z is equal to Γ. The curve C_Γ is a circle centered about the z-plane origin having radius equal to $\sqrt{\Gamma^2+\delta^2}$.

Because of the behavior of the exact resolvent r(z) in the neighborhood of the eigenfunctions of h, one has

$$\sum_{i=1}^{P} \oint_{C_i} dz\ Q(z) = 2\pi i N \quad , \tag{III.4}$$

where N is the total number of bound states of h counting degeneracy. Thus the integral (III.3) becomes

$$\int_{C_\Gamma + C_\delta} dz\ Q(z) = -2\pi i N \quad . \tag{III.5}$$

We will now evaluate the double limit

$$\lim_{\Gamma\to\infty} \lim_{\delta\to 0} \int_{C_\Gamma + C_\delta} dz\ Q(z) = -2\pi i N \quad . \tag{III.6}$$

Consider the C_δ integral first. It may be expanded as

$$\int_{C_\delta} dz\ Q(z) = 2i \int_0^\Gamma d\lambda\ \mathscr{Im}\, Q(\lambda+i\delta) + i\delta \int_{3\pi/2}^{\pi/2} d\theta\ e^{i\theta}\, Q(\delta e^{i\theta}) \ . \tag{III.7}$$

If the exact resolvent has a normalized bound state of zero energy with degeneracy n, then the exact resolvent has a simple pole at z=0 and second integral on the right becomes $-\pi i n$ in the $\delta\to 0$ limit. On the other hand, when there are no zero energy eigenstates, then this integral gives zero in the $\delta\to 0$ limit.

Let us study the first integral on the right hand side of (III.7). We shall prove

Lemma 5. For potentials such that (A) and (B) are valid then

$$\lim_{\Gamma\to\infty}\ \lim_{\delta\to 0}\ 2\int_0^{\Gamma} d\lambda\, \mathcal{Im}\, Q(\lambda+i\delta) = \int_0^{\infty} dE\left[\mathrm{tr}\ q(E) + \frac{2}{\pi}\left(\frac{\mu}{2}\right)^{3/2} \frac{\tilde{v}}{\sqrt{E}}\right] . \tag{III.8}$$

Proof: Define $D_1(\Gamma,\delta)$ and $D_2(\Gamma,\delta)$ by the expressions

$$D_1(\Gamma,\delta) = \frac{1}{\pi}\int_0^{\Gamma} d\lambda \int_0^{\infty} dE\ \frac{\delta}{(\lambda-E)^2+\delta^2}\left[\mathrm{tr}\ q(E) + \frac{2}{\pi}\left(\frac{\mu}{2}\right)^{3/2} \frac{\tilde{v}}{\sqrt{E}}\right] . \tag{III.9}$$

$$D_2(\Gamma,\delta) = c_0\int_0^{\Gamma} d\lambda \int_0^{\infty} dE\ \frac{1}{\pi\sqrt{E}}\ \frac{\delta}{(\lambda-E)^2+\delta^2} - c_0\int_0^{\Gamma} d\lambda\ \mathrm{Re}\ \frac{1}{\sqrt{\lambda+i\delta}} , \tag{III.10}$$

where $c_0 = \frac{2}{\pi}\left(\frac{\mu}{2}\right)^{3/2}\tilde{v}$. Equations (I.13), (II.34) together with (III.2) give

$$2\int_0^{\Gamma} d\lambda\ \mathcal{Im}\, Q(\lambda+i\delta) = D_1(\Gamma,\delta)+D_2(\Gamma,\delta) + \int_0^{\Gamma} d\lambda \left\{\sum_{i=1}^{N} \frac{2\delta}{|E_i+\lambda+i\delta|^2}\right\} . \tag{III.11}$$

Since E_i are the magnitudes of the negative energy eigenvalues and thus positive it is obvious that the last integral vanishes when the double limit is taken. So we need only consider the contribution from D_1 and D_2. Consider D_1. Set

$$g(E) = \mathrm{tr}\ q(E) + \frac{2}{\pi}\left(\frac{\mu}{2}\right)^{3/2} \frac{\tilde{v}}{\sqrt{E}} \tag{III.12}$$

and D_1 is

$$D_1(\Gamma,\delta) = \int_0^{\Gamma} d\lambda \int_0^{\infty} dE\ \frac{\delta\, g(E)}{\pi\left[(\lambda-E)^2+\delta^2\right]} \tag{III.13}$$

Since (B) states that $h(E)$ is L^1 and that $\left[(\lambda-E)^2+\delta^2\right]^{-1}$ is L^1 over λ for all $\delta>0$ the Fubini theorem allows us to change the order of integrations,

$$D_1(\Gamma,\delta) = \int_0^{\infty} dE \int_0^{\Gamma} d\lambda\ \frac{\delta\, g(E)}{\pi\left[(\lambda-E)^2+\delta^2\right]} . \tag{III.14}$$

The integral over $d\lambda$ is elementary and gives us

$$D_1(\Gamma,\delta) = \int_0^\infty dE\, \frac{g(E)}{\pi} \left(\tan^{-1} \frac{\Gamma - E}{\delta} + \tan^{-1} \frac{E}{\delta}\right) . \qquad \text{(III. 15)}$$

Now the E-dependent integrand is L^1 for all $\delta \geq 0$ and uniformly bounded by $|g(E)|$. The Lebesque dominated convergence theorem permits us to pass the $\delta \to 0$ limit through the integral to obtain,

$$D_1(\Gamma, 0) = \int_0^\infty dE\, \frac{g(E)}{\pi} \lim_{\delta \to 0} \left(\tan^{-1} \frac{\Gamma - E}{\delta} + \tan^{-1} \frac{E}{\delta}\right) . \qquad \text{(III. 16)}$$

Using

$$\begin{aligned} &\lim_{\delta \to 0} \tan^{-1} \frac{E}{\delta} = \frac{\pi}{2} , && \text{all } E > 0 \\ &\lim_{\delta \to 0} \tan^{-1} \left(\frac{\Gamma - E}{\delta}\right) = \begin{cases} \frac{\pi}{2} & \text{all } E < \Gamma \\ -\frac{\pi}{2} & \text{all } E > \Gamma \end{cases} \end{aligned} \qquad \text{(III. 17)}$$

we see that $\tan^{-1}$ functions give us a step function that becomes zero when $E > \Gamma$. Thus

$$D_1(\Gamma, 0) = \int_0^\Gamma dE\, g(E) \qquad \text{(III. 18)}$$

and

$$\lim_{\Gamma \to \infty} \lim_{\delta \to 0} D_1(\Gamma,\delta) = \int_0^\infty dE\, g(E) \quad . \qquad \text{(III. 19)}$$

A parallel analysis leads one to conclude

$$\lim_{\Gamma \to \infty} \lim_{\delta \to 0} D_2(\Gamma,\delta) = 0 \qquad \text{(III. 20)}$$

Thus lemma 5 is proved.

The one remaining integral in relation (III.5) that we have not yet studied is the C_Γ term. For this integral we have the result:

Lemma 6. For potentials satisfying (A), then

$$\lim_{\Gamma\to\infty} \int_{C_\Gamma} dz\, Q(z) = 0 \qquad \text{(III.21)}$$

Proof: Choose $\Gamma > \left(k_r^2 + k_i^2\right)/2\mu$. Lemma 2 states that the Born series expansion of Q(z) is uniformly convergent for $z \in C_\Gamma$. Using (II.21) to expand Q(z) we can write our integral as,

$$\int_{C_\Gamma} dz\, Q(z) = \int_{C_\Gamma} dz \sum_{n=2}^{\infty} (-1)^n \, \mathrm{Tr}\left(r_0(z)\left[V r_0(z)\right]^n\right)$$

$$= \sum_{n=2}^{\infty} (-1)^n \int_{C_\Gamma} dz\, \mathrm{Tr}\left(r_0(z)\left[V r_0(z)\right]^n\right) \quad . \qquad \text{(III.22)}$$

Equation (II.20) allows us to transform the integrand into an exact differential.

$$\int_{C_\Gamma} dz\, Q(z) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \int_{C_\Gamma} dz \frac{d}{dz} \mathrm{Tr}\left[V r_0(z)\right]^n$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \left\{ \mathrm{Tr}\left[V r_0(\Gamma + i0)\right]^n - \mathrm{Tr}\left[V r_0(\Gamma - i0)\right]^n \right\} \qquad \text{(III.23)}$$

Estimates (II.32) imply that this series is uniformly convergent in Γ. Thus the $\Gamma\to\infty$ limit may be passed through the sum. Lemma 3 shows us each term in Eq. (III.8) vanishes in the $\Gamma\to\infty$ limit. Thus Eq. (III.21) is demonstrated.

Combining the conclusions for both lemma 5 and lemma 6 together with Eq. (III.6) gives us:

Theorem. For potentials satisfying condition (A) and the trace of the time delay satisfying (B) then the following relation holds,

$$\int_0^\infty dE \left[\operatorname{tr} q(E) + \frac{2}{\pi}\left(\frac{\mu}{2}\right)^{3/2} \frac{\tilde{v}}{\sqrt{E}} \right] = -2\pi N \quad . \tag{III.24}$$

Normally, N is the total number of negative energy eigenfunctions of Hamiltonian h. If there is a zero energy bound state with degeneracy n then N is replaced by $N + \frac{1}{2} n$. If there are both positive and negative energy eigenvalues then N is the sum of both weighted by the degeneracy of each eigenvalue.

IV. ASYMPTOTIC COMPLETENESS AND LEVINSON'S THEOREM

In this section we are concerned with two aspects of our Levinson's theorem. First, we establish how one may derive the result starting from the completeness of the scattering states. Secondly, we give a physical interpretation and explanation of our result. Our aim in this section is to provide some insight into the result obtained above, rather than to supply additional rigorous proofs. Thus, we will use nonrigorous arguments which we believe convincing, even though these arguments tend to lose sight of the exact conditions on the potential for which the analysis is valid.

The derivation given above of Levinson's theorem is based on the spectral property of time delay combined with the analytic features of $\mathrm{Tr}\left[\mathrm{r}(z) - \mathrm{r}_0(z)\right]$. However, in the literature there exists another method of derivation. Jauch[12] established that the usual partial wave form of Levinson's theorem can be obtained from asymptotic completeness and certain properties of the wave operator, $\Omega^{(+)}$.

We adopt Jauch's argument to the case at hand—namely the full amplitude. The mathematical statement of asymptotic completeness is:

$$\Omega^{(+)\dagger}\,\Omega^{(+)} = \mathrm{I} \quad , \tag{IV.1}$$

$$\Omega^{(+)}\,\Omega^{(+)\dagger} = \mathrm{I} - \mathrm{P} \quad . \tag{IV.2}$$

Here I is the identity operator in $\mathscr{H}$ and P is the projection operator onto the subspace spanned by all eigenfunctions of h. We note that $\mathrm{Tr}\,\mathrm{P} = \mathrm{N}$.

The wave operator possesses a well-known representation in terms of the t-matrix. Suppose t(z) is the operator satisfying the Lippmann-Schwinger equation,

$$t(z) = V - V\,r_0(z)\,t(z) \quad . \tag{IV.3}$$

The wave operator may be expanded[12] about the identity,

$$\Omega^{(+)} = I - K \tag{IV.4}$$

where K is determined by the generalized function

$$<\vec{p}\,|K\,|\vec{p}\,'> = \frac{<\vec{p}\,|t(p'^2/2\mu + i0)\,|\vec{p}\,'>}{\vec{p}^{\,2}/2\mu - \vec{p}\,'^2/2\mu - i0} \tag{IV.5}$$

If one forms the commutator $\left[K^\dagger, K\right]$, then Eqs. (IV.1) and (IV.2) imply,

$$\left[K^\dagger, K\right] = P \quad . \tag{IV.6}$$

Levinson's theorem is obtained by taking the trace of (IV.6).

One aspect of this approach requires care. The kernel representation of K is a generalized function. As a consequence the trace needs to be computed through a limiting process. It is convenient to introduce a two parameter family of operators on $\mathscr{H}_\wedge$, $\tau(E, E')$, defined by

$$<\hat{p}\,|\tau(E, E')\,|\hat{p}'> = j(E) < E\hat{p}\,|t(E+i0)\,|E'\hat{p}'> j(E') \quad , \tag{IV.7}$$

where $|E\hat{p}>$ stands for the element $|p\hat{p}>$ and $p = \sqrt{2\mu E}$. The factor $j(E) = (2\mu^3 E)^{1/4}$. One can easily express the reduced -matrix, $s(E)$, in terms of $\tau(E, E)$, viz.

$$s(E) = e - 2\pi i\; \tau(E, E) \quad . \tag{IV.8}$$

Here e is the identity on $\mathscr{H}_\wedge$.

Construct now the trace of $\left[K^\dagger, K\right]$. Combining Eqs. (IV.7) and (IV.5) and the fact that $d\vec{p} = j^2(E)\, dE\, d\hat{p}$, we have

$$j(E) < E\hat{p}\,|\left[K^\dagger, K\right]|E'\hat{p}> j(E') =$$

$$= -\int dE_1 d\hat{p}_1 \frac{<\hat{p}_1\,|\tau(E_1, E)\,|\hat{p}>^* <\hat{p}_1\,|\tau(E_1, E')\,|\hat{p}> - <\hat{p}\,|\tau(E, E_1)\,|\hat{p}_1><\hat{p}\,|\tau(E', E_1)\,|\hat{p}_1>^*}{(E - E_1 - i0)(E_1 - E' - i0)} . \tag{IV.9}$$

Carry out the $d\hat{p}$ integration of both sides of Eq. (IV.9) and use the adjoint relation

$$<\hat{p}\,|\tau^{\dagger}(E_1, E)\,|\hat{p}_1> = <\hat{p}_1|\tau(E_1, E)|\hat{p}>^* \quad . \tag{IV.10}$$

The result is

$$\int d\hat{p}\; j(E) < E\hat{p}|\left[K^{\dagger}, K\right]|E'\hat{p}>\, j(E') =$$

$$= -\int_0^{\infty} dE_1 \frac{\mathrm{tr}\; \tau^{\dagger}(E_1, E)\,\tau(E_1, E') - \mathrm{tr}\; \tau(E, E_1)\,\tau^{\dagger}(E', E_1)}{(E - E_1 - i0)(E_1 - E' - i0)} \tag{IV.11}$$

Here, as before, tr denotes the trace on $\mathscr{H}_{\wedge}$. The diagonal element of Eq. (IV.11) is obtained by letting $E' \to E$. To carry out this limit let us recall a result established by Jauch.[13] Let $f(E, E')$ and $g(E, E')$ be complex valued functions which are differentiable in their (real) arguments. Then the following formula is valid,

$$\lim_{E' \to E} \int_0^{\infty} dE_1 \frac{f(E, E_1)\, g(E_1, E') - g(E, E_1)\, f(E_1, E')}{(E - E_1 - i0)(E_1 - E' - i0)}$$

$$= -i\pi \left[f(E, E) \frac{d}{dE} g(E, E) - g(E, E) \frac{d}{dE} f(E, E)\right] \tag{IV.12}$$

If we apply (IV.12) to Eq. (IV.11) and integrate with respect to dE, then the left hand side of Eq. (IV.12) is $\mathrm{Tr}\left[K^{\dagger}, K\right]$. Thus employing Eq. (IV.6) gives,

$$N = -i\pi \int_0^{\infty} dE\; \mathrm{tr}\left[\tau^{\dagger}(E, E) \frac{d}{dE} \tau(E, E) - \tau(E, E) \frac{d}{dE} \tau^{\dagger}(E, E)\right] \tag{IV.13}$$

This is Levinson's theorem expressed in terms of scattering amplitudes. It may be restated in terms of the reduced S-matrices, $s(E)$, by utilizing Eq. (IV.8). Simple algebra leads to

$$N = \frac{1}{4\pi} \int_0^{\infty} dE \left\{\mathrm{tr}\left[i s^{\dagger}(E) \frac{d}{dE} s(E) - i \frac{ds^{\dagger}(E)}{dE} s(E)\right] + \mathrm{tr}\left[i \frac{d}{dE}\left(s^{\dagger}(E) - s(E)\right)\right]\right\}$$

$$= \frac{1}{4\pi} \int_0^{\infty} dE \left\{-2\, \mathrm{tr}\; q(E) + i \frac{d}{dE}\, \mathrm{tr}\left(s^{\dagger}(E) - s(E)\right)\right\} \tag{IV.14}$$

What remains is to understand the behavior of the term $\mathrm{tr}\,(s^{\dagger}(E) - s(E))$. Let us define $t_2(z)$ by

$$t_2(z) = t(z) - V \quad . \qquad (\text{IV}.15)$$

Replace $t(E'+i0)$ in Eq. (IV.7) by $t_2(E'+i0)$ and denote the resultant two parameter operator by $\tau_2(E, E')$. Using Eq. (IV.5) and Eq. (IV.8) one finds

$$i \frac{d}{dE} \mathrm{tr}\left[s^{\dagger}(E) - s(E)\right] = -\frac{4}{\pi}\left(\frac{\mu}{2}\right)^{3/2} \frac{\tilde{v}}{\sqrt{E}} - 8\pi \frac{d}{dE} \mathrm{Re}\ \mathrm{tr}\ \tau_2(E, E) \quad . \qquad (\text{IV}.16)$$

Thus Eq. (IV.14) now reduces to,

$$-2\pi N = \int_0^{\infty} dE \left[\mathrm{tr}\ q(E) + \frac{2}{\pi}\left(\frac{\mu}{2}\right)^{3/2} \frac{\tilde{v}}{\sqrt{E}}\right] + 4\pi\, \mathrm{Re}\ \mathrm{tr}\ \tau_2(E, E) \Big|_{E=0}^{E=\infty} \quad . \qquad (\text{IV}.17)$$

This is Levinson's theorem when the last term is zero. The fact that the zero energy on-shell t-matrix is proportional to the scattering length, means that $\mathrm{tr}\ \tau_2(E, E)$ behaves like a const. $\times \sqrt{E}$ for small E. So, we have $\mathrm{tr}\ \tau_2(0, 0) = 0$.

All that is left to consider is the high energy limit of $\mathrm{Re}\ \mathrm{tr}\ \tau_2(E, E)$. Under assumption (A) on the potential, it is well known[13] that

$$|\langle \vec{p}\, | t(p^2/2\mu + i0) | \vec{p} \rangle - \langle \vec{p} | V | \vec{p} \rangle | = \delta(p) \quad , \qquad (\text{IV}.18)$$

and $\delta(p) \to 0$ as $p \to \infty$. Furthermore, the symmetry properties of the resolvent $r(z)$ under the transform $p \to -p$ imply that the forward scattering amplitude,

$$f(p) = -4\mu\pi^2 \langle \vec{p}\, | t(p^2/2\mu + i0) | \vec{p} \rangle \quad , \qquad (\text{IV}.19)$$

satisfies

$$f^*(p) = f(-p) \quad . \qquad (\text{IV}.20)$$

Of course only the forward scattering amplitude is needed to compute $\mathrm{Re}\ \mathrm{tr}\ \tau_2(E, E)$. The symmetry relation (IV.20) means Re f(p) is an even function of p. Thus, at infinity $\mathrm{Re}\ f(p) = O(p^{-2n})$. Now the estimate (IV.18) forces n to be a positive integer. Thus the slowest behavior possible for Re f(p) at infinity is $O(p^{-2})$.

This observation combined with the definition (IV. 7) of $\tau(E, E')$ and the trace to tr implies Re tr $\tau_2(E, E) = O(E^{-1/2})$. Thus the high energy surface term in (IV. 17) vanishes.

The details of this derivation indicate why our theorem must have the term $\tilde{v}$ present. Consider the high energy behavior of tr q(E). For sufficiently high energies we expect that the t-matrix will be dominated by the Born term. If we replace the t-matrix by v in the expression for the S-matrix then the first order contribution to tr q(E) in powers of potential is

$$\operatorname{tr} q(E) \sim -\frac{2}{\pi}\left(\frac{\mu}{2}\right)^{3/2} \frac{\tilde{v}}{\sqrt{E}} \tag{IV.21}$$

For this reason tr q(E) is not integrable at infinity with respect to E. The Tr $(r_0(z)\, V\, r_0(z))$ term in the integrand of the Levinson's theorem exactly cancels this singular behavior of tr q(E). With this singularity subtracted away it is now very reasonable to expect that condition (B) on the time delay is valid.

Since the form of our Levinson's theorem differs from the usual partial wave form it is instructive to see how the customary result can emerge from the analysis given. This is most easily understood by starting from Eq. (IV. 14). When the potential v is spherically symmetric then the angular momentum operator $\vec{L} = \vec{x} \times \vec{p}$ commutes with h and h_0, so that the reduced S-matrix and the time delay operator may be represented by

$$\langle \hat{p} | s(E) | \hat{p}' \rangle = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi}\, s_\ell(E)\, P_\ell(\hat{p}\cdot\hat{p}') \tag{IV.22}$$

$$\langle \hat{p} | q(E) | \hat{p}' \rangle = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi}\, q_\ell(E)\, P_\ell(\hat{p}\cdot\hat{p}') \tag{IV.23}$$

Here the S-matrix admits the usual phase-shift parametrization, $s_\ell(E) = e^{2i\delta_\ell(E)}$. The corresponding formula for the time delay is $q_\ell(E) = 2\,\frac{d}{dE}\,\delta_\ell(E)$.

Let us compute the contribution of a single partial wave amplitude to Eq. (IV. 14). Each bound state has a $2\ell+1$ degeneracy, so let N_ℓ denote the number of bound states with different energy. Upon substituting Eqs. (IV. 22) and (IV. 23) into Eq. (IV. 14) we have

$$N_\ell = -\frac{1}{4\pi}\int_0^\infty dE\left\{4\frac{d}{dE}\,\delta_\ell(E) - 8\pi\frac{d}{dE}\sin 2\delta_\ell(E)\right\} \tag{IV. 24}$$

The customary phase shift normalization is to set $\delta_\ell(\infty) = 0$. Thus Eq. (IV. 24) becomes

$$\pi N_\ell = \delta_\ell(0) - 2\pi \sin 2\delta_\ell(0) \tag{IV. 25}$$

and has the solution

$$\pi N_\ell = \delta_\ell(0) \quad . \tag{IV. 26}$$

This is the partial wave Levinson's theorem. For a single partial wave the terms tr q(E) and $\frac{d}{dE}$ tr $(s^\dagger(E) - s(E))$ are individually integrable. For the full amplitude case these terms are separately divergent, but when added together their divergences cancel. The mechanism for changing the behavior of these terms is the infinite sum over partial waves.

We close this section with some general comments about the results found here. One interesting aspect of statement Eq. (III. 24) of Levinson's theorem, is that it relates two observables of the scattering system. Both the time delay tr q(E) and the number of bound states N are in principle observable features of the scattering system. One nonintuitive result of the theorem concerns the behavior of time delay when resonances are present. Consider the case when at some energy, E_r, there is a very long-lived resonance. Suppose the potential is slightly perturbed so that the lifetime of the resonance increases but the number of bound states is unaltered. Then Eq. (III. 24) tells us that at energies away from the resonance there must be a corresponding decrease in the time delay since the energy integral is invariant.

A second advantage of this theorem is that it is more general than the usual Levinson's theorem in that it remains valid for scattering from a nonspherically symmetric potential. Furthermore, the general approach given here may obtain Levinson's theorems for the few-body scattering problem. We note that the spectral property of time delay has already been established for the three-body problem.[5]

So far physical applications of this theorem have not been investigated. However, one application is straightforward. The theorem may be used to predict the high temperature behavior of the second virial coefficient for a quantum gas. This will be reported on elsewhere.

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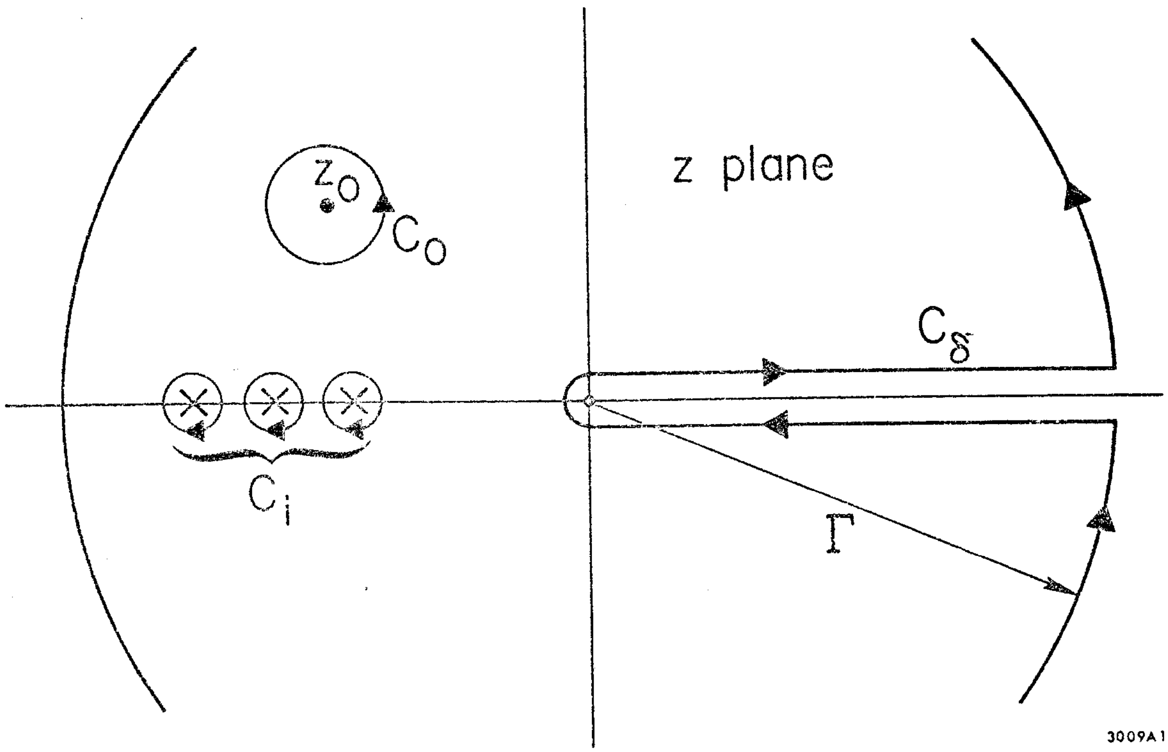

Fig. 1

The complex energy plane.