# ON ${ }^{9}$ T HOOFT BOUND STATE EQUATION: <br> A VIEW FROM TWO GAUGES* 

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#### Abstract

Two-dimensional $U(N)$ invariant Chromodynamics is canonically quantized in both the light cone and the axial gaüges. A principal value infrared cutoff is adopted. A direct Hamiltonian method leads to two different meson bound state equations in the limit of $N \rightarrow \infty, g^{2} N$ kept fixed. In the light cone gauge, ${ }^{\text {'t }}$ Hooft's equation is obtained; in the axial gauge, the corresponding equation suffers from covariance problems rooted in the severe infrared divergences of the theory。 The bosonization of the model is also presented.


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## I．COLOR ON THE LINE

It is widely believed that a color gauge theory of quarks and gluons， Quantum Chromodynamics（QCD），${ }^{1}$ provides a not manifestly wrong and eco－ nomical foundation for the dynamics of strong interactions．While much effort， both conceptual and computational，${ }^{2}$ has been spent to uncover the true infrared structure of QCD，little has yet emerged to warrant confrontation with our ex－ perimental knowledge of hadrons．However if one is willing to contemplate an unphysical world in $1+1$ dimensions such as that of $E$ 。A。Abbott？s ignorant Monarch of Lineland，${ }^{3}$ the theoretical outlook is brighter．QED and QCD ${ }^{4,5}$ manifestly confine since the attractive Coulomb potential between a fermion－ antifermion pair rises linearly with distance of particular interest are the works of ${ }^{9} t$ Hooft ${ }^{6,7}$ who studied the large $N$ limit keeping $g^{2} N$ fixed $-N$ refers to the local $\operatorname{SU}(\mathrm{N})$ group of color and g the associated group charge．For two－ dimensional QCD，referred to hereafter as TDQCD，he showed it to be solvable in this limit and by summing the set of all planar Feynman graphs for a given channel he obtained in the light cone gauge a covariant bound state equation for the mesons．This equation reveals an infinite number of finite mass color singlet bound states，equally spaced for large masses．This spectrum concurs with one＇s potential theoretic intuition in this instance of a shallow well in the weak coupling regime。

More recently，there has been a resurgence of activities concerning ＇t Hooft solution．${ }^{8}$ Restricting themselves to the light cone gauge where the model looks particularly simple，Callan et al。and Einhorn aim to test the con－ sistency of this solution．Thus Bjorken scaling，the Drell－Yan－West，as well as the Bloom－Gilman relations，are seen to be satisfied．All heralds well for the four－dimensional counterpart of ${ }^{9} t$ Hooft solution to QCD。 ${ }^{9}$ We note that the
identity in the topological structure between the $1 / \mathrm{N}$ expansion of QCD and the perturbative graphs of dual resonance models suggests that a transverse momentum cutoff must be operative in the former as it is in dual models. If this is indeed so, one may end up with having the Galilean subdynamics of the fourdimensional theory reduced to an effective TDQCD in the infinite momentum frame or more precisely on the null plane. From the viewpoint of string theories TDQCD also represents the common limiting case of the quark confining string ${ }^{10}$ and a theory with quarks constrained at the endpoints of Nambu strings. ${ }^{11}$ In short, 't Hooft TDQCD is a rich testing laboratory for bound state problems in color gauge theories.

In this work, we wish to carry into a different direction this consistency study of 't Hooft solution. We report on a Hamiltonian approach to TDQCD as an alternative $1 / \mathrm{N}$ expansion scheme to the usual diagrammatic method.

Our intentions are twofold. We formulate TDQCD in two different gauges, the light cone and the axial gauges. These choices follow respectively from the front and the instant forms of dynamics. Postulating in both instances the standard canonical free field commutation relations for the independent fields and a principal value infrared cutoff, we attempt to derive the corresponding bound state equation for quark-antiquark pairs in the $N \rightarrow \infty$ limit, $g^{2} N$ fixed。While sharing the same Lorentz invariant action there is however no a priori reason for the two forms of dynamics to be the same since they are not simply connected by a unitary transformation。 ${ }^{12}$ Moreover the infrared divergences inherent to the model have varying effects depending on the gauges chosen. Our interest in the axial gauge version of TDQCD was triggered by the work of $H_{0}$ D.I. Abarbanel et al., who fail to obtain a covariant 't Hooft's equation in ghostfree gauges other than the light cone gauge.

In our work, the $1 / \mathrm{N}$ expansion is formulated as an old-fashioned RaleighSchrodinger perturbative series in $g^{2} N=$ fixed, the perturbing potential being the particle number changing piece of our Hamiltonian. In the leading order in N , we recover in the light cone gauge 't Hooft covariant equation. However a similar calculation in the axial gauge leads to spinorial complications and a noncovariant equation. We attribute this negative result to the inadequacy of the naive principal value cutoff in the handling of the particularly severe infrared divergences in the Coulomb gauge. Possible cures for this problem are discussed.

Finally we present the bosonized equivalent of TDQCD, a form which generalizes the bosonization of the massive Schwinger model. This dual form of TDQCD will be useful in the strong coupling regime $g \gg m$, the quark mass.

Our paper is organized as follows: in Section II we define our notation, the null plane quantization of TDQCD is performed, a Hamiltonian method to get 't Hooft equation is given. In Section III a similar analysis is done in the axial gauge. In Section IV we close with writing down the bose form of TDQCD and discuss our results.

## II．TDQCD IN THE LIGHT CONE GAUGE

For definiteness，we consider the standard locally $U(N)$ invariant
Lagrangian density

$$
\begin{equation*}
\mathscr{L}=\bar{\psi}\left(\mathrm{i} \gamma^{\mu} \mathrm{D}_{\mu}-\mathrm{m}\right) \psi-\frac{1}{4} \mathrm{~T}_{\mathrm{k}}\left(\mathrm{~F}_{\mu \nu} \mathrm{F}^{\mu \nu}\right) \tag{2.1}
\end{equation*}
$$

where

$$
\begin{align*}
\mathrm{D}_{\mu}^{\psi} & =\left(\partial_{\mu}-\mathrm{igA}_{\mu}\right) \psi  \tag{2,2}\\
\mathrm{F}_{\mu \nu} & =\partial_{\mu} \mathrm{A}_{\nu}-\partial_{\nu} \mathrm{A}_{\mu}+\mathrm{g}\left[\mathrm{~A}_{\mu}, \mathrm{A}_{\nu}\right] \tag{2,3}
\end{align*}
$$

The matrix notation，$e_{\circ} g_{\circ}, A_{\mu}=A_{\mu}^{\alpha} T^{\alpha}{ }_{\circ}$ is used throughout ${ }_{\rho}$ The $T^{\alpha}$ are the matrix representatives of the generators of $\mathrm{U}(\mathrm{N}) 。 \psi=\left(\begin{array}{l}\psi_{1}^{a} \\ \frac{1}{\mathrm{a}} \\ 2\end{array}\right)$ denotes a 2 －Dirac spinor which is a N component vector in the color space。 $\mathrm{A}_{\mu}$ and $\mathrm{F}_{\mu \nu}$ are the color gauge potentials and the covariant Yang－Mills fields respectively。 Flavor indices have been deleted since only the dynamics of color is of interest here。

Following ${ }^{1} t$ Hooft，${ }^{7}$ we choose to work with the group $U(N)$ instead of $S U(N)$ ，the difference being the singlet $A_{a}^{a}$ which decouples and is a free field． To leadind order in $1 / \mathrm{N}$ ，either group leads to identical results．As is appar－ ent in $(2.1)$ we limit our treatment to the equal quark mass case，the general situation being a trivial extension．

Variation of the fields $\psi$ and $A_{\mu}$ yields the coupled set of Dirac and Yang－ Mills equations of motions

$$
\begin{align*}
& \left(\mathrm{i} \gamma^{\mu} \mathrm{D}_{\mu}-\mathrm{m}\right) \psi=0  \tag{2.4}\\
& \partial^{\nu} \mathrm{F}_{\nu \mu}=-\mathrm{g}\left(\mathrm{j}_{\mu}-\mathrm{i}\left[\mathrm{~F}_{\mu \nu}, \mathrm{A}^{\nu}\right]\right) \tag{2,5}
\end{align*}
$$

where the color current is $J_{\mu}=\bar{\psi} \gamma_{\mu} \overrightarrow{\mathrm{T}} \psi^{\circ} \overrightarrow{\mathrm{T}}$ 。
We shall also need the conserved energy momentum tensor

$$
\begin{equation*}
\mathrm{T}_{\mu \nu}=-\mathrm{g}_{\mu \nu} \mathscr{X}+\mathrm{i} \bar{\psi} \gamma_{\mu} \partial_{\nu} \psi \tag{2.6}
\end{equation*}
$$

given by Poincaré invariance through Noether ${ }^{\text {s }}$ s theorem．

We recall that TDQCD is superrenormalizable, the group charge $g$ having the dimension of a mass. Both mass and coupling renormalizations are finite . Due to the two-dimensionality of the system, there is the added bonus in any ghost-free gauge $\mathrm{n}^{\circ} \mathrm{A}=0$ that there are no nonlinear interactions among the gluons since $\left[A_{\mu}, A_{\nu}\right]=0$ 。 For this reason, we shall consider two such gauges, the light cone and axial gauges respectively.

It is well known that dynamics at infinite momentum, or the front form of dynamics, present definite computational advantages in bound states problems in relativistic theories. ${ }^{13}$ The key reason lies in the vanishing of the usually troublesome vacuum fluctuation and topologically complex graphs which are stumbling blocks in the derivation of useful integral equations for bound states in an ordinary Lorentz frame. Therefore we begin by analyzing in some detail the null plane dynamics of system (2.1) in the light cone gauge.

Our metric tensor components are $\mathrm{g}_{++}=\mathrm{g}_{--}=0, \mathrm{~g}_{+-}=\mathrm{g}_{-+}=1$ with the coordinates and the $\gamma$-matrices defined as

$$
\begin{align*}
& x^{ \pm}=x_{\mp}=\left(x^{o} \pm x^{1}\right) / \sqrt{2} \\
& \gamma^{ \pm}=\left(\gamma^{0} \pm \gamma^{1}\right) / \sqrt{2}, \gamma^{+2}=\gamma^{-2}=0  \tag{2.7}\\
& \left\{\gamma^{+}, \gamma^{-}\right\}=2, \gamma^{5}=\frac{1}{2}\left[\gamma^{-}, \gamma^{+}\right]
\end{align*}
$$

We use the Weyl representation

$$
\begin{gather*}
\gamma^{\mathrm{o}}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right) \equiv \sigma_{1} \quad, \quad \gamma^{1}=\left(\begin{array}{rr}
0 & 1 \\
-1 & 0
\end{array}\right) \equiv \mathrm{i} \sigma_{2} \\
\gamma^{5}=\gamma^{0} \gamma^{1}=\left(\begin{array}{rr}
-1 & 0 \\
0 & 1
\end{array}\right) \equiv-\sigma_{3} \tag{2,8}
\end{gather*}
$$

where the free spinor fields are such that $u(0)=\binom{1}{1}$ and $v(0)=\binom{1}{-1}$. Usually used for massless fermions, this representation is the most natural choice in
the null plane quantization which does not know about masses. ${ }^{12}$ We also make use of the Hermitian projection operators

$$
\begin{align*}
& P_{+}=\frac{1}{2} \gamma^{-} \gamma^{+}=\frac{1}{2} \gamma_{+} \gamma_{-}=\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right) \\
& P_{-}=\frac{1}{2} \gamma^{+} \gamma^{-}=\frac{1}{2} \gamma_{-} \gamma_{+}=\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) \tag{2,9}
\end{align*}
$$

fulfilling the properties

$$
\begin{align*}
& P_{+}+P_{-}=I, \quad P_{+}^{2}=P_{+}, \quad P_{-}^{2}=P_{-} \\
& P_{-} \gamma^{+} P_{+}=\gamma^{+}, P_{+} \gamma^{-} P_{-}=\gamma^{-} \tag{2,10}
\end{align*}
$$

and

$$
\begin{equation*}
P_{+} \psi^{a}=\binom{0}{\psi_{2}^{a}} \equiv \psi_{+}^{a}, \quad P_{-} \psi^{a}=\binom{\psi_{1}^{\mathrm{a}}}{0} \equiv \psi_{-}^{\mathrm{a}} \tag{2.11}
\end{equation*}
$$

By way of these $\gamma$-matrix identities in the light cone gauge $A_{-}=0$, the Dirac equation for $\psi$ splits into

$$
\begin{align*}
& 2 \mathrm{id}+_{+} \psi_{+}^{\mathrm{a}}=\mathrm{m} \gamma_{+} \psi_{-}^{\mathrm{a}}  \tag{2.12}\\
& 2 \mathrm{i} \partial_{-} \psi_{-}^{\mathrm{a}}=\mathrm{m} \gamma_{-} \psi_{+}^{\mathrm{a}} \tag{2.13}
\end{align*}
$$

With $\partial_{+}$playing the role of a "time" derivative on the null plane dynamics, $\psi_{+}^{\mathrm{a}}$ is seen as the independent variable of our problem. Eq。 $(2.13)$ plays the role of a constraint for $\psi_{-}^{a}$ and can be written as

$$
\begin{equation*}
\psi_{-}^{\mathrm{a}}(\tau, \mathrm{z})=\frac{\mathrm{m}}{4 \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d} z^{q} \in\left(\mathrm{z}-\mathrm{z}^{8}\right) \gamma_{-} \psi_{+}^{\mathrm{a}}\left(\tau, \mathrm{z}^{\prime}\right) \tag{2,14}
\end{equation*}
$$

with $\epsilon(\mathrm{x})=\left\{\begin{array}{l}1, \mathrm{x}>0 \\ -1, \mathrm{x}<0\end{array}, \tau \equiv \mathrm{x}^{+}, \mathrm{z} \equiv \mathrm{x}^{-}\right.$。Consequently (2.12) becomes

$$
\begin{equation*}
\left(\partial_{+}+\frac{\mathrm{m}^{2}}{2} \partial_{-}^{-1}\right) \psi_{+}^{\mathrm{a}}=-\mathrm{igA} \psi_{+}^{\mathrm{a}} . \tag{2.15}
\end{equation*}
$$

Since there is no physical transverse degree of freedom associated with the field in two dimensions, the Yang-Mills equations $(2,5)$

$$
\begin{equation*}
\partial_{-}^{2} \mathrm{~A}_{+}(\tau, \mathrm{z})=-\mathrm{gj} \mathbf{J}_{-}(\tau, \mathrm{z}) \tag{2,16}
\end{equation*}
$$

yield another constraint leading to a nonlocal Coulomb interaction between the quarks.

Here we recover the remarkable feature of null plane dynamics: the number of independent canonical variables is reduced to half the number present in equal time dynamics. $\psi_{+}^{\text {a }}$ is the only variable from which all the other operators of the theory can be built.

From (2.6), the null plane Hamiltonian is derived

$$
\begin{equation*}
\mathrm{H}=\int \mathrm{T}_{+}^{+} \mathrm{dz}=\mathrm{H}_{\mathrm{f}}+\mathrm{H}_{\mathrm{I}} \tag{2.17}
\end{equation*}
$$

where a separation is made of the free part $\mathrm{H}_{\mathrm{f}}$

$$
\begin{equation*}
\mathrm{H}_{\mathrm{f}}=\int \mathrm{dz} \bar{\psi}\left(\mathrm{~m}-\mathrm{i} \gamma^{-} \partial_{-}\right) \psi \tag{2.18}
\end{equation*}
$$

and the interaction part $\mathrm{H}_{\mathrm{I}}$

$$
\begin{equation*}
\mathrm{H}_{\mathrm{I}}=\int \mathrm{dz} \operatorname{Tr}\left[g J_{-} \mathrm{A}_{+}-\frac{1}{2}\left(\partial_{-} \mathrm{A}_{+}\right)^{2}\right] . \tag{2.19}
\end{equation*}
$$

Eliminating $\psi_{-}$by means of the constraint (2.14), we get the expected form

$$
\begin{equation*}
H_{f}=\frac{i m^{2}}{4} \iint \mathrm{dz} \mathrm{dz}^{p} \psi_{+}^{+}\left(\mathrm{z}^{\vee}\right)_{a} \in(\mathrm{z-z}) \psi_{+}(\mathrm{z})^{a} . \tag{2,20}
\end{equation*}
$$

Similarly, solving for $A_{+}$by use of $(2,16)$ we get the Coulomb term

$$
\begin{equation*}
H_{I}=-\frac{\mathrm{g}^{2}}{4} \iint d z d z^{q} J_{-}(z)_{a}^{b}\left|z-z^{\prime}\right| J_{-}(z)_{b}^{a} \tag{2,21}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{J}_{-}(\tau, \mathrm{z})_{\mathrm{a}}^{\mathrm{b}}=\sqrt{2} \psi_{+}^{+}(\tau, \mathrm{z})_{\mathrm{a}} \psi_{+}(\tau, \mathrm{z})^{\mathrm{b}} . \tag{2,22}
\end{equation*}
$$

The main steps leading up to $(2,21)$ are as follows. The general solution of the boundary value problem (2,16) is given as

$$
\begin{equation*}
A_{+}=-g \partial_{-}^{-2} j_{-}-E z-G \tag{2.23}
\end{equation*}
$$

where

$$
\begin{equation*}
\partial_{-}^{-2} j_{-}(z)=\frac{1}{2} \int d z^{8}\left|z-z^{8}\right| j_{-}\left(z^{\prime}\right) \tag{2.24}
\end{equation*}
$$

The constant matrix G can be gauged to zero. The E correspond to the N constant color background fields. ${ }^{5}$ They cannot affect the color singlet sector of the theory since the quark-antiquark bound states carry no dipole moments. ${ }^{8}$ Hence the $E$ can be set equal to zero in this singlet sector. By direct insertion of $(2,23)$ and $(2,24)$ in $(2,19)$ and by use of the identity $\left|z_{1}-z_{2}\right|=$ $\frac{1}{2} \int \mathrm{dz} \epsilon\left(\mathrm{z}_{1}-\mathrm{z}\right) \epsilon\left(\mathrm{z-z} \mathrm{z}_{2}\right),(2,21)$ results。

Now the null plane classical initial value problem for $\psi_{\text {_ }}$ obeying $(2.15)$ requires the additional assumption $\psi_{+}( \pm \infty, \tau)=0$ with $\tau$ held fixed ${ }^{15}$ This condition means that the physical system is local and is anyhow required for the existence of such generators as the Hamiltonian $H(2,17)$, the momentum operator $P=\int d z T_{+}^{-}$, and the total matrix charge $Q=\int_{-\infty}^{+\infty} j_{-}(z, 0) d z$. Regarding the total charge, carrying out in the light cone gauge an easy calculation analogous to that of Zumino, ${ }^{16}$ we can obtain

$$
\begin{equation*}
\left(\partial_{+}^{2}+\frac{\mathrm{g}^{2}}{\pi}\right) \mathbf{Q}=0 \tag{2.25}
\end{equation*}
$$

as a consequence of the anomaly in the current $\partial_{\mu} \mathrm{j}^{\mu}=-\frac{\mathrm{g}}{4 \pi_{1}} \epsilon^{\mu \nu} \mathrm{F}_{\mu \nu}$. Since Q cannot change in "time", (2.25) implies the constancy of the charge Q , which is true only if $Q=0$. Similarly, $Q$ is independent of the Lorentz frame only if it vanishes. However, as a quantized operator such that $[Q, \psi]=-e \psi, Q$ can only vanish weakly, i.e., the null plane quantization of TDQCD is only covariant in the singlet sector of the Hilbert space of states. In this work we shall confine
ourselves to this bound state sector of the theory.
Having eliminated all dependent fields we can proceed to quantize the theory canonically. It is natural to work in the Schrodinger picture by postulating as is usually done in null plane quantization a free field expansion for $\psi_{+}$at fixed time $\tau=0$ 。

$$
\begin{equation*}
\psi_{+}(\mathrm{z}, \tau=0)=\frac{1}{\sqrt{2 \pi}} \int \mathrm{~d} \eta \mathrm{e}^{-\mathrm{i} \eta \mathrm{z}} \mathrm{a}(\eta) \tag{2,26}
\end{equation*}
$$

with

$$
\begin{array}{r}
\left\{\mathrm{a}(\eta), \mathrm{a}^{+}\left(\eta^{\imath}\right)\right\}=\delta\left(\eta-\eta^{\vee}\right) \\
\left\{\mathrm{a}(\eta), \mathrm{a}\left(\eta^{\imath}\right)\right\}=0 \tag{2.27}
\end{array}
$$

being the covariant anticommutation relations. Equivalently in the space coordinate representation we have

$$
\begin{gather*}
\left\{\psi_{+}(\tau, \mathrm{z}), \psi^{+}\left(\tau, \mathrm{z}^{\prime}\right)\right\}=\mathrm{P}_{+} / \sqrt{2} \delta\left(\mathrm{z-z}^{8}\right)  \tag{2.28}\\
\left\{\psi_{+}(\tau, \mathrm{z}), \psi_{+}\left(\tau, \mathrm{z}^{8}\right)\right\}=0
\end{gather*}
$$

The current J_ $(z, 0)(2,22)$ then takes the form

$$
\begin{equation*}
\underset{-\mathrm{a}}{\mathrm{~b}}=\frac{1}{\sqrt{2} \pi} \iint \mathrm{~d} \eta \mathrm{~d} \eta^{\prime} \mathrm{e}^{-\mathrm{i}\left(\eta-\eta^{8}\right) \mathrm{z}}: \mathrm{a}^{+}\left(\eta^{\vee}\right)_{\mathrm{a}} \mathrm{a}(\eta)^{\mathrm{b}}: \tag{2.29}
\end{equation*}
$$

Letting $\eta-\eta^{\ell}=m$ and defining the density operators

$$
\begin{equation*}
\rho_{\mathbf{a}}^{\mathrm{b}}(\mathrm{~m})=\int \mathrm{d} \eta^{\ell}: \mathrm{a}^{+}\left(\eta^{q}\right) \mathrm{a}^{\mathrm{a}\left(\eta^{\ell}+\mathrm{m}\right)^{\mathrm{b}}:} \tag{2.29}
\end{equation*}
$$

familiar in dual theories ${ }^{17}$ and in solvable models of a one-dimensional electron gas. ${ }^{18}$ Then

$$
\begin{equation*}
J_{a}^{b}(z)=\frac{1}{\sqrt{2 \pi}} \int d m \rho_{a}^{b}(m) e^{-i m z}=\frac{1}{\sqrt{2 \pi}} \int_{m>0} d m\left(\rho^{b}(m) e^{-i m z}+\rho_{a}^{b}(m)^{+} e^{i m z}\right) \tag{2.30}
\end{equation*}
$$

Making the identification through $(2,26)$ of

$$
\begin{align*}
& a(\eta)=c(\eta) \\
& a(-\eta)=d(\eta)^{+} \tag{2,31}
\end{align*} \quad \eta=1,2, \ldots .
$$

where $\mathrm{c}^{+}$and $\mathrm{d}^{+}$are the quark and antiquark creation operators, respectively, we obtain with a little algebra

$$
\begin{equation*}
\rho_{\mathrm{a}}^{\mathrm{b}}(\mathrm{~m})=\int_{0}^{\infty} \mathrm{d} \eta\left[\mathrm{c}^{+}(\eta)_{\mathrm{a}} \mathrm{c}(\eta+\mathrm{m})^{\mathrm{b}}-\mathrm{d}^{+}(\eta)_{\mathrm{a}} \mathrm{~d}(\eta+\mathrm{m})^{\mathrm{b}}+\mathrm{d}(\mathrm{~m}-\eta)_{\mathrm{a}} \mathrm{~b}(\eta)^{\mathrm{b}}\right] \tag{2.32}
\end{equation*}
$$

and

$$
\begin{equation*}
\rho_{\mathrm{a}}^{\mathrm{b}}(\mathrm{~m})^{+}=\rho_{\mathrm{a}}^{\mathrm{b}}(-\mathrm{m}) \tag{2.33}
\end{equation*}
$$

Substituting these in the interacting Hamiltonian we obtain

$$
\begin{equation*}
\mathrm{H}_{\mathrm{I}}=\frac{\mathbf{g}^{2}}{2 \pi} \int_{0^{+}}^{\infty} \frac{\mathrm{d} \eta}{\eta}\left[\rho_{\mathrm{a}}^{\mathrm{b}}(\eta)^{+} \rho_{\mathrm{a}}^{\mathrm{b}}(\eta)+\mathrm{H}_{\circ} \mathrm{C}_{0}\right] \tag{2.34}
\end{equation*}
$$

Similarly the free Hamiltonian is given as

$$
\begin{equation*}
\mathrm{H}_{\mathrm{f}}=\int_{0_{+}}^{\infty} \mathrm{d} \eta \frac{\mathrm{~m}^{2}}{\eta^{2}}\left[\mathrm{c}^{+}(\eta) \mathrm{a} \mathrm{c}(\eta)^{\mathrm{a}}+\mathrm{d}^{+}(\eta)_{\mathrm{a}} \mathrm{~d}(\eta)^{\mathrm{a}}\right] \tag{2.35}
\end{equation*}
$$

Upon introduction of the fermion vacuum, one constructs the Hilbert space by cyclic action of the creation operators on this vacuum. When sandwiched between any vector in this Hilbert space the density operators $\rho_{a}^{b}(m)$ satisfy Bosetype commutation relations

$$
\begin{equation*}
\left[\rho(\mathrm{m}), \rho^{+}(\mathrm{m})\right]=\delta(\mathrm{m}-\mathrm{n}) \tag{2.36}
\end{equation*}
$$

$(2.36)$ is then to be understood in the weak sense of Dirac。 ${ }^{19}$
Having thus set up the above machinery, we now define our eigenvalue problem for the quark-antiquark bound states. Let

$$
\begin{equation*}
\left.\left|Q \bar{Q}>_{P}=\int_{0}^{P} d k \phi(P, k) c^{+}(\mathrm{k}) \mathrm{a}^{+}(\mathrm{P}-\mathrm{k})^{\mathrm{a}}\right| 0\right\rangle \tag{2.37}
\end{equation*}
$$

denote the quark-antiquark bound state ket-vector in the infinite momentum frame; $\phi(\mathrm{P}, \mathrm{k})$ is the amplitude for finding a quark with momentum k and an antiquark with momentum ( $\mathrm{P}-\mathrm{k}$ ), there being no spin in two dimensions. P is
the total momentum of the hadron.
The Schrodinger equation for a bound-state of invariant mass $\mu^{2}$ is then

$$
\begin{equation*}
H\left|Q \bar{Q}>_{P}=\frac{\mu^{2}}{2 P}\right| Q \bar{Q}>_{P} \tag{2.38}
\end{equation*}
$$

Before proceeding further, we first normal order the Hamiltonian, making use of the trace identity $\operatorname{Tr}\left(\delta^{\mathrm{ab}}\right)=\mathrm{N}$ and $(2.27)$. In this manner, the mass renormalization contributions are separated from the rest of the interaction; they are quadratic in $\mathrm{c}^{+} \mathrm{c}$ and $\mathrm{d}^{+} \mathrm{d}_{\text {。 }}$ We obtain

$$
\begin{aligned}
& H_{I}=\frac{g^{2}}{2 \pi} \int_{>0}^{\infty} \frac{d k}{k^{2}}\left[c^{+}(k) a^{c(k)^{a}}+d^{+}(k)_{a} d(k)^{a}\right]
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{\mathrm{g}^{2}}{2 \pi} \int_{>0}^{\infty} \frac{\mathrm{dk}}{\mathrm{k}^{2}}:\left[\rho_{\mathrm{a}}^{\mathrm{b}}(\mathrm{k}) \rho_{\mathrm{a}}^{\mathrm{b}}(\mathrm{k})+\mathrm{H} . \mathrm{C}_{\circ}{ }^{\mathrm{\jmath}}:\right.
\end{aligned}
$$

The ( $>0$ ) lowest limit of integration means the exclusion of the zero mode to be specified by a principal value cutoff. The first term yields a constant and can be dropped. The second term gives rise to mass renormalizations. The last term constitutes the actual interactions.

Regrouping again the various terms, the Hamiltonian is written as

$$
\begin{equation*}
\mathrm{H}=\mathrm{H}_{\mathrm{f}}+\mathrm{H}_{\mathrm{MR}}+\mathrm{H}^{\prime} \tag{2.40}
\end{equation*}
$$

where $H^{\prime}$ is split into a particle number conserving piece $H_{P C}$ and the rest $R$ 。

We have

$$
\begin{align*}
& H_{P C}=\frac{g^{2}}{2 \pi} \int \frac{d k}{k^{2}} \iint d k^{\prime} d k^{\prime \prime}\left[c^{+}\left(k^{8}\right)_{a} d^{+}\left(k^{\prime \prime}+k\right)^{b} c\left(k^{\prime}+k\right)_{b} d\left(k^{\prime}\right)^{a}+d^{+}\left(k^{\prime \prime}+k\right)_{a} c^{+}\left(k^{8}\right)^{b} d\left(k^{\prime \prime}\right)_{b} c\left(k^{\prime}+k\right)^{a}\right] \\
& +\frac{g^{2}}{2 \pi} \int \frac{d k}{k^{2}} \iint d k^{\prime} d k^{\prime \prime}\left[d^{+}\left(k^{\prime}\right)_{a} c^{+}\left(k^{\prime \prime}+k\right)^{b} d\left(k^{\prime}+k\right)_{b} c\left(k^{\prime \prime}\right)^{a}+c^{+}\left(k^{\prime \prime}+k\right)_{b} d^{+}\left(k^{\prime}\right)^{a} c\left(k^{\prime \prime}\right)_{a} d\left(k^{\prime}+k\right)^{b}\right] \\
& \quad+\frac{g^{2}}{2 \pi} \int \frac{d k}{k^{2}} \iint d k^{\prime \prime} d k^{\prime}\left[c^{+}\left(k^{\prime \prime}\right)^{b} d^{+}\left(k-k^{\prime}\right)_{a} d^{a}\left(k-k^{\prime \prime}\right) c\left(k^{\prime}\right)_{b}+c^{+}\left(k^{\prime \prime}\right)_{a} d^{+}\left(k-k^{\prime \prime}\right)^{b} d\left(k-k^{\prime}\right)_{b} c(k)^{a}\right] \tag{2.41}
\end{align*}
$$

We observe that the first two terms are Coulomb exchange interactions; the last term corresponds to annihilation processes which are suppressed on the light cone. ${ }^{12,14}$ Thus our bound state equation approach consists then in taking into account in the leading order in $1 / \mathrm{N}$ contribution in H due only to the mass renormalization and Coulomb exchanges. It follows that

$$
\begin{equation*}
\frac{\mu^{2}}{2 \mathrm{P}} \phi(\mathrm{P}, \mathrm{k})=\left[\frac{\mathrm{m}^{2}-\frac{\mathrm{g}^{2} N}{\pi}}{|\mathrm{k}|}+\frac{\left.\mathrm{m}^{2}-\frac{\mathrm{g}^{2} N}{\pi} \right\rvert\,}{\mid \overline{\mathrm{P}-\mathrm{k} \mid}}\right\rfloor_{\phi(\mathrm{P}, \mathrm{k})-\frac{\mathrm{Ng}^{2}}{\pi} \mathscr{P} \int_{0}^{\mathrm{P}} \mathrm{dq} \frac{\phi(\mathrm{P}, \mathrm{q})}{|\mathrm{k}-\mathrm{q}|^{2}},{ }^{2}} \tag{2.42}
\end{equation*}
$$

The symbol $\mathscr{P}$ denotes a principal value integral. Alternatively making use of the Feynman variable $\mathrm{k}=\mathrm{xP}$, and the identity

$$
\int_{0}^{1} \frac{\tilde{\phi}(x)}{|x-y|^{2}} d y=-\widetilde{\phi}(x)\left[\frac{1}{x}+\frac{1}{1-x}\right]
$$

and defining $\phi(P, x)=\widetilde{\phi}(x)$, we have 't Hooft equation

$$
\begin{equation*}
\mu_{\mathrm{t}}^{2} \widetilde{\phi}=\left(\frac{(\alpha+1)}{\mathrm{x}}+\frac{(\alpha+1)}{1+\mathrm{x}}\right) \phi(\mathrm{x})-\mathscr{P} \int_{0}^{1} \mathrm{dy} \frac{\phi(\mathrm{y})-\phi(\mathrm{x})}{(\mathrm{x}-\mathrm{y})^{2}} \tag{2,43}
\end{equation*}
$$

where $\mu_{\mathrm{t}}^{2}=\mu^{2}\left(\frac{\mathrm{~g}^{2} \mathrm{~N}}{\pi}\right)^{-1}$ is dimensionless and $\alpha=\left(\mathrm{m}^{2}-\frac{\mathrm{g}^{2} \mathrm{~N}}{\pi}\right)\left(\frac{\mathrm{g}^{2} \mathrm{~N}}{\pi}\right)^{-1}$. In the above Schrodinger equation approach, higher order $1 / \mathrm{N}$ corrections are in principle calculable by way of old-fashioned perturbation theory in the particle number changing potential $R$. It can be verified without calculation that a second order perturbation theory in $R$ to the energy eigenvalues in $(2.43)$ is proportional to
$\left(g^{2} \mathrm{~N}\right)^{2} / \mathrm{N}$ on dimensional grounds.
We shall not go into any of the details regarding the solutions to Eq。(2.43). They have been treated by 't Hooft and will be the topics of a forthcoming work of Hanson et al., ${ }^{20}$ who apply a powerful method of numerical analysis to several one-dimensional bound state equations of QCD and string theory in different coupling regimes.

## III. TDQCD IN THE AXIAL GAUGE

We now proceed to the quantization of TDQCD in the axial gauge in a manner entirely analogous to that done for the light cone gauge. From (2.1) the Hamiltonian density is

$$
\begin{equation*}
\mathrm{H}=\bar{\psi}\left(\mathrm{m}+\mathrm{i} \gamma^{1} \partial_{1}\right) \psi+\mathrm{gj}_{0} \mathrm{~A}_{0}-\frac{1}{2}\left(\partial_{1} \mathrm{~A}_{0}\right)^{2} \tag{3.1}
\end{equation*}
$$

with $A_{1}=0$; the Euler-Lagrangian equations for the gauge fields are

$$
\begin{equation*}
\partial_{1}^{2} \mathrm{~A}_{0}=-\mathrm{gj} \mathrm{j}_{0} \tag{3.2}
\end{equation*}
$$

They have the nature of a constraint. The general solution of (3.2) is

$$
\begin{equation*}
A_{0}=-g \partial_{1}^{-2} \cdot j_{0}-E x-G \tag{3.3}
\end{equation*}
$$

For the same reasons given in Section II, G=0. The background color fields E can be set to equal zero in the color-singlet sector.

By direct substitution of (3.3) in H , we get

$$
\begin{equation*}
\mathrm{H}=\int \mathrm{dx} \bar{\psi}\left(\mathrm{~m}+\mathrm{i} \gamma^{1} \partial_{1} \psi-\frac{\mathrm{g}^{2}}{4} \iint \mathrm{dx} \mathrm{dx} \mathrm{x}^{\prime} \mathrm{j}_{0}(\mathrm{x})\left|\mathrm{x}-\mathrm{x}^{\prime}\right| \mathrm{j}_{0}(\mathrm{x})\right. \tag{3.4}
\end{equation*}
$$

Just as in Section II, we assume a free-field expansion for the 2-spinor

$$
\begin{equation*}
\psi=\frac{1}{\sqrt{2 \pi}} \int d p\left[A(p) u(p) e^{i p x}+B^{+}(p) v(p) e^{-i p x}\right] \tag{3.5}
\end{equation*}
$$

The spinors satisfy

$$
\begin{gathered}
(\alpha \mathrm{p}+\mathrm{m} \beta) \mathrm{u}=\mathrm{E}_{\mathrm{p}} \mathrm{u}, \quad(-\alpha \mathrm{p}+\mathrm{m} \beta) \mathrm{v}=-\mathrm{E}_{\mathrm{p}} \mathrm{v} \\
\mathrm{E}_{\mathrm{p}}=\sqrt{\mathrm{p}^{2}+\mathrm{m}^{2}}
\end{gathered}
$$

and

$$
\begin{equation*}
\left\{A(p), A^{+}\left(p^{\prime}\right)\right\}_{+}=\left\{B(p), B^{+}\left(p^{\prime}\right)\right\}_{+}=\delta\left(p-p^{\prime}\right) \tag{3.6}
\end{equation*}
$$

are the postulated equal time canonical commutation relations.

For $\gamma$-matrix representation, we choose

$$
\begin{gather*}
\gamma_{0}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right), \quad \gamma_{1}=\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right)  \tag{3.7}\\
\gamma_{5}=\gamma_{0} \gamma_{1}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)
\end{gather*}
$$

In terms of the quark ( $\mathrm{A}, \mathrm{A}^{+}$) and antiquarks $\left(\mathrm{B}, \mathrm{B}^{+}\right)$creation and annihilation operators we can rewrite

$$
\begin{gather*}
H=H_{0}+H_{I} \\
H_{0}=\int d p E_{p}\left[A^{+}(p)^{a} A(p)^{a}+B^{+}(p)^{a} B(p)^{a}\right] \tag{3.8}
\end{gather*}
$$

In order to reexpress $H_{I}$ in terms of these oscillators, we must replace $J_{0}$ in (3.4) by its normal ordered current ${ }^{21}$ (omitting the $U(n)$ matrix label)

$$
\begin{align*}
: J_{0}:= & \frac{1}{2 \pi} \int d p d p^{\prime}\left\{A^{+}\left(p^{\prime}\right) A(p) u^{+}\left(p^{\prime}\right) u(p) e^{-i\left(p^{\prime}-p\right) x}\right. \\
& +B^{+}(p) B\left(p^{\prime}\right) v^{+}\left(p^{\prime}\right) v(p) e^{i\left(p^{\prime}-p\right) x} \\
& +A^{+}\left(p^{\prime}\right) B^{+}(p) u^{+}\left(p^{\prime}\right) v(p) e^{-i\left(p^{\prime}+p\right) x} \\
& \left.+B\left(p^{\prime}\right) A(p) v^{+}\left(p^{\prime}\right) u(p) e^{i\left(p^{\prime}+p\right) x}\right\} \tag{3.9}
\end{align*}
$$

Just as in the light cone case, we set up the eigenvalue equation for the meson bound states

$$
\begin{equation*}
H\left|Q \bar{Q}>_{\mathrm{P}}=\sqrt{\mu^{2}+\mathrm{P}^{2}}\right| Q \bar{Q}>_{\mathrm{P}} \tag{3.10}
\end{equation*}
$$

Similarly we shall only take into account in H terms which do not change the number of particles. After normal ordering and dropping the particle number changing terms, we are left with

$$
\begin{equation*}
\mathrm{H}=\mathrm{H}_{0}+\mathrm{H}_{\mathrm{MR}}+\mathrm{H}_{\mathrm{c}}+\mathrm{H}_{\mathrm{a}} \tag{3.11}
\end{equation*}
$$

where the mass renormalization piece is ( $\mathscr{P}$ stand for principal value)

$$
\begin{align*}
\mathrm{H}_{\mathrm{MR}}= & \frac{\mathrm{g}^{2} \mathrm{~N}}{4 \pi} \iint d p d q \frac{\mathscr{P}}{(p-q)^{2}}\left\{u^{+}(p) u(q) u^{+}(q) u(p) A^{+a}(p) A^{a}(p)\right. \\
& \left.+v^{+}(p) v(q) v^{+}(q) v(p) B^{+a}(q) B^{a}(q)\right\} \\
& -\frac{g^{2}}{4 \pi} \int d p d p^{\prime} d q d q^{\prime} \frac{\mathscr{P}}{\left(p-p^{\prime}\right)}{ }^{2} \delta\left(p-p^{\prime}+q-q^{\prime}\right) \times \\
& \times\left\{u^{+}(p) u^{\prime}\left(p^{\prime}\right) u^{+}(q) u\left(q^{\prime}\right) A^{+a}(p) A^{+b}(q) A^{b}\left(p^{\prime}\right) A^{a}\left(q^{\prime}\right)\right. \\
& \left.+v^{+}(p) v\left(p^{\prime}\right) v^{+}(q) v\left(q^{\prime}\right) B^{+b}(p) B^{+a}(q) B^{a}\left(p^{\prime}\right) B^{b}\left(q^{\prime}\right)\right\} \tag{3.12}
\end{align*}
$$

the Coulomb interaction piece

$$
\begin{align*}
H_{c}=\frac{-g^{2}}{4 \pi} \int & d p d p^{\prime} d q d q^{\prime} \frac{\mathscr{P}}{\left(p-p^{\prime}\right)^{2}} \delta\left(p-p^{\prime}+q^{\prime}-q\right) \times \\
& \times \int^{u^{+}}(p) u^{\prime}\left(p^{\prime}\right) v^{+}(q) v\left(q^{\prime}\right) A^{+a}(p) B^{+a}\left(q^{\prime}\right) A^{b}\left(p^{\prime}\right) B^{b}(q) \\
+ & v^{+}(p) v\left(p^{\prime}\right) u^{+}(q) u\left(q^{\prime}\right) B^{+b}\left(p^{\prime}\right) A^{+b}(q) B^{a}(p) A^{a}\left(q^{\prime}\right) \tag{3.13}
\end{align*}
$$

the annihilation piece

$$
\begin{align*}
& H_{a}=\frac{-g^{2}}{4 \pi} \int d p d p^{\prime} d q d q^{\prime} \delta\left(p+p^{\prime}-q-q^{\prime}\right) \frac{\mathscr{P}}{\left(p+p^{\prime}\right)^{2}} \times \\
& \times\left\{u^{+}(p) v\left(p^{\prime}\right) v^{+}(q) u\left(q^{\prime}\right) A^{+a}(p) B^{+b}(p) B^{b}(q) A^{a}\left(q^{\prime}\right)\right. \\
&+\left.v^{+}(p) u\left(p^{\prime}\right) u^{+}(q) v\left(q^{\prime}\right) A^{+b}(q) B^{+a}\left(q^{\prime}\right) B^{a}(p) A^{b}\left(p^{\prime}\right)\right\} \\
&- \frac{{N g^{2}}_{4 \pi}^{2} \iint d p d q \frac{\mathscr{P}}{(p+q)^{2}} v^{+}(p) u(q) u^{+}(q) v(p) \times}{} \\
& \times\left\{B^{+a}(p) B^{a}(p)+A^{+a}(q) A^{a}(q)\right\} \tag{3.14}
\end{align*}
$$

By further inspection, we observe that the second group of terms in (3.12) is down by a factor of N , compared to the first group, therefore can be neglected.

Regrouping our interaction Hamiltonian again, we get

$$
\begin{align*}
H_{M R}= & \frac{N g^{2}}{4 \pi} \iint d p d q \frac{\mathscr{P}}{(p-q)^{2}}\left\{\left[u^{+}(p) u(q) u^{+}(q) u(p)\right] A^{+a}(p) A^{a}(p)\right. \\
& \left.+\left[v^{+}(p) v(q) v^{+}(q) v(p)\right] B^{+a}(q) B^{a}(q)\right\} \\
& -\frac{N g^{2}}{4 \pi} \iint d p d q \frac{\mathscr{P}}{(p+q)^{2}} v^{+}(p) u(q) u^{+}(q) v(p) \times \\
& \times\left\{A^{+a}(q) A^{a}(q)+B^{+a}(p) B^{a}(p)\right\} \tag{3.15}
\end{align*}
$$

and

$$
\begin{align*}
H_{a}=-\frac{g^{2}}{4 \pi} \int & d p d p^{\prime} d q d q^{8} \delta\left(p+p^{\prime}-q-q^{\prime}\right) \frac{\mathscr{P}}{\left(p+p^{\prime}\right)} \times \\
& \times\left\{u^{+}(p) v\left(p^{\prime}\right) v^{+}(q) u\left(q^{\prime}\right) A^{+a}(p) B^{+b}\left(p^{\prime}\right) B^{b}(q) A^{a}\left(q^{\prime}\right)\right. \\
+ & \left.v^{+}(p) u\left(p^{\prime}\right) u^{+}(q) v\left(q^{\prime}\right) A^{+b}(q) B^{+a}(q) B^{a}(p) A^{b}\left(p^{\prime}\right)\right\} \tag{3.16}
\end{align*}
$$

So the truncated Hamiltonian is $\mathrm{H}_{\mathrm{t}}=\mathrm{H}_{0}+\mathrm{H}_{\mathrm{MR}}+\mathrm{H}_{\mathrm{c}}+\mathrm{H}_{\mathrm{a}}$. We then compute $H|Q \bar{Q}\rangle_{P}=E_{P}|Q \bar{Q}\rangle_{P}$ taking as our bound state representation

$$
|Q \bar{Q}\rangle_{P}=\int_{0}^{\infty} d k \phi(P, k) A^{+a}(\mathrm{k}) B^{+\mathrm{a}}(\mathrm{P}-\mathrm{k})|0\rangle
$$

which is consistent with our ansatz of a free spinor expansion; here $\phi$ is chosen to be an even function of k . Calculating each term separately, the results are

$$
\begin{align*}
& \left.\mathrm{H}_{0}\left|Q \overline{\mathrm{Q}}>_{\mathrm{P}}=\int_{0}^{\infty} \mathrm{dk}\left(\mathrm{E}_{\mathrm{k}}+\mathrm{E}_{\mathrm{P}-\mathrm{k}}\right) \phi(\mathrm{P}, \mathrm{k}) \mathrm{A}^{+\mathrm{a}}(\mathrm{k}) \mathrm{B}^{+\mathrm{a}}(\mathrm{P}-\mathrm{k})\right| 0\right\rangle  \tag{3.17}\\
& \mathrm{H}_{\mathrm{MR}}{ }^{\mid Q \bar{Q}>_{\mathrm{P}}}=\frac{\mathrm{Ng}^{2}}{4 \pi} \int_{0}^{\infty} \mathrm{dk} \phi(\mathrm{P}, \mathrm{k}) \\
& \mathscr{P} \int_{0}^{\infty} \mathrm{dq}\left[\frac{\mathrm{U}(\mathrm{k}, \mathrm{q})}{(\mathrm{k}-\mathrm{q})^{2}}+\frac{\mathrm{V}(\mathrm{q}, \mathrm{P}-\mathrm{k})}{(\mathrm{k}+\mathrm{q}-\mathrm{P})^{2}}-\frac{\mathrm{W}(\mathrm{q}, \mathrm{P}-\mathrm{k})}{(\mathrm{k}+\mathrm{q})^{2}}-\frac{\mathrm{W}(\mathrm{q}, \mathrm{P}-\mathrm{k})}{(\mathrm{q}-\mathrm{k}+\mathrm{P})^{2}}\right] A^{+\mathrm{a}}(\mathrm{k}) \mathrm{B}^{+\mathrm{a}}(\mathrm{P}-\mathrm{k})|0\rangle \tag{3.18}
\end{align*}
$$

where

$$
\begin{align*}
& U(p, q)=u^{+}(p) u(q) u^{+}(q) u(p) \\
& V(p, q)=v^{+}(p) v(q) v^{+}(q) v(p)  \tag{3.19}\\
& W(p, q)=v^{+}(p) u(q) u^{+}(q) v(p)
\end{align*}
$$

Notice that in the representation we are working the quark propagator, which appears as the middle term above, is

$$
\begin{align*}
u(q) u^{+}(q) & =\frac{1}{E_{q}}\left(E_{q} \gamma_{0}-q \gamma_{1}+m\right) \gamma^{o} \\
& =\frac{1}{E_{q}}\left(E_{q}+q \gamma_{5}+m \gamma_{0}\right) \tag{3.20}
\end{align*}
$$

Thus the first term in (3.18) is

$$
\begin{equation*}
\frac{\mathrm{Ng}^{2}}{4 \pi} \int_{0}^{\infty} \mathrm{dk} \phi(\mathrm{P}, \mathrm{k}) \mathscr{P} \int_{0}^{\infty} \mathrm{dq} \mathrm{u}{ }^{+}(\mathrm{k})\left(\frac{\mathrm{E}_{\mathrm{q}}+\mathrm{q} \gamma_{5}+\mathrm{m} \gamma_{0}}{\mathrm{E}_{\mathrm{q}}}\right) \mathrm{u}(\mathrm{k}) \times \frac{1}{(\mathrm{k}-\mathrm{q})^{2}} \tag{3.21}
\end{equation*}
$$

Using the explicit representation for the free spinors

$$
\begin{equation*}
u(k)=\frac{1}{\sqrt{2 \mathrm{E}(\mathrm{E}+\mathrm{m})}}\binom{\mathrm{E}+\mathrm{m}}{\mathrm{k}} \tag{3.22}
\end{equation*}
$$

this becomes

$$
\begin{equation*}
\frac{\mathrm{Ng}^{2}}{4 \pi} \int_{0}^{\infty} \mathrm{dk} \phi(\mathrm{P}, \mathrm{k}) \int_{0}^{\infty} \mathrm{dq}\left[\frac{\mathrm{E}_{\mathrm{k}} \mathrm{E}_{\mathrm{q}}+\mathrm{kq}+\mathrm{m}^{2}}{\mathrm{E}_{\mathrm{k}} \mathrm{E}_{\mathrm{q}}}\right] \frac{\mathscr{P}}{(\mathrm{k}-\mathrm{q})^{2}} \tag{3.21a}
\end{equation*}
$$

The sign of the first term in the integrand may be changed freely because its contribution to the integral is zero. So the numerator vanishes quadratically when $\mathrm{p}=\mathrm{q}$, therefore the principal value symbol can be dropped. Thus (3.21a) becomes

$$
\begin{equation*}
\frac{\mathrm{Ng}^{2}}{4 \pi} \int_{0}^{\infty} \mathrm{dk} \frac{\phi(\mathrm{P}, \mathrm{k})}{\mathrm{E}_{\mathrm{k}}} \int_{0}^{\infty} \mathrm{dq} \frac{1}{\mathrm{E}_{\mathrm{q}}}\left(\mathrm{~m}^{2}+\mathrm{kq}-\mathrm{E}_{\mathrm{k}} \mathrm{E}_{\mathrm{q}}\right) \frac{1}{(\mathrm{k}-\mathrm{q})^{2}} \tag{3.21b}
\end{equation*}
$$

Introducing the new variables, $\mathrm{q}=\mathrm{m} \sinh \mathrm{x}$ and $\mathrm{k}=\mathrm{m} \sinh \mathrm{c}$, the q -integral is done easily, and the result is

$$
\begin{equation*}
-\frac{\mathrm{Ng}^{2}}{2 \pi} \int_{0}^{\infty} \mathrm{dk} \frac{\phi(\mathrm{P}, \mathrm{k})}{\mathrm{E}_{\mathrm{k}}} \tag{3.22}
\end{equation*}
$$

Using the same integrations for the other three terms in (3.18) we get zero, for W-terms, and a similar result for the second term. So finally

$$
\begin{equation*}
\left.\mathrm{H}_{\mathrm{MR}}{ }^{(\mathrm{Q} \overline{\mathrm{Q}})_{\mathrm{P}}}=-\frac{\mathrm{Ng}^{2}}{2 \pi} \int_{0}^{\infty} \mathrm{dk}\left(\frac{1}{\mathrm{E}_{\mathrm{k}}}+\frac{1}{\mathrm{E}_{\mathrm{P}-\mathrm{k}}}\right) \phi(\mathrm{P}, \mathrm{k}) \times \mathrm{A}^{+\mathrm{a}}(\mathrm{k}) \mathrm{B}^{+\mathrm{a}}(\mathrm{P}-\mathrm{k}) \right\rvert\, 0> \tag{3.23}
\end{equation*}
$$

Moving on to the Coulomb term in (3.13), we have

$$
\begin{gather*}
H_{c} \left\lvert\, Q \bar{Q}>=\frac{N g^{2}}{4 \pi} \int_{0}^{\infty} d p d p^{\prime} d q d q^{\prime} \frac{\mathscr{P}}{\left(p-p^{\prime}\right)^{2}}\left\{\int_{0}^{\infty} d k \delta\left(p^{\prime}-p^{\prime}+q^{\prime}-q\right) \delta\left(p^{\prime}-k\right) \times\right.\right. \\
\times \delta\left(p^{\prime}-k\right) \delta(q+k-P) u^{+}(p) u\left(p^{\prime}\right) v^{+}(q) v\left(q^{\prime}\right) \phi(k) A^{+a}(p) B^{+a}\left(q^{\prime}\right) \mid 0> \\
-\int_{0}^{\infty} d k \delta\left(p-p^{\prime}+q^{\prime}-q\right) \delta\left(q^{\prime}-k\right) \delta(p+k-P) v^{+}(p) v\left(p^{\prime}\right) u^{+}(q) u\left(q^{\prime}\right) \times \\
\left.\times \phi(k) B^{+a}\left(p^{\prime}\right) A^{+a}(q) \mid 0>\right\} \tag{3.24}
\end{gather*}
$$

which simplifies to

$$
\begin{gather*}
\mathrm{H}_{\mathrm{c}}|Q \bar{Q}\rangle_{\mathrm{P}}=\frac{\mathrm{Ng}^{2}}{2 \pi} \int_{0}^{\infty} \mathrm{dp} \int_{0}^{\infty} \mathrm{dk} \frac{\mathscr{P}}{\left(\mathrm{p}-\mathrm{k}^{2}\right)} \phi(\mathrm{P}, \mathrm{k})\left[\mathrm{u}^{+}(\mathrm{p}) \mathrm{u}(\mathrm{k}) \mathrm{v}^{+}(\mathrm{P}-\mathrm{k}) \mathrm{v}(\mathrm{P}-\mathrm{p})\right] \\
\mathrm{A}^{+\mathrm{a}}(\mathrm{p}) \mathrm{B}^{+\mathrm{a}}(\mathrm{P}-\mathrm{p})|0\rangle \tag{3.25}
\end{gather*}
$$

Noting that the meson representation $\mid Q Q>_{P}$ sets a constraint on the variable $\mathrm{p}, 0<\mathrm{p}<\mathrm{P}$, we get after a change of variable the form

$$
\begin{equation*}
\mathrm{H}_{\mathrm{c}}|\mathrm{QQ}\rangle_{\mathrm{P}}=\frac{\mathrm{Ng}^{2}}{2 \pi} \int_{0}^{\infty} \mathrm{dk} \mathscr{P} \int_{0}^{\infty} \mathrm{dp} \frac{\mathrm{~K}(\mathrm{P}, \mathrm{k}, \mathrm{p})}{(\mathrm{p}-\mathrm{k})^{2}} \phi(\mathrm{P}, \mathrm{k}) \mathrm{A}^{+\mathrm{a}}(\mathrm{k}) \mathrm{B}^{+\mathrm{a}}(\mathrm{P}-\mathrm{k})|0\rangle \tag{3.26}
\end{equation*}
$$

with

$$
K(P, k, p)=u^{+}(p) u(k) v^{+}(P-k) v(P-p)
$$

While the annihilation term is down by a factor of $1 / \mathrm{N}$ compared to we give for the sake of completeness the corresponding equation (3.16).

$$
\begin{align*}
& \mathrm{H}_{\mathrm{a}} \left\lvert\, \mathrm{QQ}>=\frac{\mathrm{g}^{2}}{4 \pi} \int_{0}^{\infty} \mathrm{dp} \int_{0}^{\infty} \mathrm{dk} \frac{\mathscr{P}}{(\mathrm{p}+\mathrm{k})^{2}} \mathrm{u}^{+}(\mathrm{p}) \mathrm{v}(\mathrm{P}-\mathrm{p}) \mathrm{v}^{+}(\mathrm{P}-\mathrm{k}) \mathrm{u}(\mathrm{k})\right. \\
& \phi(\mathrm{P}, \mathrm{k}) \mathrm{A}^{+\mathrm{a}}(\mathrm{p}) \mathrm{B}^{+\mathrm{a}}(\mathrm{P}-\mathrm{p}) \mid 0> \\
&-\frac{\mathrm{g}^{2}}{4 \pi} \int_{0}^{\infty} \mathrm{dq} \int_{0}^{\infty} \mathrm{dk} \frac{\mathscr{P}}{(\mathrm{q}+\mathrm{k})^{2}} \mathrm{v}^{+}(\mathrm{P}-\mathrm{k}) \mathrm{u}(\mathrm{k}) \mathrm{u}^{+}(\mathrm{q}) \mathrm{v}(\mathrm{P}-\mathrm{q}) \\
& \mathrm{A}^{+\mathrm{a}}(\mathrm{q}) \mathrm{B}^{+\mathrm{a}}(\mathrm{P}-\mathrm{q}) \phi(\mathrm{P}, \mathrm{k}) \mid 0> \tag{3.27}
\end{align*}
$$

In the event that the group is $U(1)$, so that (1) become the massive Schwinger model $H_{a}$ can be neglected on the grounds that it is of order $\hbar^{2}$. In this case, the weak coupling limit has been studied by Coleman ${ }^{22}$ who made use of semiclassical approximations while preserving relativistic kinematics.

Gathering all contributions (3.17), (3.19) and (3.26), we reach the eigenvalue equation

$$
\begin{gather*}
\mathrm{E}_{\mathrm{P}} \phi(\mathrm{P}, \mathrm{k})=\left[\left(\mathrm{E}_{\mathrm{k}}+\mathrm{E}_{\mathrm{P}-\mathrm{k}}\right)-\frac{\mathrm{Ng}^{2}}{2 \pi}\left(\frac{1}{\mathrm{E}_{\mathrm{k}}}+\frac{1}{\mathrm{E}_{\mathrm{P}-\mathrm{k}}}\right)\right] \phi(\mathrm{P}, \mathrm{k}) \\
+\frac{\mathrm{Ng}^{2}}{2 \pi} \mathscr{P} \int_{0}^{\infty} \mathrm{dp} \frac{\mathrm{~K}(\mathrm{P} ; \mathrm{k}, \mathrm{p})}{(\mathrm{p}-\mathrm{k})^{2}} \phi(\mathrm{P}, \mathrm{p}) \tag{3.28}
\end{gather*}
$$

where

$$
\begin{aligned}
& E_{k}=\sqrt{k^{2}+m^{2}} \\
& K(P ; k, p)=u^{+}(p) u(k) v^{+}(P-k) v(P-p)
\end{aligned}
$$

By using the equality

$$
\begin{equation*}
\sqrt{\mathrm{E}_{\mathrm{k}}-\frac{\mathrm{Ng}^{2}}{\pi}}=\mathrm{E}_{\mathrm{k}}-\frac{\mathrm{Ng}^{2}}{2 \pi} \frac{1}{\mathrm{E}_{\mathrm{k}}}+\mathscr{O}\left(\mathrm{Ng}^{2}\right)^{2} \tag{3.29}
\end{equation*}
$$

which is good in the weak coupling limit, (3.28) can be cast into the following form:

$$
\begin{gather*}
\mathrm{E}_{\mathrm{P}} \phi(\mathrm{P}, \mathrm{k})=\left[\sqrt{\mathrm{k}^{2}+\left(\mathrm{m}^{2}-\frac{\mathrm{Ng}^{2}}{\pi}\right)}+\sqrt{(\mathrm{P}-\mathrm{k})^{2}+\left(\mathrm{m}^{2}-\frac{\mathrm{Ng}^{2}}{\pi}\right)}\right] \phi(\mathrm{P}, \mathrm{k}) \\
+\frac{\mathrm{Ng}^{2}}{2 \pi} \mathscr{P} \int_{0}^{\infty} \mathrm{dp} \frac{\mathrm{~K}(\mathrm{P} ; \mathrm{k}, \mathrm{p})}{(\mathrm{p}-\mathrm{k})^{2}} \phi(\mathrm{P}, \mathrm{p}) \tag{3.30}
\end{gather*}
$$

which exposes the "mass renormalization" explicitly.
It is obvious that this equation is not covariant, due to the explicit dependence on $P$, the momentum of the bound state. Of all the frames, the most convenient one is clearly the center-of-mass frame, where $\mathrm{P}=0$ :

$$
\begin{align*}
\mu \phi(\mathrm{O}, \mathrm{k})=2 & \sqrt{\mathrm{k}^{2}+\left(\mathrm{m}^{2}-\frac{\mathrm{Ng}^{2}}{\pi}\right)} \phi(\mathrm{O}, \mathrm{k}) \\
& +\frac{\mathrm{Ng}^{2}}{2 \pi} \mathscr{P} \int_{0}^{\infty} \mathrm{dp} \frac{\mathrm{~K}(\mathrm{O} ; \mathrm{p}, \mathrm{k})}{(\mathrm{p}-\mathrm{k})^{2}} \phi(\mathrm{O}, \mathrm{p}) \tag{3.31}
\end{align*}
$$

where, now

$$
\begin{aligned}
K(P ; p, k) & =u^{+}(p) u(k) v^{+}(-k) v(-p) \\
& =\left[\frac{1}{2 E_{p}\left(E_{p}+m\right) 2 E_{k}\left(E_{k}+m\right)}\right] \times\left[\left(E_{p}+m\right)\left(E_{k}+m\right)+p k\right]^{2}
\end{aligned}
$$

The spectrum of this particular equation is being numerically studied by Hanson et al. ${ }^{20}$

## IV. BOSONIZATION AND CONCLUDING REMARKS

What have we learnt from the foregoing exercise? First we have seen a key feature of null plane quantization of TDQCD, namely its lack of sensitivity to the specific form of the infrared cutoff. It has been known that the massive Schwinger model and hence also TDQCD suffer from very severe infrared divergences induced by the bare quark masses. ${ }^{23}$ There appears to be a softening of these divergences in the $A_{-}=0$ gauge reflected in the indifference to cutoff, possibly due to the peculiarities of the null plane quantization. ${ }^{12}$ It does not know about masses and its vacuum is the bare vacuum. These features account for the simplest spinor kinematics, the intuitive picture of constituents in a relativistic bound state.

In the axial (or Coulomb) gauge, none of the above properties are available. Thus the interaction kernel (3.28) is plagued with mass dependent kinematic factors. Nonrelativistic approximations need to be made to obtain a more tractable bound state equation. No clear picture of a relativistic bound state is available and our assumption of a free field expansion at fixed time is thereby suspect. Indeed the Coulomb gauge computation performed here should be taken only heuristically. It is known ${ }^{24}$ that no bona fide quark field operators can be constructed in this gauge due to the linearly rising potential. Since a simple principal value cutoff fails, a more involved technique such as a mass and coupling dependent cutoff procedure could be tried. A more illuminating approach, we believe, would be that of Lowenstein and Swieca ${ }^{24}$ who get to the Coulomb gauge via a limiting procedure starting from covariant gauges. This is a difficult problem we are presently studying.

However irrespective of possible covariance problems, our method of attack of the bound state problem in the $1 / \mathrm{N}$ expansion is a straightforward one. The
usual apparatus of Schrodinger perturbation theory can be in principle applied to computed higher $1 / \mathrm{N}$ corrections and to handle problem of bound state scattering and the analysis of form factors etc. ${ }^{8}$ In the instance of the massive Schwinger model recently studied by Coleman in the axial gauge, our results in Section II carry over provided the coupling is weak, $\mathrm{g} \ll \mathrm{m}$. The null plane quantization of the massive Schwinger model gives in our opinion a more tractable as well as attractive resolution of the weak coupling structure of the model, when contrasted with the spinor complexities of the axial gauge formalism.

Finally another limit of interest to TDQCD is the strong coupling limit believed to be of genuine relevance to the infrared problem. In two dimensions, a handle on this strong coupling regime is possible without going to a lattice thanks to a correspondence between say the massive Schwinger model and the Sine Gordon theory. The strong coupling limit in the first model corresponds remarkably to the weak coupling limit in the second and hence mades it computable. Recent works on the bosonization of the massive SU(N) Thirring models ${ }^{25}, 26$ allow us to write down the bosonic equivalent of (2.1). We shall be very brief. For more details, the reader is referred to the quoted literature.

We work in the interaction representation and in the axial gauge $A_{1}^{a}=0$ of TDQCD.

It is known ${ }^{25}$ that the $\mathrm{SU}(\mathrm{N})$ invariant free massive Thirring theory

$$
\begin{equation*}
\mathscr{L}=\sum_{\mathrm{i}=1}^{\mathrm{N}} \bar{\psi}^{\mathrm{i}}(\mathrm{i} \not \emptyset-\mathrm{m}) \psi_{\mathrm{i}} \tag{4.1}
\end{equation*}
$$

is equivalent to a theory of $N$ Bose fields $\phi_{i}$

$$
\begin{equation*}
\mathrm{H}_{\mathrm{e}}=\mathrm{N}_{\mathrm{m}} \sum_{\mathrm{i}=1}^{\mathrm{N}}\left(\frac{1}{2} \pi_{\mathrm{i}}^{2}+\frac{1}{2}\left(\partial_{1} \phi_{\mathrm{i}}\right)^{2}-\mathrm{cm}^{2} \cos 2 \sqrt{\pi} \phi_{\mathrm{i}}\right) \tag{4.2}
\end{equation*}
$$

$\mathrm{N}_{\mathrm{m}}$ denotes normal ordering with respect to the mass m , c is a constant of no relevance to our consideration.

The boson forms of the N -two component spinors are ${ }^{26}$

$$
\begin{gather*}
\psi_{L, R}^{a}(\mathrm{x}, \mathrm{t})=\frac{1}{\sqrt{\Omega}}: \mathrm{e}^{\mp 2 \mathrm{i} \sqrt{\pi} \Phi_{\mathrm{L}, \mathrm{R}}^{\mathrm{a}}(\mathrm{x}, \mathrm{t})}:{\chi_{\mathrm{L}, \mathrm{R}}}_{\mathrm{a}}^{\mathrm{a}=1, \ldots, \mathrm{~N}}
\end{gather*}
$$

$\Omega$ is the quantization box length, the $\chi_{\mathrm{L}, \mathrm{R}}^{\mathrm{a}}$ are nondynamical anticommuting operators which Banks et al. ${ }^{26}$ need to introduce. Their properties are $\chi_{a}=\chi_{a}^{+}$, $\left\{\chi_{\mathrm{a}}, \chi_{\beta}\right\}=2 \delta_{\alpha \beta}$ and $\mathrm{P}_{\chi_{1}} \mathrm{P}^{+}=\chi_{2}$ where P is the parity operator. Then the color current $\mathrm{J}_{0}=\mathrm{J}_{0}^{\mathrm{a}} \mathrm{T}^{\mathrm{a}}$ is given

$$
\begin{align*}
J_{0}^{a}= & \sum_{a \neq b}^{N} \frac{\lambda_{a b}^{(i)}}{2 \Omega} \chi_{1}^{a} \chi_{1}^{b}: e^{2 i \sqrt{\pi}\left(\Phi_{L}^{a}-\Phi_{L}^{b}\right)}:+\chi_{2}^{a} \chi_{2}^{b}: e^{-2 i \sqrt{\pi}\left(\Phi_{R}^{a}-\Phi_{R}^{a}\right)}: \\
& +\frac{1}{\sqrt{\pi}} \sum_{a=1}^{N} \frac{\lambda_{a a}^{(i)}}{2} \partial_{x} \Phi^{a} \quad i=1, \ldots, N^{2}-1 \tag{4.4}
\end{align*}
$$

Since in the charge zero sector, the axial gauge Hamiltonian is

$$
\begin{equation*}
\mathscr{H}=\sum_{\mathrm{i}}^{\mathrm{N}} \bar{\psi}^{\mathrm{i}}\left(\mathrm{i} \gamma_{1} \partial_{1}+\mathrm{m}\right) \psi_{\mathrm{i}}+\frac{1}{2}\left(\mathrm{~F}_{01}^{\mathrm{a}}\right)^{2} \tag{4.5}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathrm{F}_{01}^{\mathrm{a}}=-\mathrm{g} \partial_{1}^{-1} \mathrm{~J}_{0}^{\mathrm{a}} \tag{4.6}
\end{equation*}
$$

the bosonization, TDQCD is readily accomplished using (4.2) and (4.4). For $\mathrm{G}=\mathrm{U}(1)$, it reduces to the bosonization of the massive Schwinger model. ${ }^{22}$ For the non-Abelian case, the Bose form looks rather intricate. Its study connected with the strong coupling limit $\mathrm{g}^{2} \gg \mathrm{~m}^{2}$ of TDQCD will be the object of another work.

Note: After we completed this work, M. K. Prasad kindly informed us of a paper by M. S. Marinov, A. M. Perelomov and M. V. Terent'ev, ZhETF Pis. Red 20, 7 (1974) 494 [JETP Lett. 20, 7 (1974) 225]. These authors proposed the same method as ours to obtain the spectrum of 't Hooft model in the $A_{-}=0$ gauge four Section II). However, their resulting bound state equation is incorrect since it lacks the mass renormalization contributions present in 't Hooft equation and in ours.

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