SLAC-PUB-1784
July 1976
(T)

RENORMALISATION GROUP CALCULATIONS FOR REGGEON FIELD THEORY ON A TRANSVERSE LATTICE*

John L. Cardy†

Department of Physics
University of California, Santa Barbara CA 93106

and

Stanford Linear Accelerator Center
Stanford University, Stanford CA 94305

ABSTRACT

We describe a numerical investigation of an analogue quantum spin lattice model for Reggeon field theory, when the bare Pomeron intercept α_0 is close to its critical value α_{0c}. For $\alpha_0 \leq \alpha_{0c}$ we derive the usual scaling laws and calculate approximate values for the critical exponents. When $\alpha_0 > \alpha_{0c}$, the Reggeon field ψ does not gain a nonzero vacuum expectation value, and there is no spontaneous symmetry breaking. Instead, there appears a state $|1\rangle$ which is degenerate with the ground state $|0\rangle$. The order parameter, σ, is proportional to the transition matrix element $\langle 0|\psi|1\rangle$. As $\alpha_0 \to \alpha_{0c}$ from above, σ vanishes like $(\alpha_0 - \alpha_{0c})^\beta$, where β is a new exponent. The opacity of hadronic matter at infinite energy is roughly proportional to σ^2.

(Submitted to Nucl. Phys. B.)

---

*Work supported in part by the Energy Research and Development Administration and the National Science Foundation.

†Address after 1 September 1976: CERN, Geneva.

Recently, a quantum spin lattice system, which has the same critical behaviour as Reggeon field theory, has been proposed [1, 2]. Reggeon field theory [3] is a theory of quasiparticles which reside in impact parameter space and propagate in rapidity. Its critical behaviour claims to describe diffraction scattering at high energies and small momentum transfers, when the leading vacuum singularity has an intercept near one. Several methods have been employed to investigate this critical behaviour. The pure field theoretic approach works well when the number D of dimensions of impact parameter space is near four [4, 5], or exactly equal to zero [6]. Classical analogue spin systems, which discretise both impact parameter space and rapidity, have also been used [7, 8]. While they lead to some interesting results, they tend to be too involved for practical calculations, and, moreover, give little insight into the nature of the phase transition.

The new quantum analogue spin system has the advantage of having one less dimension, being a Hamiltonian formulation on a lattice in impact parameter space. It is therefore possible to carry out practical renormalisation group calculations, even in D=2. More importantly, it gives considerable insight into the nature of the ordered phase, when the bare Pomeron intercept α_0 is greater than its critical value α_{0c}. A previous investigation of this question, by Abarbanel et al. [9], using field theory and based on an analogy with real ϕ^4 theory, came to the paradoxical conclusion that Lorentz invariance of the scattering amplitude was spontaneously broken. Recently, in an important paper, Amati et al. [1] have come to different, more satisfactory, conclusions using the quantum spin model. Their results are based on variational calculations and mean field theory, and will be described shortly.

Since the quantum spin Hamiltonian appears to be too difficult to solve exactly, even in D=1, we investigated it using numerical methods, by constructing an approximate renormalisation group transformation which is an explicit realisation of the ideas of Kadanoff [10] and Wilson [11]. These same methods have recently been applied to an Ising model in a transverse magnetic field (which is the quantum spin analogue of real ϕ^4 field theory), to obtain reasonable values for the critical exponents [12].

Our calculations confirm and amplify most of the results of Amati et al. [1]. While the quantitative results for the exponents are not very good, we believe the qualitative features of the calculations to be correct. We have tested this by making different approximations. The qualitative features are preserved.

The main conclusions, which largely reproduce those of Amati et al. [1] are as follows:

1) In both D=1 and D=2 the system undergoes a second order phase transition, characterised by the onset of long range order. At the critical point the Green's functions satisfy the scaling laws suggested by the ϵ-expansion [4,5]. For $\alpha_0<\alpha_{0c}$ correlations are short-range, and there is an energy gap Δ (which represents the distance of the intercept of the physical Pomeron pole below one). Δ is related to $(\alpha_{0c}-\alpha_0)$ by an exponent: $\Delta \propto (\alpha_{0c}-\alpha_0)^{\nu}$.

2) For $\alpha_0>\alpha_{0c}$, in the ordered phase, the system differs considerably from ϕ^4 theory and a conventional ferromagnet. The ground state remains the original ground state $|0\rangle$ which is annihilated by the bare quanta of the Pomeron field ψ. In addition, there is (at least in the analogue spin theory) an isolated zero-momentum state $|1\rangle$ which remains degenerate with $|0\rangle$ for $\alpha_0>\alpha_{0c}$. The matrix elements $\langle 0|\psi|1\rangle$ and $\langle 1|\psi|1\rangle$ are nonzero in the ordered phase. In the disordered phase $(\alpha_0<\alpha_{0c})$, they are zero.

In ϕ^4 theory, in the same approach [12, 13] one also finds degenerate states $|0\rangle$ and $|1\rangle$, but $|0\rangle$ is not the unperturbed ground state, and $\langle 0|\phi|1\rangle = \langle 1|\phi|0\rangle \neq 0$, $\langle 0|\phi|0\rangle = \langle 1|\phi|1\rangle = 0$. As a result, one can, in this case, define new ground states $|0\rangle_{\pm} = (1/\sqrt{2})(|0\rangle \pm |1\rangle)$, which have the property that there are no nonzero transition matrix elements between them. One is then free to choose either as the physical vacuum. In these states, the field operator ϕ has a nonzero expectation value, and the original symmetry $\phi \to -\phi$ is spontaneously broken. This expectation value is the analogue of the magnetisation in a ferromagnet, and plays the role of an order parameter.

In Reggeon field theory, because $\langle 1|\psi|0\rangle = \langle 0|\psi|0\rangle = 0$, one cannot make such a construction. Also, because it turns out that

$$\langle 0|\psi|1\rangle = \langle 1|\bar{\psi}|0\rangle \tag{1.1}$$

and

$$\langle 1|\psi|1\rangle = \langle 1|\bar{\psi}|1\rangle \tag{1.2}$$

the $\psi \leftrightarrow \bar{\psi}$ symmetry is unbroken in the states $|0\rangle$ and $|1\rangle$. The relevant order parameter, $\sigma \propto \langle 0|\psi|1\rangle$, cannot be expressed as a vacuum expectation value. Nevertheless, it possesses all the usual features of an order parameter. As $\alpha_0 \to \alpha_{0c}$ from above, it obeys a power law

$$\sigma \propto (\alpha_0 - \alpha_{0c})^{\beta} \tag{1.3}$$

where β is a new exponent, which turns out to be related to the other three principal exponents γ, z, κ by a scaling relation. As $\alpha_0 \to \infty$, $\sigma \to 1$ (in appropriate units). The behaviour of σ, as calculated numerically, is shown in fig. 1.

3) Because $|1\rangle$ can occur as an intermediate state in the propagation of the Pomeron (since $\langle 0|\psi|1\rangle \neq 0$), it dominates at large rapidities. As a consequence, we find that the elastic amplitude at large rapidity has the asymptotic

behaviour

$$A(Y,B) \rightarrow \text{is } \beta_a \beta_b \sigma^2 \tag{1.4}$$

where β_a, β_b are renormalised couplings to the external particles, and depend weakly on α_0 near α_{0c}. Thus the opacity of hadronic matter at infinite energy vanishes like σ^2 as $\alpha_0 \rightarrow \alpha_{0c}$, and for $\alpha_0 \rightarrow \infty$, hadrons become increasingly black. The behaviour of A(Y, B) for $Y \lesssim B$ is outside the scope of our approximations. Amati et al. [1] suggest that $A \rightarrow 0$ for $Y \lesssim B$, leading to an expanding grey disc picture which satisfies the Froissart bound.

The results of our approximate calculations of the exponents are shown in Table 1. Obviously they are not very good. However, except for the exponent z, the results for the two-site cells (which, for reasons that will appear, are more trustworthy) are of the expected order of magnitude. The exponent z, which measures the anomalous dimension between impact parameter and rapidity, is probably estimated badly because the renormalisation group transformation treats them in quite different ways. The same thing happens in the Ising model, where z should equal two, but turns out quite large [12]. The results for the D=1 Ising model in a transverse field (which is equivalent to the D=2 classical Ising model), taken from the analogous work of Jafarey and Stoeckly [12], are shown in Table 2, and compared with the exact values.

The layout of this paper is as follows. After a brief description of the derivation of the quantum spin model in Section 2, we construct an explicit renormalisation group transformation in D=1. This turns out to have nearly all the salient features of the more complicated D=2 calculations, which we have relegated to an Appendix. Thus all our discussion will be based on the D=1 model, and we shall merely point out when a particular result is peculiar to that dimension. In Section 4 we show how to derive the usual scaling laws for the Green's functions

and discuss the nature of the spectrum for $\alpha_0 < \alpha_{0c}$. In Section 5 we investigate the nature of the ordered phase, and the order parameter, and explicitly construct the state $|1\rangle$. This turns out to be particularly simple in the limits $\alpha_0 \to +\infty$ or $r_0 \to 0$. By considering a typical Green's function, we show that $|1\rangle$ is an isolated state, at least for the quantum spin model, and not connected with any $k \neq 0$ spectrum. In Section 6 we discuss the implications of this picture for the asymptotic scattering amplitude. Finally, we discuss the limitations of our treatment, and the considerable possibilities available for its improvement.

2. QUANTUM SPIN MODEL

For the purposes of orientation and establishing notation, we first give a brief derivation of the quantum spin model. For further details, the reader is referred to refs. [1, 2]. The method parallels that of Stoeckly and Scalapino [13] for ϕ^4 theory. It relies heavily on the detailed analysis of the D=0 problem [6].

The Hamiltonian for a Reggeon field theory with a triple-Regge coupling is

$$H = \int d^D x \left[\alpha_0' \nabla\bar{\psi} \cdot \nabla\psi + \Delta_0 \bar{\psi}\psi + \frac{i r_0}{2} \bar{\psi}(\psi + \bar{\psi})\, \psi \right] \tag{2.1}$$

where the field operators $\psi, \bar{\psi}$ satisfy canonical commutation relations

$$\left[\psi(x), \bar{\psi}(x')\right] = \delta^{(D)}(x - x') \tag{2.2}$$

On going to a lattice in impact parameter space of spacing b, (2.1) can be replaced by

$$H = \sum_j H_{0j} + \frac{\alpha_0'}{b^2} \sum_{\langle ij \rangle} (\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j) \tag{2.3}$$

where

$$H_{0j} = \Delta_0 \bar{\psi}_j \psi_j + \frac{1}{2} i r_0 b^{-D/2} \bar{\psi}_j (\psi_j + \bar{\psi}_j)\, \psi_j \tag{2.4}$$

and we have rescaled the fields

$$\psi(x_j) = b^{-D/2} \psi_j \quad , \qquad \bar{\psi}(x_j) = b^{-D/2} \bar{\psi}_j \tag{2.5}$$

so that

$$\left[\psi_i, \bar{\psi}_j\right] = \delta_{ij} \tag{2.6}$$

In (2.3) the second term represents a sum over pairs of nearest neighbors i, j. If we first consider the case $\alpha_0'=0$, we have a system of uncoupled D=0 systems. For r_0 small enough, it is known [14, 15] that the phase transition occurs at a large negative value of $\Delta_0=1-\alpha_0$. In that regime, Bronzan et al. and Jengo [6] have shown that the spectrum of H_{0j} is simple. It consists of a ground state, $|0\rangle_j$, which is the state annihilated by ψ_j, and a low-lying first excited state, $|1\rangle_j$, with energy

$$\epsilon \approx \sqrt{\frac{2}{\pi}}\, \frac{\Delta_0^2 b^{D/2}}{|r_0|} \exp\left(-\frac{2\Delta_0^2 b^D}{r_0^2}\right) \tag{2.7}$$

All the other states have much higher energy (of $0(|\Delta_0|)$). Therefore, if α_0' is small, we expect that only product states of $|0\rangle_j$ and $|1\rangle_j$ will have significant matrix elements with the low lying states of the full Hamiltonian H, which are relevant to high energy diffraction scattering. We therefore neglect all the high lying states of H_{0j}, and proceed to write H in a basis consisting of product states of $|0\rangle_j$ and $|1\rangle_j$.

Although this truncation is an approximation, it is in a sense an arbitrarily accurate one for calculating the exponents. The reason is that changing α_0' does not change the exponents, but only the value of α_{0c}. By choosing α_0' and r_0 sufficiently small, we can drive the neglected states to arbitrarily high energies. In the case of real ϕ^4 theory in two dimensions, the analogous procedure [13] gives the one-dimensional Ising model in a transverse magnetic field, which can be solved exactly to give the correct exponents [13, 16].

In order to write H in the truncated basis, we need the matrix elements of ψ_j and $\bar{\psi}_j$ between $|0\rangle_j$ and $|1\rangle_j$. From the work of Bronzan et al. [6] one finds

$$_j\langle 0|\psi_j|1\rangle_j = {}_j\langle 1|\bar{\psi}_j|0\rangle_j = 2|\Delta_0|b^{D/2}/r_0 \tag{2.8}$$

$$_j\langle 1|\psi_j|1\rangle_j = {}_j\langle 1|\bar{\psi}_j|1\rangle_j = 2i|\Delta_0|b^{D/2}/r_0 \tag{2.9}$$

It is interesting to note how the non-Hermiticity of H_0 manifests itself in (2.9).

If we introduce matrices

$$c = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad a = \begin{pmatrix} 0 & 1 \\ 0 & i \end{pmatrix} \quad \bar{a} = \begin{pmatrix} 0 & 0 \\ 1 & i \end{pmatrix} \tag{2.10}$$

we can write

$$\begin{aligned} H_{0j} &= \epsilon c_j \\ \psi_j &= \left[2|\Delta_0|b^{D/2}/r_0\right] a_j \\ \bar{\psi}_j &= \left[2|\Delta_0|b^{D/2}/r_0\right] \bar{a}_j \end{aligned} \tag{2.11}$$

Using the fact that $\bar{a}_j a_j = 0$, H finally has the form

$$H = \epsilon \sum_j c_j - \rho^2 \sum_{<ij>} (\bar{a}_i a_j + \bar{a}_j a_i) \tag{2.12}$$

where

$$\rho^2 = 4\alpha_0' b^{D-2} (\Delta_0/r_0)^2 \tag{2.13}$$

Equations (2.10), (2.12) give the form of H in which we shall use it. Obviously it can be expressed in terms of the Pauli matrices, which is why it represents an interacting system of spin one-half objects. It turns out that (2.12) is more convenient for renormalisation group calculations.

If H has a phase transition it will occur at a critical value of the ratio $T = \epsilon/\rho^2$, since it is only this which determines the nature of the spectrum. This will determine, via eqs. (2.7), (2.13) the critical value Δ_{0c}. Note that this

quantity will not have the logarithmic dependence on b which it is known to have [14, 15] for small b in D=2. This is because the two-level approximation breaks down as the lattice spacing goes to zero, since α_0'/b^2 becomes large.

There is a subtle point in using the quantum spin Hamiltonian. As can be seen explicitly, the states $|0\rangle_j$ and $|1\rangle_j$ do not saturate the commutation relations (2.6), so we cannot use the matrix representations (2.10) to derive equations of motion for the dynamical variables ψ_j, $\bar{\psi}_j$. Rather we must go back to the correct equations of motion and take matrix elements between $|0\rangle_j$ and $|1\rangle_j$. As another example, we see that the matrix a satisfies

$$a^n = i^{n-1}a \quad \text{for} \quad n \geq 1 \ . \tag{2.14}$$

However, in Appendix B, we prove that $\langle 0|\psi_j^n|1\rangle \neq i^{n-1}\langle 0|\psi_j|1\rangle$ in general.

3. RENORMALISATION GROUP METHOD

We now restrict ourselves to the case D=1. Two possible approximations in D=2 are considered in Appendix A. We begin by grouping the sites of the one dimensional lattice into pairs. Each pair forms a cell. We seek to write a Hamiltonian involving only quantities defined for each cell which will have the same form as H and exactly the same matrix elements. Obviously this is impossible to do exactly, since each site has two states, and each cell has four states. By solving the Hamiltonian for an isolated cell exactly, we find that there are two states which always lie lowest. We make the approximation of neglecting the other two cell states, and demand that our new cell Hamiltonian H' have the same matrix elements between the states we retain as did the initial Hamiltonian H. We find that, by making a small generalisation of the form of H, that this is possible. This then constitutes a renormalisation group transformation in the sense of Wilson [11], since H' is equivalent to H, but has different parameters, and has all lengths scaled by a factor of two.

Obviously the truncation of states described above is on a quite different footing from that involved in deriving the quantum spin model in the first place. We shall see, however, that it is a very good approximation for the low lying states for $\alpha_0 >> \alpha_{0c}$, and moderately good at the critical point.

The type of real space renormalisation group transformation we use is rather different from those considered by Niemeijer and van Leeuwen [17], and Kadanoff and Houghton [18], for classical spin systems, and applied to Reggeon field theory in ref. [7]. These methods seem difficult to apply to quantum systems. Recently Friedman [19] has developed a method which involves projections onto the ground state, which works well for the Ising model in a transverse field. Unfortunately it fails for Reggeon field theory, since the ground state turns out to be a renormalisation group invariant.

Before constructing the transformation, we generalise the parameter space by writing

$$H = \epsilon \sum_j c_j - \sum_{<ij>} \left[\left(\rho_1 \sigma_j^+ + i\rho_2 c_j\right)\left(\rho_1 \sigma_i^- + i\rho_2 c_i\right) + (i \leftrightarrow j) \right] \tag{3.1}$$

where

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad c = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.2}$$

This reduces to (2.12) for $\rho_1 = \rho_2 = \rho$. We note that there is no underlying symmetry which requires $\rho_1 = \rho_2$. However our initial Hamiltonian should always have this form.

The Hamiltonian for a single isolated cell with sites j, j+1 is

$$H^{cell} = \epsilon \left(c_j + c_{j+1}\right) - \left[\left(\rho_1 \sigma_j^+ + \rho_2 c_j\right)\left(\rho_1 \sigma_{j+1}^- + \rho_2 c_{j+1}\right) + (+ \leftrightarrow -) \right] \tag{3.3}$$

With respect to the basis $\left\{|0\rangle_j|0\rangle_{j+1},\ |0\rangle_j|1\rangle_{j+1},\ |1\rangle_j|0\rangle_{j+1},\ |1\rangle_j|1\rangle_{j+1}\right\}$ it has the representation

$$H^{cell} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon & -\rho_1^2 & -i\rho_1\rho_2 \\ 0 & -\rho_1^2 & \epsilon & -i\rho_1\rho_2 \\ 0 & -i\rho_1\rho_2 & -i\rho_1\rho_2 & 2\epsilon+2\rho_2^2 \end{pmatrix} \tag{3.4}$$

H^{cell} is a complex symmetric matrix. This is true for an arbitrarily large cell, since σ^+ is the transpose of σ^-, and c is symmetric.

The eigenvalues are

$$\lambda = \begin{cases} 0 \\ \epsilon + \rho_1^2 \\ \frac{1}{2}\left\{3\epsilon+2\rho_2^2-\rho_1^2 \pm \left[\epsilon^2+4\epsilon\rho_2^2+2\epsilon\rho_1^2+\left(2\rho_2^2-\rho_1^2\right)^2\right]^{1/2}\right\} \end{cases} \tag{3.5}$$

The lowest state is just $|0\rangle_j|0\rangle_{j+1}$. The first excited state has energy

$$\epsilon' = \frac{1}{2}\left\{3\epsilon+2\rho_2^2-\rho_1^2-\left[\epsilon^2+4\epsilon\rho_2^2+2\epsilon\rho_1^2+\left(2\rho_2^2-\rho_1^2\right)^2\right]^{1/2}\right\} \tag{3.6}$$

For $\epsilon \ll \rho_1^2, \rho_2^2$ and $\rho_1 = \rho_2$, this state is nearly degenerate with the ground state and has energy $\epsilon' = 0(\epsilon^2)$. For $\epsilon >> \rho_1^2, \rho_2^2$ it is nearly degenerate with the state with energy $\epsilon+\rho_1^2$. In the intermediate region, it lies somewhere between. We denote this state by $|1\rangle_{cell}$. It has the form

$$|1\rangle_{cell} = A|0\rangle_j|1\rangle_{j+1} + A|1\rangle_j|0\rangle_{j+1} + iB|1\rangle_j|1\rangle_{j+1} \tag{3.7}$$

where

$$A = \frac{\rho_1\rho_2}{\left[2\rho_1^2\rho_2^2-\left(\rho_1^2+\epsilon'-\epsilon\right)^2\right]^{1/2}} \tag{3.8}$$

$$B = \frac{\rho_1^2 + \epsilon' - \epsilon}{\left[2\rho_1^2\rho_2^2 - \left(\rho_1^2 + \epsilon' - \epsilon\right)^2\right]^{1/2}} \tag{3.9}$$

A subtle point arises here, due to the non-Hermiticity of H^{cell}. If we want ${}_{cell}\langle 1|$ to be a left eigenstate of H^{cell}, we must define it to be

$${}_{cell}\langle 1| = A\ {}_j\langle 0|\ {}_{j+1}\langle 1| + A\ {}_j\langle 1|\ {}_{j+1}\langle 0| + iB\ {}_j\langle 1|\ {}_{j+1}\langle 1| \tag{3.10}$$

although A, B are real. The correct normalisation of (3.7) is then

$$2A^2 - B^2 = 1 \tag{3.11}$$

which is satisfied by (3.8), (3.9). We must therefore expect to meet some states with negative or zero norm.

We now want to write a new Hamiltonian H' which will have the same form as (3.1) where i, j now represent cells, and has the same matrix elements as H between all product states formed from $|0\rangle_{cell}$ and $|1\rangle_{cell}$. To do this, we note that

$$\begin{aligned}
{}_{cell}\langle 0|\sigma_j^+|1\rangle_{cell} &= A \\
{}_{cell}\langle 1|\sigma_j^-|0\rangle_{cell} &= A \\
{}_{cell}\langle 1|\sigma_j^+|1\rangle_{cell} &= iAB \\
{}_{cell}\langle 1|\sigma_j^-|1\rangle_{cell} &= iAB \\
{}_{cell}\langle 1|c_j|1\rangle_{cell} &= A^2 - B^2
\end{aligned} \tag{3.12}$$

with all other matrix elements zero.

Therefore, with respect to the cell basis $(\rho_1\sigma_j^{\pm} + \rho_2 c_j)$ has the same matrix elements as $(\rho_1'\sigma_{j'}^{\pm} + \rho_2' c_{j'})$, where $\sigma_{j'}^{\pm}$ and $c_{j'}$ are cell variables, as long as

$$\rho_1' = A\rho_1$$

$$\rho_2' = (A^2 - B^2)\rho_2 + AB\rho_1 \tag{3.13}$$

If we express H' in terms of the new cell variables and the new parameters ϵ', ρ_1', ρ_2', we are then assured that it has the same matrix elements as H between all product cell states. Equations (3.6), (3.13), with A, B defined as in (3.8), (3.9), define our approximate renormalisation group (RG) transformation. We remark that if $\rho_1 = \rho_2$ it is not necessary that $\rho_1' = \rho_2'$. This justifies our initial enlargement of the parameter space.

Fixed points and numerical results

Since only the ratios ϵ/ρ_1^2, ρ_2^2/ρ_1^2 are relevant in determining the spectrum, it is convenient to map the three-dimensional parameter space onto a triangle, shown in fig. 2, such that the distances from the labelled sides are respectively proportional to ϵ, ρ_1^2 and ρ_2^2. Our initial Hamiltonian corresponds to a point on the line $\rho_1^2 = \rho_2^2$, and the variable $T = \epsilon/\rho_1^2$ denotes its position on that line. There are two stable fixed points of the transformation which can be found analytically:

$$S_1: \quad \epsilon \to \text{constant}, \ \rho_2 \to 0, \ \rho_1/\rho_2 \to 0 \ .$$

$$S_2: \quad \rho_1 \sim \rho_2 \to \text{constant}, \ \epsilon \to 0 \ .$$

In addition, there is a fixed point U, located approximately at

$$\left(\epsilon/\rho_1^2\right)^* = 1.26 \ , \quad \left(\rho_2^2/\rho_1^2\right)^* = .80 \tag{3.14}$$

This point is unstable in ϵ/ρ_1^2 and stable in ρ_2^2/ρ_1^2. U lies on a critical line such that any point on this line is driven to U. As this happens, ϵ, ρ_1^2, ρ_2^2 all tend to zero, in a fixed ratio when near U. The critical line intersects the initial line at $T = T_c = 1.14733$. (Note that numerically one can determine T_c very accurately, but the fixed point values (3.14) are harder to pin down, since the trajectory starting at any given initial T always diverges before it quite gets to the fixed point.)

The point $T=T_c$ corresponds to a critical Pomeron, $\alpha_0=\alpha_{0c}$. Any trajectory starting at $T>T_c$ ($\alpha_0<\alpha_{0c}$) ultimately ends up at S_1, while a trajectory starting at $T<T_c$ ($\alpha_0>\alpha_{0c}$) ends up at S_2.

We remark parenthetically that there are other fixed points around, including a stable one at $\rho_2=0$, $\epsilon/\rho_1^2 \rightarrow$ constant. When $\rho_2=0$ (3.1) is equivalent to a free fermion theory in D=1, and is exactly solvable. Unfortunately, this possibility of exactly calculating the D=1 exponents is quashed by the numerical fact that this fixed point is unattainable from the initial line.

Several trajectories are shown schematically in fig. 2. The picture we have described is a classic example of the general structure of the parameter space which corresponds to a second order phase transition, as discussed by Wilson [11].

4. SCALING LAWS AND EXPONENTS

The object of interest is the two-point function

$$G(x,Y,H) = \langle 0|\psi_i\, e^{-HY}\, \bar{\psi}_j|0\rangle \tag{4.1}$$

where $x=|i-j|$, and $|0\rangle$ is the ground state of H. By iterating the RG transformation until the cell is as large as the whole system, we see that $|0\rangle = \prod_j |0\rangle_j$. So the ground state of H is that state annihilated by the bare quanta ψ_j. For $\alpha_0 \leq \alpha_{0c}$, this is due to the nonrelativistic nature of the theory, and the absence of vacuum processes. It persists for $\alpha_0>\alpha_{0c}$. This behaviour is different from that exhibited in ϕ^4 theory.

In terms of the matrices (3.2), G(x, Y, H) is, apart from a constant,

$$\bar{G}\left(x, \epsilon Y, \epsilon/\rho_1^2, \rho_2^2/\rho_1^2\right) = \langle 0|\sigma_i^+\, e^{-HY}\, \sigma_j^-|0\rangle \tag{4.2}$$

where we have used the fact that, on the right hand side, Y appears only in the combination HY. Let us now insert a complete set of cell states to the right of

σ_i^+, and to the left of σ_j^-. We denote the cell containing i by i', and the cell containing j by j'. Also we use $|\tilde{1}\rangle_{i'}$ to denote the state of the whole system in which the cell i' is in the state $|1\rangle_{i'}$, and all the other cells are in their ground states. Equation (4.2) is then equal to

$$\langle 0|\sigma_i^+|\tilde{1}\rangle_{i'}\ {}_{i'}\langle\tilde{1}|e^{-HY}|\tilde{1}\rangle_{j'}\ {}_{j'}\langle\tilde{1}|\sigma_j^-|0\rangle \tag{4.3}$$

$$= A^2\ {}_{i'}\langle\tilde{1}|e^{-H'Y}|\tilde{1}\rangle_{j'} \tag{4.4}$$

since H' has the same matrix elements as H between cell states. Equation (4.4) can now be written as

$$A^2\ \langle 0|\sigma_{i'}^+|\tilde{1}\rangle_{i'}\ {}_{i'}\langle\tilde{1}|e^{-H'Y}|\tilde{1}\rangle_{j'}\ {}_{j'}\langle\tilde{1}|\sigma_{j'}^-|0\rangle \tag{4.5}$$

$$= A^2\ \langle 0|\sigma_{i'}^+ e^{-H'Y}\sigma_{j'}^-|0\rangle \tag{4.6}$$

which has the same form as (4.2) with H replaced by H', and the distance $|i-j|$ scaled by a factor of 2. Therefore

$$\bar{G}\left(x, \epsilon Y, \epsilon/\rho_1^2, \rho_2^2/\rho_1^2\right) = A^2\bar{G}\left(x/2, \epsilon' Y, \epsilon'/\rho_1'^2, \rho_2'^2/\rho_1'^2\right) \tag{4.7}$$

Now suppose we start off at $T=T_c$. After a large but finite number of iterations we will be in the vicinity of the fixed point U. Suppose, at the fixed point,

$$A^2 = A^{*2} \equiv 2^{-z_3} \tag{4.8}$$

and

$$\epsilon'/\epsilon = 2^{-z_2} \tag{4.9}$$

The expression on the right hand side of (4.7) is related to $G(x, Y, H')$, where H' differs from H by a factor 2^{-z_2} only, since the ratios ϵ/ρ_1^2, ρ_2^2/ρ_1^2 are unchanged at the fixed point. Therefore, at the fixed point

$$G\left(x, Y, 2^{-z_2}H\right) = G\left(x, 2^{-z_2}Y, H\right) \tag{4.10}$$

and so

$$G(x, Y, H) = 2^{-z_3}\, G\left(x/2, 2^{-z_2} Y, H\right) \tag{4.11}$$

Repeating the RG transformation an arbitrary number of times

$$G(x, Y, H) = \lambda^{-z_3}\, G\left(x/\lambda, \lambda^{-z_2} Y, H\right) \tag{4.12}$$

from which we obtain the familiar scaling law for the two-point function [4, 5] (in D=1)

$$G(x, Y, H) = Y^{-\gamma - z/2}\, f\,(x^2/Y^z) \tag{4.13}$$

where

$$\gamma + \frac{1}{2} z = z_3/z_2$$

$$z = 2/z_2 \tag{4.14}$$

Actually, because of our approximation, $z_3 = z_2$. This can be seen by noting that ϵ and ρ_1^2 scale in a fixed ratio near the fixed point, and that $\rho_1'^2/\rho_1^2 = A^{*2}$, by eq. (3.13). This happens because only one site in a given cell participates in the interaction with another cell. If we go to D=2, or if we include next-nearest neighbour interactions, it is no longer true. An example of this can be seen in the Appendix.

We can also derive a scaling law when $\epsilon/\rho_1^2 \neq \left(\epsilon/\rho_1^2\right)^*$, which corresponds to $\alpha_0 \neq \alpha_{0c}$. Writing $K = \epsilon/\rho_1^2$ then

$$K' - K^* = 2^{z_1}(K - K^*) \tag{4.15}$$

where $z_1 > 0$, since the fixed point is unstable in K. Going through the same argument as before,

$$G(x, Y, K) = \delta^{z_3/z_1}\, f\left(x\delta^{1/z_1},\ Y\delta^{z_2/z_1}\right) \tag{4.16}$$

where $\delta = |K - K^*|$. Unfortunately, since the fixed point U is outside the space of allowed parameters, δ is not simply proportional to $|\alpha_0 - \alpha_{0c}|$ or $|T - T_c|$.

Determining δ as a function of $|T-T_c|$ numerically is impractical since we cannot determine K* with sufficient accuracy. Therefore we cannot relate z_2/z_1 to the exponent κ of ref. [9]. Instead, we use another relation to determine κ.

$\underline{\alpha_0 < \alpha_{0c}}$ -

If we start with $T > T_c$, after a large but finite number of steps we arrive in the vicinity of S_1. If we iterate an infinite number of times, so the cell is the whole system, we obtain a Hamiltonian with $\rho_1=\rho_2=0$, and $\epsilon=\Delta$, say. In doing this we wash out all states of the initial Hamiltonian with nonzero momentum, since the cell size ultimately becomes larger than their wavelength. We are left with a single excited state of zero momentum and energy Δ. This must be the energy gap of the initial Hamiltonian, since, by construction, $H^{initial}$ and H^{final} have the same matrix elements between the states we retain.

If we plot Δ as a function of $(T-T_c)$ on doubly logarithmic scales, we find a curve indistinguishable from a straight line for

$$10^{-4} \lesssim \frac{T-T_c}{T_c} \lesssim 10^{-1} \tag{4.17}$$

(The deviations at the lower end are due to the fact that we know T_c to only 5 decimal places.) This indicates the existence of an exponent ν where

$$\Delta \propto (T-T_c)^{\nu} \propto (\alpha_{0c}-\alpha_0)^{\nu} \tag{4.18}$$

The exponent is related to the κ of ref. [9] by

$$\nu = 1/(1-\kappa) \tag{4.19}$$

If we stop the iterations after a finite number of steps n we retain the states with momentum $|k| < k_0$ where

$$k_0 b \lesssim 2^{-n} \tag{4.20}$$

In this regime, ρ_1 and ρ_2 are small, and $\epsilon \sim \Delta$, so we can do perturbation theory in the intercell couplings to obtain a dispersion relation of the form

$$E = \Delta + \alpha' k^2 \tag{4.21}$$

which corresponds to a renormalised Pomeron pole below one, with a linear trajectory.

As we increase α_0 towards α_{0c}, the trajectory will spend an increasing length of time in the vicinity of the unstable fixed point U before diverging towards S_1. In that case the scaling law (4.16) is applicable. Thus we get a smooth transition to the critical Pomeron as $\alpha_0 \to \alpha_{0c}$.

5. THE ORDERED PHASE

If we start with $T < T_c$ ($\alpha_0 > \alpha_{0c}$), the trajectory goes quite rapidly to S_2. This is because $\epsilon' = 0(\epsilon^2)$ for ϵ small. Since $\epsilon = 0$ for a very large cell, the system has no energy gap, and there is a state degenerate with the ground state $|0\rangle$. This is of course just the state $|1\rangle$, which is defined as the limit of the states $|\tilde{1}\rangle_{cell}$ as the cells become larger. By its construction, it is translationally invariant and has zero momentum.

In the case T=0, the state $|1\rangle$ can be constructed explicitly. For then the trajectory begins at S_2, and remains there. Equation (3.7) becomes

$$|1\rangle_{cell} = |1\rangle_j |0\rangle_{j+1} + |0\rangle_j |1\rangle_{j+1} + i |1\rangle_j |1\rangle_{j+1} \tag{5.1}$$

Defining

$$|\phi\rangle_j = |0\rangle_j + i |1\rangle_j \tag{5.2}$$

and $|0\rangle_{cell}$ analogously, (5.1) can be written

$$|\phi\rangle_{cell} = |\phi\rangle_j |\phi\rangle_{j+1} \tag{5.3}$$

Therefore the state $|\phi\rangle = |0\rangle + i |1\rangle$ is simply a product state

$$|\phi\rangle = \prod_j |\phi\rangle_j \tag{5.4}$$

It is easy to check that $\bar{a}_j |\phi\rangle = 0$, so $|\phi\rangle$ is an exact eigenstate when $\epsilon = 0$. Knowing $|\phi\rangle$, we can then determine $|1\rangle$. Note that $|\phi\rangle$ has zero norm, but $|1\rangle$ does not.

$|\phi\rangle$ is only an exact product state at T=0 ($\alpha_0 \to +\infty$). At finite T, it is approximately a product state over large cells (since $A, B \sim 1$ at that level), but it is much more complicated at the single site level.

The question arises as to whether $|1\rangle$ is an isolated zero momentum state, or merely the zero momentum component of a continuous spectrum with zero gap. We shall argue that the former is true, at least at the level of the quantum spin Hamiltonian, by considering the Green's function $G(x, Y, H)$. Near the fixed point S_2 we can derive a scaling law in the same way as at U. In this case, since A=1 and $\epsilon'/\epsilon = 0$ at S_2, it is simply

$$G(x, Y, H) = G(x/\lambda, 0, H) \tag{5.5}$$

Therefore $G(x, Y, H)$, for large x and Y, is simply a constant. This is due to the presence of $|1\rangle$ as an intermediate state in the propagator. If there were any low energy, nonzero momentum states, they would modify the behaviour of $G(x, Y, H)$ at large x, Y.

Further insight into the nature of the phase transition can be obtained by considering the quantity.

$$\sigma = \langle 0 | a_j | 1\rangle = \langle 0 | \sigma_j^+ | 1\rangle \tag{5.6}$$

which is related to the matrix element of the original field operator ψ by

$$\langle 0 | \psi | 1\rangle = (2 | \Delta_0 | / r_0)\sigma \tag{5.7}$$

The quantity σ can be determined by our RG method. We have, by (3.12)

$$_{\text{cell}}\langle 0 | \sigma_j^+ | 1\rangle_{\text{cell}} = A^{(0)} \tag{5.8}$$

where $A^{(0)}$ is the function A evaluated at the starting point of the iterations. Continuing the iterations indefinitely

$$<0|\sigma_j^+|1> = \prod_{n=0}^{\infty} A^{(n)} \tag{5.9}$$

One can check, analytically, that the infinite product exists. When $\alpha_0 < \alpha_{0c}$, $A^{(n)} \to 0$, and so $<0|\sigma_j^+|1> = 0$. When $\alpha_0 > \alpha_{0c}$, $A^{(n)} \to 1$, and $<0|\sigma_j^+|1>$ is a constant which can be evaluated numerically as a function of $T_c - T$. Plotting σ against $T_c - T$ shows a power law behaviour which defines a new exponent which we call β, by analogy with statistical mechanics.

$$\sigma \propto (T_c - T)^{\beta} \propto (\alpha_0 - \alpha_{0c})^{\beta} \qquad (\alpha_0 \gtrsim \alpha_{0c}) \tag{5.10}$$

In the particular D=1 approximation we are using, σ is just equal to the final value of ρ_1, if $\rho_1^{\text{initial}} = 1$, since they scale by an amount $A^{(n)}$ each time. This is not true in general, as we show in Appendix A. In general $\rho_1, \rho_2 \to \infty$ at S_2, but σ still tends to a constant. However it is still true that $\rho_1/\rho_2 \to 1$ at S_2. Also the ratio ${}_{\text{cell}}<1|a_j|1>_{\text{cell}} / {}_{\text{cell}}<0|a_j|1>_{\text{cell}}$ stays equal to $i\rho_2/\rho_1$ as the cells get larger. We then find that

$$<1|a_j|1> = i\sigma \tag{5.11}$$

is true in general, not just in D=1, where it is trivially equal to $i\rho_2^{\text{final}} = \rho_1^{\text{final}}$.

The other matrix elements can be worked out to give

$$<0|\psi|1> = <1|\bar{\psi}|0> = (2|\Delta_0|/r_0)\sigma \tag{5.12}$$

$$<1|\psi|1> = <1|\bar{\psi}|1> = i(2|\Delta_0|/r_0)\sigma \tag{5.13}$$

As $T \to 0$, $A^{(n)} \to 1$ for all n, and so $\sigma \to 1$. The numerical calculations of σ versus T are plotted in fig. 1.

We see then that σ plays the role of an order parameter, vanishing identically for $T \geq T_c$, and tending to a constant as $T \to 0$ and the system is completely ordered.

As explained in the introduction, σ cannot be written as a ground state expectation value, unlike ϕ^4 theory.

6. SCATTERING AMPLITUDE FOR $\alpha_0 > \alpha_{0c}$

The elastic amplitude A(Y, B) is, at least for $\alpha_0 \leq \alpha_{0c}$

$$A(Y,B) = -is \sum_{n,m=1}^{\infty} i^{n+m} g_a^{(n)} g_b^{(m)} \langle 0 | \psi(B)^n e^{-HY} \bar{\psi}(0)^m | 0 \rangle \tag{6.1}$$

where $g_a^{(n)}$ is the real coupling of n Pomerons to the external particle a. Since $|0\rangle$ remains a (degenerate) ground state for $\alpha_0 > \alpha_{0c}$, it natural to assume (6.1) as the definition of A(Y, B) for $\alpha_0 > \alpha_{0c}$. (Note that derivations of (6.1) hold only for $\alpha_0 < 1$; above this value we must use it as a definition. One already does this for the critical Pomeron, when $\alpha_0 = \alpha_{0c} > 1$.)

Let us first consider the case n=m=1. As $Y \to \infty$ only the intermediate state $|1\rangle$ contributes, and we obtain

$$A^{(1,1)} \sim is\, g_a^{(1)} g_b^{(1)} (2|\Delta_0|/r_0)^2 \sigma^2 \tag{6.2}$$

For the higher Pomeron couplings, the answer depends on whether the sources are localised at a single lattice point, or spread out over many. In the first case we have to calculate matrix elements of the form

$$\langle 0 | \psi_j^n | 1 \rangle = {}_j\langle 0 | \psi_j^n | 1 \rangle_j \; {}_j\langle \tilde{1} | 1 \rangle = {}_j\langle 0 | \psi_j^n | 1 \rangle_j \, \sigma \tag{6.3}$$

We show how to evaluate the single site matrix element on the right hand side in Appendix B. It cannot be evaluated in closed form. However, it depends relatively weakly on the input parameters near $\alpha_0 = \alpha_{0c}$ (which is the only region where the simplified model has any meaning). We notice then that each term in (6.1) is proportional to σ^2, and vanishes like $(\alpha_0 - \alpha_{0c})^{2\beta}$ as $\alpha_0 \to \alpha_{0c}$. From

the Appendix, $a_n \sim \Gamma\left(\frac{1}{2}n\right)$ as $n \to \infty$, so that if $g_a^{(n)}, g_b^{(n)}$ have the eikonal forms

$$g_a^{(n)} = g_a^n/n! \ , \qquad g_b^{(n)} = g_b^n/n! \tag{6.4}$$

the sum in (6.1) certainly converges. However, in this limit of highly localised sources, the sum (6.1) does not have the eikonal form, even though the bare couplings $g_a^{(n)}, g_b^{(n)}$ do have.

Now consider the opposite extreme, when the sources are very extended in impact parameter. The appropriate matrix elements will then be of the form $<0|\psi_{j_1}\psi_{j_2}\ldots\psi_{j_n}|1>$ where the sites $j_1, \ldots, j_n$ all lie within the source. This matrix element cannot be calculated for arbitrary n except in the limit $\alpha_0 \to +\infty$ (or $r_0 \to 0$). In that case we can write

$$<0|\psi_{j_1}\ldots\psi_{j_n}|1> = -i<0|\psi_{j_1}\ldots\psi_{j_n}|\phi> \simeq -i\left[<0|\psi_j|\phi>\right]^n$$

$$= i^{n-1}\left[<0|\psi_j|1>\right]^n \tag{6.5}$$

which is true because $|\phi>$ is then a product of single site states, and we can neglect configurations in which two of the j_r are equal. Then

$$A^{(n,m)} \sim is\,(-1)^{n+m} g_a^{(n)} g_b^{(m)} (2|\Delta_0|/r_0)^{n+m} \tag{6.6}$$

as obtained in ref. [1]. For eikonal couplings, we can perform the sum over n, m to obtain

$$A(Y,B) \sim is\left(1-e^{-2g_a|\Delta_0|/r_0}\right)\left(1-e^{-2g_b|\Delta_0|/r_0}\right) \sim is \tag{6.7}$$

So, in this limit, the scatterer appears black. Note that to derive this we assume both r_0 small, and completely uncorrelated emission of the Pomerons. This approximation is, no doubt, more suited to nuclei than hadrons. As $\alpha_0 \to \alpha_{0c}$, eikonalisation breaks down, even though the bare multi-Pomeron couplings have eikonal form. This is because the site spins are then highly correlated in the state $|1>$.

To illustrate how one can still calculate the higher order terms in (6.1) in this regime, we consider the matrix element $<0|\psi_j\psi_{j+1}|1>$, a contribution to the terms in (6.1) proportional to $g_a^{(2)}$. If j' is the cell containing j and j+1, and defining $|\tilde{1}>_{j'}$ as in Section 4,

$$<0|\psi_j\psi_{j+1}|\tilde{1}> = <0|\psi_j\psi_{j+1}|\tilde{1}>_{j'} \; {}_{j'}<\tilde{1}|1> \tag{6.8}$$

Using (3.7), this is

$$iB^{\text{initial}} \; {}_j<0|\psi_j|1>_j \; {}_{j+1}<0|\psi_{j+1}|1>_{j+1} \; <0|a_{j'}|1> \tag{6.9}$$

But

$$<0|a_j|1> = \sigma = A^{\text{initial}} <0|a_{j'}|1> \tag{6.10}$$

so

$$<0|\psi_j\psi_{j+1}|1> = \left(2|\Delta_0|b^{D/2}/r_0\right)^2 i\sigma\,(B/A)^{\text{initial}} \tag{6.11}$$

At $\alpha_0=\alpha_{0c}$ we find numerically that $(B/A)^{\text{initial}} \approx 0.5$. As $\alpha_0 \to +\infty$ it increases towards one (as does σ), and we recover our previous result.

It is clear from this example that we can calculate any matrix element $<0|\psi_{j_1}\ldots\psi_{j_n}|1>$ by iterating enough times so that $j_1,\ldots,j_n$ are contained in the same cell. If the hadron sources are of finite extent in impact parameter, this will happen after a finite number of iterations. The result will then be some numerical factor, finite at $\alpha_0=\alpha_{0c}$, multiplied by σ.

We conclude that in general, for $\alpha_0 \gtrsim \alpha_{0c}$, as $Y\to\infty$ at fixed B

$$A(Y,B) \to \text{is}\; \beta_a\beta_b\sigma^2 \tag{6.12}$$

where β_a, β_b describe composite couplings to the external hadrons, and depend only weakly on α_0. The factorisation occurs of course because only one intermediate state dominates.

If (6.12) holds also for large impact parameters B>>Y, the truncated model will not satisfy the Froissart bound. In that case three possibilities can arise in the full theory:

a) It has a spectrum like that of the truncated theory, with a degenerate state $|1\rangle$ and a gap. Amati et al. [1] suggest that the excitations above the gap enter in such a way as to cancel the unwanted behaviour for B>>Y. For this it is necessary to have eikonal-type multi-Pomeron couplings. This seems an unfortunate restriction on the theory, and it is difficult to see how it will then satisfy t-channel unitarity.

b) The full theory has a degenerate state $|1\rangle$ but there is no gap. The Green's functions of the theory will then make sense, no assumptions on multi-Pomeron couplings need be made, and the theory will manifestly satisfy t-channel unitarity.

c) The full theory does not make sense for $\alpha_0 > \alpha_{0c}$.

As $\alpha_0 \to \alpha_{0c}$ the scaling law (4.16) should hold. From our work on $\alpha_0 < \alpha_{0c}$ we know that it can be written in terms of $|\alpha_0 - \alpha_{0c}|$ rather than δ:

$$A(Y,B) \to |\alpha_0 - \alpha_{0c}|^{\beta'} f\left(Y|\alpha_0 - \alpha_{0c}|^{\nu}\right) \tag{6.13}$$

where we have neglected the B dependence. From (6.11) we see that $\beta' = 2\beta$. Since, to obtain a finite result as $\alpha_0 \to \alpha_{0c}$, f(x) must behave as $x^{-\beta'/\nu}$ as $x \to 0$, we obtain, at $\alpha_0 = \alpha_{0c}$,

$$A(Y,B) \to Y^{-2\beta/\nu} \tag{6.14}$$

We have thus derived the scaling relation for the exponents

$$\beta = \frac{\nu}{2}(\gamma + Dz/2) \tag{6.15}$$

In our approximate calculations this is seen to be very well satisfied for D=1, and approximately so for D=2, at least with the 2-site cell. It is also satisfied by the

mean field theory values at D=4, as expected. Equation (6.15) is just the same exponent as found in ref. [9], although the order parameter is quite different.

7. DISCUSSION

Our numerical investigations have been directed towards two aims: to see whether practical calculations of the exponents can be made in the quantum spin lattice model, and to elucidate the properties of Reggeon field theory for $\alpha_0 > \alpha_{0c}$.

We have seen that the first aim is achieved, although our approximation methods need considerable refinement if reliable values for the exponents are to be obtained. Since we have argued that the truncation necessary to derive the quantum spin model does not affect the exponents, the only approximation we have made is in the truncation of the cell states at each iteration. Clearly we need some way of mixing in these neglected states, possibly by using perturbation theory to take into account second order transitions between the lowest states. Unfortunately, any such scheme will be numerically more complicated. The numerical calculations described in this paper were performed on a simple on-line system. More sophisticated versions will probably require numerical diagonalisation of the Hamiltonian at each step.

Perhaps more interesting is the insight we have gained into the properties of the theory for $\alpha_0 > \alpha_{0c}$. In this case, the Reggeon field does not gain a vacuum expectation value, as assumed at the outset by Abarbanel et al. [9]. Instead, the relevant order parameter σ is a transition matrix element $\langle 0 | \psi | 1 \rangle$ to a state $|1\rangle$ degenerate with the ground state. This was first pointed out by Amati et al. [1]. Since the Pomeron can propagate in the zero energy state $|1\rangle$, amplitudes at fixed impact parameter tend to constants at high energies, proportional to σ^2. $\sigma \to 0$ as $\alpha_0 \to \alpha_{0c}$, and there is a smooth transition to the critical Pomeron.

This behaviour, if it persists to impact parameters $\gg Y$, is of course a disaster, for the theory will not satisfy the Froissart bound. Amati et al. [1] have suggested that the higher excitations (neglected in the truncation to obtain the quantum spin Hamiltonian) will provide a sufficient cutoff,* and support this with calculations for $\alpha_0 \gg \alpha_{0c}$ where one can use classical methods. Until this question is definitely settled it is not clear that Reggeon field theory makes sense for $\alpha_0 > \alpha_{0c}$. Even if it does, the question of satisfaction of t-channel unitarity remains open, and until this is demonstrated, the theory for $\alpha_0 > \alpha_{0c}$ will merely be one among many which satisfy the simpler constraints of s-channel unitarity.

Acknowledgements

This work could not have been carried out without the help of numerous discussions with S. Jafarey, D. J. Scalapino, B. Stoeckly and R. L. Sugar. The author would also like to thank S. Jafarey for assistance with the computing, and D. Amati and R. Brower for the early communication of their results.

*The CERN group have now discussed this question in the context of the quantum spin model. See ref. [20].

APPENDIX A: TWO TRANSVERSE DIMENSIONS

We have used two different approximation schemes for studying this problem. They are the same in principle as that described in the text for D=1, and differ only in the choice of basic cell.

1) 3-site cell. For this case we take a triangular lattice, and divide it into cells, as shown in fig. 3. The Hamiltonian for a single cell is represented by an 8×8 matrix in the single site basis. As well as the ground state $|000\rangle$ there is a first excited state $|1\rangle_{cell}$ with energy ϵ' which happens to be given by the same formula (3.6) as in D=1. This state has the form

$$|1\rangle_{cell} = A\,(|100\rangle + |010\rangle + |001\rangle)$$
$$+ iB\,(|011\rangle + |101\rangle + |011\rangle) + C\,|111\rangle \tag{A.1}$$

in an obvious notation. Here A, B, C are real and normalised in accordance with our convention by

$$3A^2 - 3B^2 + C^2 = 1 \tag{A.2}$$

The analogues of (3.12) are

$$_{cell}\langle 0\,|\sigma_j^+|\,1\rangle_{cell} = {}_{cell}\langle 1\,|\sigma_j^-|\,0\rangle_{cell} = A \tag{A.3}$$

$$_{cell}\langle 1\,|\sigma_j^+|\,1\rangle_{cell} = {}_{cell}\langle 1\,|\sigma_j^-|\,1\rangle_{cell} = 2iAB + iBC \tag{A.4}$$

$$_{cell}\langle 1\,|c_j|\,1\rangle_{cell} = A^2 - 2B^2 + C^2 \tag{A.5}$$

In choosing ρ_1', ρ_2' so that the intercell interactions in H' have the same matrix elements as those in H, we must take into account that between each pair of cells there are two intersite interactions. This is clear from the figure. We therefore take

$$\rho_1' = \sqrt{2}\,A\rho_1$$
$$\rho_2' = \sqrt{2}\,(A^2 - 2B^2 + C^2)\rho_2 + \sqrt{2}\,(2AB + BC)\rho_1 \tag{A.6}$$

Iterating the RG transformation, we find fixed points U, S_1, S_2 as before. T_c is approximately 4.65, but this must be multiplied by a factor 2/3 for comparison with a square lattice, which has 4 nearest neighbours per site compared with 6 for a triangular lattice. Scaling laws near T_c can be derived as before, remembering that lengths are scaled by a factor $\sqrt{3}$ at each step.

At S_2 we find that A=B=-C=1. Thus ρ_1 and ρ_2 do not tend to constants there, but the order parameter defined as in the text does not have the factor $\sqrt{2}$, and is finite.

2) 2-site cell. The above approximation gives bad values for the exponents. This may well be because, at each step, we discard six out of the eight states in each cell. It is therefore preferable to choose a 2-site cell, shown in fig. 4, where we only discard half the states. Unfortunately, with this shape of cell, the cell lattice has a different shape from the site lattice. One can take this into account by enlarging the parameter space to include unequal couplings in the two directions. It is easier, however, to align the cells in different directions at each successive iteration, in order to get back to the same lattice, scaled by a length factor of 2, after two iterations. Since we are interested only in the fixed points we just consider a single iteration, in which lengths are scaled by $\sqrt{2}$ each time, and there are $\sqrt{2}$ intersite interactions between each pair of cells. In this way we avoid enlarging the parameter space. Thus, the only difference from the D=1 calculation is that a factor $2^{1/4}$ appears on the right hand side of eqs. (3.13), and lengths are scaled by $\sqrt{2}$ rather than 2.

The fixed points appear as before, and scaling laws and exponents are derived as in the text.

APPENDIX B

We evaluate the single site matrix element ${}_j\langle 0|\psi_j^n|1\rangle_j$, where the single site Hamiltonian is (dropping the suffix j)

$$H_0 = \Delta_0 \bar{\psi}\psi + \frac{ig_0}{2}(\bar{\psi}^2\psi + \bar{\psi}\psi^2) \tag{B.1}$$

where

$$g_0 = r_0/b^{D/2}$$

We have

$$\left[\psi^n, H_0\right] = n\Delta_0\psi^n + \frac{ig_0}{2}\left[2n\,\bar{\psi}\psi^n + n(n-1)\,\psi^{n-1} + n\,\psi^{n+1}\right] \tag{B.2}$$

Taking matrix elements between $\langle 0|$ and $|1\rangle$, and writing $a_n = \langle 0|\psi^n|1\rangle$,

$$\epsilon a_n = n\Delta_0 a_n + \frac{1}{2} ig_0 n(n-1)\, a_{n-1} + \frac{1}{2} ig_0 n\, a_{n+1} \tag{B.3}$$

Even when $|\Delta_0|/g_0 \to \infty$, $\epsilon \to 0$, we cannot find a closed form for a_n. We note that, as $n \to \infty$, $a_n \sim \Gamma(n/2)$. For $n \ll |\Delta_0|/g_0$, $a_n \sim (2i\Delta_0/g_0)^n$, which is the eikonal result.

References

[1] D. Amati, M. le Bellac, G. Marchesini and M. Ciafaloni, Ref. TH 2152-CERN, submitted to Nucl. Phys. (1976).

[2] R. C. Brower, M. A. Furman and K. Subbarao, University of California, Santa Cruz preprint (1976).

[3] H.D.I. Abarbanel, J. B. Bronzan, R. L. Sugar and A. R. White, Phys. Rep. C 21 (1975) 119.

[4] A. A. Migdal, A. M. Polyakov and K. A. Ter-Martirosyan, Zh. Eksp. Teor. Fiz. 67 (1974) 848 [Sov. Phys.-JETP 40 (1974) 420].

[5] H.D.I. Abarbanel and J. B. Bronzan, Phys. Rev. D 9 (1974) 2397.

[6] J. B. Bronzan, J. A. Shapiro and R. L. Sugar, Phys. Rev., to be published; R. Jengo, Trieste preprint (1976).

[7] J. L. Cardy and R. L. Sugar, Phys. Rev. D 12 (1975) 2514.

[8] R. C. Brower, J. Ellis, R. Savit and W. J. Zakrzewski, Nucl. Phys. B 94 (1975) 460;
R. C. Brower, J. Ellis, R. Savit and J. Zinn-Justin, CERN TH 2125 (1976).

[9] H.D.I. Abarbanel, J. B. Bronzan, A. Schwimmer and R. L. Sugar, Phys. Rev., to be published.

[10] L. P. Kadanoff, Physics (Long Island City, New York) 2 (1966) 263.

[11] K. G. Wilson, Phys. Rev. B 4 (1971) 3174, 3189.

[12] S. Jafarey and B. Stoeckly, private communication. S. D. Drell, M. Weinstein, S. Yankielowicz, R. Pearson, S. Jafarey, D. J. Scalapino and B. Stoeckly, in preparation.

[13] B. Stoeckly and D. J. Scalapino, Phys. Rev. B 11 (1975) 205.

[14] R. L. Sugar and A. R. White, Phys. Rev. D 10 (1974) 4074.

[15] J. L. Cardy, Nucl. Phys., to be published.

[16] P. Pfeuty, Ann. Phys. (N.Y.) 57 (1970) 79.

[17] Th. Niemeijer and J.M.J. van Leeuwen, Physica 71 (1974) 17.

[18] L. P. Kadanoff and A. Houghton, Phys. Rev. B 11 (1975) 377.

[19] Z. Friedman, Phys. Rev. Lett. 36 (1976) 1326.

[20] D. Amati, G. Marchesini, M. Ciafaloni and G. Parisi, Ref. TH 2185-CERN.

Table 1

Calculated Values for Exponents

D	Type of cell	$-\gamma$	z	ν	κ	β	T_c
1	2-site	0.27	2.5	1.3	0.22	0.64	1.15
2	Triangular	2.0	8.0	0.27	-2.7	1.0	3.10*
2	2-site	1.4	4.8	1.06	0.06	1.06	2.87
Mean field theory		0	1	1	0	1	2D

*adjusted for comparison with square lattice

Table 2

Calculated Values for Ising Exponents, taken from ref. [12]

	β	ν	z
Estimated (2-site cell)	0.37	0.9	3.6
Exact	1/8	1	2

Figure Captions

1. Calculated values for order parameter σ versus $T = \epsilon/\rho^2$ in $D = 1$.

2. Map of the parameter space.

3. Construction of cells for a triangular lattice.

4. Construction of 2-site cells for a square lattice.

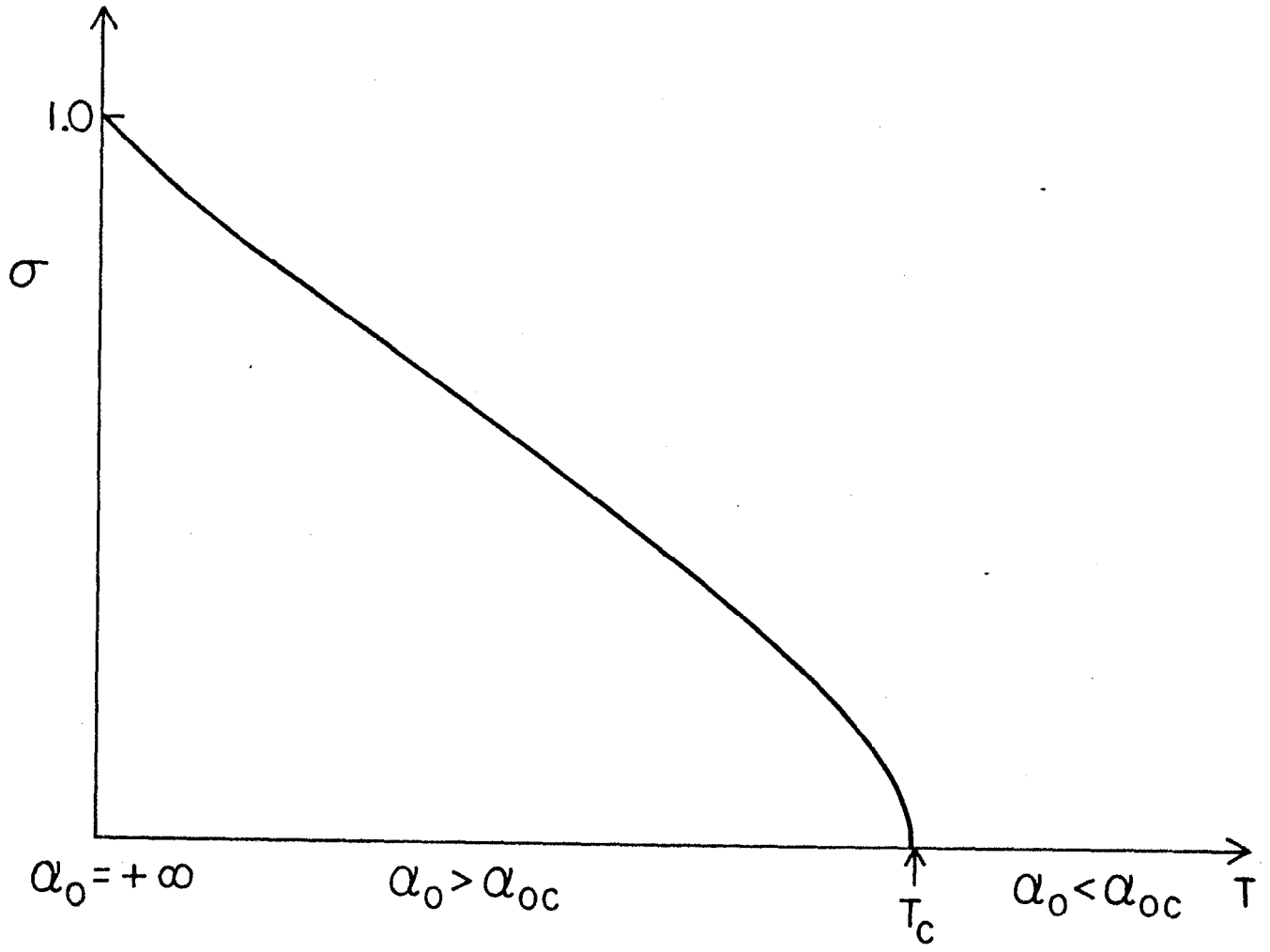

Fig. 1

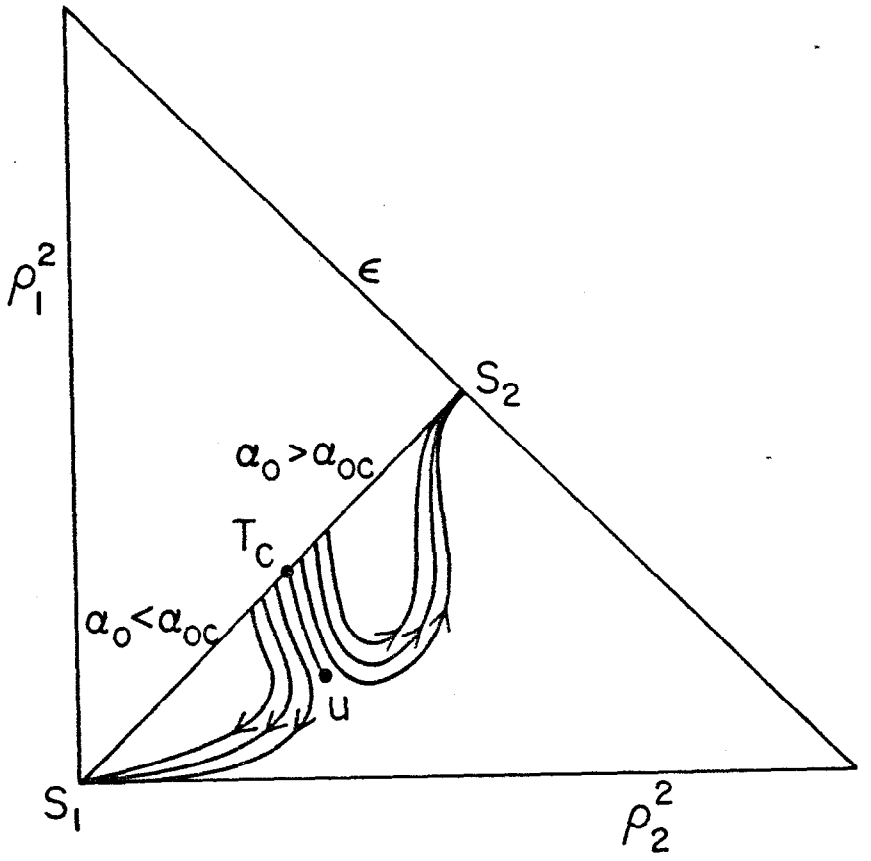
ρ_1^2
ϵ
S_2
$\alpha_o > \alpha_{oc}$
T_c
$\alpha_o < \alpha_{oc}$
u
S_1
ρ_2^2

Fig. 2

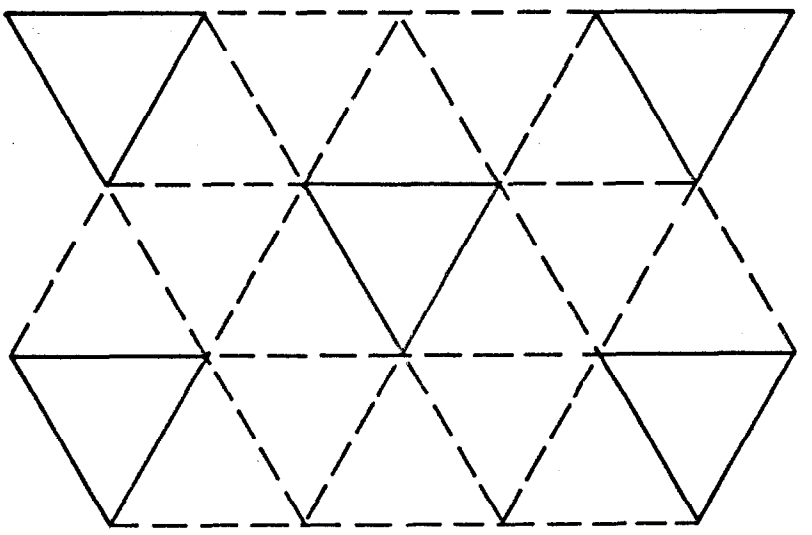

Fig. 3

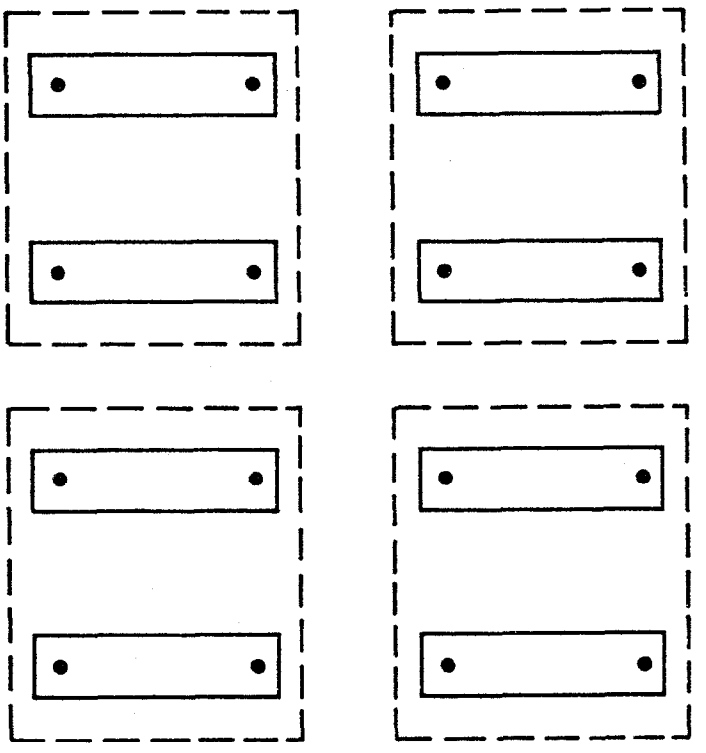

Fig. 4