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DYNAMICAL GENERATION OF EXTENDED STRUCTURES IN FIELD THEORY*

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ABSTRACT

In this paper we study the generation of extended structures in field theory through the agency of dynamical symmetry breaking. We generalize Gorkov's derivation of the Ginzburg-Landau theory of superconductivity to relativistic systems by working entirely at the level of the quantum-mechanical action in the presence of a local space-time dependent mass source. Our approach provides a compact and elegant derivation of the results of Eguchi and Sugawara and also permits an analysis of the utility and range of application of their work.

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I. INTRODUCTION

There is currently much interest in extended structures in particle physics. Though it is not yet clear whether such an approach will ultimately be related to problems such as quark confinement, it does at least provide an alternative to canonical perturbation theory and is worthy of study in its own right. As such extended models isolate particularly useful collective coordinates which summarize some of the main nonperturbative features of the many-body problem, perhaps the most well known example being the vortices of Type II superconductors. While there has been much study recently of relativistic analogs of the purely classical vortices there has only been a limited analysis[1, 2] of underlying dynamical quantum-mechanical mechanisms which would generate the vortices in the first place. This latter approach is of course very powerful since it eliminates the need for arbitrary fundamental scalar tachyons, and is consequently far more predictive. In this paper we shall present a new method for studying dynamical mechanisms which may serve to complement the analyses of Refs. 1 and 2.

Following Eguchi and Sugawara we set out to develop a relativistic generalization of Gorkov's derivation of the Ginzburg-Landau equations of superconductivity and so for the benefit of the reader we quickly review the Ginzburg-Landau theory. In 1951 Ginzburg and Landau presented a purely phenomenological macroscopic theory of superconductivity. In this theory the superconducting state was to be described by an order parameter $\phi(\vec{x})$. The reason for this was twofold. Firstly because this was a general approach which Landau had developed for all ordered phases in the solid state, and secondly because of London's idea of a supercurrent which through the presumed rigidity of the many-body wave function would allow a macroscopically observable quantum-mechanical state. Ginzburg and Landau did not question further the origin of this c-number macroscopic order parameter

but intuited that near the critical temperature its dynamics would be described in the presence of an external magnetic field by a (Higgs) Lagrangian with a wrong sign mass term,

$$\mathscr{L} = -\frac{1}{2} |(\vec{\partial} - 2ie\vec{A})\phi|^2 + \frac{1}{2}\mu^2 |\phi|^2 - \frac{1}{4}\lambda |\phi|^4 \quad . \tag{1}$$

Abrikosov then studied this theory and found that it admits of vortex type solutions in which $\phi(\vec{x})$ approaches its vacuum value of $(\mu^2/\lambda)^{1/2}$ at spatial infinity while vanishing at the origin with the magnetic field being localized entirely within the vortex, thus producing inhomogeneities in the vacuum.

In 1957 Bardeen, Cooper and Schrieffer (BCS) identified the basic microscopic agency responsible for superconductivity, namely the existence of correlated Cooper pairs of electrons. By studying the quantum fluctuations of the reduced BCS Hamiltonian in a nonperturbative and self-consistent manner they were able to show that in the ground state the energy gap

$$\Delta = \langle S | \psi\psi | S \rangle \tag{2}$$

was nonzero with the state $|S\rangle$ possessing lower energy than the normal state $|N\rangle$ where Δ vanishes. Since $|S\rangle$ was taken to be the translationally invariant vacuum we note that for the moment Δ has no space dependence.

In 1958 Gorkov realized that the Ginzburg-Landau order parameter would correspond to a space dependent gap parameter and was then able to derive Eq. (1) starting from the microscopic BCS theory. Though Gorkov himself never used the language of coherent states, in the modern terminology of bag models we would interpret the order parameter as

$$\Delta(\vec{x}) = \langle C | \psi(\vec{x})\ \psi(\vec{x}) | C \rangle \tag{3}$$

where $|C\rangle$ is built on $|S\rangle$ using a space-dependent Bogoliubov transform.[3] Gorkov's actual method of obtaining Eq. (1) was to perturb around the constant

value of Eq. (2) by expanding in the gap equation itself, i.e., in the equation of motion, and then restricting to the case of slowly varying order parameters so that higher order gradient terms do not appear in Eq. (1). In cases where higher gradient terms are important (i.e., at temperatures well below the critical point) Gorkov's method becomes intractable and for this reason we have developed an alternative approach which expands in the action. However in this paper we shall restrict our study to relativistic theories and leave the nonzero temperature case to the future.

Gorkov's method thus allows us to look for coherent states once we have first established that the vacuum has undergone dynamical symmetry breaking, and thus the natural relativistic theory to study is the four-Fermi model of Nambu and Jona-Lasinio.[4] We study the model in Section II and recover the results of Ref. 1. We shall contrast the results in four space-time dimensions by applying the same analysis in Section III to the same model in two dimensions. This will enable us to establish the range of validity of the work of Ref. 1. In Section IV we discuss briefly the stability properties of the extended solutions. Finally in Section V we assemble together the main advantages of our approach and make some general comments.

II. EXTENDED STRUCTURES FROM AN ACTION PRINCIPLE

Before we begin to look for extended structures we first review the Hartree-Fock method for finding the self-consistent vacuum of the cutoff chiral invariant four-Fermi theory in four dimensions,

$$\mathscr{L} = \frac{1}{2} i\left(\bar{\psi}\gamma_\mu \overleftrightarrow{\partial}_\mu \psi\right) - \frac{1}{2} g\left[(\bar{\psi}\psi)^2 + \left(\bar{\psi} i\gamma_5 \psi\right)^2\right] \quad , \tag{4}$$

discussed originally by Nambu and Jona-Lasinio.[4] We rewrite the Lagrangian as

$$\begin{aligned}\mathscr{L} &= \frac{1}{2} i\left(\bar{\psi}\gamma_\mu \overleftrightarrow{\partial}_\mu \psi\right) - m\bar{\psi}\psi + \frac{m^2}{2g} - \frac{1}{2} g\left[\left(\bar{\psi}\psi - \frac{m}{g}\right)^2 + \left(\bar{\psi} i\gamma_5 \psi\right)^2\right] \\ &= \mathscr{L}_D + \frac{m^2}{2g} + \mathscr{L}_R = \tilde{\mathscr{L}}_D + \mathscr{L}_R\end{aligned} \tag{5}$$

where $\mathscr{L}_D$ is diagonal (and will eventually include the constant term), and $\mathscr{L}_R$ is known as the residual interaction. Here for the moment m is just a conventional space-time independent mass term. The vacuum energy density of $\mathscr{L}_D$ is easily calculable as[5]

$$\begin{aligned}\epsilon(m) &= i\int \frac{d^4p}{(2\pi)^4} \operatorname{Tr} \ln\left(\frac{1}{(\not{p}+i\epsilon)}(\not{p} - m + i\epsilon)\right) \\ &= -\frac{m^4}{32\pi^2}\left[4\frac{\Lambda^2}{m^2} - 1 - 2\ln\left(\frac{\Lambda^2}{m^2 - i\epsilon}\right)\right]\end{aligned} \tag{6}$$

which is a multiple valued function of m^2. If we go round the branch point at $m^2=0$ we determine the function above the cut in the m^2 plane to be

$$\epsilon(m) = -\frac{m^4}{32\pi^2}\left[\frac{4\Lambda^2}{m^2} - 1 - 2\ln\left(\frac{\Lambda^2}{m^2 + i\epsilon}\right)\right] + i\frac{m^4}{8\pi} \tag{7}$$

which has a completely unacceptable imaginary part and thus we have to choose the sheet on which $\epsilon(m)$ is real to be the physical sheet. [It will become clear in Section III why we have made this apparently pedantic analysis.] From Eq. (6)

we can then determine the vacuum expectation value of $\bar{\psi}\psi$ as

$$\epsilon'(m) = \langle S|\bar{\psi}\psi|S\rangle = -\frac{m\Lambda^2}{4\pi^2} + \frac{m^3}{4\pi^2}\ln\frac{\Lambda^2}{m^2} \quad . \tag{8}$$

So far everything is formalism. The physics comes by requiring that $\mathscr{L}_R$ vanish in the state $|S\rangle$ in the one loop approximation, i.e.,

$$\begin{aligned} \langle S|\left(\bar{\psi}\psi - \frac{m}{g}\right)^2|S\rangle &= \langle S|\left(\bar{\psi}\psi - \frac{m}{g}\right)|S\rangle^2 = 0 \\ \langle S|(\bar{\psi}i\gamma_5\psi)^2|S\rangle &= \langle S|(\bar{\psi}i\gamma_5\psi)|S\rangle^2 = 0 \end{aligned} \tag{9}$$

The physical mass M is then that particular value of m which satisfies the constraints of Eq. (9), i.e., which satisfies the gap equation

$$-\frac{M\Lambda^2}{4\pi^2} + \frac{M^3}{4\pi^2}\ln\frac{\Lambda^2}{M^2} = \frac{M}{g} \tag{10}$$

This is the Hartree-Fock method. Since $\epsilon(M) < \epsilon(0)$ in the cutoff theory we thus see that the nontrivial solution to Eq. (10) is energetically favored, and so the self-consistent vacuum lies lower than the normal one.

We can also calculate the energy of $\tilde{\mathscr{L}}_D$ in the state $|S\rangle$. We find

$$\tilde{\epsilon}(m) = \epsilon(m) - \frac{m^2}{2g} = \frac{m^4}{16\pi^2}\ln\frac{\Lambda^2}{m^2} - \frac{m^2M^2}{8\pi^2}\ln\frac{\Lambda^2}{M^2} + \frac{m^4}{32\pi^2} \quad , \tag{11}$$

so that the imposition of the gap equation automatically renders $\tilde{\epsilon}(m)$ to be logarithmically divergent in contrast to the quadratically divergent $\epsilon(m)$, without any need to adjust the counter-term by hand. With this renormalization we then note that the self-consistent vacuum is found as the variational state in which

$$\tilde{\epsilon}'(M) = 0 \quad , \tag{12}$$

i.e., we have imposed the gap equation as a stationarity condition. Since $\tilde{\epsilon}(M) < 0$, $\tilde{\epsilon}''(M) > 0$ we then confirm that we have found the ground state of $\mathscr{L}$ in the Hartree-Fock approximation.

Having set up the formalism to find the vacuum we can now proceed to look for extended solutions. We want now to find coherent states in which the residual interaction again vanishes. To do this we simply introduce a space-time dependent mass term m(x) in Eq. (5) and study $\mathscr{L}_D$ again with a mass term $m(x)\,\bar{\psi}(x)\,\psi(x)$. We can then calculate the energy of $\mathscr{L}_D$ in the coherent state (when we ultimately take the static limit in m(x)) by noting that m(x) acts as an external source to the kinetic energy term and defines a vacuum functional

$$-\int d^4x\, W(m(x)) = \sum \frac{1}{n!} \int d^4x_1 \ldots d^4x_n\, m(x_1) \ldots m(x_n)\, G^{(n)}(x_1, \ldots x_n); \tag{13}$$

here the $G^{(n)}$ are the connected Green's functions of the $\bar{\psi}\psi$ composite calculated in the absence of the source, i.e., in a translationally invariant basis, which in this case is also in fact a massless basis. Thus though the eigenstates of $\mathscr{L}_D$ are no longer translationally invariant the expansion of Eq. (13) enables us to isolate the specific nontranslationally invariant terms explicitly and continue to use known forms for the $G^{(n)}$. On Fourier transforming Eq. (13) and expanding about the point in momentum space where all momenta vanish, we obtain

$$-\int d^4x\, W(m(x)) = \sum \frac{1}{n!} \int d^4x_1 \ldots d^4x_n\, m(x_1) \ldots m(x_n)\, d^4p_1 \ldots d^4p_n\, e^{ip_1 \cdot x_1} \ldots e^{ip_n \cdot x_n}$$
$$\times (2\pi)^4\, \delta^4\left(\sum p_i\right) \left[G^{(n)}(p_i=0) + \sum p_i p_j \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_j} G^{(n)}(p_k) \Big|_{p_k=0} + \ldots \right] \tag{14}$$

We define $\epsilon(m(x))$ and $\Pi\left(q^2, m(x)\right)$ through the graphical infinite sums of Figs. 1 and 2. After calculating some nontrivial combinatoric factors we can then set

$$-\int d^4x\, W(m(x)) = \int d^4x \left\{ -\epsilon(m(x)) + \frac{1}{2} (\partial_\mu m)^2\, Z(m(x)) + \ldots \right\} \tag{15}$$

where

$$Z(m(x)) = -\frac{d}{dq^2}\,\Pi(q^2, m(x))\Big|_{q^2=0} \tag{16}$$

We now note that the series of Figs. 1 and 2 can be summed analytically. Figure 1, of course, gives Eq. (6) where the parameter m is to be replaced by m(x) after the p-space integration has been performed. Also the infinite summation of Fig. 2 of graphs with soft insertions taken in the normal phase (massless propagators) sums up into one Feynman graph taken in the ordered phase (massive propagator), i.e.,

$$\Pi(q^2, m(x)) = -i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left(\frac{1}{(\not p-m)}\,\frac{1}{(\not p+\not q-m)}\right)\Big|_{m=m(x)} \tag{17}$$

Moreover we can generate the higher gradient terms in Eq. (15) in a similar manner by defining

$$V(q^2, m(x)) = -i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left(\frac{1}{(\not p-m)}\,\frac{1}{(\not p+\not q-m)}\,\frac{1}{(\not p-m)}\,\frac{1}{(\not p+\not q-m)}\right)\Big|_{m=m(x)} \tag{18}$$

and so on. $\Pi''(0)$ gives the coefficient of $(\Box m)^2$ and $V''(0)$ gives the coefficient of $(\partial_\mu m)^4$, etc. Thus each coefficient in the expansion of Eq. (15) will be given by an appropriate derivative of a Feynman graph calculated first with a conventional constant mass with that mass parameter then being replaced by a space-time dependent mass in the resulting mathematical expression. At no stage does m(x) appear inside the Feynman integrations.

Calculating Z(m(x)) we then find that

$$Z(m(x)) = \frac{1}{8\pi^2}\left(\ln\frac{\Lambda^2}{m^2(x)} - \frac{5}{3}\right) \tag{19}$$

Collecting everything together and incorporating the space-time dependent counter term we then find that

$$-\int d^4x\, \tilde{W}(m(x)) = \int d^4x \left\{ \frac{1}{8\pi^2} \ln \frac{\Lambda^2}{M^2} \left[\frac{1}{2} (\partial_\mu m)^2 + m^2(x)\, M^2 - \frac{1}{2} m^4(x) \right] + \text{U.V. Finite} \right\} \tag{20}$$

where "U.V. Finite" includes terms of order $\ln(M^2/m^2(x))$ and higher gradient terms. We shall see later that there is a lot of physics buried in these terms but for the moment if we only keep the terms which depend on the cutoff we then obtain the wrong sign Higgs result of Eguchi and Sugawara, with the stationarity condition on Eq. (20) leading to

$$\Box m(x) - 2m(x)\, M^2 + 2m^3(x) = 0 \quad . \tag{21}$$

[The extension to include a gauge field is given in the appendix. Note that Eq. (21) is an equation of constraint and should not be thought of as an Euler-Lagrange equation of motion since $\partial_\mu m$ is not an independent degree of freedom. This remark will become relevant when we need to include the higher gradient terms in Eq. (20).

Before beginning to discuss the stability and significance of solutions to Eq. (21) we turn first to see what happens to the same analysis in two dimensions.

III. THE HARTREE-FOCK APPROXIMATION TO THE THIRRING MODEL

We proceed exactly as before and calculate the vacuum energy density in the cutoff Thirring model in one loop.

$$\begin{aligned}\epsilon(m) &= i\int \frac{d^2p}{(2\pi)^2}\,\mathrm{Tr}\,\ln\left(\frac{1}{(\not p+i\epsilon)}(\not p-m+i\epsilon)\right)\\ &= -\frac{m^2}{4\pi}\left[\ln\left(\frac{\Lambda^2}{m^2-i\epsilon}\right)+1\right]\end{aligned} \tag{22}$$

where m is now space-time independent. If we go round the branch point at $m^2=0$ we find that above the cut

$$\epsilon(m) = -\frac{m^2}{4\pi}\left[\ln\left(\frac{\Lambda^2}{m^2+i\epsilon}\right)+1\right]-i\,\frac{m^2}{2} \tag{23}$$

whose imaginary part is negative. In sharp contrast to the situation in four dimensions we this time have to conclude that the self-consistent Hartree-Fock state is not the ground state since it decays. It cannot of course decay to the normal perturbative vacuum since that state lies above the Hartree-Fock vacuum (Re $\epsilon(m) < \epsilon(0)$). Thus there must be some other state in the theory to which the Hartree-Fock state can decay. In fact by putting the theory on a lattice Drell, Weinstein and Yankielowicz[6] have constructed a configuration space variational wave function which does have lower energy than our momentum space trial function. This then resolves the conflict between the self-consistent field mechanism and the absence of spontaneous breakdown of continuous symmetries in two dimensions,[7] and we see that the Hartree-Fock method contains sufficient information in it to provide a warning signal against interpreting the calculation as evidence of dynamical symmetry breaking. Unfortunately it is not yet apparent as to how we may use Eq. (23) to find the state into which the Hartree-Fock state does decay and also its γ_5 structure. This then makes the lattice

approach of Ref. 6 more useful at least for some ranges of values of the coupling constant.

If we ignore the imaginary part in Eq. (23) we can then build coherent states on the Hartree-Fock states and look for extended solutions as before. This is then a useful testing ground for extended model ideas,[2] though these states will also eventually decay via the imaginary part of $\epsilon(m)$, so their ultimate physical interpretation remains open at the present time.

Thus proceeding as before we obtain a gap equation in two dimensions

$$-\frac{M}{2\pi}\ln\frac{\Lambda^2}{M^2} = \frac{M}{g} \quad , \tag{24}$$

so that for coherent states

$$\tilde{\epsilon}(m(x)) = -\frac{m^2(x)}{4\pi}\left\{\ln\frac{M^2}{m^2(x)} + 1\right\} \tag{25}$$

Further

$$\Pi(q^2, m(x)) = -\frac{1}{2\pi}\left[\ln\frac{\Lambda^2}{m^2(x)} - \sqrt{\frac{4m^2(x) - q^2}{-q^2}}\ \ln\left\{\frac{\sqrt{4m^2(x)-q^2} + \sqrt{-q^2}}{\sqrt{4m^2(x)-q^2} - \sqrt{-q^2}}\right\}\right] \tag{26}$$

so that

$$Z(m(x)) = \frac{1}{12\pi m^2(x)} \tag{27}$$

Thus the effective action is

$$-\int d^2x\ \tilde{W}(m(x)) = \int d^2x\left\{\frac{1}{24\pi m^2(x)}(\partial_\mu m)^2 + \frac{m^2(x)}{4\pi}\ln\frac{M^2}{m^2(x)} + \frac{m^2(x)}{4\pi} + \ldots\right\} \tag{28}$$

and unlike Eq. (20) has no simple wrong sign mass term interpretation (though there is of course a double-well structure). Thus the simple structure of Eq. (20) is not a property of only keeping the graphs of Figs. 1 and 2, but rather of

subsequently giving physical significance to the cutoff, so that the results of Ref. 1 are very restricted. In the present case moreover there is no justification for stopping at the first gradient and so we have to include more terms. Moreover since we would like to look for solutions in which m(x) vanishes somewhere we see that Z(m(x)) has an infrared divergence at the core of the vortex. Since W(m(x)) of Eq. (13) is infrared finite we then learn that we cannot terminate the expansion of Eq. (28) at a finite point and hope to investigate extended solutions. Unfortunately though we have developed a method for obtaining all the coefficients of Eq. (28) we have not yet found a way to perform their sum analytically so we are unable to study this point completely. However Dashen, Hasslacher and Neveu[2] have developed an analytic method to which we return below.

Before discussing their work we note that it is possible to obtain an analytic expression for the behavior of a static extended solution in the wings of the vortex, i.e., how m(x) approaches M at spatial infinity. It is tempting to anticipate that this will be given by the approximate stationarity condition

$$\tilde{\epsilon}'(m(x)) - Z(m(x))\, m''(x) - \frac{1}{2} Z'(m(x))\, m'(x)^2 = 0 \tag{29}$$

However in the most likely case where the falloff is exponential, so that we may even ignore the m'^2 type terms, we are unable to disregard terms of the form $m''''(x)$, etc. These equally leading asymptotic terms are generated entirely by the higher derivatives of $\Pi(q^2, m(x))$, so that we again have to sum an infinite series. Fortunately this series is readily summable being generated by the action

$$\int d^2x \left\{ -\tilde{\epsilon}(m(x)) - \frac{1}{2} m(x) \left[\Pi\left(-\partial_\mu \partial^\mu, m(x)\right) - \Pi(0, m(x)) \right] m(x) \right\} \quad ;$$

thus for static solutions Eq. (29) is replaced by

$$\tilde{\epsilon}'(m(x)) + \sum \frac{1}{n!} \frac{d^n}{d(q^2)^n} \Pi(q^2, m(x)) \Big|_{q^2=0} \left(\frac{\partial}{\partial x}\right)^{2n} m(x) = 0 \quad . \tag{30}$$

Setting

$$m(x) = M(1 - A\, e^{-ax}) \tag{31}$$

we obtain

$$\tilde{\epsilon}'(m(x)) = -M\, A\, e^{-ax} \left[\Pi(0, m(x)) - \Pi(a^2, m(x))\right] \quad , \tag{32}$$

so that to lowest order in e^{-ax}

$$\begin{aligned} \tilde{\epsilon}''(M) &= \Pi(0, M) - \Pi(a^2, M) \\ &= \epsilon''(M) - \frac{1}{M}\, \epsilon'(M) \end{aligned} \tag{33}$$

using the definition of $\tilde{\epsilon}(M)$. Since by definition the second derivative of $\epsilon(M)$ in the presence of the mass is $\Pi(0, M)$ we obtain finally

$$\Pi(a^2, M) - \frac{1}{M}\, \epsilon'(M) = 0 \tag{34}$$

From Eq. (26) we then find that $a = 2M$ so that the leading behavior at spatial infinity is given by

$$m(x) = M(1 - A\, e^{-2Mx}) \tag{35}$$

with A undetermined. The factor of 2M has a direct physical interpretation. It can be thought of as arising from the two particle threshold in $\Pi(q^2, M)$ or as the mass of the (Cooper-type) fermion antifermion pair, i.e., the inverse of the coherence length of $\langle C | \bar{\psi}(x)\, \psi(x) | C \rangle$. To proceed beyond this point and calculate the nonleading terms as we come into the core from the wings is for the moment beyond our computational ability, and so we turn now to an alternative method of solution.

The alternative to developing the action in powers of the gradients of the order parameter is to diagonalize $\widetilde{W}(m(x))$ directly in terms of its eigenstates. Defining the Dirac problem in a space dependent potential

$$\left[-i\,\alpha_x \frac{\partial}{\partial x} + \beta\, m(x)\right] u_n(x) = E_n\, u_n(x)$$
$$\left[-i\,\alpha_x \frac{\partial}{\partial x} + \beta\, m(x)\right] v_n(x) = -E_n\, v_n(x) \tag{36}$$

we find that[8]

$$\int d^2x\, W(m(x)) = i\ \mathrm{Tr}\ \ln\left\{\frac{(i\not{\partial} - m(x))}{i\not{\partial}}\right\} = -T \sum_n \left\{E_n - E_n(M=0)\right\} \tag{37}$$

summed over the negative energy states. [T is the time volume.] After including the counter term we find that the stationarity condition on $\widetilde{W}(m(x))$ leads to

$$\sum_n \bar{v}_n(x)\, v_n(x) = \frac{m(x)}{g} = \frac{m(x)}{M}\, \epsilon'(M) \tag{38}$$

Equations (36) and (38) define a self-consistent problem in which the fermion moves in a potential generated by its own negative energy sea. However rather than solving Eqs. (36) and (38) (say using Eq. (35) as a good input trial wave function) the authors of Ref. 2 noted that it is more convenient to work with $\int d^2x\, \widetilde{W}(m(x))$ directly using the inverse scattering method which reexpresses the counter term $m^2(x)/2g$ in terms of the scattering data for the Dirac problem of Eq. (36). The remarkable achievement of Dashen, Hasslacher and Neveu was that they were then able to obtain an analytic solution to the above coupled equations, the kink of Ref. 8.[2] [We discuss their other solutions in Section IV.]

Though the method of Ref. 2 is constructive and elegant it suffers from the fact that it is difficult to apply in four dimensions (difficult technically rather than conceptually) since little is known about the inverse scattering method for the Dirac problem in three space dimensions, and so for the moment we can only

use our local source methods which provide some physical insight. Solving in four dimensions we can again obtain the analytic behavior of the extended solution far from its core. The analysis leading to Eq. (34) is exactly as before where now the coordinate x of Eq. (31) is the radial coordinate for either spherical or cylindrical (stringlike) structures. From Eq. (17) we then find again that a=2M for both types of structures with m(x) itself having no dependence on the cutoff even though the constraint equation for m(x) is cutoff dependent. We also note that these structures are then static solutions of the Eguchi and Sugawara equation of motion, Eq. (21). Thus in the wings the leading behavior is obtained separately both for the cutoff dependent and cutoff independent parts of $\widetilde{W}(m(x))$, and has the same physical interpretation as in the two dimensional case.

At this stage we can now discuss the limitations of Eq. (21). We have seen that it does correctly describe the asymptotic behavior of the spherical or cylindrical vortices. Suppose we would like such vortices to exist nonasymptotically and have a core where m(x) vanishes. In that case the "U.V. Finite" term of Eq. (20) becomes infrared divergent and hence we must still make the complete expansion in Eq. (20) as was done in the two dimensional case, despite the presence of the cutoff. Thus Eq. (21) becomes unreliable as we go toward the core of the vortex.

Though we have seen that we cannot truncate the vacuum functional at a finite point in relativistic field theories the situation is somewhat different in nonrelativistic theories at finite temperatures. In superconductivity the term Z(m(x)) of Eq. (27) behaves as T^{-2}, so that the expansion of $\widetilde{W}(m(x))$ will be truncatable provided we stay away from absolute zero. Further very close to $T=T_c$ $\widetilde{\epsilon}(m(x))$ does behave like a wrong sign Higgs potential[9] as was already known from Gorkov's

work. Thus in superconductivity as we lower the temperature below T_c more and more gradient terms come in in a controllable and calculable fashion so that we can go beyond the Ginzburg-Landau theory to any required degree of accuracy, being only required to sum the series to all orders at T=0. Thus our formalism is particularly well suited to superconductors when T is substantially below the critical point.

IV. STABILITY PROPERTIES OF THE EXTENDED STRUCTURES

Up to now we have only studied stationarity conditions and have not yet discussed the problem of the stability of our extended solutions. There are two main ways of obtaining stability, dynamical (e.g., the SLAC bag[3]) and topological (e.g., the kink[8]). We discuss first the topologically stable structures. For such structures the potential energy of the state only depends on the modulus of the order parameter so that if the modulus only varies from its vacuum value over a finite region of space the total energy will be finite, provided of course that the energy density is finite. Since the coherent state is constructed by creating an infinite number of pairs out of the vacuum (by Bogoliubov transform) we thus see that the extended structures are remarkable in that they possess an infinite number of modes and yet a finite energy. If further the phase of the order parameter varies over surfaces at spatial infinity, the order parameter then interpolates between different degenerate vacua, so that the overlap of the coherent state with any particular vacuum is zero (i.e., vanishing tunelling probability in the limit of an infinite number of degrees of freedom). The coherent state thus lies in a different Hilbert space than any of the degenerate vacua and is hence topologically stable. This can then provide a basis for confinement of quantum numbers as an analog to flux trapping, and general topological structures have recently been classified (see e.g., Refs. 10 and 11). Thus though the two

dimensional kink of Ref. 2 is stable the four dimensional cylindrical and spherical structures of Eq. (21) have to be rejected, the spherical since it is not topologically stable (there being no topological relation between the U(1) chiral phase group of Eq. (4) and SO(3)), and the cylindrical, because, like the usual Type II vortices of superconductivity, there is still a need for an extra gauge field in order to yield finite (kinetic) energy density. [In the original work of Eguchi and Sugawara there is an effective gauge field $<C|\bar{\psi}\gamma_\mu\gamma_5\psi|C>$ (see appendix) since they break Lorentz invariance spontaneously which is rather unphysical and difficult to interpret.]

The second kind of stability is dynamical in which the coherent state itself is not eigenstate (thus eliminating difficulties associated with the translation mode of Ref. 8), but rather the state $b_0^+|C>$ in which we put the lowest positive energy fermion into the coherent state is stable. This is achieved by mutually balancing the fermion energy against the bag pressure.[3] [In this case the phase of the order parameter need not vary over surfaces at spatial infinity.] Thus looking now for states in which

$$<C|b_0(\bar{\psi}(x)\psi(x))b_0^+|C> = \frac{m_0(x)}{g} \tag{39}$$

then leads to the equation of constraint

$$Z(m_0(x))\nabla^2 m_0(x) + \frac{1}{2}Z'(m_0(x))\left(\vec{\nabla} m_0(x)\right)^2 - \tilde{\epsilon}'(m_0(x)) + \ldots$$
$$= \frac{m_0(x)}{M}\epsilon'(M) - \sum_n \bar{v}_n(x)v_n(x) = \bar{u}_0(x)u_0(x) \tag{40}$$

where the dots denote the higher gradients. Equation (40) is thus seen to be the dynamical analog of the equation of motion of Ref. 3 (Eq. (3.35)) which uses a fake scalar tachyon coupled to the localized fermion. The two new features of Eq. (40) are that we need to include higher gradient terms and also need to

include the summation over the occupied negative energy states which produce the dynamical bag pressure in the first place. In this case although there is localization in space of the bound state wave function $u_0(x)$ there is no confinement of quantum numbers since the composite structure $b_0^+|C\rangle$ of the fermion localized in its own self-consistent potential has the same quantum numbers as the basic fermion. [If anything we confine the wrong vacuum in which $\langle\bar{\psi}\psi\rangle = 0$.] There is also an important difference with our previous case of calculating in the state $|C\rangle$. Now the potential $m_0(x)$ of Eq. (40) in which the fermion moves is determined by the fact that the first positive energy level E_0 is occupied. This is not the same as defining the potential $m(x)$ of Eq. (38) via the coherent state $|C\rangle$ and then simply looking at its bound states. [Equation (38) does not even possess any nontopologically stable solutions.[2]] In other words the potential adjusts itself to the fact that the fermion is there. Moreover the wave function for a bag with two fermions in it has to be self-consistently determined afresh and bears no relation to the second excited state of the potential of Eq. (40).

Despite the fact that we must now consider positive energy states as well we note that asymptotically $\bar{u}_0(x)\, u_0(x)$ vanishes much faster than the potential $m_0(x)$, since the fermion is bound, and hence the behavior of Eq. (35) is still obtained asymptotically in both two and four dimensions. Thus the asymptotic behavior of the order parameter is not influenced by the stabilizing fermion so we will again have a good input trial wave function for the variational calculation of Eq. (40). In passing we also note that we recover the analytic result found in Ref. 2, since asymptotically the exact solution behaves as predicted by Eq. (35) in the case where there is one positive energy state and only one species of fermion.

V. GENERAL REMARKS

In this section we collect together the main advantages of generating extended structures dynamically. The most important feature of course is that the classical equations that are obtained are output to the underlying quantum field theory and arise because of nonperturbative infrared effects of quantum fluctuations at the fundamental fermion level. There is never any need to quantize the output classical equations. Consequently the classical equations are not restricted to have the usual structure of renormalizable field theories and moreover can have any number of higher gradient terms. Despite the fact that the classical equations can even (and usually do) possess an infinite number of derivatives the underlying microscopic theory is still completely local, and hence there is no loss of locality. Moreover the coefficients in the classical equations are determined completely by the microscopic dynamics, so there are no adjustable parameters. The potential energy either yes or no has a double-well structure (and it will have if the vacuum undergoes dynamical symmetry breaking), and there is no freedom to change the sign of a mass term arbitrarily by hand. Once dynamical symmetry breaking takes place there will always be extended solutions as well (since $\Pi(q^2, m(x))$ cannot vanish), though their stability properties need to be investigated separately in each individual case.

The second important feature is that the order parameter while obviously not a fundamental scalar field is not in fact a dynamical tachyon either. It is a pure c-number mass term, and couples to $\bar{\psi}(x)\,\psi(x)$ as an external source. It is not a bound state pole in the fermion antifermion scattering amplitude. Indeed the (nontranslation invariant) equation of constraint for the order parameter bears no relation to the (translation invariant) bound state Bethe-Salpeter equation that any prospective dynamical tachyon would satisfy. Whether or not there is a

bound state depends on the dynamics of the residual interaction. It is not forced by the existence of a self-consistent order parameter. Thus we can envisage a situation in which there is an order parameter while at the same time the Bethe-Salpeter kernel does not generate any bound states; typically this would occur if the kernel is non-Fredholm as in finite quantum electrodynamics (see e.g., Ref. 5). In this case even though there is no bound state any external gauge field coupled minimally to the order parameter would still acquire a mass as an analog to the penetration depth of a Type II superconductor. Thus the method gives "Higgs without Higgs." We are currently investigating whether this is the dynamics of W mesons.

The final advantage of our approach is that it is easily generalizable to interacting theories since we can start dressing the loops of Figs. 1 and 2 with some new interaction such as the exchange of a meson. Provided we do not include any additional fermion loops within the basic loop the formalism goes through untouched and can hence be applied in theories such as loopwise summed finite quantum electrodynamics. In fact we have recently analyzed finite quantum electrodynamics this way and find that the Poincare stresses needed to stabilize a completely dynamical electron arise as a bag pressure because of a phase transition in the vacuum with the electron becoming an extended object. This work will be discussed in detail in a forthcoming publication.[12]

After this work was completed we became aware of some recent rather formal derivations of the results of Ref. 1 which use a path-integral formulation.[13,14]

APPENDIX

In this appendix we discuss briefly the extension of Eq. (21) to include a gauge field. Suppose we extend Eq. (4) to a local chiral gauge invariance by adding an interaction $g_A \bar{\psi}\gamma_\mu\gamma_5 A_{\mu 5}\psi$, so that the theory is invariant under

$$\psi \to e^{ig_A\gamma_5\lambda(x)}\psi, \qquad A_{\mu 5} \to A_{\mu 5} + \partial_\mu \lambda(x) \tag{A.1}$$

Introducing the convenient parameter

$$\phi = \bar{\psi}\psi - i\bar{\psi}i\gamma_5\psi \tag{A.2}$$

we may then rewrite the four-Fermi interaction as $-\frac{1}{2}g\phi^*\phi$, with ϕ having the property that it transforms as

$$\phi \to e^{2ig_A\lambda(x)}\phi \tag{A.3}$$

under the chiral transformation. Consequently

$$\left[\left(\partial_\mu - 2ig_A A_{\mu 5}\right)\phi\right]^* \left[\left(\partial^\mu - 2ig_A A^{\mu 5}\right)\phi\right]$$

is invariant under the local chiral transformation. Thus any mechanism which induces a term $\partial_\mu\langle\phi^*\rangle\partial^\mu\langle\phi\rangle$ will automatically induce the minimal coupling to $A_{\mu 5}$ with a strength $2g_A$. [This is familiar of course from the theory of superconductivity, there being a factor of 2e in Eq. (1).] Thus minimal coupling to the fundamental fermion implies minimal coupling to the order parameter.

In the four-Fermi theory calculation we can now introduce $A_{\mu 5}$ either as an external gauge field (as was done by Konisi, Saito and Shigemoto[15]) so that there is no breaking of Lorentz invariance, or have it emerge as a collective coordinate $A_{\mu 5} = \langle\bar{\psi}\gamma_\mu\gamma_5\psi\rangle$ by treating the interaction $\frac{1}{2}g_A(\bar{\psi}\gamma_\mu\gamma_5\psi)^2$ in the Hartree-Fock approximation (see Eguchi[16]). [In this latter case the residual interaction must then generate a Goldstone boson in order to restore the Lorentz symmetry.] Either method then extends Eq. (20) to include coupling to a c-number axial gauge

field in the form originally given in Ref. 1, viz.

$$-\int d^4x\, \widetilde{W}\left(<\phi(x)>, A_{\mu 5}(x)\right) = \int d^4x \left\{ \frac{1}{8\pi^2} \ln \frac{\Lambda^2}{M^2} \left[\frac{1}{2} \left| \left(\partial_\mu - 2ig_A A_{\mu 5}\right) <\phi(x)> \right|^2 \right.\right.$$

$$\left.\left. + \left|<\phi(x)>\right|^2 M^2 - \frac{1}{2} \left|<\phi(x)>\right|^4 - \frac{1}{6} g_A^2 F_{\mu\nu 5} F^{\mu\nu 5} \right] + \text{U.V. Finite} \right\} \qquad \text{(A.4)}$$

Graphically this amounts to calculating the one-loop Feynman graphs with appropriate external $<\phi>$ and $A_{\mu 5}$ sources in the action, and we have checked that the cutoff dependent parts of the resulting coefficients are indeed in the above relative weights required for minimal coupling. Also the axial vacuum polarization is renormalized by the fermion loop and we have calculated the coefficient of $F_{\mu\nu 5} F^{\mu\nu 5}$ using a Pauli-Villars regulator of mass Λ, in order to obtain Eq. (A.4).

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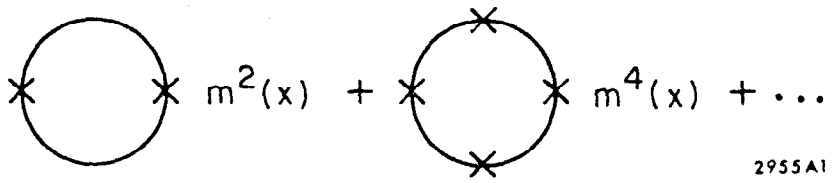

Fig. 1

The infinite series of massless graphs with soft insertions used to calculate $\epsilon(m(x))$.

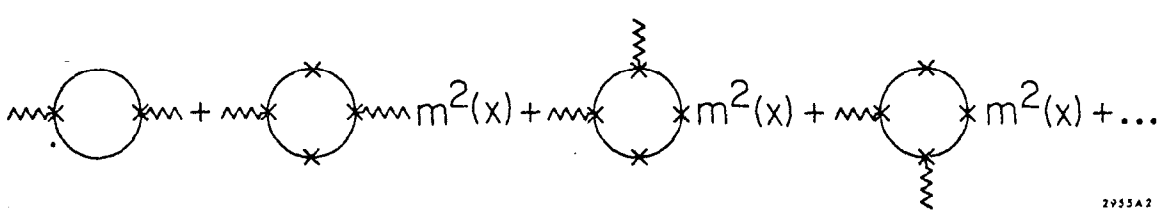

Fig. 2

The infinite series of massless graphs with two external insertions carrying momentum q_μ and all possible combinations of soft insertions used to calculate $\Pi(q^2, m(x))$. The squiggles indicate momentum q_μ and the crosses indicate momentum zero.