# IN AN ELECTRON STORAGE RING* 

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#### Abstract

The distribution function of particles in a plasma can often be found by solving the Fokker-Planck equation. This technique can also be used to find the average particle distribution in an electron storage ring in the presence of coupling between the particle coordinates and momenta. Using a smooth approximation the beam distribution parameters, such as the transverse beam sizes, can be described in terms of the eigenvalues and eigenvectors of the coupling matrix. For some cases, analytic expressions for these quantities can be obtained directly. This method offers us a straightforward means to find the functional dependence of the beam shape parameters upon various machine parameters which are essential for storage ring design.


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## I. INTRODUCTION

The stationary particle distribution for an electron beam in a storage ring in the presence of linear couplings between the oscillations of the transverse motion in $x$ and $y$ and energy deviation, $\delta=\frac{\Delta \mathrm{E}}{\mathrm{E}_{0}}$, can be written as $\psi_{S} \propto \exp \left(\sum_{i} \sum_{j} x_{i} A_{i j} x_{j}\right)$ with $x_{i}$ denoting the canonical coordinates $\left(x, p_{x}, y, p_{y}\right.$, $\delta, \mathrm{p}_{\delta}$ ) and $\mathrm{A}_{\mathrm{ij}}$ 's some constants. This Gaussian distribution function can be obtained by solving the time-independent Fokker-Planck equation. ${ }^{1,2}$ Under certain conditions in the presence of perturbation, the particle distribution $\psi$ is nonstationary and is different from $\psi_{S}$. After the perturbation is terminated at $t=0$, $\psi$ leaves its initial form $\psi_{0}$, and changes toward its final form $\psi_{s}$. This transient behavior of $\psi$ can be obtained by solving the time-dependent Fokker-Planck equation with the boundary condition $\psi=\psi_{0}$ at $\mathrm{t}=0$. The time-dependent particle distribution can also be expressed in a similar form as $\psi_{S}$ but with the values of $A_{i j}$ time-dependent.

In this paper, a general method for finding the matrix A for a stored electron beam will be described, taking into account the radiation damping, quantum fluctuation and linear coupling effects. Since the average $\left\langle x_{i} x_{j}\right\rangle$ is equal to the $i j-t h$ element of the matrix $-\frac{1}{2} A^{-1}$, this procedure offers us a systematic method for obtaining the dependence of these quantities upon the coupling element strengths and machine operation conditions. From the expressions for $\left\langle x_{i} x_{j}\right\rangle$ some physical quantities can be derived.

As an example, the above method will be applied to obtain the dependence of the stationary transverse beam sizes and the tilt angle of the beam profile upon the strength of rotated quadrupoles and solenoid magnets in a storage ring. In a smooth approximation ${ }^{3}$ for weak coupling it will be shown that the beam sizes resulting from betatron oscillations, $\sigma_{\mathrm{x} \beta}$ and $\sigma_{\mathrm{y} \beta}$, satisfy an
invariant condition: the value of

$$
\frac{\alpha_{\mathrm{x}} \sigma_{\mathrm{x} \beta}^{2}}{\beta_{\mathrm{x}}}+\frac{\alpha_{\mathrm{y}} \sigma_{\mathrm{y} \beta}^{2}}{\beta_{\mathrm{y}}}
$$

is independent of coupling strengths; with $\beta_{\mathrm{x}, \mathrm{y}}$ the betatron functions ${ }^{4}$. and $\alpha_{\mathrm{x}, \mathrm{y}}$ the damping rates. A generalized definition of the aspect ratio which characterizes the emittance transfers between $x$ - and $y$-motions as caused by coupling will be presented. Some of these results have been confirmed by experimental observations in SPEAR. ${ }^{5}$

The transient behavior of the horizontal beam width for an injected electron beam will be studied as an illustration of the time-dependent solution. These results can be useful for studying the effects of the injection system parameters upon the beam width for an electron storage ring.

## II. EQUATIONS OF MOTION

The equation of motion for each of the three modes of oscillation ( $\mathrm{i}=\mathrm{x}, \mathrm{y}, \delta$ ) is given by ${ }^{2,6}$

$$
\begin{equation*}
\ddot{u}_{i}+2 \alpha_{i} \dot{u}_{i}+\omega_{i}^{2} u_{i}=\frac{\text { quantum }}{\text { excitation }}+\text { coupling } \tag{1}
\end{equation*}
$$

with $\alpha_{i}$ the damping rate and $\omega_{i}$ the frequency of oscillation. If we introduce the canonical variables $x_{i}{ }^{\prime}$ s for ( $x, p_{x}, y, p_{y}, \delta, p_{\delta}$ ), the equations of motion reduce to a system of six first order differential equations. In particular, for a storage ring having lattice elements which only produce linear coupling between $x_{i}$ and $x_{j}$ the system of equations are:

$$
\begin{equation*}
\frac{d x_{i}}{d t}=\sum_{j} C_{i j} x_{j}+d_{i} \xi(t), \quad i=1 \text { to } 6 \tag{2}
\end{equation*}
$$

with the values of damping rates, oscillation frequencies and coupling coefficients given by the matrix $C$; the quantum excitation effects are given by a succession of
sudden random variations of $\mathrm{x}_{\mathrm{i}}$ and are specified by a stochastic function $\xi(\mathrm{t})$ representing energy jumps caused by photon emissions. In the smooth approximation, $C_{i j}$ 's and $d_{i}$ 's are taken to be: $d_{1,3}=-\eta x, y^{\prime} / E_{0}, d_{5}=-E_{0}^{-1}, d_{2}=d_{4}=d_{6}=0$ with $\eta_{\mathrm{x}, \mathrm{y}}$ the energy dispersion functions, $\mathrm{E}_{0}$ the ideal particle energy and

## III. PARTICLE DISTRIBUTION FUNCTION

Under the above assumptions the particle distribution function, $\psi$, satisfies the Fokker-Planck equation ${ }^{1,2}$

$$
\begin{equation*}
\sum_{i} C_{i i} \psi+\sum_{i} \sum_{j} C_{i j} x_{j} \frac{\partial \psi}{\partial x_{i}}-\sum_{i} \sum_{j} D_{i j} \frac{\partial^{2} \psi}{\partial x_{i} \partial x_{j}}=-\frac{\partial \psi}{\partial t} \tag{3}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathrm{D}_{\mathrm{ij}}=\frac{1}{2}\left\langle\mathscr{D} \mathrm{~d}_{\mathrm{i}} \mathrm{~d}_{\mathrm{j}}\right\rangle \mathrm{B} \tag{4}
\end{equation*}
$$

where < > $\quad$ Beans averaging over all bending magnets; $\mathscr{D}$ is the product of the mean square photon energy and the mean rate whose value ${ }^{2,6}$ is given by $\frac{55}{24 \sqrt{3}} \mathrm{r}_{\mathrm{e}} \hbar \mathrm{mc} \mathrm{c}^{4} \gamma^{7} / \rho^{3}$, 'with $\mathrm{r}_{\mathrm{e}}$ the classical electron radius, c the speed of light,市 the Planck's constant, $\rho$ the radius of curvature, $\gamma \mathrm{mc}^{2}$ the particle energy, and $m$ the electron rest mass.

We will first describe a time-dependent transient solution of $\psi$ which satisfies a certain initial condition at time $t=0$. The stationary particle distribution $\psi_{s}$ is
then simply obtained by letting $t \rightarrow \infty$. At $t=0$, we assume that the initial distribution is Gaussian (the solution of non-Gaussian cases will be described later):

$$
\begin{equation*}
\psi_{0}=F(0)^{-1} \exp \left\{\sum_{i} \sum_{j} A_{i j 0}\left(x_{i}-\bar{x}_{i 0}\right)\left(x_{j}-\bar{x}_{j 0}\right)\right\} \tag{5}
\end{equation*}
$$

$F(0), A_{i j 0}$ and $\vec{x}_{\mathrm{i} 0}$ are known constants. Under this assumption the distribution function at $t>0$ is taken to be:

$$
\begin{equation*}
\psi=F(t)^{-1} \exp \left\{\sum_{i} \sum_{j} A_{i j}(t)\left(x_{i}-\bar{x}_{i}(t)\right)\left(x_{j}-\bar{x}_{j}(t)\right)\right\} \tag{6}
\end{equation*}
$$

with $\bar{x}_{i}(t)$ and $A_{i j}(t)$ some unknown functions of time. From symmetry, $A_{i j}(t)=A_{j i}(t)$. The normalization factor $F(t)$ is determined by

$$
\begin{equation*}
\int \psi d x_{1} \ldots \mathrm{dx}_{6}=1 \tag{7}
\end{equation*}
$$

Substituting Eq. (6) into Eq. (3) and equating the coefficients for the terms containing the same factors yields the following conditions: from the $\left(x_{i}-\bar{x}_{i}\right)\left(x_{j}-\bar{x}_{j}\right)$ terms

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{dt}} \mathrm{~A}=4 \mathrm{ADA}-\mathrm{AC}-\widetilde{\mathrm{C}} \mathrm{~A} \tag{8}
\end{equation*}
$$

from the $\left(x_{i}-\bar{x}_{i}\right)$ terms

$$
\begin{equation*}
\frac{d}{d t} \bar{X}(t)=C \bar{X}(t) \tag{9}
\end{equation*}
$$

and from the remaining terms

$$
\begin{equation*}
\frac{1}{\mathrm{~F}} \frac{\mathrm{dF}}{\mathrm{dt}}=\operatorname{trace}(\mathrm{C})-2 \operatorname{trace}(\mathrm{DA}) \tag{10}
\end{equation*}
$$

The initial values of $A_{i j}$ and $\bar{x}_{i}$ are $A_{i j 0}$ and $\bar{x}_{i 0}$, respectively. A tilde is used to denote the transpose of a matrix; $A(t), C$ and $D$ are matrices whose elements are $A_{i j}(t), C_{i j}$ and $D_{i j}$, respectively; $\bar{X}(t)$ is a column vector whose elements are $\bar{X}_{i}(t)$.

The number of independent conditions in Eq. (8) is equal to the number of unknown functions, $A_{i j}(t)$, which is given by $m(2 m+1)$ with $m$ the number of modes
of oscillation. Equation (9) are 2 m coupled differential equations of an m dimensional damped harmonic oscillator whose solution can be found independently from the $A_{i j}$ 's. It can be shown that Eq. (10) is automatically satisfied if $\mathrm{A}(\mathrm{t})$ is a solution of Eq. (8). To obtain the particle distribution function, we therefore only have to solve Eqs. (8) and (9) independently.

Since $\tilde{A}=A$, from Eq. (8) we obtain an equation for the inverse of $A$ :

$$
\begin{equation*}
\frac{d}{d t} A^{-1}=A^{-1} \widetilde{C}+C A^{-1}-4 D \tag{11}
\end{equation*}
$$

Now we assume that C can be diagonalized, then

$$
\begin{equation*}
\mathrm{C}=\mathrm{E} \lambda \mathrm{E}^{-1} \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
\widetilde{\mathrm{C}}=\widetilde{\mathrm{E}}^{-1} \lambda \widetilde{\mathrm{E}}, \tag{13}
\end{equation*}
$$

with $\lambda$ a diagonal matrix whose elements are the eigenvalues of $C$, and $E$ is composed of the corresponding eigenvectors; i.e., the ith column of E is given by the eigenvector corresponding to the eigenvalue $\lambda_{\mathrm{ii}}$. It may be noted that for the single particle motion to be bounded, we must have $\operatorname{Re}\left(\lambda_{\mathrm{ii}}\right)<0$. The solution of Eq. (11) can be shown to be

$$
\begin{equation*}
\mathrm{A}^{-1}=\mathrm{EB} \widetilde{\mathrm{E}} \tag{14}
\end{equation*}
$$

with

$$
\begin{equation*}
B_{i j}(t)=B_{i j}(0) e^{\Lambda_{i j}{ }^{t}}+\frac{4}{\Lambda_{i j}}\left(E^{-1} D \widetilde{E}^{-1}\right)_{i j}\left(1-e^{\Lambda_{i j} t}\right), \tag{15}
\end{equation*}
$$

which satisfies

$$
\begin{equation*}
\frac{d}{d t} B=B \lambda+\lambda B-4 F^{-1} D \widetilde{F}^{-1} \tag{16}
\end{equation*}
$$

and the initial conditions

$$
\begin{equation*}
\mathrm{B}_{\mathrm{ij}}(0)=\mathrm{E}^{-1} \mathrm{~A}^{-1}(0) \widetilde{\mathrm{E}}^{-1} \tag{17}
\end{equation*}
$$

where we have used $\Lambda_{i j}=\lambda_{i i}+\lambda_{j j}$.

The other unknown functions, $\bar{x}_{\mathrm{i}}(\mathrm{t})$, must satisfy Eq. (9) and the initial conditions $\bar{x}_{i}(0)=\bar{x}_{i 0}$. The solution is

$$
\begin{equation*}
\bar{X}(t)=E e^{\lambda t} E^{-1} \bar{X}_{0} \tag{18}
\end{equation*}
$$

where $\bar{X}_{0}$ is a column matrix whose $i$-th element is $\bar{x}_{i 0}$.
Note that the above analysis could be extended for a non-Gaussian initial distribution function. In particular, for a delta function initial distribution $\delta\left(\mathrm{x}_{1}-\mathrm{x}_{10}\right) \ldots \delta\left(\mathrm{x}_{6}-\mathrm{x}_{60}\right)$ the transient solution, $\psi_{\delta}$, is the same as that for the Gaussian initial distribution but with $\mathrm{B}_{\mathrm{ij}}(0)=0$. The transient solution for a general initial distribution $\psi_{0}$ is given by $\int \psi_{0} \psi_{\delta} \mathrm{dx}_{10} \ldots \mathrm{dx}_{60}$.

The stationary distribution $\psi_{S}=\exp (\widetilde{\mathrm{X}} A X)$ can be obtained by letting $\mathrm{t} \rightarrow \infty$ in the above calculation. Thus the corresponding matrix A is determined by Eq. (14) with

$$
\begin{equation*}
B_{i j}=\frac{4}{\Lambda_{i j}}\left(E^{-1} D \widetilde{E}^{-1}\right)_{i j} \tag{19}
\end{equation*}
$$

The stationary distribution obtained is independent of initial conditions.
From Eq. (6), it follows that the first and second moments of the distribution are given by

$$
\begin{equation*}
\left\langle x_{i}\right\rangle=\bar{x}_{i}(t) \tag{20}
\end{equation*}
$$

and

$$
\begin{align*}
\left\langle\left(x_{i}-\left\langle x_{i}\right\rangle\right)\left(x_{j}-\left\langle x_{j}\right\rangle\right)\right\rangle & =-\frac{1}{2}\left[A^{-1}(t)\right]_{i j}  \tag{21}\\
& =-\frac{1}{2}[E B(t) \tilde{E}]_{i j}
\end{align*}
$$

respectively. The trajectory of the center of this distribution $\psi(t)$ therefore coincides with the trajectory of a damped harmonic oscillator in the presence of coupling and the rms distribution widths are readily obtained from Eq. (21). For stationary cases, $\left\langle\mathrm{x}_{\mathrm{i}}\right\rangle=0$.

## IV. TRANSVERSE BEAM SHAPE WITH COUPLING MAGNETS

The transverse beam shape in the absence of perturbations is determined by coupling elements such as rotated quadrupole and solenoid magnets. The equation of motion obtained by smooth approximation near the coupling resonance $\omega_{\mathrm{x}} \approx \omega_{\mathrm{y}} \approx \omega$ can be shown to be given by Eq. (2) with the coupling matrix ${ }^{5}$

$$
\mathrm{C}=\left[\begin{array}{cccc}
0 & 1 & -\mathrm{k}_{2} & 0  \tag{22}\\
-\omega_{\mathrm{x}}^{2} & -2 \alpha_{\mathrm{x}} & -\mathrm{k}_{1} & -\mathrm{k}_{2} \\
\mathrm{k}_{2} & 0 & 0 & 1 \\
-\mathrm{k}_{1} & \mathrm{k}_{2} & -\omega_{\mathrm{y}}^{2} & -2 \alpha_{\mathrm{y}}
\end{array}\right]
$$

The energy oscillation is nonessential and has been ignored. The betatron oscillation coordinates $x$ and $y$ have been normalized by $(\nu \beta / R)_{x, y^{\prime}}^{1 / 2}$ with $R$ the machine radius and $\nu$ the betatron wave number ${ }^{4}$ given by $\mathrm{R} \omega / \mathrm{c}$. The coefficients $\mathrm{k}_{1}$ and $\mathrm{k}_{2}$ are defined to be

$$
\mathrm{k}_{1}=\frac{2 \mathrm{Q}_{1} \nu \mathrm{c}^{2}}{\mathrm{R}^{2}}
$$

and

$$
\begin{equation*}
\mathrm{k}_{2}=-\frac{\mathrm{Q}_{2} \mathrm{c}}{\mathrm{R}} \tag{23}
\end{equation*}
$$

For weak coupling case, the coupling coefficient $Q$ is given by ${ }^{7}$ :

$$
\begin{align*}
\mathrm{Q} & =\mathrm{Q}_{1}+\mathrm{i} Q_{2} \\
& =\frac{1}{4 \pi \mathrm{R}} \int_{\theta}^{\theta}{ }_{\theta}^{\theta+2 \pi} \mathrm{~d} \theta \beta_{\mathrm{x}}^{1 / 2} \beta_{\mathrm{y}}^{1 / 2}\left\{\mathrm{~K}-\frac{1}{4} \mathrm{M}\left(\frac{\frac{\mathrm{~d}}{\mathrm{~d} \theta} \beta_{\mathrm{x}}}{\beta_{\mathrm{x}}}-\frac{\frac{\mathrm{d}}{\mathrm{~d} \theta} \beta_{\mathrm{y}}}{\beta_{\mathrm{y}}}\right)-\frac{\mathrm{i}}{2} \mathrm{MR}\left(\frac{1}{\beta_{\mathrm{x}}}+\frac{1}{\beta_{\mathrm{y}}}\right)\right\} \\
& \exp \left[\mathrm{i} \int_{\theta_{\theta}}^{\theta} \mathrm{d} \theta\left(\frac{\mathrm{R}}{\beta_{\mathrm{x}}}-\frac{\mathrm{R}}{\beta_{\mathrm{y}}}-\Delta \nu\right)\right], \tag{24}
\end{align*}
$$

where

$$
\mathrm{K}(\theta)=\frac{\mathrm{R}^{2}}{\mathrm{~B} \rho} \frac{\partial \mathrm{~B}_{\mathrm{x}}}{\partial \mathrm{x}}(\theta)
$$

and

$$
\begin{equation*}
\mathrm{M}(\theta)=\frac{\mathrm{R}}{\mathrm{~B} \rho} \mathrm{~B}_{\mathrm{z}}(\theta) \tag{25}
\end{equation*}
$$

are the strengths of rotated quadrupole and solenoid magnets, respectively, $\theta_{\theta}$ is the azimuthal position of the observation point, $B \rho$ is the particle rigidity and $\Delta \nu=\nu_{\mathrm{x}}-\nu_{\mathrm{y}}$ is the split in betatron wave numbers.

As shown in the previous section, the stationary particle distribution is given by

$$
\begin{equation*}
\psi=\exp \left(\sum_{i j} x_{i} A_{i j} x_{j}\right) \tag{26}
\end{equation*}
$$

For the present special example, it is possible to solve the problem without diagonalizing the $C$ matrix. Indeed, we can obtain the symmetric matrix $A^{-1}$ by directly solving the ten independent linear equations contained in Eq. (11) with the left hand side replaced by zero. The diffusion matrix $D$ is given by

$$
\mathrm{D}=\frac{1}{2 \mathrm{E}_{0}^{2}}\left[\begin{array}{cccc}
\left\langle\mathscr{D} \eta_{\mathrm{x}}^{2} \mathrm{~B}\right. & 0 & <\mathscr{D} \eta_{\mathrm{x}} \eta_{\mathrm{y}}{ }^{\rangle} \mathrm{B} & 0  \tag{27}\\
0 & 0 & 0 & 0 \\
\left.<\mathscr{D} \eta_{\mathrm{x}} \eta_{\mathrm{y}}\right\rangle_{\mathrm{B}} & 0 & \left\langle\mathscr{D} \eta_{\mathrm{y}^{\prime} \mathrm{B}}^{2}\right. & 0 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

Since $\eta_{\mathrm{y}}$ as caused by coupling elements oscillates rapidly around the ring, it contributes very little to an average over all mending magnets; for this example, we will neglect $\left\langle\mathscr{D} \eta_{\mathrm{x}} \eta_{\mathrm{y}}>\mathrm{B}\right.$.

By explicitly solving Eq. (11) and using Eq. (21), we can obtain the beam shape parameters $\left\langle\mathrm{x}^{2}\right\rangle,\left\langle\mathrm{y}^{2}\right\rangle$ and $\langle\mathrm{xy}\rangle$ in terms of the parameters $\left\langle\mathrm{x}^{2}\right\rangle_{0}$ and $\left\langle\mathrm{y}^{2}\right\rangle_{0}$ without coupling:

$$
\begin{align*}
& \left\langle x^{2}\right\rangle=\frac{1}{1+A_{y}^{2}}\left\langle x^{2}\right\rangle_{0}+\frac{A_{x}^{2}}{1+A_{x}^{2}} \frac{\alpha}{\alpha_{x}}\left\langle y^{2}\right\rangle_{0} \\
& \left\langle y^{2}\right\rangle=\frac{A_{y}^{2}}{1+A_{y}^{2}} \frac{\alpha_{x}}{\alpha}\left\langle x^{2}\right\rangle_{0}+\frac{1}{1+A_{x}^{2}}\left\langle y^{2}\right\rangle_{0} \tag{28}
\end{align*}
$$

and

$$
\langle\mathrm{xy}\rangle=\frac{\alpha_{\mathrm{x}} \alpha \mathrm{y} \Delta \nu \mathrm{Q}_{1}\left(\left\langle\mathrm{x}^{2}\right\rangle_{0}-\left\langle\mathrm{y}^{2}\right\rangle_{0}\right)}{\left(\alpha_{\mathrm{x}}+\alpha_{\mathrm{y}}\right)^{2}|\mathrm{Q}|^{2}+\alpha_{\mathrm{x}} \alpha_{\mathrm{y}} \Delta \nu^{2}}
$$

where $A_{x}$ and $A_{y}$ are the generalized aspect ratios:

$$
\begin{equation*}
A_{\mathrm{x}}=\left\{\frac{\alpha_{\mathrm{x}}\left(\alpha_{\mathrm{x}}+\alpha_{\mathrm{y}}\right)|Q|^{2}}{\alpha_{\mathrm{y}}\left[\left(\alpha_{\mathrm{x}}+\alpha_{\mathrm{y}}\right)|\mathrm{Q}|^{2}+\alpha_{\mathrm{x}} \Delta \nu^{2}\right]}\right\}^{1 / 2} \tag{29}
\end{equation*}
$$

and

$$
A_{y}=\left\{\frac{\alpha_{y}\left(\alpha_{x}+\alpha_{y}\right)|Q|^{2}}{\alpha_{x}\left[\left(\alpha_{x}+\alpha_{y}\right)|Q|^{2}+\alpha_{y} \Delta \nu^{2}\right]}\right\}^{1 / 2}
$$

It can be seen from Eq. (28) that the value of $\left\langle\mathrm{x}^{2}\right\rangle$ is jointly determined by the contributions from $\left\langle\mathrm{x}^{2}\right\rangle_{0}$ and $\left\langle\mathrm{y}^{2}\right\rangle_{0}$. The value of $\mathrm{A}_{\mathrm{y}}^{2}$ determines the portion contributed from $\left\langle\mathrm{x}^{2}\right\rangle_{0}$, and a similar interpretation for $\mathrm{A}_{\mathrm{x}}^{2}$. In the derivation of these results, we have assumed slow damping rates, keeping only the leading order in $\alpha_{x, y}^{-1}$ and have consistently neglected higher order terms in $Q_{1}, Q_{2}$, and $\Delta \nu$. The natural beam shape due to betatron oscillations in the absence of ${ }^{\prime}$
coupling is characterized by

$$
\begin{equation*}
\left\langle\mathrm{x}^{2}\right\rangle_{0}=\frac{\left\langle\mathscr{D} \eta_{\mathrm{x}}^{2} \mathrm{~B}\right.}{4 \alpha_{\mathrm{x}} \mathrm{E}_{0}^{2}}, \quad\left\langle\mathrm{y}^{2}\right\rangle_{0}=\frac{\left\langle\mathscr{D} \eta_{\mathrm{y}}^{2} \mathrm{~B}\right.}{4 \alpha_{\mathrm{y}}^{\mathrm{E}_{0}^{2}}}, \quad\langle\mathrm{xy}\rangle_{0}=0 \tag{30}
\end{equation*}
$$

Equation (28) also shows that the normalized transverse beam sizes due to betatron oscillations satisfy the invariance condition

$$
\begin{equation*}
\alpha_{\mathrm{x}}\left\langle\mathrm{x}^{2}\right\rangle+\alpha_{\mathrm{y}}\left\langle\mathrm{y}^{2}\right\rangle=\frac{1}{4 \mathrm{E}_{0}^{2}}\left\langle\mathscr{D}\left(\eta_{\mathrm{x}}^{2}+\eta_{\mathrm{y}}^{2}\right)\right\rangle \mathrm{B} \tag{31}
\end{equation*}
$$

which is independent of the coupling coefficients. For the special case of equal damping rates (i.e., $\alpha_{x}=\alpha_{y}$ ), this invariant reduces to the well known condition in Ref. 6.

The tilting angle of the beam profile due to betatron oscillations $\phi$, can be obtained from

$$
\begin{align*}
\tan 2 \phi & =\frac{2\langle\mathrm{xy}\rangle}{\left\langle\mathrm{x}^{2}\right\rangle-\left\langle\mathrm{y}^{2}\right\rangle}  \tag{32}\\
& =\frac{2 \mathrm{Q}_{1}}{\Delta \nu}
\end{align*}
$$

This result coincides with Ref. 8 for the tilt of the principle axes of the normal modes for a uniform proton storage ring without diffusion effects.

The aspect ratios of the beam, defined in Eq. (29), become equal when $\alpha_{x}=\alpha_{y}$; i.e., $A_{x}=A_{y}=A$ with

$$
\begin{equation*}
A=\left\{\frac{2|Q|^{2}}{2|Q|^{2}+\Delta v^{2}}\right\}^{1 / 2} \tag{33}
\end{equation*}
$$

independent of the damping rates. Furthermore, if we neglect the contribution from $\eta_{y}$, we obtain the usual expressions ${ }^{6}$ :

$$
\begin{equation*}
\left\langle x^{2}\right\rangle=\frac{1}{1+A^{2}}\left\langle x^{2}\right\rangle \tag{34}
\end{equation*}
$$

and

$$
\left\langle y^{2}>=\frac{A^{2}}{1+A^{2}}\left\langle x^{2}\right\rangle_{0}\right.
$$

In practice, the transverse beam shape is determined not just by the coupled betatron oscillations but also by the values of the energy dispersion functions $\left(\eta_{\mathrm{x}, \mathrm{y}_{\theta}}\right)^{\text {at the point of observation and the value of energy spread, } \delta \text {. For a beam }}$ without bunch lengthening,

$$
\begin{equation*}
\left\langle\delta^{2}\right\rangle=\frac{\langle\mathscr{D}\rangle_{\mathrm{B}}}{4 \alpha_{\delta} \mathrm{E}_{0}^{2}} \tag{35}
\end{equation*}
$$

where $\alpha_{\delta}$ is the damping rate in synchrotron oscillation. In addition, the beam sizes due to the betatron motions must be scaled by the betatron functions at the observation point, $\left(\beta_{\mathrm{x}, \mathrm{y}}\right)_{\theta}$. When all these factors are included, the observable transverse beam parameters become ${ }^{9}$

$$
\begin{align*}
& \left\langle\mathrm{x}_{\theta}^{2}\right\rangle=\frac{\nu \mathrm{x} \beta_{\mathrm{x} \theta}}{\mathrm{R}}\left\langle\mathrm{x}^{2}\right\rangle+\eta_{\mathrm{x} \theta}^{2}\left\langle\delta^{2}\right\rangle \\
& \left\langle\mathrm{x}_{\theta} \mathrm{y}_{\theta}\right\rangle=\left(\frac{\nu_{\mathrm{x}} \beta_{\mathrm{x} \theta} \nu_{\mathrm{y}} \beta_{\mathrm{y} \theta}}{\mathrm{R}^{2}}\right)^{1 / 2}\langle\mathrm{x} \mathrm{y}\rangle+\eta_{\mathrm{x} \theta} \eta_{\mathrm{y} \theta}\left\langle\delta^{2}\right\rangle \tag{36}
\end{align*}
$$

and

$$
\left\langle\mathrm{y}_{\theta}^{2}\right\rangle=\frac{\nu_{\mathrm{y}} \beta_{\mathrm{y} \theta}}{\mathrm{R}}\left\langle\mathrm{y}^{2}\right\rangle+\eta_{\mathrm{y} \theta}^{2}\left\langle\delta^{2}\right\rangle
$$

The observable tilt angle, $\phi_{\theta}$, is given by

$$
\begin{equation*}
\tan 2 \phi_{\theta}=\frac{2\left\langle\mathrm{x}_{\theta} \mathrm{y}_{\theta}\right\rangle}{\left\langle\mathrm{x}_{\theta}^{2}\right\rangle-\left\langle\mathrm{y}_{\theta}^{2}\right\rangle} \tag{37}
\end{equation*}
$$

The horizontal and vertical beam sizes, as well as the tilting angle of the transverse beam profile, have been measured in SPEAR as a function of the strength of a rotated quadrupole magnet. The results are in reasonable
agreement with Eqs. (36) and (37), ${ }^{5}$ which are useful expressions for storage ring deerign studies.

## V. TRANSIENT DISTRIBUTION AFTER INJECTION

As an illustration of the time-dependent solution, we consider the behavior of the horizontal beam width for an injected beam. For this example we assume no coupling between the horizontal motion and energy deviation and neglect the vertical motion. The matrices C and D are given by:

$$
C=\left[\begin{array}{cccc}
0 & 1 & 0 & 0  \tag{38}\\
-\omega_{\mathrm{x}}^{2} & -2 \alpha_{\mathrm{x}} & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -\omega_{\delta}^{2} & -2 \alpha_{\delta^{\prime}}
\end{array}\right]
$$

and

$$
\mathrm{D}=\frac{1}{2 \mathrm{E}_{0}^{2}}\left|\begin{array}{cccc}
<\mathscr{D} \eta_{\mathrm{x}}^{2}>\mathrm{B} & 0 & <\mathscr{D} \eta_{\mathrm{x}}>\mathrm{B} & 0  \tag{39}\\
0 & 0 & 0 & 0 \\
<\mathscr{D} \eta_{\mathrm{x}}>\mathrm{B} & 0 & <\mathscr{D}\rangle_{\mathrm{B}} & 0 \\
0 & 0 & 0 & 0
\end{array}\right|
$$

For simplicity we assume that the initial distribution at $t=0$ is a delta function of the form

$$
\begin{equation*}
\psi_{0}=\delta\left(\mathrm{x}-\mathrm{x}_{0}\right) \delta\left(\mathrm{p}_{\mathrm{x}}-\mathrm{p}_{\mathrm{x} 0}\right) \delta\left(\delta-\delta_{0}\right) \delta\left(\mathrm{p}_{\delta}-\mathrm{p}_{\delta 0}\right) \tag{40}
\end{equation*}
$$

The corresponding boundary conditions for the $B$ matrix are $B_{i j}=0$ at $t=0$. By using the procedure described in Eqs. (14), (15) and (17), we obtain the matrix $A(t)^{-1}$ as well as expressions for $\bar{x}(t)=\bar{x}_{1}(t)$ and $\bar{\delta}(t)=\bar{x}_{3}(t)$. From these results, we can describe the transient behavior of the beam by finding the horizontal position for the particle distribution center and the rms horizontal width about this
center. The distribution center is described by

$$
\begin{equation*}
\left\langle\mathrm{x}_{\mathrm{H}}\right\rangle=\overline{\mathrm{x}}(\mathrm{t})+\eta_{\mathrm{x} \theta} \bar{\delta}(\mathrm{t}) \tag{41}
\end{equation*}
$$

with

$$
\begin{equation*}
\bar{x}(t)=\left[x_{0} \cos \omega_{x} t+\frac{p_{x 0}}{\omega_{x}} \sin \omega_{x} t\right] e^{-\alpha x^{t}} \tag{42}
\end{equation*}
$$

and

$$
\bar{\delta}(t)=\left[\delta_{0} \cos \omega_{\delta} t+\frac{p_{\delta 0}}{\omega_{\delta}} \sin \omega_{\delta} \mathrm{t}\right] \mathrm{e}^{-\alpha \delta^{t}}
$$

where $\eta_{\mathrm{x} \theta}$ is the energy dispersion function at the observation point. Slow damping rates $\alpha_{x, y}$ have been assumed in the calculation. The beam width is found to be

$$
\begin{equation*}
\sigma_{H}^{2}=\left\langle x^{2}\right\rangle_{0}\left(1-e^{-2 \alpha_{x} t}\right)+\eta_{x \theta}^{2}\left\langle\delta^{2}\right\rangle_{0}\left(1-e^{-2 \alpha_{\delta} t}\right) \tag{43}
\end{equation*}
$$

where the stationary values $\left\langle\mathrm{x}^{2}\right\rangle_{0}$ and $\left\langle\delta^{2}\right\rangle_{0}$ are given by Eqs. (30) and (35), respectively.

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9. In these modifications, we have used the fact that both <x $\delta>$ and $<y \delta>$ vanish, which can be proven by repeating the analysis given in this section with the energy oscillation included.

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