

Ken Wilson: Solving the Strong Interactions

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ABSTRACT

Ken Wilson's ideas on the renormalization group were shaped by his attempts to build a theory of the strong interactions based on the concepts of quantum field theory. I describe the development of his ideas by reviewing four of Wilson's most important papers.

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1 Introduction

Ken Wilson is best known for his contributions to statistical mechanics. His breakthroughs in this field, including the computation of critical exponents and the solution of the Kondo problem, have had wide influence. Wilson began his career, however, as an elementary particle physicist. His ambition was to “solve the strong interactions”, that is, to find a predictive theory of the subnuclear strong interactions. The ideas that he developed profoundly influenced our understanding of that problem, just as they provided tools and insights for problems in statistical physics.

In this article, I will review the development of Wilson’s ideas on the strong interactions through a review of four of his most important papers [1–4]. I recommend these papers to all students of theoretical physics. All four read like explorations of realms previously unknown. They give insight into the problems Wilson sought to address with his initial concepts of the renormalization group. And, both for particle physicists and for condensed matter physicists, they illustrate how issues in each domain gave insight into the other.

2 The Fixed Source Problem

Quantum Field Theory (QFT) had some of its greatest successes in the late 1940’s, with the development of Quantum Electrodynamics and the successful explanation of the electron magnetic moment and the Lamb shift. The resulting euphoria led to the idea that QFT could be used to build a theory of the strong interactions based on a Lagrangian for pion-nucleon interactions. The new technology of Feynman diagrams assisted calculation (as Feynman recalled memorably in his Nobel Prize lecture [5]). However, it did not produce a better understanding of the nuclear forces. Pion exchange did not lead in any clear way to the observed phenomenology of nucleon-nucleon scattering. It could not account for the nucleon and meson resonances that began to be discovered.

Most importantly, the theory had few concrete predictions. It was stymied by the fact that the strong interactions are strong, while the methods developed for Quantum Electrodynamics relied on weak-coupling perturbation theory. Strong coupling in QFT implies that states with an arbitrarily large number of interacting quanta play an essential role. Feynman diagrams, which introduce additional quanta one by one, cannot easily give insight into this strong-coupling limit.

A relatively simple problem that encapsulated the difficulties of QFT is the fixed source problem. This is the problem of a static or infinitely heavy nucleon with two states

$$|p\rangle \quad |n\rangle \tag{1}$$

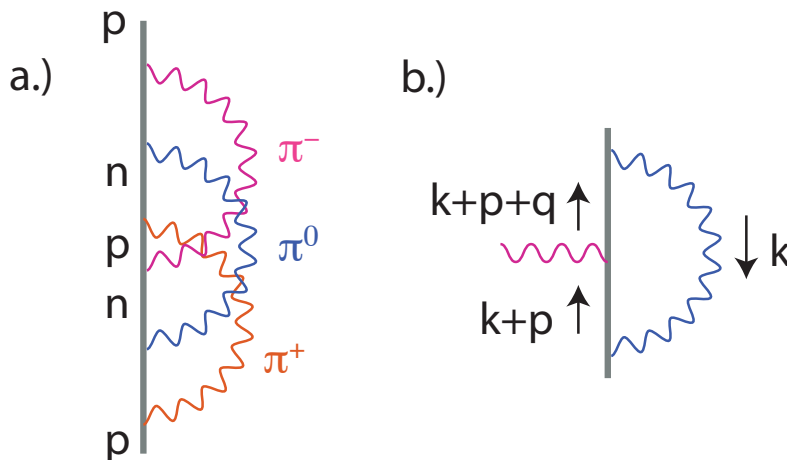


Figure 1: Feynman diagrams for the fixed source problem: (a) a diagram illustrating the complexity of the problem; (b) a divergent one-loop diagram.

interacting at its location $\vec{x} = 0$ with a pion field

$$\pi^a(x) = (\pi^+(x), \pi^0(x), \pi^-(x)) . \quad (2)$$

T. D. Lee showed that a truncated version of this model, with only the π^+ field, could be solved exactly [8]. However, the full problem allows complex intermediate states with many virtual pions, shown in Fig. 1(a) as loops coupling to the nucleon. Further, each loop has an ultraviolet divergence. The diagram shown in Fig. 1(b) has the value

$$-ig^3 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^0 + p^0)(k^0 + p^0 + q^0)(k^2 - m_\pi^2)} , \quad (3)$$

and is logarithmically divergent. So there is no clear way even to compute the first loop sensibly, much less to limit the number of loops relevant to the final answer.

By the mid-1950's, the search for the theory of the strong interactions had turned away from QFT to other methods, essentially phenomenological techniques such as dispersion relations and more fundamental proposals based on the analytic properties of scattering amplitudes. Geoffrey Chew proposed that there was a unique analytic S-matrix that could be discovered by deep analysis. As late as 1968, he stated:

“There exists at present no mechanical framework consistent with both quantum and relativistic principles. The chief candidate is local Lagrangian field theory, but countless theoretical studies have suggested insuperable pathologies in the concept of interaction between fields at a point of space-time.” [6]

Even for those who tried to build up the theory of strong interactions from symmetry principles, the infinities of quantum field theory posed a barrier to taking this theory completely literally. For example, Murray Gell-Mann’s paper that introduced the method of current algebra—one of the most important theoretical methods used in particle physics in the 1960’s— includes the following statement:

“... we use the method of abstraction from a Lagrangian field theory model. In other words, we construct a mathematical theory of strongly interacting particles, which may or may not have anything to do with reality, find suitable algebraic relations that hold in the model, postulate their validity, and then throw away the model. We may compare this process to a method sometimes employed in French cuisine: a piece of pheasant is cooked between two slices of veal, which are then discarded.” [7]

Ken Wilson was Gell-Mann’s student at Caltech from 1957 to 1961. He chose the fixed source problem described above as the topic of his thesis research. In his thesis, he threw at this problem the full arsenal of mathematical methods developed in the 1950’s, with minimal success. This investigation proved Wilson’s skills and promise, but it not make much headway toward the solution.

Many first-rank theorists find their thesis problem overreaching and frustratingly difficult. Usually, the solution is to pick another problem that yields more easily to their talents. This was not Wilson’s style. He would continue to struggle with the fixed-source problem for many more years.

3 Momentum Slicing

Wilson’s first published work on the fixed-source problem did not appear until 1965. It is the first of the four papers that are the subject of this review:

“Model Hamiltonians for Local Quantum Field Theory”, *Phys. Rev.* **140**, B445 (1965) [1]

This paper had no immediate impact, because it enunciated a point of view that ran against the main current of theoretical particle physics in the 1960’s. Wilson’s boldly stated attitude is that there is no mysticism about QFT. The way to understand its issues to reduce problems in QFT to ordinary quantum-mechanical problems that can be solved by the standard methods of atomic physics. As Wilson writes in this paper:

“The Hamiltonian formulation of quantum mechanics has been essentially abandoned in the investigations of the interactions of π mesons, nucleons, and strange particles. This is a pity. The Hamiltonian approach has several advantages over the kind of approach (using dispersion relations) presently in use. One advantage is that all properties of a system are uniquely determined ... A second advantage is the existence of many approximation schemes ... A third advantage is that one can often analyze a Hamiltonian intuitively.” [1]

For the neglect of QFT, Wilson blamed the problem of infinities. To rectify this, what was needed was a direct assault on that problem.

The infinities of QFT arise from the fact that the quantum excitations represented by the legs of Feynman diagrams may have any momenta and, in particular, momenta taking arbitrarily high values. Wilson’s approach to the infinities was to lay out these momenta in an orderly set of regions that could be analyzed one by one. He called this concept “momentum slicing”.

In [1], the full momentum space available to pions in the fixed source problem is replaced by the set of intervals

$$0 < |k| < m_\pi, \quad \frac{1}{2}\Lambda < |k| < \Lambda, \quad \frac{1}{2}\Lambda^2 < |k| < \Lambda^2, \quad \dots, \quad \frac{1}{2}\Lambda^n < |k| < \Lambda^n, \quad (4)$$

The slicing of momentum space is illustrated in Fig. 2. This severe reduction of the allowed phase space not only sharpens the problem of infinities but also reverses the standard viewpoint. At first sight, it is the low-momentum degrees of freedom that are the most important for the physics of pion-nucleon interactions encoded by the fixed-source problem. The appearance of high momenta is an intrusion that needs to be controlled. However, if the system defined by (4) is studied using the standard approximation schemes of quantum mechanics, the opposite is true. The most important terms in the Hamiltonian are those in the highest momentum interval. This part of the Hamiltonian must be diagonalized before lower-momentum intervals can be studied.

Standard ideas of quantum-mechanical perturbation theory dictate how this diagonalization should be done. The problem of pions in the highest momentum interval should be solved first, and the ground state of this system found. This ground state configuration of the pion modes with momenta of order Λ^n can then be used as the starting point for an analysis of the pion modes with momenta of order Λ^{n-1} . It is only at the end of this process that momenta of the order of m_π come into play.

The diagonalization of the Hamiltonian is then naturally structured as an iteration. The modes at $|k| \sim \Lambda^n$ primarily affect the modes at $|k| \sim \Lambda^{n-1}$ by modifying their coupling to the nucleon. This gives a recursion equation

$$g_{n-1} = f(g_n) \quad (5)$$

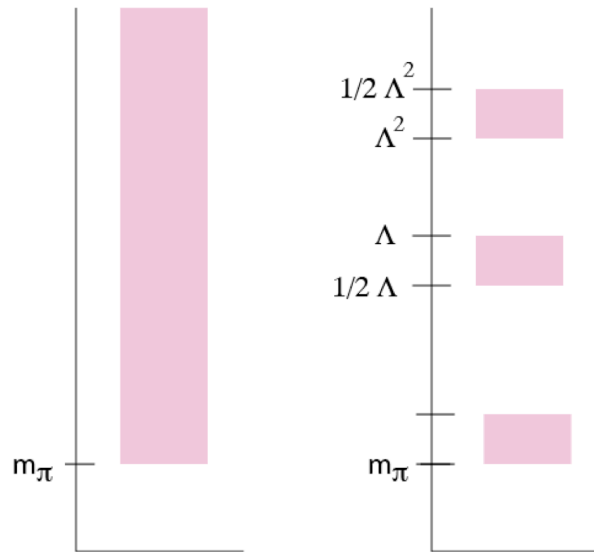


Figure 2: The slicing of momentum space used by Wilson in [1]. The vertical axis gives the energies associated with the selected momentum states, on a logarithmic scale.

The effect of modes at large momentum scales on the pion modes at $|k| \sim m_\pi$ is then encapsulated in the evolution of the coupling constant from scale to scale that results from this evolution. At each stage of the evolution, the higher momenta are said to be “integrated out” and are removed from Hamiltonian. This is the essence of Wilson’s concept of the renormalization group.

The restriction of momenta to the domains (4) is of course an extreme truncation of the original problem. To solve the fixed source problem quantitatively by momentum slicing, it is necessary to consider regions that are contiguous, without gaps, and to integrate out each high-momentum region down to its boundary in an accurate way. In [9], Wilson addressed this question as a matter of principle, proving that the method led to a solution to the fixed source problem with all infinities eliminated.

However, Wilson did not stop there. It happens that the fixed source problem is related to a puzzle that appeared in the theory of magnetism, the Kondo problem [10]. This is the problem of a fixed magnetic impurity coupling to a free gas of electrons. For ferromagnetic coupling, the impurity behaves as a weakly coupled free spin interacting with the electrons. But for antiferromagnetic coupling, however weak in the underlying theory, the ground state contains a strong binding of the impurity to an electron that essentially quenches its magnetism. By introducing additional operators and corresponding couplings that transform under a multiparameter recursion equation, Wilson was able to integrate out shells in the electron momentum sufficiently accurately to identify the transition energy from weak to strong coupling. The full



Figure 3: Generation of a composite operator by integrating out high-momentum degrees of freedom.

story of this calculation is outside the scope of this review, but it is described lucidly in [11]. Wilson’s calculation of the strong-coupling scale in this model

$$\frac{T_K}{4\pi T_0} = 0.1032 \pm 0.0005 \quad (6)$$

was later verified by an exact solution of the Kondo problem, using Bethe’s ansatz, by Andrei and Lowenstein [12].

4 The Operator Product Expansion

The idea of integration out has a more general consequence for the description of operators in QFT. Integrating out momentum modes with $|k| \sim \Lambda$ corresponds to the solution of the quantum theory for point separations of order $|x - y| \sim \pi/\Lambda$. Local operators placed more closely together than this distance cannot be considered separately after integration out of this momentum shell. They must merge to become single operators located at some intermediate point, as shown in Fig. 3.

Formalizing this intuition gives Wilson’s Operator Product Expansion. The idea of the operator product expansion appeared fully formed in the literature in the second paper of this review

“Non-Lagrangian Models of Current Algebra”, Phys. Rev. **179**, 1499 (1969). [2]

The concept is expressed by the statement that all expectation values of a pair of QFT operators located at points x and y together with operators located at points z far from x, y can be computed by the replacement

$$\mathcal{O}_A(x)\mathcal{O}_B(y) = \sum_C \mathcal{C}_{ABC}(x-y)\mathcal{O}_C(y) , \quad (7)$$

where the sum over C runs over all operators in the QFT with appropriate quantum numbers, and $\mathcal{C}_{ABC}(x-y)$ is a c-number coefficient function.

The relation is especially simple in theories in which the QFT dynamics at distances smaller than $|x-y|$ is independent of any intrinsic length scale. Then, for scalar operators

$$\mathcal{C}_{ABC}(x-y) = \frac{c_{ABC}}{|x-y|^{d_A+d_B-d_C}} , \quad (8)$$

where c_{ABC} and d_A, d_B, d_C are numbers. For operators with spin, c_{ABC} is replaced by a number times an appropriate Lorentz structure.

The quantities d_i are called the dimensions of the operators and reflect the scaling of operator matrix elements with changes of distance scale. In Wilson's original conception, these dimensions were integers, as in free field theory. An anonymous referee (now known to be Arthur Wightman [13]) pointed out that the exact solution of the Thirring model in (1+1) dimension gave an example in which these dimensions could be arbitrary real numbers, prompting Wilson to make a serious study of that model. In [2], the idea that the values of operator dimensions could have a nontrivial influence on physical phenomena was presented for the first time.

The paper [2] applied these ideas to one of the most important problems being considered at that time, the nature of products of currents. Such products appear in the structure of the weak interactions, in analyses of the properties of pions and kaons using the methods introduced in [7], and in the analysis of deep inelastic electron scattering. In 1967, the results of the SLAC-MIT electron scattering experiment and their interpretation by Bjorken [14] and Feynman [15] pointed to models of the structure of the proton with free-field behavior at short distances for the proton constituents. Theoretical analysis of these experiments required the high-momentum asymptotic behavior of a pair of electromagnetic currents.

One of the properties of the Thirring model that was striking to Wilson in this context is that the current algebra of the model remains unchanged as the operator spectrum of the model is distorted by the effects of strong interactions. In our (3+1)-dimensional world, conserved currents would remain operators of dimension $d = 3$ while the dimensions of other operators would shift. Typically, results derived from Gell-Mann's current algebra for strong interaction matrix elements depend not only on the algebra but also on the behavior of short-distance limits. The operator product expansion gave a systematic way to analyze this issue.

The result of the paper that seems most striking from our modern point of view is the explanation Wilson gives for the $\Delta I = \frac{1}{2}$ rule, the fact that $\Delta I = \frac{1}{2}$ weak decays of K mesons and strange baryons, for example $K^0 \rightarrow \pi^+\pi^-$, go more than 100 times faster than $\Delta I = \frac{3}{2}$ decays such as $K^+ \rightarrow \pi^+\pi^0$. Wilson suggested that different operators, \mathcal{O}_C in (7), in the product of W boson currents contributed to these two amplitudes, and that the difference in the amplitudes arises from the different factors

$$(m_K/m_W)^{6-d_C} . \tag{9}$$

in their operator product coefficients. This was the first suggestion of a qualitative effect on physics caused by dynamically-generated differences in operator dimensions. In 1974, Gaillard and Lee analyzed the operator product of W boson currents in the gauge theory of strong interactions QCD, to be described below, and showed that this effect does account for a large part, if not all, of the $\Delta I = \frac{1}{2}$ enhancement [16].

5 Scale Invariance at Short Distances

Wilson discussed the results of the paper [2] in the context of a vision for the structure of a QFT description of strong interactions. The infinities of the theory would be tamed by the principle that the recursion described in Section 3 would have converged to a set of couplings that did not change with scale. Wilson describes a “skeleton theory” for strong interactions that is exactly scale invariant. To build a realistic model with nonzero hadron masses, this theory would be perturbed by mass terms or other operators with dimensionful coefficients. Wilson notes the idea of Kastrup [17] and Mack [18] that these terms might arise from the spontaneous breaking of a scale symmetry.

I have already pointed out that the idea that the strong interactions are described at short distances by a scale-invariant free field theory was very much in the air at this time. Both current algebra and the parton description of deep inelastic scattering rested on this foundation. However, it was recognized that this foundation could not be realized in any interacting QFT model.

Wilson’s ideas cut through the haze surrounding this question. They suggested a framework for a model that could actually arise from QFT. However, the exact form of that model was still obscure. Wilson ends the paper [2] with the statement:

“It is hard to imagine that one could have a complete formula ... without having a complete solution of the hadron skeleton theory. The prospects for obtaining such a solution seem dim at present.” [2]

6 The Paper with Three Errors

If all one knows about the underlying scale-invariant theory of strong interactions is that it arises from renormalization group recursion, one can at least analyze the possibilities for the structure of such a theory by examining the renormalization group equations more closely. Wilson presented such an analysis in the third paper reviewed here,

“Renormalization Group and Strong Interactions”, Phys. Rev. **D3**, 1818 (1970) [3].

This paper concentrates on the case of one coupling constant evolving according to a continuous renormalization group equation. In modern notation, this equation is

$$\frac{dg(\mu)}{d\log\mu} = \beta(g(\mu)) , \quad (10)$$

where μ is a momentum scale, $g(\mu)$ is the dimensionless coupling constant of the strong interaction theory, and $\beta(g)$ describes its evolution with scale. Wilson did not consider this equation familiar to his audience. Rather, much of the paper is devoted to deriving the equation in perturbative QFT, beginning from the original treatment of Gell-Mann and Low [19]. Wilson gives as his primary reference for the renormalization group the textbook of Bogoliubov and Shirkov [20], though his explanation is a highly processed version of the one found there [21].

Wilson’s approach to (10) was to analyze it as the equation for a general dynamical system. Possible asymptotic behaviors for a dynamic system include a fixed point and a limit cycle. Wilson described fixed-point solutions to (10) with the example of a β function of the form shown in Fig. 4. At momenta M high enough that mass parameters could be ignored, the coupling constant g would take some value g_M on the horizontal axis. For values $g_M < g_1$, the value would then increase at larger scales, coming close asymptotically to the value g_1 , which would be a fixed point of the renormalization group. If the value of g_M were in the range $g_1 < g_M < g_2$. the value of $g(\mu)$ would decrease to the fixed point g_1 .

An alternative picture discussed by Wilson is one in which, at very high momentum scales, the weak and electromagnetic couplings become as large as the strong interaction coupling. The high-momentum value of g is then determined by properties of this unified theory. At lower scales, where the strong interactions can be treated in isolation from the weak and electromagnetic interactions, g takes on the renormalization group evolution described by $\beta(g)$. In that case, the low-momentum behavior might be evolution to an infrared-asymptotic fixed point of the renormalization group, such as g_2 . The observed strong interaction coupling would be modified slightly from g_2 by effects of the mass terms.

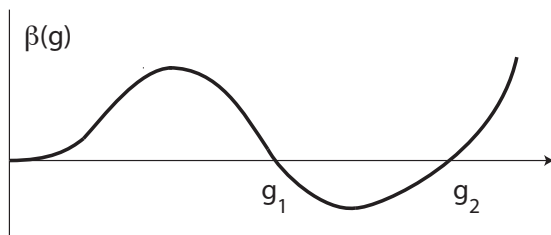


Figure 4: An example of a renormalization group function $\beta(g)$.

Wilson analyzes one more possibility. In principle, if there is more than one coupling constant, the high-energy asymptotic behavior of the renormalization group equation might be a limit cycle. In that case, the values of the coupling constants would perpetually oscillate, with a regular period in $\log \mu$. This would be directly observable as an oscillating behavior of the cross section for e^+e^- annihilation to hadrons.

This paper was eye-opening for many theorists at the time [22]. It introduced the idea of qualitative analysis of a QFT through visualization of its renormalization group flows, an idea that is now a standard method both in particle physics and in statistical mechanics.

Still, Wilson is said to have referred to this work as “the paper with three errors”. In hindsight, the omissions are easy to find. Limit cycles of the renormalization group have never played a role in particle physics (though examples do exist for discrete renormalization group transformations [23]). The idea that the low energy value of the strong interaction coupling constant would be an infrared-stable fixed point of the renormalization group was also not realized. This idea might still be relevant in the theory of the top quark mass [24,25]. Most importantly, though, Wilson assumed that the β function must be positive in the low- g region of Fig. 4, where it is computed in perturbation theory. He writes that, negative $\beta(g)$ “violates the Källén-Lehmann representation for the photon propagator” [3]. The last secret of the strong interactions was hidden here, as I will explain in a moment.

7 Statistical Mechanics and Quantum Field Theory

It was in this same period that Wilson became involved with problems of the theory of phase transitions. Wilson’s interaction with David Mermin, Michael Fisher, Ben Widom, and other statistical mechanics experts at Cornell will be covered by other contributions to this volume. The street ran both ways. The converse of the idea that statistical mechanics problems can be modelled by QFT is the idea that

QFT can be given a foundation by constructions taken from statistical mechanics.

From our previous discussion, the ingredients are all in place. A lattice with spacing a in d dimensions can represent the space that results when quantum states at large momenta are integrated out down to $|k| \sim \pi/a$. Integration out potentially leads to a complicated Hamiltonian with many nonzero operator coefficients. However, most of these operators have high dimension and so do not affect physics at energy scales of the order of particle masses.

The cleanest connection of this type is between lattice statistical mechanics problems on a d -dimensional lattice and Euclidean QFT in d dimensions. It follows from the axioms of QFT [26] that operator expectation values on Lorentzian spacetime can be analytically continued to a Euclidean spacetime with

$$x^2 = (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 . \quad (11)$$

Continuation to Euclidean space carries the time translation operator

$$U(t) = e^{-iHt} \rightarrow T(t_E) = e^{-L_E t_E} , \quad (12)$$

where L_E can be identified with the Lagrangian of the analytically continued problem. The object on the right is the transfer matrix of a statistical mechanics problem. Setting $t_E = a$ gives the evolution from one lattice spacing to the next. The complete partition function is

$$Z = \text{tr}[T(a)^N] \quad (13)$$

for a lattice of length Na .

Wilson's student Ashok Suri worked out these connections in detail and explained them in his 1969 Ph.D. thesis [27]. That thesis became a basic reference document for the developments to follow.

8 Quantum Chromodynamics

The missing piece in the story of the scale invariance of strong interactions popped out in the spring of 1973 with the announcement by Politzer, Gross, and Wilczek that non-Abelian Yang-Mills theory is asymptotically free [28,29]. By this, I mean that the renormalization group equation in this theory has a negative β function at small values of g , causing the coupling constant to run to zero for large momenta. In (3+1)-dimensions, non-Abelian gauge theories are absolutely exceptional in allowing negative values of the β function [30,31]. The indefinite metric spaces used in the quantization of gauge fields allow them to slip through a crack in the argument that the β function is always positive.

A theory with asymptotic freedom would be asymptotically scale-invariant, and, in fact, free, at short distances. However, in certain observables, one could still find large effects of operator dimensions, now scaling with powers of logarithms of momenta

$$\left(\log \frac{m_K^2}{\Lambda^2} / \log \frac{m_W^2}{\Lambda^2}\right)^{\gamma_A + \gamma_B - \gamma_C} \quad (14)$$

rather than (9). The combination of these two features in the same package made this an ideal solution to all of the problems of constructing a theory of strong interactions. Almost immediately, the idea of building interactions from gauge fields converged with other aspects of strong interaction phenomenology to pick out the Yang-Mills gauge group $SU(3)$, with quarks belonging to the fundamental $\mathbf{3}$ representation [32]. This theory was named Quantum Chromodynamics (QCD). Today, a wealth of evidence supports the claim that QCD is the fundamental description of the strong interactions.

QCD included the picture of strongly interacting particles as bound states of more fundamental spin $\frac{1}{2}$ particles, quarks. The quark model explained the mass spectrum and quantum numbers of the baryons and mesons. It gave a basis for current algebra and the structure of hadronic weak interactions. But, together with these successes came a puzzle: The quark model required the electric charge assignments $+\frac{2}{3}$ for the u quark and $-\frac{1}{3}$ for the d and s quarks. Yet, no particles with fractional electric charge had been seen in nature. By 1973, extensive searches had been done, all with negative results [33].

9 Lattice Gauge Theory

Wilson had not been studying gauge theories, or any weak-coupling proposal for the nature of the skeleton theory of strong interactions. However, now a question arose that he was uniquely positioned to answer: How do non-Abelian gauge theories behave when their coupling constants are taken to be strong? The answer to this question was given in the final paper for this review

“Confinement of Quarks”, Phys. Rev. **D10**, 2445 (1974) [4]

That answer turned out to be profound. The following important results were summarized in the introduction to this paper:

“A new mechanism which keeps quarks bound will be proposed in this paper. The mechanism applies to gauge theories only.”

“By analogy to the solid-state situation one can think of the transition from zero to nonzero photon mass as a change of phase.”

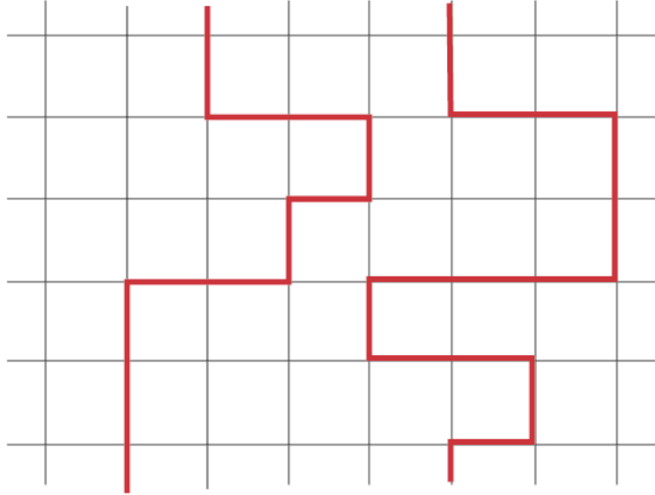


Figure 5: Path of a heavy quark-antiquark pair in a lattice space-time. Euclidean time runs upward.

“... the strong-coupling expansion ... has the same general structure as the relativistic string model of hadrons ...” [4]

The correspondence between continuum QFT and lattice statistical mechanics was the crucial tool for this investigation. By now Wilson had thoroughly assimilated the idea of lattice-regulated QFT. He writes:

“The model discussed in this paper is a gauge theory set up on a four-dimensional Euclidean lattice. The inverse of the lattice spacing serves as an ultraviolet cutoff. The use of a Euclidean space ... instead of a Lorentz space is not a serious restriction.”

Other members of our particle physics community took decades to get used to this idea. Today, gauge theory on a Euclidean lattice is a proven numerical tool for calculations in the low-momentum region of QCD [34].

To begin the description of a local gauge symmetry on a lattice, we might start from the description of a matter particle in a lattice QFT. A path of a heavy particle in Euclidean spacetime has the form shown in Fig. 5. A quantum particle travelling on paths of this type can be realized by a scalar field transforming under a global symmetry, which we associate with the particle number,

$$\phi_n \rightarrow e^{i\alpha} \phi_n , \tag{15}$$

where α is a global parameter. A discretized derivative of the field can be defined as

$$\Delta_\mu \phi_n = \frac{1}{a}(\phi_{n+a\hat{\mu}} - \phi_n) \quad (16)$$

A lattice QFT with the Lagrangian

$$L_E = \sum_n a^3 \left[\frac{1}{2T} |\Delta_\mu \phi_n|^2 + \frac{1}{2} m^2 |\phi_n|^2 \right] \quad (17)$$

is invariant under the symmetry. Expanding the partition function in the parameter $1/T$ is a controlled expansion in the lattice-regulated theory. This expansion is analogous to the high-temperature expansion of lattice statistical models. It corresponds to a sum of graphs with paths of the form shown in the figure. A similar treatment can be given for fermions on the lattice. To implement this, Wilson described the integral over fermionic variables invented by Berezin [35], still, at that time, quite unfamiliar as a QFT tool.

The generalization to a local gauge symmetry raises new issues. The symmetry transformation is now

$$\phi_n \rightarrow e^{i\alpha_n} \phi_n, \quad (18)$$

where α_n is independent at each lattice site. The derivative (16) now no longer has a linear transformation law under the symmetry group.

The remedy for this problem is to generalize the lattice derivative with an additional element

$$\Delta_\mu \phi_n = \frac{1}{a}(\phi_{n+a\hat{\mu}} - U_{n+a\hat{\mu},n} \phi_n), \quad (19)$$

where U_{n_1, n_2} , with (n_1, n_2) neighboring lattice sites, has the transformation

$$U_{n_1, n_2} \rightarrow e^{i\alpha_{n_1}} U_{n_1, n_2} e^{-i\alpha_{n_2}} \quad (20)$$

Minimally, U_{n_1, n_2} can be taken to be a unitary matrix representing an element of the gauge group, with the identification $U_{n_2, n_1} = U_{n_1, n_2}^\dagger$. The statistical sum over U_{n_1, n_2} is an integral over the gauge group for each link of the lattice.

In terms of continuum variables, a quantity with the same transformation law as U_{n_1, n_2} is the exponential of the line integral of the vector potential

$$U_{n_1, n_2} \equiv \exp \left[ig \int_{n_2}^{n_1} dx^\mu A_\mu \right] \quad (21)$$

Making this identification and expanding for the case in which $A_\mu(x)$ varies slowly over a lattice spacing, we find

$$\Delta_\mu \phi = (\partial_\mu - igA_\mu)\phi, \quad (22)$$

the standard gauge-covariant derivative. A gauge-invariant quantity built purely from the U_{n_1, n_2} is

$$\text{tr}[V_{n, \mu, \nu}] = \text{tr}[U_{n, n+\hat{\nu}} U_{n+\hat{\nu}, n+\hat{\mu}+\hat{\nu}} U_{n+\hat{\mu}+\hat{\nu}, n+\hat{\mu}} U_{n+\hat{\mu}, n}] \quad (23)$$

In the continuum, $V_{n, \mu, \nu}$ can be identified as the exponential of a line integral around the elementary square,

$$V_{n, \mu\nu} \equiv \exp \left[ig \oint dx^\mu A_\mu \right] = \exp \left[ig \int d^2 s^{\mu\nu} F_{\mu\nu} \right] \quad (24)$$

I have written these formulae for the case of an Abelian gauge group, but they go through with only minor modifications for a non-Abelian gauge group. The lattice QFT action

$$L_E = \sum_{n, \mu, \nu} \frac{1}{g^2} \text{tr}[V_{n, \mu, \nu} + V_{n, \mu, \nu}^\dagger] \quad (25)$$

then gives the usual gauge field action $\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2$. Similar lattice QFTs were constructed by Wegner [36], for the case of discrete gauge symmetry, and by Polyakov [37].

The lattice Lagrangian (25) has the amazing property of possessing a straightforward expansion in powers of $1/g^2$. One simply needs to expand the exponential of (25) in a power series. Each factor of $\text{tr}[V_n]$ brings down four factors of $U_{n, n+\hat{\mu}}$, which are then integrated over the gauge group. This $1/g^2$ expansion realized one of Wilson's longstanding goals, to directly compute the structure of a QFT in the limit of strong coupling.

To understand the implications of this expansion, go back to Fig. 5. In the gauge theory, each link of the path of the heavy quark and antiquark acquires a factor $U_{n, n+\hat{\mu}}$. Each term $\text{tr}[V_n]$ is a product of four factors of $U_{n, n+\hat{\mu}}$ arranged around an elementary square of the lattice. We might imagine this as a tile placed on the square. These factors must come together to prevent the integrals over the gauge group from giving a zero result. Indeed,

$$\int dg U(g)_{ij} = 0 \quad \text{while} \quad \int dg U(g)_{ij} U^\dagger(g)_{k\ell} = c \delta_{i\ell} \delta_{kj} . \quad (26)$$

A term in the strong-coupling perturbation theory is nonzero only if each link has a matching number of factors of U and U^\dagger . The nonzero terms correspond to tilings of the region between the quark and antiquark paths, as shown in Fig. 6.

We find that the amplitude for propagation of a heavy quark-antiquark pair is nonzero only if the entire region between the paths of these particles is spanned by tiles. If the quark and antiquark are far apart, there must be gauge excitation covering every interval between them. This led Wilson to the conclusion that the strong-coupling gauge theory gives a potential between quarks and antiquarks of the form

$$V(|\vec{x}_q - \vec{x}_{\bar{q}}|) \sim k |\vec{x}_q - \vec{x}_{\bar{q}}| \quad (27)$$

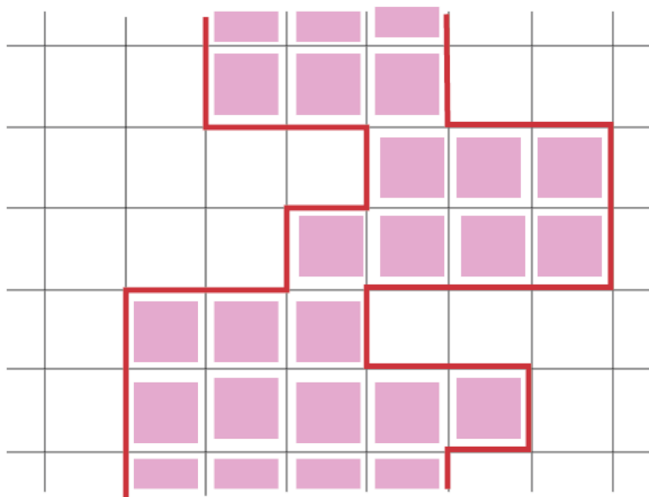


Figure 6: Path of a heavy quark and antiquark in lattice space-time, with the leading terms from the gauge theory at strong coupling.

rising linearly with distance.

For general values of the coupling, the qualitative behavior of this potential would depend on the long-range order in the gauge degrees of freedom. At strong coupling

$$\langle U_{n,\hat{\mu}} \rangle = 0, \quad (28)$$

and we find the quark-confining potential (27). At weak coupling, at least in electrodynamics, there is an expansion about

$$U_{n,\hat{\mu}} \approx 1 \quad (29)$$

that leads to the usual Coulomb potential. In non-Abelian gauge theories, it is plausible that the renormalization-group flow makes the coupling strong enough, at sufficiently large distances, that the strong coupling region is reached and the theory is confining. This statement is not yet proven rigorously, but it is supported by a wealth of numerical data [34].

10 Afterword

I am ending this review almost exactly at the point where my career intersects the story. I came to Cornell as a graduate student in the fall of 1973. The first particle theory seminar that I attended was Wilson's seminar at Cornell on the lattice gauge theory results that I have just described. A year later, the discovery of the J/ψ

resonance led to striking evidence for all aspects of the picture I have described here: the quark model, asymptotic freedom at short distances, a linear confining potential at large distances, even the direct experimental verification of the number 3 in the $SU(3)$ gauge group [38]. The rout of QCD was on.

By the time I finished graduate school, QCD was already an established theory. I became interested in what I felt would be the next problem ripe for solution, the physics of the spontaneous breaking of symmetry responsible for the properties of the subnuclear weak interactions. Thirty-five years later, that problem is still an open one, although the recent discovery of the Higgs boson at the Large Hadron Collider [39,40] surely provides an important piece of the puzzle.

Every student of physics seeks to emulate his or her thesis advisor. Having Ken Wilson as an advisor sets a very high standard.

The experience of working with Ken instilled some values that continue to guide my approach to physics. First, even when approaching the fundamental equations of nature, a physicist should dismiss mysticism. The universe is essentially mechanical. There is a Hamiltonian; solve it. For better or worse, I find the current Standard Model of particle physics too lacking in explanatory power, and too lacking in specific mechanisms that might explain the fact and the consequences of its spontaneous symmetry breaking.

Second, a physicist should have a vision, and pursue it to the end. We are not all as blessed with genius as Ken Wilson, but the mountains are there nevertheless. Ken always climbed straight up.

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