

Where is \hbar Hiding in Entropic Gravity?

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The entropic gravity scenario recently proposed by Erik Verlinde reproduced the Newton's law of purely classical gravity yet the key assumptions of this approach all have quantum mechanical origins. This is atypical for emergent phenomena in physics, where the underlying, more fundamental physics often reveals itself as corrections to the leading classical behavior. So one naturally wonders: where is \hbar hiding in entropic gravity? To address this question, we first revisit the idea of holographic screen as well as entropy and its variation law in order to obtain a self-consistent approach to the problem. Next we argue that when dealing with quantum gravity issues the generalized uncertainty principle (GUP) should be the more appropriate foundation. Indeed based on GUP it has been demonstrated that the black hole Bekenstein entropy area law must be modified not only in the strong but also in the weak gravity regime. In the weak gravity limit, such a GUP modified entropy exhibits a logarithmic correction term. When applying it to the entropic interpretation, we demonstrate that the resulting gravity force law does include sub-leading order correction terms that depend on \hbar . Such deviation from the classical Newton's law may serve as a probe to the validity of the entropic gravity postulate.

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I. INTRODUCTION

The issue of how gravity and thermodynamics are correlated has been studied for decades, triggered by the seminal discovery by Bekenstein[1, 2] on the area-law of black hole (BH) entropy and temperature. After Hawking's discovery of the BH evaporation and the interpretation of the its temperature as the thermal temperature of blackbody radiation[3], considerable efforts have been made to find the statistical interpretation of the proportionality of black hole entropy and its horizon area. See [4] and [5], for example, for a review. By now a well-accepted view is that the black hole entropy is associated with the external thermal state perceived by an observer outside the event horizon who has no access to the BH interior. Namely, the correlation between the degrees of freedom on opposite sides of the horizon results in a mixed state for observation from the outside, i.e., the 'entanglement entropy'[6, 7], which depends upon the boundary properties and will be discussed more in the later sections of this paper.

The inversion of the logic that describes gravity as an emergent phenomenon was first proposed by Sakharov [8], who suggested that gravity is induced by quantum field fluctuations. Invoking the area scaling property of entanglement entropy, Jacobson in 1995 [9] used basic laws of thermodynamics to derive Einstein equations. In

his perspective Einstein equations are now an equation of state rather than a fundamental theory. More ideas on emergent gravity have been recently proposed (See, for example, [10–13]).

Similar to Jacobson's derivation of Einstein equations through thermodynamic, Verlinde treated gravity as an entropic force analogous to the restoring force of a stretched elastic polymer driven by the system's tendency towards the maximization of entropy [13], and interestingly the Newton's law of gravitation was shown to arise. To arrive at the Newton's force law of gravity through the first law of thermodynamic $Fdx = TdS$, Verlinde first invoked the Compton wavelength of the test particle to find the change of entropy with respect to its displacement. He then invoked the holographic principle [14–16] and the equipartition theorem to define the temperature experienced by the test particle. One cannot but notices that all these building blocks have quantum mechanical origin, or more specifically the presence of \hbar . Yet all the \hbar 's just get subtly cancelled and at the end a purely classical Newton's law has emerged. This is rather atypical for emergent phenomena in physics, where the underlying, more fundamental physics often reveals itself as corrections to the leading classical behavior. So one naturally wonders: where is \hbar hiding in entropic gravity?

There have been previous works aiming at finding the entropic corrections to Newton's law but however unsatisfactory: Santos et al. [17] and Ghosh [18] suggested the corrected to Newton's gravity force law when the dependence on the uncertainty in position is included, which is bothersome; both Modesto [19] and Setare [20] suggested the sub-leading terms of force law rather than an

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exact form while the former considered only the corrections to entropic variation without noting that the information content and therefore the temperature is also affected, and the latter failed to introduce the right form of GUP corrected entropy into consideration. We argue that when dealing with quantum gravity issues the generalized uncertainty principle (GUP) should be the more appropriate foundation. Indeed based on GUP it has been demonstrated that the black hole Bekenstein entropy area law must be modified not only in the strong but also in the weak gravity regime [21]. In the weak gravity limit, such a GUP modified Bekenstein entropy exhibits a logarithmic correction. Such a log-correction is consistent with similar conclusions drawn from string theory, AdS/CFT correspondence, and loop quantum gravity considerations [22–25]. When applying it to the entropic derivations, we demonstrate that the resulting entropic gravity does include sub-leading order correction terms that depend on \hbar .

The organization of this paper is as follows. To address the question we posted, we first revisit the ideas of holographic screen as well as entropy and its variation law in order to obtain a self-consistent approach to the problem in section II. We set up the key ingredients of entropic gravity framework toward the derivation of weak-field-limit gravity force law. Holographic principle, which we deem misapplied by Verlinde, is illustrated and the concept of entanglement entropy is introduced to support the validity of the derivation. We then revisit Verlinde’s previous work and make some elaboration for a more concrete foundation of this entropic gravity, clarification of the idea of holographic screen and re-derivation of entropy variation law are made especially. In section III we bring in the generalized uncertainty principle, which leads to a corrected form of black hole temperature and entropy. The modification of the information content so provided by GUP will result in the revision of the force law. In section IV we repeat the steps of Verlinde’s, but with the entropy variation law and the temperature form redefined by the GUP corrected entropy. We arrive at an exact force law of gravity at the end, and this exact force law recovers not only the classical Newton’s law but also the sub-leading order quantum correction terms in the weak-field limit. In section V, conclusions and comments are made about the implications of our findings. We suggest that the resulting deviation from the classical Newton’s law may serve as a probe to the validity of the entropic gravity postulate.

II. ENTROPIC GRAVITY

A. Holographic principle and holographic entanglement entropy

1. Holographic principle

Holographic principle [14, 15], which is developed from black hole thermodynamics, plays a key role in Verlinde’s entropic interpretation of gravity. It dictates that the degrees of freedom in a $d+2$ dimensional quantum gravity system can be seen as encoded on its $d+1$ dimensional boundary like a holographic image. This principle originates from the Bekenstein-Hawking formula where the BH entropy is proportional to the surface area of its event horizon:

$$S_B = \frac{4\pi k_B G M^2}{\hbar c} = 4\pi \frac{M^2}{M_p^2} = \frac{k_B c^3}{4\hbar G} A. \quad (1)$$

Here A is the surface area of black hole event horizon and $M_p = \sqrt{\hbar c/G}$ is the Planck mass.

For the second law of thermodynamics to hold, that is, the entropy in the universe to be non-decreasing, the holographic principle suggests that the information content S of an enclosed spacetime region should be no larger than the Bekenstein-Hawking entropy defined by its surface area [2, 16]:

$$S \leq \frac{k_B}{4L_p^2} A = \frac{k_B c^3}{4\hbar G} A = S_B. \quad (2)$$

Here $L_p = \sqrt{G\hbar/c^3}$ is the Planck length. This implies that the number of fundamental degrees of freedom is related to the surface area in spacetime. Note that the black hole information paradox, that is, the information seems to be lost as all physical states evolve into the same final state of black hole, is also resolved by this principle. An extensive illustration of the holographic principle is given by Bousso [16].

2. Holographic entanglement entropy

In the derivation of the entropic gravity force law, the maximum value of the holographic entropy is used rather than the inequality, which is unjustified. To treat the problem more properly, one should instead invoke the concept of entanglement entropy [26–30]. The entanglement entropy is a quantum mechanical quantity that measures the correlation between a subsystem A and its complementary subsystem B . When the world is divided into two subsystems, the total Hilbert space can be written as $\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B$. If an observer can access the entire system, then the total entropy of the system is the quantum version of the classical Shannon entropy, $H = -k_B \sum_i P_i \ln(P_i)$, here P_i the probability for a given state i , i.e., the von Neumann entropy

for a statistical state in \mathcal{H}_{tot} with density matrix ρ_{tot} : $S(\rho_{\text{tot}}) = -k_B \text{Tr}(\rho_{\text{tot}} \ln \rho_{\text{tot}})$ [26]. For an observer who can only access the information of subsystem A, she will feel as if the state is described by a reduced density matrix $\rho_A = \text{Tr}_B \rho_{\text{tot}}$, where the trace is over all eigenstates in \mathcal{H}_B for the total density matrix. The entanglement entropy is thus defined as the von Neumann entropy for the reduced density matrix ρ_A : $S_A = -k_B \text{Tr}_A(\rho_A \ln \rho_A)$. If the total state is entangled, that is, if it is not factorizable as $|\Psi_{\text{tot}}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$, then the entanglement entropy is non-vanishing even if the total state is a pure state with zero entropy [27].

It can be shown by straight-forward calculations that the entanglement entropy of subsystem A is equal to that of subsystem B if the total state is pure [27]. Srednicki [4] pointed out that with the property $S_A = S_B$, the entanglement entropy for a pure state, which we often referred to as the unique ground state of the total system, should only depend on the properties shared by the two regions. Therefore, it is expected that the leading behavior for pure ground state of a quantum field system scales as the boundary area rather than the volume of subsystems. This area-scaling leading behavior of the entanglement entropy has been revealed in various physical systems such as the quantum critical phenomena [31], explicit calculations of quantum field systems [6], and the AdS/CFT correspondence in string theory [32, 33]. This property of entanglement entropy is referred to as the holographic entanglement entropy: for a gravity field theory, a given boundary ∂B that divides the field theory into B and B' components, the entanglement entropy for the ground state of the field is

$$S_E = \frac{\text{Area}(\partial B) c^3 k_B}{4\hbar G} + \text{subleading terms.} \quad (3)$$

The entanglement entropy can be renormalized by fixing the cutoff length of the theory at Planck Length L_p . Here $\text{Area}(\partial B)$ is the area of the minimal surface on the boundary ∂B , and G is the Newton's constant (see also [34] for a review). Because this entropy of entanglement is associated with the quantum ground state, some refer to it as the entropy of the fundamental degrees of freedom for the underlying quantum field theory across the boundary, others may call it the entanglement entropy on the boundary surface.

The original motivation for the entanglement entropy was to give a statistical explanation for Bekenstein entropy in black hole thermodynamics. The entanglement entropy in quantum gravity has been known as the quantum corrections to black hole entropy from matter fields [6, 7, 15, 34]. Some further pointed out that the black hole entropy is a pure entanglement entropy if the entire gravitational action is 'induced' by the quantum fluctuations inside and outside the event horizon [7, 15, 34]. Thus the black hole entropy is provided by this correlation between the degrees of freedoms on opposite sides of the horizon. An observer outside the event horizon without the access to what happens inside will experience a

thermal state associated to this entanglement entropy.

We should note that the entanglement entropy is not exactly proportional to the area; only the leading order term follows the Bekenstein's law: $S = A/4 L_p^2$. The correction terms for the entanglement will be discussed later in section III, and the fact that simple entropy-area relation is only valid in the leading order will be emphasized to retrieve the missing \hbar factor in weak field entropic gravity hypothesis. In this section we will only treat the entropy following Bekebstein's law without any extra terms, as is the case in Verlinde's scenario.

B. Verlinde's entropic gravity scenario

In this subsection we briefly review how Verlinde arrives at classical Newton's law of gravity through thermodynamics and the holographic principle, and we will clarify some points in his approach.

1. Meaning of holographic screen

In Verlinde's picture, there is a spherical screen with radius R which centers at the massive source M and separates the universe into two components, one inside the sphere and the other stays outside. A particle of mass m is placed just outside the spherical screen, see FIG. 1. The spirit of this entropic gravity system is that for the test particle outside the sphere, it will interact thermodynamically with the screen on which the information of the massive source is registered. If the variation in the entropy occurs as the test particle moves, the test particle will then confronted a restoring force according to the first law of thermodynamics: $Fdx = TdS$. To find the form of this restoring force caused by the system's tendency toward the maximization of entropy, one first has to know how the entropy varies in response to the displacement of the test particle. If the temperature can also be determined, then putting these together one can arrive at the entropic force law.

Verlinde called this spherical screen a 'holographic screen', on which the information content obeys holographic principle. He uses this holographic principle argument to make the suggestion that the information inside the screen is distributed over the number of bits proportional to its surface area. The use of holographic principle here, however, is unclear and misleading. As discussed earlier, the holographic principle only suggests an inequality in the information content. According to this principle only the black hole horizon would saturate the upper-bound and recovers Bekenstein's area law. We believe what Verlinde really intended to convey is that the microscopic degrees of freedom can be represented holographically on the boundary. Thus the appropriate terminology should be 'the holographic formula for entanglement entropy' rather than the holographic principle itself. We therefore clarify the meaning of 'holographic

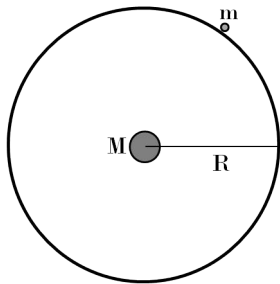


FIG. 1: Verlinde's system: a massive source M is encoded by a spherical screen with radius R , and test particle m is placed just outside the screen.

graphic screen' as the 'minimal surface in the quantum gravity spacetime' [35], on which the holographic property of entanglement entropy holds as in Eq.(3), and the degrees of freedom is now proportional to the area of this minimal surface. The entropy on the screen is the entanglement entropy associated with the separation of the spacetime regions defined by this screen.

2. Entropy variation law

With the meaning of the holographic screen and the information content clarified we now proceed to see how the entanglement entropy changes as the test particle moves. We first review Verlinde's conjecture of the entropy variation law, in which we find some inconsistencies. Then we review the consistency check made by Fursaev [36] through the metric calculation where two infinite surfaces as a whole serve as the holographic screen. We then make our calculation for spherical holographic screens following the same spirit as Fursaev. We will demonstrate these three approaches to entropy variation law by Verlinde, Fursaev and us, respectively.

a. Verlinde's approach Motivated by Bekenstein's argument "When a particle is one Compton wavelength from the horizon, it is considered to be part of the black hole," Verlinde proposed that the entropy on the screen decreases by $2\pi k_B$ when the test particle m moves one Compton wavelength away from the screen. Further assuming that the entropy varies linearly within small distance, Verlinde assumes that the variation of entropy associated with a small displacement Δx of the test particle m away from the screen is

$$\Delta S = -2\pi k_B \frac{\Delta x}{\lambda_m} = -2\pi k_B \frac{mc}{\hbar} \Delta x. \quad (4)$$

One difficulty with this argument is that Bekenstein's idea involves the nature of quantum uncertainty in distance. That is, when the particle is located within one Compton wavelength from the horizon, quantum mechanics dictates that it is indistinguishable whether the particle is right on the horizon or one Compton wave-

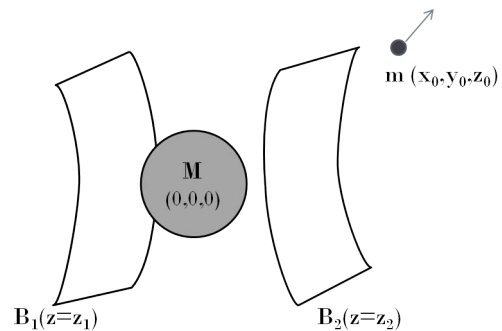


FIG. 2: Fursaev's system: two infinite surface B_1 and B_2 , with their z coordinates fixed, are placed around a massive source M . Outside the sphere is a test particle m whose displacement will affect the area of the surfaces.

length away. How, therefore, would the horizon react to the infinitesimal displacement within this uncertainty?

A more crucial issue of this argument toward the entropy variation law has to do with the possible inconsistency in Verlinde's approach. There are two equations corresponding to the nature of entropy. One is the entropy variation law in Eq.(4) and the other is the Bekenstein law $S = A/4L_p^2$, which was implicitly used through the holographic formula of entanglement. A priori, these two formulas may or may not be compatible. This tension was also been pointed out by others [19, 20]. While Verlinde conjectured the entropy variation law independently of the definition of entropy itself, we should point out that these two equations must be compatible since both of them are tightly related to the underlying form of entropy. In other words,

$$\Delta S = \frac{k_B c^3}{4\hbar G} \Delta A = -2\pi k_B \frac{mc}{\hbar} \Delta x \quad (5)$$

must hold under the entropic gravity framework. Therefore, the change in surface area should be described as

$$\Delta A = -8\pi \frac{mG}{c^2} \Delta x \quad (6)$$

when the test particle m moves a distance Δx away from the normal to the screen, if the entropy variation is correct.

Whereas Verlinde gave the argument associated with entropy without guaranteeing the compatibility, we will show that Eq.(6) can be calculated straight-forwardly in the weak gravity limit based on the knowledge of the spacetime, without the

b. Fursaev's approach Fursaev studied the dynamics of the holographic minimal surface by the displacement of test particle under the gravity in the weak field limit [35]. In his approach two infinite surfaces B_1 and B_2 are located at $z = z_1$ and $z = z_2$, respectively. These two surfaces separate the universe into two regions: a subspace between the two surfaces and its complement on the outside. The massive source M is placed between

the spheres while a test particle is on the outside near one of the surfaces, see FIG. 2.

The spacetime metric with a massive source placed at the origin, in the weak gravity limit, is

$$\begin{aligned} ds^2 &= - \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 + \frac{2GM}{rc^2}\right) dr^2 + r^2 d\Omega^2 \\ &= - \left(1 - \frac{2GM}{\rho c^2}\right) c^2 dt^2 + \left(1 + \frac{2GM}{\rho c^2}\right) (dx^2 + dy^2 + dz^2), \end{aligned} \quad (7)$$

where $\rho = r(1 - GM/rc^2) = \sqrt{x^2 + y^2 + z^2}$.

Now in Fursaev's system there are a massive source M at $(x, y, z) = (0, 0, 0)$ and a test particle m at $(0, 0, r_0)$. The presence of the test particle should affect the above metric, which can be viewed as a back-reaction. The resultant metric becomes

$$\begin{aligned} ds^2 &= - \left(1 - \frac{2GM}{c^2 \rho} - \frac{2Gm}{c^2 \rho_0}\right) c^2 dt^2 \\ &\quad + \left(1 + \frac{2GM}{c^2 \rho} + \frac{2Gm}{c^2 \rho_0}\right) (dx^2 + dy^2 + dz^2). \end{aligned} \quad (8)$$

Here $\rho = r(1 - GM/rc^2) = \sqrt{x^2 + y^2 + z^2}$ and $\rho_0 = \sqrt{(x - x_0)^2 + (y - x_0)^2 + (z - z_0)^2}$. This new metric makes intuitive sense based on the equivalence principle consideration.

A small segment of area on one infinite surface B_k , with $k = 1, 2$, is described as $da^2 = g_{xx} dx^2 g_{yy} dy^2$. The total area of surface B_k is therefore

$$A_k = \iint dx dy \left(1 + \frac{2GM}{c^2 \rho_k} + \frac{2Gm}{c^2 \rho_{k,0}}\right). \quad (9)$$

Here $\rho_{k,0} = \sqrt{(x - x_0)^2 + (y - x_0)^2 + (z_k - z_0)^2}$, $\rho_k = \sqrt{x^2 + y^2 + z_k^2}$.

As the distance between the test particle and the surface changes by an amount $\Delta r \approx \Delta z_0$, the surface area will vary by an amount, to the leading order,

$$\begin{aligned} \Delta A_k &= \frac{2Gm \Delta z_0}{c^2} \iint dx dy \frac{\partial(1/\rho_{k,0})}{\partial z_0} \\ &= - \frac{2Gm \Delta z_0}{c^2} \iint dx dy \frac{(z_0 - z_k)}{\rho_{k,0}^3} \\ &= - \frac{2Gm \Delta z_0}{c^2} \int_{u=0}^{\infty} \int_{\psi}^{2\pi} u du d\psi \frac{(z_0 - z_k)}{(u^2 + (z_k - z_0)^2)^{3/2}} \\ &= - \frac{4\pi Gm \Delta z_0}{c^2} = - \frac{4\pi Gm \Delta r}{c^2}, \end{aligned} \quad (10)$$

where the change of variables with $x - x_0 = u \cos \psi$ and $y - y_0 = u \sin \psi$ have been made. The total variation of the area is then equal to

$$\Delta A = \Delta A_1 + \Delta A_2 = - \frac{8\pi Gm \Delta r}{c^2}. \quad (11)$$

With this, the entropy variation law Eq.(4) is successfully reproduced.

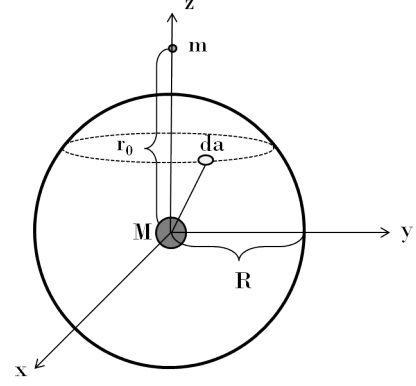


FIG. 3:

c. Our approach Although Fursaev has successfully reproduced the entropy variation law that is consistent with the Bekenstein law of entropy, his derivation is only valid for the special case of infinite surfaces. The more physically relevant geometry should be a sphere. We therefore follow Fursaev's approach but apply it to the variation of the surface area of a sphere, i.e., a spherical holographic screen, to recheck Verlinde's set up of the entropic gravity.

Consider a gravitational source of mass M located at $r = 0$, the spacetime metric in the weak field approximation is

$$\begin{aligned} ds^2 &= - \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 + \frac{2GM}{rc^2}\right) dr^2 + r^2 d\Omega^2 \\ &= - \left(1 - \frac{2GM}{\rho c^2}\right) c^2 dt^2 + \left(1 + \frac{2GM}{\rho c^2}\right) (d\rho^2 + \rho^2 d\Omega^2), \end{aligned} \quad (12)$$

where $\rho = r(1 - GM/rc^2)$ and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$.

In the system of interest, we have a massive source M at $(x, y, z) = (0, 0, 0)$ and a test particle m at $(0, 0, r_0)$. A sphere of radius $r = R$ is a minimal surface that cuts the universe into two complementary regions and also separates M and m into these two regions of the universe, see FIG.3. Similar to Fursaev's case, the area of the surface no longer equals to $4\pi R^2$ because of the slight warpage of the metric induced by the presence of the test particle. The metric in this system becomes

$$\begin{aligned} ds^2 &= - \left(1 - \frac{2GM}{\rho c^2} - \frac{2Gm}{c^2 \sqrt{\rho_0^2 + \rho^2 - 2\rho_0 \rho \cos \theta}}\right) c^2 dt^2 \\ &\quad + \left(1 + \frac{2GM}{\rho c^2} + \frac{2Gm}{c^2 \sqrt{\rho_0^2 + \rho^2 - 2\rho_0 \rho \cos \theta}}\right) (d\rho^2 + \rho^2 d\Omega^2), \end{aligned} \quad (13)$$

where $\rho = R(1 - GM/Rc^2)$ and $\rho_0 = r_0(1 - GM/r_0 c^2)$.

The surface area of the sphere is therefore

$$A = \int \rho^2 \sin\theta d\theta d\phi \left(1 - \frac{2GM}{c^2\rho} - \frac{2Gm}{c^2\sqrt{\rho_0^2 + \rho^2 - 2\rho_0\rho\cos\theta}} \right) \\ = 4\pi\rho^2 - \frac{8\pi GM\rho}{c^2} + A_m . \quad (14)$$

A_m is the surface area deviation due to the metric correction induced by the test particle m :

$$A_m = - \int_0^{2\pi} d\phi \int_{-1}^1 d\cos\theta \frac{2Gm\rho^2}{c^2\sqrt{\rho_0^2 + \rho^2 - 2\rho_0\rho\cos\theta}} \\ = \frac{-4\pi Gm\rho^2}{c^2} \left(\frac{-1}{\rho_0\rho} \sqrt{\rho_0^2 + \rho^2 - 2\rho_0\rho\cos\theta} \right)_{-1}^1 \\ = \begin{cases} \frac{8\pi Gm\rho^2}{c^2\rho_0}, & \rho_0 > \rho , \\ \frac{8\pi Gm\rho}{c^2}, & \rho_0 < \rho . \end{cases} \quad (15)$$

Keeping the leading order in GmR/c^2 and GMR/c^2 , we find

$$r_0 > R : A = 4\pi R^2 + \frac{8\pi GmR^2}{c^2 r_0} , \\ r_0 < R : A = 4\pi R^2 - \frac{8\pi GmR}{c^2} . \quad (16)$$

While to the leading order the surface area A is equal to $4\pi R^2$, the leading behavior of the surface area variation induced by an infinitesimal displacement of the test particle at r_0 is instead contributed by the $8\pi GmR^2/c^2 r_0$ term. (Here we assume that the test particle is outside the sphere.) Now we would like to see how this infinitesimal displacement of the test particle m affects the surface area of the sphere due to the spacetime warpage. When the test particle makes a small displacement Δr_0 away from the sphere, the area will change by an amount

$$\frac{\partial A}{\partial r_0} \Delta r_0 = - \frac{8\pi Gm r^2}{c^2 r_0^2} \Delta r_0 . \quad (17)$$

Therefore if the entropy on the surface follows the Bekenstein's law, Eq.(1), then the entropy variation given by the displacement of the test particle should be

$$\Delta S = k_B \frac{\Delta A}{4l_p^2} = - \frac{2\pi k_B r^2}{r_0^2} \frac{mc}{\hbar} \Delta r_0 . \quad (18)$$

When the center of the test particle is just outside the sphere, that is, $R \approx r_0$, with $R - r_0 \gg Gm/c^2$ to satisfy the weak field condition, the entropy variation on the sphere becomes

$$\Delta S = -2\pi k_B \frac{mc}{\hbar} \Delta r_0 = -2\pi k_B \frac{\Delta r_0}{\lambda_m} , \quad (19)$$

with $\lambda_m = \hbar/mc$ the Compton wavelength of the particle. We have thus obtained the entropy variation law suggested by Verlinde explicitly and consistently without invoking the ambiguous Compton wavelength argument.

3. Temperature

Once the entropy variation with the displacement is determined, we only need to define the temperature as the final step towards the entropic gravity force law. Verlinde utilized the idea of hologram by assuming that the total number of bits of information is proportional to the area of the holographic screen A , which is a consequence of the entanglement entropy obeying Bekenstein law. That is, the degrees of freedom on the screen is proportional to its surface area

$$N = \frac{Ac^3}{G\hbar} . \quad (20)$$

We comment again that this equation is satisfied because of the holographic formula for entanglement entropy, not by taking the upper limit of the holographic bound.

The temperature on the surface is determined by the equipartition rule where the rest energy of mass M is distributed evenly over the occupied bits,

$$E = Mc^2 = \frac{1}{2} N k_B T . \quad (21)$$

The underlying physics for this equipartition-determined temperature derivation is that the test particle m outside the screen interacts with the massive source M indirectly through thermodynamic effects near the screen, with the information of the mass M being registered on this boundary surface [35]. We now arrive at the expression of the temperature

$$T = \frac{2Mc^2}{Nk_B} = \frac{2G\hbar M}{ck_B A} . \quad (22)$$

When the test particle makes a small displacement Δx from the screen, the entropy on the screen changes by an amount ΔS and therefore the test particle will face an effective macroscopic restoring force originated from the system's tendency to increase its entropy. This "entropic force law" should thus follow the first law of thermodynamics:

$$F\Delta x = T\Delta S . \quad (23)$$

With Eqs.(4)–(23) and that the area of the spherical screen equals to $4\pi R^2$, we finally obtain the entropic force law which is identical to Newton's force law of gravity,

$$F = - \frac{GMm}{R^2} . \quad (24)$$

The minus sign in this force law shows that the entropic force is oriented opposite to the direction of distance's increase, just as in Newton's view of the gravitational force that is attractive between two massive sources.

While Newton's force law of gravitation seems to appear elegantly through this entropic derivation, we should emphasize again that both the entropy variation formula and the temperature formula involve an \hbar ,

which indicates their quantum origin. Both these two \hbar 's are originated from the information content of the holographic screen, where one comes out of the direct calculation of the entropy formula while the other emerges from the distribution of the number of bits on the surface. The complete cancellation between these two \hbar 's was due to the coincidence that both the number of bits, N , and the Bekenstein law are straight-forwardly proportional to the surface area of the holographic screen, which was fortuitous. We will argue in the next section that the entropy of entanglement is not exactly proportional to the area. As demonstrated in Ref.[21], the generalized uncertainty principle (GUP) implies a corrected formula for entanglement entropy not only in the strong gravity but also in the weak gravity regime.

III. GENERALIZED UNCERTAINTY PRINCIPLE

That the standard Heisenberg uncertainty principle, which is deduced under the flat, Minkowski spacetime, must be modified, or generalized, when the effect of gravity becomes strong, or the spacetime becomes warped, has long been suggested since 1950s. Since 1980s, GUP acquires additional theoretical support from the string theory's perspective [37–41]. One important implication of GUP is that the standard forms of Bekenstein entropy and Hawking temperature no longer hold as the size of black hole approaches the Planck length [21]. A direct consequence of this GUP modified BH entropy is that the BH evaporation process will come to a stop when its Schwarzschild radius approaches the Planck length. As a result there should be a BH remnant at Planck mass and size left behind.

Based on GUP, it was found that the modified BH temperature is of the form [21]

$$T_{\text{GUP}} = \frac{Mc^2}{4\pi k_B} \left[1 - \sqrt{1 - \frac{M_p^2}{M^2}} \right] \quad (25)$$

for a Schwarzschild black hole of mass M . In the large mass limit, i.e., $M_P/M \ll 1$, the BH temperature is

$$T_{\text{GUP}} = \frac{c^2 M_P^2}{8\pi k_B M} \left[1 + \frac{M_P^2}{4M^2} + \frac{M_P^4}{8M^4} + \dots \right], \quad (26)$$

which agrees with the standard Hawking temperature in leading order.

As the black hole temperature has been modified, the entropy obtained by $S = \int c^2 T dM$ is also corrected to the form different from the simple Bekenstein entropy expression:

$$S_{\text{GUP}} = 2\pi k_B \left\{ \frac{M^2}{M_p^2} \left(1 - \frac{M_p^2}{M^2} + \sqrt{1 - \frac{M_p^2}{M^2}} \right) - \text{Log} \left[\frac{M}{M_p} \left(1 + \sqrt{1 - \frac{M_p^2}{M^2}} \right) \right] \right\}. \quad (27)$$

The integration constant of the integral is fixed by setting the entropy to zero at the final remnant state. Thus in the large mass limit we have

$$\begin{aligned} S_{\text{GUP}} &= 4\pi k_B \frac{M^2}{M_p^2} - \pi k_B \text{Log} \left(\frac{M^2}{M_p^2} \right) + \text{const.} + \dots, \\ &= k_B \frac{A}{4L_p^2} - \pi k_B \text{Log} \left(\frac{A}{L_p^2} \right) + \text{const.} + \dots, \end{aligned} \quad (28)$$

which recovers Bekenstein entropy as M_P/M goes to zero.

The general correction terms to the semiclassical area law of black hole entropy has been extensively studied. A logarithmic term as the leading correction for black hole entropy has been deemed universal by considerations in string theory and loop quantum gravity approaches, see for example [22–25]. We now treat GUP as a fundamental assumption to provide the correct form of entropy, which is in agreement with the well-supported logarithmic subleading behavior.

As BH entropy is just the entanglement entropy on the BH horizon, we assert that under GUP the area-dependence of entanglement entropy is now expressed in the correct form as

$$\begin{aligned} S_{\text{GUP}} &= \frac{Ak_B}{8L_p^2} \left[1 - \frac{16\pi L_p^2}{A} + \sqrt{1 - \frac{16\pi L_p^2}{A}} \right] \\ &\quad - 2\pi k_B \text{Log} \left[\frac{A}{4\sqrt{\pi} L_p} \left(1 + \sqrt{1 - \frac{16\pi L_p^2}{A}} \right) \right], \end{aligned} \quad (29)$$

and in the asymptotic limit of large black hole, its perturbative expansion behaves as Eq.(28).

IV. QUANTUM EFFECTS IN ENTROPIC GRAVITY

In Verinde's entropic gravity scenario, the purely classical Newton's force law of gravitation is derived based on a quantum-mechanical and thermodynamical setup. By keeping track of the underlying quantum dynamics, we now invoke generalized uncertainty principle to uncover the missing quantum contribution in entropic gravity.

Again we consider a spherical holographic screen, whose information content is defined by the GUP corrected entanglement entropy, encoding a massive source M at the center and a test particle m placed just outside this spherical surface of radius R . The restoring force acting on the test particle m as it moves a little bit through the first law of thermodynamics will be found.

First of all, the entropy variation law is directly affected by the GUP corrected form. Under GUP, the entropy varies with the surface area as

$$\Delta S = \frac{\partial S_{\text{GUP}}}{\partial A} \Delta A, \quad (30)$$

while the definition of ΔA remains the same as in Eq.(6).

Next we determine the temperature on the screen. We see that Eq.(20) is proposed on the prerequisite that the information of the space is proportional to the surface area of the screen. Under the GUP framework the entropy is no longer proportional to the area, so we compare Eq.(20) and Bekenstein entropy to arrive at the form for the number of bits on the screen as

$$N = \frac{S_{\text{GUP}}}{4k_B} . \quad (31)$$

We again apply the equipartition formula Eq.(21) to determine the temperature on the screen and find

$$T = \frac{2Mc^2}{Nk_B} = \frac{Mc^2}{2S_{\text{GUP}}} . \quad (32)$$

Finally, using the first law of thermodynamics we arrive at the modified gravity force law:

$$F_{\text{GUP}} = F_N \frac{2[\alpha(1+\eta) - 2(2+\eta)]}{\eta(1+\eta) \left\{ -4 + \alpha(1+\eta) - 4\text{Log} \left[\frac{1}{2}\alpha(1+\eta) \right] \right\}} . \quad (33)$$

Here $F_N = GmM/R^2$ is Newton's gravitational force law, and we have introduced symbols $\eta = \sqrt{1 - 4G\hbar/c^3 R^2}$ and $\alpha = G\hbar/c^3 R^2$ to simplify the expression.

In the large distance limit where $R \gg L_p = \sqrt{G\hbar/c^3}$ and therefore $\alpha = G\hbar/c^3 R^2 \ll 1$, we can expand the force to the third order of α as

$$F_{\text{GUP}} = F_N \left\{ 1 + \alpha [2 - \text{Log}\alpha] + \alpha^2 [4 - 5\text{Log}\alpha + (\text{Log}\alpha)^2] + \alpha^3 [7 - 18\text{Log}\alpha + 8(\text{Log}\alpha)^2 - (\text{Log}\alpha)^3] + \dots \right\} . \quad (34)$$

It is clear that this GUP-based force law recovers the classical Newton's gravitational force law in the infinite distance limit, while some subleading quantum corrections is present as long as α is finite. On the other hand these correction terms go to zero in the classical limit as \hbar vanishes. These α -dependent terms, we conclude, are where \hbar is hiding in entropic gravity.

V. CONCLUSION AND DISCUSSIONS

In this paper we raised the question about where \hbar is hiding in entropic gravity. Through the reanalysis of the fundamental building blocks of entropic gravity, in particular the meaning of holographic screen and its associated entanglement entropy, we argued that the perfect cancellation of \hbar among all the quantum mechanically originated inputs is broken if the more exact form of the BH entanglement entropy based on GUP is to replace the Bekenstein area law. Based on this we found, in the weak gravity limit, the hidden \hbar 's in the form of logarithmic corrections to the classical Newton's law, in Eq.(34).

In our attempt of seeking the missing \hbar 's, we reinvestigated all the fundamental assumptions in the existing derivations of entropic gravity. Within Verlinde's entropic gravity derivation, two ingredients involving entropy formula have been invoked without the guarantee of their mutual compatibility. We applied Fursaev's consistency-check procedure to reproduce the leading order entropy variation in Verlinde's setup of spherical holographic screen. While our approach manages to avoid the compatibility issue, there is a price to pay. In our alternative approach we have introduced the concept of spacetime metric and its deformation due to the presence of a massive object, which implicitly assumed the knowledge of general relativity, the standard theory of classical gravity. Yet the very attempt of entropic gravity is to deduce it from quantum mechanics and statistical physics alone without any prior knowledge of gravity. We are therefore at risk of a circular logic in our approach if gravity is to be interpreted as an emergent phenomenon. In this regard a more cogent and consistent argument without involving any gravity-related concept is needed towards an alternative entropy variation law, in order to assert the validity of the entropic framework of gravity as an emergent phenomena. By the similar token, the existing derivations of entropic gravity also faces the similar issue since Newton's constant has been invoked as a fundamental constant from the outset instead of being a secondary, derived parameter from the theory as it should if gravity is to be an emergent phenomenon.

Under this light one can instead view our derivation of the entropic gravity not as an emergent phenomenon but as a means to deduce the 'quantum gravity force law' via the quantization of the information content on the minimal surfaces in units of Planck area provided by GUP as well as the spacetime warpage effect in the presence of a massive particle provided by general relativity.

Although there are still rooms to improve in this line of approach to gravity, we have provided an exact form of quantum corrected entropic gravity force law based on the assumption of GUP as a fundamental input. Such quantum corrections, though minute, may serve as a probe to examine the concreteness of the entropic gravity interpretation in the the experimentally measurable scale of large distance and weak gravity limit.

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