SLAC–PUB–15132

Single-valued harmonic polylogarithms and the multi-Regge limit

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Abstract

We argue that the natural functions for describing the multi-Regge limit of six-gluon scattering in planar $\mathcal{N}=4$ super Yang-Mills theory are the single-valued harmonic polylogarithmic functions introduced by Brown. These functions depend on a single complex variable and its conjugate, (w, w^*). Using these functions, and formulas due to Fadin, Lipatov and Prygarin, we determine the six-gluon MHV remainder function in the leading-logarithmic approximation (LLA) in this limit through ten loops, and the next-to-LLA (NLLA) terms through nine loops. In separate work, we have determined the symbol of the four-loop remainder function for general kinematics, up to 113 constants. Taking its multi-Regge limit and matching to our four-loop LLA and NLLA results, we fix all but one of the constants that survive in this limit. The multi-Regge limit factorizes in the variables (ν, n) which are related to (w, w^*) by a Fourier-Mellin transform. We can transform the single-valued harmonic polylogarithms to functions of (ν, n) that incorporate harmonic sums, systematically through transcendental weight six. Combining this information with the four-loop results, we determine the eigenvalues of the BFKL kernel in the adjoint representation to NNLLA accuracy, and the MHV product of impact factors to N^{3}LLA accuracy, up to constants representing beyond-the-symbol terms and the one symbol-level constant. Remarkably, only derivatives of the polygamma function enter these results. Finally, the LLA approximation to the six-gluon NMHV amplitude is evaluated through ten loops.

Submitted to Journal of High Energy Physics (JHEP)

Work supported by US Department of Energy contract DE−AC02−76SF00515.

1 Introduction

Enormous progress has taken place recently in unraveling the properties of relativistic scattering amplitudes in four-dimensional gauge theories and gravity. Perhaps the most intriguing developments have been in maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory, in the planar limit of a large number of colors. Many lines of evidence suggest that it should be possible to solve for the scattering amplitudes in this theory to all orders in perturbation theory. There are also semi-classical results based on the AdS/CFT duality to match to at strong coupling [1]. The scattering amplitudes in the planar theory can be expressed in terms of a set of dual (or region) variables x_i^μ, which are related to the usual external momentum four-vectors k_i^μ by $k_i = x_i - x_{i+1}$. Remarkably, the planar $\mathcal{N} = 4$ super-Yang-Mills amplitudes are governed by a dual conformal symmetry acting on the x_i [1, 2, 3, 4, 5, 6, 7]. This symmetry can be extended to a dual superconformal symmetry [8], which acts on supermultiplets of amplitudes that are packaged together by using an $\mathcal{N} = 4$ on-shell superfield and associated Grassmann coordinates [9, 10, 11, 12].

Due to infrared divergences, amplitudes are not invariant under dual conformal transformations. Rather, there is an anomaly, which was first understood in terms of polygonal Wilson loops rather than amplitudes [7]. (For such Wilson loops the anomaly is ultraviolet in nature.) A solution to the anomalous Ward identity for maximally-helicity violating (MHV) amplitudes is to write them in terms of the BDS ansatz [13],

$$A_n^{\rm MHV} = A_n^{\rm BDS} \times \exp(R_n), \tag{1.1}$$

where R_n is the so-called *remainder function* [14, 15], which is fully dual-conformally invariant.

For the four- and five-gluon scattering amplitudes, the only dual-conformally invariant functions are constants, and because of this fact the BDS ansatz is exact and the remainder function vanishes to all loop orders, $R_4 = R_5 = 0$. For six-gluon amplitudes, dual conformal invariance restricts the functional dependence to have the form $R_6(u_1, u_2, u_3)$, where the u_i are the unique invariant cross ratios constructed from distances x_{ij}^2 in the dual space:

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}}\,, \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} = \frac{s_{23}s_{56}}{s_{234}s_{456}}\,, \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2} = \frac{s_{34}s_{61}}{s_{345}s_{561}}\,. \tag{1.2}$$

The need for a nonzero remainder function R_n for Wilson loops was first indicated by the strong-coupling behavior of polygonal loops corresponding to amplitudes with a large number of gluons n [6]. At the six-point level, investigation of the multi-Regge limits of $2 \to 4$ gluon scattering amplitudes led to the conclusion that R_6 must be nonvanishing at two loops [16]. Numerical evidence was found soon thereafter for a nonvanishing two-loop coefficient $R_6^{(2)}$ for generic non-singular kinematics [14], in agreement with the numerical values found simultaneously for the corresponding hexagonal Wilson loop [15].

Based on the Wilson line representation [15], and using dual conformal invariance to take a quasi-multi-Regge limit and simplify the integrals, an analytic result for $R_6^{(2)}$ was derived [17, 18] in terms of Goncharov's multiple polylogarithms [19]. Making use of properties of the *symbol* [20, 21, 22, 23, 24] associated with iterated integrals, the analytic result for $R_6^{(2)}$ was then simplified to just a few lines of classical polylogarithms [23].

A powerful constraint on the structure of the remainder function at higher loop order is provided by the operator product expansion (OPE) for polygonal Wilson loops [25, 26, 27]. At three loops, this constraint, together with symmetries, collinear vanishing, and an assumption about the final entry of the symbol, can be used to determine the symbol of $R_6^{(3)}$ up to just two constant parameters [28]. Another powerful technique for determining the remainder function is to exploit an infinite-dimensional Yangian invariance [29, 30] which includes the dual superconformal generators. These symmetries are anomalous at the loop level (or alternatively one can say that the algebra has to be deformed) [31]. However, the symmetries imply a first order linear differential equation for the ℓ-loop n-point amplitude, and the anomaly dictates the inhomogenous term in the differential equation, in terms of an integral over an $(\ell-1)$-loop $(n+1)$-point amplitude [32, 33]. Using this differential equation, a number of interesting results were obtained in ref. [33]. In particular, the result for the symbol of $R_6^{(3)}$ found in ref. [28] was recovered and the two previously-undetermined constants were fixed.

In principle, the method of refs. [32, 33] works to arbitrary loop order. However, it requires knowing lower-loop amplitudes with an increasing number of external legs, for which the number of kinematical variables (the dual conformal cross ratios) steadily increases. Although the symbol of the two-loop remainder function $R_n^{(2)}$ is known for arbitrary n [34], the same is not true of the three-loop seven-point remainder function, which would feed into the four-loop six-point remainder function — one of the subjects of this paper.

In this article, we focus on features of the six-point kinematics that allow us to push directly to higher loop orders for this amplitude, without having to solve for amplitudes with more legs. In fact, most of our paper is concerned with a special limit of the kinematics in which we can make even more progress: multi-Regge kinematics (MRK), a limit which has already received considerable attention in the context of $\mathcal{N}=4$ super-Yang-Mills theory [16, 28, 35, 36, 37, 38, 39, 40, 41, 42, 43]. In the MRK limit of $2\to 4$ gluon scattering, the four outgoing gluons are widely-spaced in rapidity. In other words, two of the four gluons are emitted far forward, with almost the same energies and directions of the two incoming gluons. The other two outgoing gluons are also well-separated from each other, and have smaller energies than the two far-forward gluons.

The MHV amplitude possesses a unique limit of this type. For definiteness, we will take legs 3 and 6 to be incoming, legs 1 and 2 to be the far-forward outgoing gluons, and legs 4 and 5 to be the other two outgoing gluons. Neglecting power-suppressed terms, helicity must be conserved along the high-energy lines. In the usual all-outgoing convention for labeling helicities, the helicity configuration can be taken to be $(++-++-)$. For generic $2\to 4$ scattering in four dimensions there are eight kinematic variables. Dual conformal invariance reduces the eight variables down to just the three dual conformal cross ratios u_i. Taking the multi-Regge limit essentially reduces the amplitude to a function of just two variables, w and w^*, which turn out to be the complex conjugates of each other.

We will argue that the function space relevant for this limit has been completely characterized by Brown [44]. We call the functions *single-valued harmonic polylogarithms* (SVHPLs). They are built from the analytic functions of a single complex variable that are known as harmonic polylogarithms (HPLs) in the physics literature [45]. These functions have branch cuts at $w=0$ and $w=-1$. However, bilinear combinations of HPLs in w and in w^* can be constructed [44] to cancel the branch cuts, so that the resulting functions are single-valued in the (w, w^*) plane. The single-valued property matches perfectly a physical constraint on the remainder function in the multi-Regge limit. SVHPLs, like HPLs, are equipped with an integer transcendental *weight*. The required weight increases with the loop order. However, at any given weight there is only a finite-dimensional vector space of available functions. Thus, once we have identified the proper function space, the problem of solving for the remainder function in MRK reduces simply to determining a set of rational numbers, namely the coefficients multiplying the allowed SVHPLs at a given weight.

In order to further appreciate the simplicity of the multi-Regge limit, we recall that for generic six-point kinematics there are nine possible choices for the entries in the symbol for the remainder function $R_6(u_1, u_2, u_3)$ [23, 28]:

$$\{u_1, u_2, u_3, 1-u_1, 1-u_2, 1-u_3, y_1, y_2, y_3\}\,, \tag{1.3}$$

where

$$\begin{aligned} y_i &= \frac{u_i - z_+}{u_i - z_-}\,, &(1.4)\\ z_\pm &= \frac{-1+u_1+u_2+u_3 \pm \Delta}{2}\,, &(1.5)\\ \Delta &= (1-u_1-u_2-u_3)^2 - 4u_1u_2u_3\,. &(1.6) \end{aligned}$$

The first entry of the symbol is actually restricted to the set $\{u_1, u_2, u_3\}$ due to the location of the amplitude's branch cuts [27]; the integrability of the symbol restricts the second entry to the set $\{u_i, 1-u_i\}$ [27, 28]; and a "final-entry condition" [28, 34] implies that there are only six, not nine, possibilities for the last entry. However, the remaining entries are unrestricted. The large number of possible entries, and the fact that the y_i variables are defined in terms of square-root functions of the cross ratios (although the u_i can be written as rational functions of the y_i [28]), complicates the task of identifying the proper function space for this problem.

So in this paper we will solve a simpler problem. The MRK limit consists of taking one of the u_i, say u_1, to unity, and letting the other two cross ratios vanish at the same rate that $u_1 \to 1$: $u_2 \approx x(1-u_1)$ and $u_3 \approx y(1-u_1)$ for two fixed variables x and y. To reach the Minkowski version of the MRK limit, which is relevant for $2 \to 4$ scattering, it is necessary to analytically continue u_1 from the Euclidean region according to $u_1 \to e^{-2\pi i}|u_1|$, before taking this limit [16]. Although the square-root variables y_2 and y_3 remain nontrivial in the MRK limit, all of the square roots can be rationalized by a clever choice of variables [38]. We define w and w^* by

$$x \equiv \frac{1}{(1+w)(1+w^*)}, \qquad y \equiv \frac{w\,w^*}{(1+w)(1+w^*)}\,. \tag{1.7}$$

Then the MRK limit of the other variables is

$$u_1 \to 1, \qquad y_1 \to 1, \qquad y_2 \to \tilde{y}_2 = \frac{1+w^*}{1+w}\,, \qquad y_3 \to \tilde{y}_3 = \frac{(1+w)w^*}{w(1+w^*)}\,. \tag{1.8}$$

Neglecting terms that vanish like powers of $(1-u_1)$, we expand the remainder function in the multi-Regge limit in terms of coefficients multiplying powers of the large logarithm $\log(1-u_1)$ at each loop order, following the conventions of ref. [28],

$$R_6(u_1,u_2,u_3)|_{\mathrm{MRK}} \;=\; 2\pi i \sum_{\ell=2}^{\infty} \sum_{n=0}^{\ell-1} a^\ell \, \log^n(1-u_1) \, \left[g_n^{(\ell)}(w,w^*) + 2\pi i \, h_n^{(\ell)}(w,w^*) \right] \,, \tag{1.9}$$

where the coupling constant for planar $\mathcal{N}=4$ super-Yang-Mills theory is $a = g^2 N_c/(8\pi^2)$.

The remainder function R_6 is a transcendental function with weight 2ℓ at loop order ℓ. Therefore the coefficient functions $g_n^{(\ell)}$ and $h_n^{(\ell)}$ have weight $2\ell-n-1$ and $2\ell-n-2$ respectively. As a consequence of eqs. (1.7) and (1.8), their symbols have only four possible entries,

$$\{w, 1+w, w^*, 1+w^*\} \,. \tag{1.10}$$

Furthermore, w and w^* are independent complex variables. Hence the problem of determining the coefficient functions factorizes into that of determining functions of w whose symbol entries are drawn from $\{w, 1+w\}$ — a special class of HPLs — and the complex conjugate functions of w^*.

On the other hand, not every combination of HPLs in w and HPLs in w^* will appear. When the symbol is expressed in terms of the original variables $\{x, y, \tilde{y}_2, \tilde{y}_3\}$, the first entry must be either x or y, reflecting the branch-cut behavior and first-entry condition for general kinematics. Also, the full function must be a single-valued function of x and y, or equivalently a single-valued function of w and w^*. These conditions imply that the coefficient functions belong to the class of SVHPLs defined by Brown [44].

The MRK limit (1.9) is organized hierarchically into the leading-logarithmic approximation (LLA) with $n=\ell-1$, the next-to-leading-logarithmic approximation (NLLA) with $n=\ell-2$, and in general the N^{k}LL terms with $n=\ell-k-1$. Just as the problem of DGLAP evolution in x space is diagonalized by transforming to the space of Mellin moments N, the MRK limit can be diagonalized by performing a Fourier-Mellin transform from (w,w^*) to a new space labeled by (ν,n). In fact, Fadin, Lipatov and Prygarin [38, 40] have given an all-loop-order formula for R_6 in the multi-Regge limit, in terms of two functions of (ν,n): The eigenvalue $\omega(\nu,n)$ of the BFKL kernel in the adjoint representation, and the (regularized) MHV impact factor $\Phi_{\mathrm{Reg}}(\nu,n)$. Each function can be expanded in a, and each successive order in a corresponds to increasing k by one in the N^{k}LLA. The leading term in the impact factor is just one, while the leading BFKL eigenvalue $E_{\nu,n}$ was found in ref. [35]. The NLL term in the impact factor was found in ref. [38], and the NLL contribution to the BFKL eigenvalue in ref. [40].

With this information it is possible to compute the LLA functions $g_{\ell-1}^{(\ell)}$, NLLA functions $g_{\ell-2}^{(\ell)}$ and $h_{\ell-2}^{(\ell)}$, and even the real part at NNLLA, $h_{\ell-3}^{(\ell)}$. All one needs to do is perform the inverse Fourier-Mellin transform back to the (w,w^*) variables. At the three-loop level, this was carried out at LLA for $g_2^{(3)}$ and $h_1^{(3)}$ in ref. [38], and at NLLA for $g_1^{(3)}$ and $h_0^{(3)}$ in ref. [40]. Here we will use the SVHPL basis to make this step very simple. The inverse transform contains an explicit sum over n, and an integral over ν which can be evaluated via residues in terms of a sum over a second integer m. For low loop orders we can perform the double sum analytically using harmonic sums [46, 47, 48, 49, 50, 51]. For high loop orders, it is more efficient to simply truncate the double sum. In the (w, w^*) plane this truncation corresponds to truncating the power series expansion in $|w|$ around the origin. We know the answer is a linear combination of a finite number of SVHPLs with rational-number coefficients. In order to determine the coefficients, we simply compute the power series expansion of the generic linear combination of SVHPLs and match it against the truncated double sum over m and n. We can now perform the inverse Fourier-Mellin transform, in principle to all orders, and in practice through weight 10, corresponding to 10 loops for LLA and 9 loops for NLLA.

Furthermore, we can bring in additional information at fixed loop order, in order to obtain more terms in the expansion of the BFKL eigenvalue and the MHV impact factor. In ref. [40], the NLLA results for $g_1^{(3)}$ and $h_0^{(3)}$ confirmed a previous prediction [28] based on an analysis of the multi-Regge limit of the symbol for $R_6^{(3)}$. In this limit, the two free symbol parameters mentioned above dropped out. The symbol could be integrated back up into a function, but a few more "beyond-the-symbol" constants entered at this stage. One of the constants was fixed in ref. [40] using the NLLA information. As noted in ref. [40], the result from ref. [28] for $g_0^{(3)}$ can be used to determine the NNLLA term in the impact factor. In this paper, we will use our knowledge of the space of functions of (w, w^*) (the SVHPLs) to build up a dictionary of the functions of (ν, n) (special types of harmonic sums) that are the Fourier-Mellin transforms of the SVHPLs. From this dictionary and $g_0^{(3)}$ we will determine the NNLLA term in the impact factor.

We can go further if we know the four-loop remainder function $R_6^{(4)}$. In separate work [52], we have heavily constrained the symbol of $R_6^{(4)}(u_1, u_2, u_3)$ for generic kinematics, using exactly the same constraints used in ref. [28]: integrability of the symbol, branch-cut behavior, symmetries, the final-entry condition, vanishing of collinear limits, and the OPE constraints (which at four loops are a constraint on the triple discontinuity). Although there are millions of possible terms before applying these constraints, afterwards the symbol contains just 113 free constants (112 if we apply the overall normalization for the OPE constraints). Next we construct the multi-Regge limit of this symbol, and apply all the information we have about this limit:

- Vanishing of the super-LLA terms $g_n^{(4)}$ and $h_n^{(4)}$ for $n = 4, 5, 6, 7$;
- LLA and NLLA predictions for $g_n^{(4)}$ and $h_n^{(4)}$ for $n = 2, 3$;
- the NNLLA real part $h_1^{(4)}$, which is also predicted by the NLLA formula;
- a consistency condition between $g_1^{(4)}$ and $h_0^{(4)}$.

Remarkably, these conditions determine all but one of the symbol-level parameters in the MRK limit. (The one remaining free parameter seems highly likely to vanish, given the complicated way it enters various formulae, but we have not yet proven that to be the case.)

We then extract the remaining four-loop coefficient functions, $g_1^{(4)}$, $h_0^{(4)}$ and $g_0^{(4)}$, introducing some additional beyond-the-symbol parameters at this stage. We use this information to determine the NNLLA BFKL eigenvalue and the N^{3}LLA MHV impact factor, up to these parameters.

Although our general dictionary of functions of (ν, n) contains various multiple harmonic sums, we find that the key functions entering the multi-Regge limit can all be expressed just in terms of certain rational combinations of ν and n, together with the polygamma functions ψ, ψ', ψ'', etc. (derivatives of the logarithm of the Γ function) with arguments $1 \pm i\nu + |n|/2$.

As a byproduct, we find that the SVHPLs also describe the multi-Regge limit of the one remaining helicity configuration for six-gluon scattering in $\mathcal{N} = 4$ super-Yang-Mills theory, namely the next-to-MHV (NMHV) configuration with three negative and three positive gluon helicities. It was shown recently [43] that in LLA the NMHV and MHV remainder functions are related by a simple integro-differential operator. This operator has a natural action in terms of the SVHPLs, allowing us to easily extend the NMHV LLA results of ref. [43] from three loops to 10 loops.

This article is organized as follows. In Section 2 we review the structure of the six-point MHV remainder function in the multi-Regge limit. Section 3 reviews Brown's construction of single-valued harmonic polylogarithms. In Section 4 we exploit the SVHPL basis to determine the functions $g_n^{(\ell)}$ and $h_n^{(\ell)}$ at LLA through 10 loops and at NLLA through 9 loops. Section 5 determines the NMHV remainder function at LLA through 10 loops. In Section 6 we describe our construction of the functions of (ν, n) that are the Fourier-Mellin transforms of the SVHPLs. Section 7 applies this knowledge, plus information from the four-loop remainder function [52], in order to determine the NNLLA MHV impact factor and BFKL eigenvalue, and the N^{3}LLA MHV impact factor, in terms of a handful of (mostly) beyond-the-symbol constants. In Section 8 we report our conclusions and discuss directions for future research.

We include two appendices. Appendix A collects expressions for the SVHPLs (after diagonalizing the action of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry), in terms of HPLs through weight 5. It also gives expressions before diagonalizing one of the two $\mathbb{Z}_2$ factors. Appendix B gives a basis for the function space in (ν, n) through weight 5, together with the Fourier-Mellin map to the SVHPLs. In addition, for the lengthier formulae, we provide separate computer-readable text files as ancillary material. In particular, we include files (in `Mathematica` format) that contain the expressions for the SVHPLs in terms of ordinary HPLs up to weight six, decomposed into an eigenbasis of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, as well as the analytic results up to weight ten for the imaginary parts of the MHV remainder function at LLA and NLLA and for the NMHV remainder function at LLA. Furthermore, we include the expressions for the NNLL BFKL eigenvalue and impact factor and the N^{3}LL impact factor in terms of the building blocks in the variables (ν, n) constructed in Section 6, as well as a dictionary between these building blocks and the SVHPLs up to weight five.

2 The six-point remainder function in the multi-Regge limit

The principal aim of this paper is to study the six-point MHV amplitude in $\mathcal{N} = 4$ super Yang-Mills theory in multi-Regge kinematics. This limit is defined by the hierarchy of scales,

$$s_{12} \gg s_{345},\ s_{456} \gg s_{34},\ s_{45}\,,s_{56} \gg s_{23},\ s_{61},\ s_{234}\,. \tag{2.1}$$

In this limit the cross ratios (1.2) behave as

$$1-u_1,\; u_2,\; u_3 \sim 0\,, \tag{2.2}$$

together with the constraint that the following ratios are held fixed,

$$x \equiv \frac{u_2}{1-u_1} = \mathcal{O}(1) \quad\text{and}\quad y \equiv \frac{u_3}{1-u_1} = \mathcal{O}(1)\,. \tag{2.3}$$

In the following it will be convenient [38] to parametrize the dependence on x and y by a single complex variable w,

$$x \equiv \frac{1}{(1+w)(1+w^*)} \quad\text{and}\quad y \equiv \frac{w\,w^*}{(1+w)(1+w^*)}\,. \tag{2.4}$$

Any function of the three cross ratios can then develop large logarithms $\log(1-u_1)$ in the multi-Regge limit, and we can write generically,

$$F(u_1,u_2,u_3) = \sum_i \log^i(1-u_1)\, f_i(w,w^*) + \mathcal{O}(1-u_1)\,. \tag{2.5}$$

Let us make at this point an important observation which will be a recurrent theme in the rest of the paper: If $F(u_1,u_2,u_3)$ represents a physical quantity like a scattering amplitude, then F should only have cuts in physical channels, corresponding to branch cuts starting at points where one of the cross ratios vanishes. Rotation around the origin in the complex w plane, i.e. $(w,w^*) \to (e^{2\pi i}w, e^{-2\pi i}w^*)$, does not correspond to crossing any branch cut. As a consequence, the functions $f_i(w,w^*)$ should not change under this operation. More generally, the functions $f_i(w,w^*)$ must be *single-valued* in the complex w plane.

Let us start by reviewing the multi-Regge limit of the MHV remainder function $R(u_1,u_2,u_3) \equiv R_6(u_1,u_2,u_3)$ introduced in eq. (1.1). It can be shown that, while in the Euclidean region the remainder function vanishes in the multi-Regge limit, there is a Mandelstam cut such that we obtain a non-zero contribution in MRK after performing the analytic continuation [16]

$$u_1 \to e^{-2\pi i}\,|u_1|\,. \tag{2.6}$$

After this analytic continuation, the six-point remainder function can be expanded into the form given in eq. (1.9), which we repeat here for convenience,

$$R|_{\text{MRK}} = 2\pi i \sum_{\ell=2}^{\infty}\sum_{n=0}^{\ell-1} a^\ell \,\log^n(1-u_1)\,\left[g_n^{(\ell)}(w,w^*) + 2\pi i\, h_n^{(\ell)}(w,w^*)\right]\,. \tag{2.7}$$

The functions $g_n^{(\ell)}(w,w^*)$ and $h_n^{(\ell)}(w,w^*)$ will in the following be referred to as the *coefficient functions* for the logarithmic expansion in the MRK limit. The imaginary part $g_n^{(\ell)}$ is associated with a single discontinuity, and the real part $h_n^{(\ell)}$ with a double discontinuity, although both functions also include information from higher discontinuities, albeit with accompanying explicit factors of π^2.

The coefficient functions are single-valued pure transcendental functions in the complex variable w, of weight $2\ell - n - 1$ for $g_n^{(\ell)}$ and weight $2\ell - n - 2$ for $h_n^{(\ell)}$. They are left invariant by a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry acting via complex conjugation and inversion,

$$w \leftrightarrow w^* \quad \text{and} \quad (w, w^*) \leftrightarrow (1/w, 1/w^*)\,. \tag{2.8}$$

The complex conjugation symmetry arises because the MHV remainder function has a parity symmetry, or invariance under $\Delta \to -\Delta$, which inverts $\tilde{y}_2$ and $\tilde{y}_3$ in eq. (1.8). The inversion symmetry is a consequence of the fact that the six-point remainder function is a totally symmetric function of the three cross ratios u_1, u_2 and u_3. In particular, exchanging $\tilde{y}_2 \leftrightarrow \tilde{y}_3$ is the product of conjugation and inversion. The inversion symmetry is sometimes referred to as target-projectile symmetry [37]. Finally, the vanishing of the six-point remainder function in the collinear limit implies the vanishing of $g_n^{(\ell)}(w, w^*)$ and $h_n^{(\ell)}(w, w^*)$ in the limit where $(w, w^*) \to 0$. Clearly the functions $g_n^{(\ell)}$ and $h_n^{(\ell)}$ are already highly constrained on general grounds.

In ref. [38, 40] an all-loop integral formula for the six-point amplitude in MRK was presented[1],

$$e^{R+i\pi\delta}|_{\text{MRK}} = \cos \pi\omega_{ab} + i\,\frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{w}{w^*}\right)^{\frac{n}{2}} \int_{-\infty}^{+\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} \,|w|^{2i\nu}\, \Phi_{\text{Reg}}(\nu, n) \left(-\frac{1}{\sqrt{u_2\, u_3}}\right)^{\omega(\nu,n)}. \tag{2.9}$$

The first term is the Regge pole contribution, with

$$\omega_{ab} = \frac{1}{8}\,\gamma_K(a)\,\log\frac{u_3}{u_2} = \frac{1}{8}\,\gamma_K(a)\,\log|w|^2\,, \tag{2.10}$$

and $\gamma_K(a)$ is the cusp anomalous dimension, known to all orders in perturbation theory [53],

$$\gamma_K(a) \;=\; \sum_{\ell=1}^{\infty} \gamma_K^{(\ell)}\, a^\ell \;=\; 4\,a - 4\,\zeta_2\, a^2 + 22\,\zeta_4\, a^3 - (\tfrac{219}{2}\,\zeta_6 + 4\,\zeta_3^2)\, a^4 + \cdots\,. \tag{2.11}$$

The second term in eq. (2.9) arises from a Regge cut and is fully determined to all orders by the BFKL eigenvalue $\omega(\nu, n)$ and the (regularized) impact factor $\Phi_{\text{Reg}}(\nu, n)$. The function δ appearing in the exponent on the left-hand side is the contribution from a Mandelstam cut present in the BDS ansatz, and is given to all loop orders by

$$\delta = \frac{1}{8}\,\gamma_K(a)\,\log(xy) = \frac{1}{8}\,\gamma_K(a)\,\log\frac{|w|^2}{|1+w|^4}\,. \tag{2.12}$$

In addition, we have

$$\frac{1}{\sqrt{u_2\, u_3}} = \frac{1}{1-u_1}\,\frac{|1+w|^2}{|w|}\,. \tag{2.13}$$

[1]There is a difference in conventions regarding the definition of the remainder function. What we call R is called $\log(R)$ in refs. [38, 40]. Apart from the zeroth order term, the first place this makes a difference is at four loops, in the real part.

The BFKL eigenvalue and the impact factor can be expanded perturbatively,

$$\begin{aligned}\omega(\nu,n) &= -a\left(E_{\nu,n} + a\,E^{(1)}_{\nu,n} + a^2\,E^{(2)}_{\nu,n} + \mathcal{O}(a^3)\right)\,,\\ \Phi_{\rm Reg}(\nu,n) &= 1 + a\,\Phi^{(1)}_{\rm Reg}(\nu,n) + a^2\,\Phi^{(2)}_{\rm Reg}(\nu,n) + a^3\,\Phi^{(3)}_{\rm Reg}(\nu,n) + \mathcal{O}(a^4)\,.\end{aligned} \tag{2.14}$$

The BFKL eigenvalue is known to the first two orders in perturbation theory [40, 35],

$$E_{\nu,n} = -\frac{1}{2}\frac{|n|}{\nu^2+\frac{n^2}{4}} + \psi\left(1+i\nu+\frac{|n|}{2}\right) + \psi\left(1-i\nu+\frac{|n|}{2}\right) - 2\psi(1)\,, \tag{2.15}$$

$$\begin{aligned}E^{(1)}_{\nu,n} = &-\frac{1}{4}\Bigg[\psi''\left(1+i\nu+\frac{|n|}{2}\right) + \psi''\left(1-i\nu+\frac{|n|}{2}\right)\\ &- \frac{2i\nu}{\nu^2+\frac{n^2}{4}}\left(\psi'\left(1+i\nu+\frac{|n|}{2}\right) - \psi'\left(1-i\nu+\frac{|n|}{2}\right)\right)\Bigg] - \zeta_2\,E_{\nu,n} - 3\zeta_3 - \frac{1}{4}\frac{|n|\left(\nu^2-\frac{n^2}{4}\right)}{\left(\nu^2+\frac{n^2}{4}\right)^3}\,,\end{aligned} \tag{2.16}$$

where $\psi(z) = \frac{d}{dz}\log\Gamma(z)$ is the digamma function, and $\psi(1) = -\gamma_E$ is the Euler-Mascheroni constant. The NLL contribution to the impact factor is given by [37]

$$\Phi^{(1)}_{\rm Reg}(\nu,n) = -\frac{1}{2}E^2_{\nu,n} - \frac{3}{8}\frac{n^2}{(\nu^2+\frac{n^2}{4})^2} - \zeta_2\,. \tag{2.17}$$

The BFKL eigenvalues and impact factor in eqs. (2.15), (2.16) and (2.17) are enough to compute the six-point remainder function in the Regge limit in the leading and next-to-leading logarithmic approximations (LLA and NLLA). Indeed, we can interpret the integral in eq. (2.9) as a contour integral in the complex ν plane and close the contour at infinity. By summing up the residues we then obtain the analytic expression of the remainder function in the LLA and NLLA in MRK. This procedure will be discussed in greater detail in Section 4. Some comments are in order about the integral in eq. (2.9):

1. The contribution coming from $n = 0$ seems ill-defined, as the integral in eq. (2.9) diverges. After closing the contour at infinity, our prescription is to take only half of the residue at $\nu = n = 0$ into account.

2. We need to specify the Riemann sheet of the exponential factor in the right-hand side of eq. (2.9). We find that the replacement

$$\left(-\frac{1}{\sqrt{u_2\,u_3}}\right)^{\omega(\nu,n)} \to e^{-i\pi\omega(\nu,n)}\left(\frac{1}{\sqrt{u_2\,u_3}}\right)^{\omega(\nu,n)} \tag{2.18}$$

   gives the correct result.

The $i\pi$ factor in the right-hand side of eq. (2.18) generates the real parts $h^{(\ell)}_n$ in eq. (2.7). It is easy to see that the $g^{(\ell)}_n$ and $h^{(\ell)}_n$ functions are not independent, but they are related. For example, at LLA and NLLA we have,

$$\begin{aligned}
h^{(\ell)}_{\ell-1}(w,w^*) &= 0\,,\\
h^{(\ell)}_{\ell-2}(w,w^*) &= \frac{\ell-1}{2}\,g^{(\ell)}_{\ell-1}(w,w^*) + \frac{1}{16}\,\gamma_K^{(1)}\,g^{(\ell-1)}_{\ell-2}(w,w^*)\,\log\frac{|1+w|^4}{|w|^2}\\
&\quad -\frac{1}{2}\sum_{k=2}^{\ell-2} g^{(k)}_{k-1}g^{(\ell-k)}_{\ell-k-1}\,, \qquad \ell>2,
\end{aligned} \tag{2.19}$$

where $\gamma_K^{(1)} = 4$ from eq. (2.11). (Note that the sum over k in the formula for $h^{(\ell)}_{\ell-2}$ would not have been present if we had used the convention for R in refs. [38, 40].) Similar relations can be derived beyond NLLA, i.e. for $n < \ell - 2$.

So far we have only considered $2 \to 4$ scattering. In ref. [39] it was shown that if the remainder function is analytically continued to the region corresponding to $3 \to 3$ scattering, then it takes a particularly simple form. The analytic continuation from $2 \to 4$ to $3 \to 3$ scattering can be obtained easily by performing the replacement

$$\log(1-u_1) \to \log(u_1-1) - i\pi \tag{2.20}$$

in eq. (2.9). After analytic continuation the real part of the remainder function only gets contributions from the Regge pole and is given by [39]

$$\mathrm{Re}\left(e^{R_{3\to3}-i\pi\delta}\right) = \cos\pi\omega_{ab}\,. \tag{2.21}$$

It is manifest from eq. (2.9) that eq. (2.21) is automatically satisfied if the relations among the coefficient functions derivable by tracking the $i\pi$ from eq. (2.18) (e.g. eq. (2.19)) are satisfied in $2 \to 4$ kinematics.

So far we have only reviewed some general properties of the six-point remainder function in MRK, but we have not yet given explicit analytic expressions for the coefficient functions. The two-loop contributions to eq. (2.9) in LLA and NLLA were computed in refs. [37, 38], while the three-loop contributions up to the NNLLA were found in refs. [28, 37]. In all cases the results have been expressed as combinations of classical polylogarithms in the complex variable w and its complex conjugate w^*, with potential branching points at $w = 0$ and $w = -1$. As discussed at the beginning of this section, all the branch cuts in the complex w plane must cancel, i.e., the function must be single-valued in w. The class of functions satisfying these constraints has been studied in full generality in the mathematical literature, as will be reviewed in the next section.

3 Harmonic polylogarithms and their single-valued analogues

3.1 Review of harmonic polylogarithms

In this section we give a short review of the classical and harmonic polylogarithms, one of the main themes in the rest of this paper. The simplest possible polylogarithmic functions are the so-called *classical* polylogarithms, defined inside the unit circle by a convergent power series,

$$\mathrm{Li}_m(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^m}\,, \qquad |z| < 1\,. \tag{3.1}$$

They can be continued to the cut plane $\mathbb{C}\backslash[1,\infty)$ by an iterated integral representation,

$$\mathrm{Li}_m(z) = \int_0^z dz'\, \frac{\mathrm{Li}_{m-1}(z')}{z'}\,. \tag{3.2}$$

For $m = 1$, the polylogarithm reduces to the ordinary logarithm, $\mathrm{Li}_1(z) = -\log(1-z)$, a fact that dictates the location of the branch cut for all m (along the real axis for $z > 1$). It also determines the discontinuity across the cut,

$$\Delta\mathrm{Li}_m(z) = 2\pi i\, \frac{\log^{m-1} z}{(m-1)!}\,. \tag{3.3}$$

It is possible to define more general classes of polylogarithmic functions by allowing for different kernels inside the iterated integral in eq. (3.1). The *harmonic* polylogarithms (HPLs) [45] are a special class of generalized polylogarithms whose properties and construction we review in the remainder of this section. To begin, let w be a word formed from the letters x_0 and x_1, and let e be the empty word. Then, for each w, define a function $H_w(z)$ which obeys the differential equations,

$$\frac{\partial}{\partial z} H_{x_0 w}(z) = \frac{H_w(z)}{z} \qquad \text{and} \qquad \frac{\partial}{\partial z} H_{x_1 w}(z) = \frac{H_w(z)}{1-z}\,, \tag{3.4}$$

subject to the following conditions,

$$H_e(z) = 1, \qquad H_{x_0^n}(z) = \frac{1}{n!}\log^n z, \qquad \text{and} \qquad \lim_{z\to 0} H_{w\neq x_0^n}(z) = 0\,. \tag{3.5}$$

There is a unique family of solutions to these equations, and it defines the HPLs. Note that we use the term "HPL" in a restricted sense[2] – we only consider poles in the differential equations (3.4) at $z = 0$ and $z = 1$. (In our MRK application, we will let $z = -w$, so that the poles are at $w = 0$ and $w = -1$.)

The *weight* of an HPL is the length of the word w, and its *depth* is the number of x_1's[3]. HPLs of depth one are simply the classical polylogarithms, $H_n(z) = \mathrm{Li}_n(z)$. Like the classical polylogarithms, the HPLs can be written as iterated integrals,

$$H_{x_0 w}(z) = \int_0^z dz'\, \frac{H_w(z')}{z'} \qquad \text{and} \qquad H_{x_1 w} = \int_0^z dz'\, \frac{H_w(z')}{1-z'}\,. \tag{3.7}$$

[2]In the mathematical literature, these functions are sometimes referred to as *multiple polylogarithms in one variable.*

[3]For ease of notation, we will often impose the replacement $\{x_0 \to 0, x_1 \to 1\}$ in subscripts. In some cases, we will use the collapsed notation where a subscript m denotes $m-1$ zeroes followed by a single 1. For example, if $w = x_0 x_0 x_1 x_0 x_1$,

$$H_w(z) = H_{x_0x_0x_1x_0x_1}(z) = H_{0,0,1,0,1}(z) = H_{3,2}(z)\,. \tag{3.6}$$

In the collapsed notation, the *weight* is the sum of the indices, and the *depth* is the number of nonzero indices.

Weight	Lyndon words	Dimension
1	0, 1	2
2	01	1
3	001, 011	2
4	0001, 0011, 0111	3
5	00001, 00011, 00101, 00111, 01011, 01111	6

Table 1: All Lyndon words $\text{Lyndon}(x_0, x_1)$ through weight five

The structure of the underlying iterated integrals endows the HPLs with an important property: they form a *shuffle algebra*. The shuffle relations can be written,

$$H_{w_1}(z)\, H_{w_2}(z) = \sum_{w \in w_1 ш w_2} H_w(z)\,, \tag{3.8}$$

where $w_1 ш w_2$ is the set of mergers of the sequences w_1 and w_2 that preserve their relative ordering. Equation (3.8) may be used to express all HPLs of a given weight in terms of a relatively small set of basis functions and products of lower-weight HPLs. One convenient such basis [54] of irreducible functions is the *Lyndon* basis, defined by $\{H_w(z) : w \in \text{Lyndon}(x_0, x_1)\}$. The Lyndon words $\text{Lyndon}(x_0, x_1)$ are those words w such that for every decomposition into two words $w = uv$, the left word is lexicographically smaller than the right, $u < v$. Table 1 gives the first few examples of Lyndon words.

All HPLs are real whenever the argument z is less than 1, and so, in particular, the HPLs are analytic in a neighborhood of $z = 0$. The Taylor expansion around $z = 0$ is particularly simple and involves only a special class of harmonic numbers [45, 48] (hence the name *harmonic* polylogarithm),

$$H_{m_1,\ldots,m_k}(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^{m_1}} Z_{m_2,\ldots,m_k}(l-1)\,, \qquad m_i > 0\,, \tag{3.9}$$

where $Z_{m_1,\ldots,m_k}(n)$ denote the so-called Euler-Zagier sums [46, 47], defined recursively by

$$Z_{m_1}(n) = \sum_{l=1}^{n} \frac{1}{l^{m_1}} \quad \text{and} \quad Z_{m_1,\ldots,m_k}(n) = \sum_{l=1}^{n} \frac{1}{l^{m_1}} Z_{m_2,\ldots,m_k}(l-1)\,. \tag{3.10}$$

Note that the indexing of the weight vectors $m_1, \ldots, m_k$ in eqs. (3.9) and (3.10) is in the collapsed notation.

Another important property of HPLs is that they are closed under certain transformations of the arguments [45]. In particular, using the integral representation (3.7), it is easy to show that the set of all HPLs is closed under the following transformations,

$$z \mapsto 1-z, \quad z \mapsto 1/z, \quad z \mapsto 1/(1-z), \quad z \mapsto 1-1/z, \quad z \mapsto z/(z-1)\,. \tag{3.11}$$

If we add to these mappings the identity map $z \mapsto z$, we can identify the transformations in eq. (3.11) as forming a representation of the symmetric group S_3. In other words, the vector space spanned by all HPLs is endowed with a natural action of the symmetric group S_3.

Finally, it is evident from the iterated integral representation (3.7) that HPLs can have branch cuts starting at $z = 0$ and/or $z = 1$, i.e., HPLs define in general multi-valued functions on the complex plane. In the next section we will define analogues of HPLs without any branch cuts, thus obtaining a single-valued version of the HPLs.

3.2 Single-valued harmonic polylogarithms

Before reviewing the definition of single-valued harmonic polylogarithms in general, let us first review the special case of single-valued classical polylogarithms. The knowledge of the discontinuities of the classical polylogarithms, eq. (3.3), can be leveraged to construct a sequence of real analytic functions on the punctured plane $\mathbb{C}\backslash\{0,1\}$. The idea is to consider linear combinations of (products of) classical polylogarithms and ordinary logarithms such that all the branch cuts cancel. Although the space of single-valued functions is unique, the choice of basis is not unique, and there have been several versions proposed in the literature. As an illustration, consider the functions of Zagier [55],

$$D_m(z) = \mathfrak{R}_m \left\{ \sum_{k=1}^{m} \frac{(-\log|z|)^{m-k}}{(m-k)!} \mathrm{Li}_k(z) + \frac{\log^m |z|}{2\, m!} \right\} , \tag{3.12}$$

where $\mathfrak{R}_m$ denotes the imaginary part for m even and the real part for m odd. The discontinuity of the function inside the curly brackets is given by

$$2\pi i \sum_{k=1}^{m} \frac{(-\log|z|)^{m-k}}{(m-k)!} \frac{\log^{k-1} z}{(k-1)!} = 2\pi \frac{i^m}{(m-1)!} (\arg z)^{m-1} . \tag{3.13}$$

Since eq. (3.13) is real for even m and pure imaginary for odd m, $D_m(z)$ is indeed single-valued. For the special case $m = 2$, we reproduce the famous Bloch-Wigner dilogarithm [56],

$$D_2(z) = \mathrm{Im}\{\mathrm{Li}_2(z)\} + \arg(1-z)\log|z| . \tag{3.14}$$

Just as there have been numerous proposals in the literature for single-valued versions of the classical polylogarithms, there are many potential choices of bases for single-valued HPLs. On the other hand, if we choose to demand some reasonable properties, it turns out that a unique set of functions emerges. Following ref. [44], we require the single-valued HPLs to be built entirely from holomorphic and anti-holomorphic HPLs. Specifically, they should be a linear combination of terms of the form $H_{w_1}(z)H_{w_2}(\bar{z})$, where w_1 and w_2 are words in x_0 and x_1 or the empty word e. The single-valued classical polylogarithms obey an analogous property, and it can be understood as the condition that the single-valued functions are the proper extensions of the original functions. The remaining requirements are simply the analogs of the conditions used to construct the ordinary HPLs.

Define a function $\mathcal{L}_w(z)$, which is a linear combination of functions $H_{w_1}(z)H_{w_2}(\bar{z})$ and which obeys the differential equations

$$\frac{\partial}{\partial z}\mathcal{L}_{x_0 w}(z) = \frac{\mathcal{L}_w(z)}{z} \qquad \text{and} \qquad \frac{\partial}{\partial z}\mathcal{L}_{x_1 w}(z) = \frac{\mathcal{L}_w(z)}{1-z} , \tag{3.15}$$

subject to the conditions,

$$\mathcal{L}_e(z) = 1\,, \qquad \mathcal{L}_{x_0^n}(z) = \frac{1}{n!}\log^n|z|^2 \qquad \text{and} \qquad \lim_{z\to 0}\mathcal{L}_{w\neq x_0^n}(z) = 0\,. \tag{3.16}$$

In ref. [44] Brown showed that there is a unique family of solutions to these equations that is single-valued in the complex z plane, and it defines the single-valued HPLs (SVHPLs). The functions $\mathcal{L}_w(z)$ are linearly independent and span the space. That is to say, every single-valued linear combination of functions of the form $H_{w_1}(z)H_{w_2}(\bar{z})$ can be written in terms of the $\mathcal{L}_w(z)$. In ref. [44] an algorithm was presented that allows for the explicit construction of all SVHPLs as linear combinations of (products of) ordinary HPLs. We present a short review of this algorithm in Section 3.3.

The SVHPLs of ref. [44] share all the nice features of their multi-valued analogues. First, like the ordinary HPLs, they obey shuffle relations,

$$\mathcal{L}_{w_1}(z)\,\mathcal{L}_{w_2}(z) = \sum_{w\in w_1 ш w_2}\mathcal{L}_w(z), \tag{3.17}$$

where again $w_1 ш w_2$ represents the shuffles of w_1 and w_2. As a consequence, we may again choose to solve eq. (3.17) in terms of a Lyndon basis. It follows that if we want the full list of all SVHPLs of a given weight, it is enough to know the corresponding Lyndon basis up to that weight.

Furthermore, the space of SVHPLs is also closed under the S_3 action defined by eq. (3.11). Indeed, if we extend the action to the complex conjugate variable $\bar{z}$, then the closure of the space of all ordinary HPLs implies the closure of the space spanned by all products of the form $H_{w_1}(z)H_{w_2}(\bar{z})$, and, in particular, the closure of the subspace of SVHPLs. For the SVHPLs, it is possible to enlarge the symmetry group to $\mathbb{Z}_2 \times S_3$, where the $\mathbb{Z}_2$ subgroup acts by complex conjugation, $z \leftrightarrow \bar{z}$.

It turns out that the functions $\mathcal{L}_w(z)$ can generically be decomposed as

$$\mathcal{L}_w(z) = \big(H_w(z) - (-1)^{|w|}H_w(\bar{z})\big) + [\text{products of lower weight}]\,, \tag{3.18}$$

where $|w|$ denotes the weight. As such, it is convenient to consider the even and odd projections, i.e., the decomposition into eigenfunctions of the $\mathbb{Z}_2$ action,

$$L_w(z) = \frac{1}{2}\left(\mathcal{L}_w(z) - (-1)^{|w|}\,\mathcal{L}_w(\bar{z})\right) \qquad \text{and} \qquad \overline{L}_w(z) = \frac{1}{2}\left(\mathcal{L}_w(z) + (-1)^{|w|}\mathcal{L}_w(\bar{z})\right)\,. \tag{3.19}$$

The basis defined by $\mathcal{L}_w(z)$ was already complete, and yet here we have doubled the number of potential basis functions. Therefore $L_w(z)$ and $\overline{L}_w(z)$ must be related to one another. Writing $L_w(z) = \mathfrak{R}_{|w|}(\mathcal{L}_w(z))$, we see that it has the same parity as Zagier's single-valued versions of the classical polylogarithms given in eq. (3.12). Therefore we might expect the $L_w(z)$ to form a complete basis on their own. Indeed this turns out to be the case, and the $\overline{L}_w(z)$ can be expressed as products of the functions $L_w(z)$,

$$\overline{L}_w(z) = [\text{products of lower weight } L_{w'}(z)]\,. \tag{3.20}$$

Hence we will not consider the functions $\overline{L}_w(z)$ any further and will concentrate solely on the functions $L_w(z)$.

The functions $L_w(z)$ do not automatically form simple representations of the S_3 symmetry. For the current application, we will mostly be concerned with the $\mathbb{Z}_2 \subset S_3$ subgroup generated by inversions $z \leftrightarrow 1/z$. The functions $L_w(z)$ can easily be decomposed into eigenfunctions of this $\mathbb{Z}_2$, and, furthermore, these eigenfunctions form a basis for the space of all SVHPLs. The latter follows from the observation that,

$$L_w(z) - (-1)^{|w|+d_w} L_w\Big(\frac{1}{z}\Big) = \text{[products of lower weight]}, \tag{3.21}$$

where $|w|$ is the weight and d_w is the depth of the word w. We will denote these eigenfunctions of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by,

$$L_w^{\pm}(z) \equiv \frac{1}{2}\left[L_w(z) \pm L_w\Big(\frac{1}{z}\Big)\right], \tag{3.22}$$

and present most of our results in terms of this convenient basis. For low weights, Appendix A gives explicit representations of these basis functions in terms of HPLs. The expressions through weight six can be found in the ancillary files.

We have seen in the previous section that in the multi-Regge limit the six-point amplitude is described to all loop orders by single-valued functions of a single complex variable w satisfying certain reality and inversion properties. It turns out that the SVHPLs we just defined are particularly well-suited to describe these multi-Regge limits. This description will be the topic of the rest of this paper.

3.3 Explicit construction

The explicit construction of the functions $\mathcal{L}_w(z)$ is somewhat involved so we take a brief detour to describe the details. Let X^* be the set of words in the alphabet $\{x_0, x_1\}$, along with the empty word e. Define the Drinfel'd associator $Z(x_0, x_1)$ as the generating series,

$$Z(x_0, x_1) = \sum_{w \in X^*} \zeta(w) w, \tag{3.23}$$

where $\zeta(w) = H_w(1)$ for $w \neq x_1$ and $\zeta(x_1) = 0$. Using the collapsed notation for w, these $\zeta(w)$ are the familiar multiple zeta values.

Next, define an alphabet $\{y_0, y_1\}$ (and a set of words Y^*) and a map $\tilde{}: Y^* \to Y^*$ as the operation that reverses words. The alphabet $\{y_0, y_1\}$ is related to the alphabet $\{x_0, x_1\}$ by the following relations:

$$\begin{aligned} y_0 &= x_0 \\ \tilde{Z}(y_0, y_1) y_1 \tilde{Z}(y_0, y_1)^{-1} &= Z(x_0, x_1)^{-1} x_1 Z(x_0, x_1). \end{aligned} \tag{3.24}$$

The inversion operator is to be understood as a formal series expansion in the weight $|w|$. Solving eq. (3.24) iteratively in the weight yields a series expansion for y_1. The first few terms are,

$$\begin{aligned} y_1 = x_1 - \zeta_3 \,(&2x_0x_0x_1x_1 - 4x_0x_1x_0x_1 + 2x_0x_1x_1x_1 + 4x_1x_0x_1x_0 \\ &-6x_1x_0x_1x_1 - 2x_1x_1x_0x_0 + 6x_1x_1x_0x_1 - 2x_1x_1x_1x_0) + \ldots \end{aligned} \tag{3.25}$$

Letting $\phi : Y^* \to X^*$ be the map that renames y to x, i.e. $\phi(y_0) = x_0$ and $\phi(y_1) = x_1$, define the generating functions

$$L_X(z) \;=\; \sum_{w\in X^*} H_w(z) w\,, \qquad \tilde{L}_Y(\bar{z}) \;=\; \sum_{w\in Y^*} H_{\phi(w)}(\bar{z})\tilde{w}\,. \tag{3.26}$$

In the following, we use a condensed notation for the HPL arguments, in order to improve the readability of explicit formulas:

$$H_w \equiv H_w(z) \;\text{ and }\; \overline{H}_w \equiv H_w(\bar{z})\,. \tag{3.27}$$

Then we can write

$$\begin{aligned} L_X(z) \;&= 1 + H_0 x_0 + H_1 x_1 \\ &\qquad + \; H_{0,0} x_0 x_0 + H_{0,1} x_0 x_1 + H_{1,0} x_1 x_0 + H_{1,1} x_1 x_1 \\ &\qquad + \; H_{0,0,0} x_0 x_0 x_0 + H_{0,0,1} x_0 x_0 x_1 + H_{0,1,0} x_0 x_1 x_0 + H_{0,1,1} x_0 x_1 x_1 \\ &\qquad + \; H_{1,0,0} x_1 x_0 x_0 + H_{1,0,1} x_1 x_0 x_1 + H_{1,1,0} x_1 x_1 x_0 + H_{1,1,1} x_1 x_1 x_1 \;+\; \ldots\,, \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} \tilde{L}_Y(\bar{z}) \;&= 1 + \overline{H}_0 y_0 + \overline{H}_1 y_1 \\ &\qquad + \; \overline{H}_{0,0} y_0 y_0 + \overline{H}_{0,1} y_1 y_0 + \overline{H}_{1,0} y_0 y_1 + \overline{H}_{1,1} y_1 y_1 \\ &\qquad + \; \overline{H}_{0,0,0} y_0 y_0 y_0 + \overline{H}_{0,0,1} y_1 y_0 y_0 + \overline{H}_{0,1,0} y_0 y_1 y_0 + \overline{H}_{0,1,1} y_1 y_1 y_0 \\ &\qquad + \; \overline{H}_{1,0,0} y_0 y_0 y_1 + \overline{H}_{1,0,1} y_1 y_0 y_1 + \overline{H}_{1,1,0} y_0 y_1 y_1 + \overline{H}_{1,1,1} y_1 y_1 y_1 \;+\; \ldots \\ &= 1 + \overline{H}_0 x_0 + \overline{H}_1 x_1 \\ &\qquad + \; \overline{H}_{0,0} x_0 x_0 + \overline{H}_{0,1} x_1 x_0 + \overline{H}_{1,0} x_0 x_1 + \overline{H}_{1,1} x_1 x_1 \\ &\qquad + \; \overline{H}_{0,0,0} x_0 x_0 x_0 + \overline{H}_{0,0,1} x_1 x_0 x_0 + \overline{H}_{0,1,0} x_0 x_1 x_0 + \overline{H}_{0,1,1} x_1 x_1 x_0 \\ &\qquad + \; \overline{H}_{1,0,0} x_0 x_0 x_1 + \overline{H}_{1,0,1} x_1 x_0 x_1 + \overline{H}_{1,1,0} x_0 x_1 x_1 + \overline{H}_{1,1,1} x_1 x_1 x_1 \;+\; \ldots\;. \end{aligned} \tag{3.29}$$

In the last step of eq. (3.29) we used $y_0 = x_0$ and $y_1 = x_1$. Note that the latter only holds through weight three, as is clear from eq. (3.25). Finally, we are able to construct the SVHPLs as a generating series,

$$\mathcal{L}(z) = L_X(z)\tilde{L}_Y(\bar{z}) \equiv \sum_{w\in X^*} \mathcal{L}_w(z) w. \tag{3.30}$$

Indeed, taking the product of eq. (3.28) with eq. (3.29) and keeping terms through weight three, we obtain,

$$\begin{aligned} \sum_{w\in X^*} \mathcal{L}_w(z) w \;&=\; 1 + \; \mathcal{L}_0(z)\, x_0 + \mathcal{L}_1(z)\, x_1 \\ &\quad + \mathcal{L}_{0,0}(z)\, x_0 x_0 + \mathcal{L}_{0,1}(z)\, x_0 x_1 + \mathcal{L}_{1,0}(z)\, x_1 x_0 + \mathcal{L}_{1,1}(z)\, x_1 x_1 \\ &\quad + \mathcal{L}_{0,0,0}(z)\, x_0 x_0 x_0 + \mathcal{L}_{0,0,1}(z)\, x_0 x_0 x_1 + \mathcal{L}_{0,1,0}(z)\, x_0 x_1 x_0 + \mathcal{L}_{0,1,1}(z)\, x_0 x_1 x_1 \\ &\quad + \mathcal{L}_{1,0,0}(z)\, x_1 x_0 x_0 + \mathcal{L}_{1,0,1}(z)\, x_1 x_0 x_1 + \mathcal{L}_{1,1,0}(z)\, x_1 x_1 x_0 + \mathcal{L}_{1,1,1}(z)\, x_1 x_1 x_1 + \ldots\;, \end{aligned} \tag{3.31}$$

where the SVHPL's of weight one are,

$$\mathcal{L}_0(z) \;=\; H_0+\overline{H}_0\,, \quad \mathcal{L}_1(z) \;=\; H_1+\overline{H}_1, \tag{3.32}$$

the SVHPL's of weight two are,

$$\begin{aligned}
\mathcal{L}_{0,0}(z) &= H_{0,0}+\overline{H}_{0,0}+H_0\overline{H}_0\,,\\
\mathcal{L}_{0,1}(z) &= H_{0,1}+\overline{H}_{1,0}+H_0\overline{H}_1\,,\\
\mathcal{L}_{1,0}(z) &= H_{1,0}+\overline{H}_{0,1}+H_1\overline{H}_0\,,\\
\mathcal{L}_{1,1}(z) &= H_{1,1}+\overline{H}_{1,1}+H_1\overline{H}_1\,,
\end{aligned} \tag{3.33}$$

and the SVHPL's of weight three are,

$$\begin{aligned}
\mathcal{L}_{0,0,0}(z) &= H_{0,0,0}+\overline{H}_{0,0,0}+H_{0,0}\overline{H}_0+H_0\overline{H}_{0,0}\,,\\
\mathcal{L}_{0,0,1}(z) &= H_{0,0,1}+\overline{H}_{1,0,0}+H_{0,0}\overline{H}_1+H_0\overline{H}_{1,0}\,,\\
\mathcal{L}_{0,1,0}(z) &= H_{0,1,0}+\overline{H}_{0,1,0}+H_{0,1}\overline{H}_0+H_0\overline{H}_{0,1}\,,\\
\mathcal{L}_{0,1,1}(z) &= H_{0,1,1}+\overline{H}_{1,1,0}+H_{0,1}\overline{H}_1+H_0\overline{H}_{1,1}\,,\\
\mathcal{L}_{1,0,0}(z) &= H_{1,0,0}+\overline{H}_{0,0,1}+H_{1,0}\overline{H}_0+H_1\overline{H}_{0,0}\,,\\
\mathcal{L}_{1,0,1}(z) &= H_{1,0,1}+\overline{H}_{1,0,1}+H_{1,0}\overline{H}_1+H_1\overline{H}_{1,0}\,,\\
\mathcal{L}_{1,1,0}(z) &= H_{1,1,0}+\overline{H}_{0,1,1}+H_{1,1}\overline{H}_0+H_1\overline{H}_{0,1}\,,\\
\mathcal{L}_{1,1,1}(z) &= H_{1,1,1}+\overline{H}_{1,1,1}+H_{1,1}\overline{H}_1+H_1\overline{H}_{1,1}\,.
\end{aligned} \tag{3.34}$$

The y alphabet differs from the x alphabet starting at weight four. Referring to eq. (3.25), we expect the difference to generate factors of ζ_3. To illustrate this effect, we list here the subset of weight-four SVHPLs with explicit ζ terms:

$$\begin{aligned}
\mathcal{L}_{0,0,1,1}(z) &= H_{0,0,1,1}+\overline{H}_{1,1,0,0}+H_{0,0,1}\overline{H}_1+H_0\overline{H}_{1,1,0}+H_{0,0}\overline{H}_{1,1}-2\zeta_3\,\overline{H}_1\,,\\
\mathcal{L}_{0,1,0,1}(z) &= H_{0,1,0,1}+\overline{H}_{1,0,1,0}+H_{0,1,0}\overline{H}_1+H_0\overline{H}_{1,0,1}+H_{0,1}\overline{H}_{1,0}+4\zeta_3\,\overline{H}_1\,,\\
\mathcal{L}_{0,1,1,1}(z) &= H_{0,1,1,1}+\overline{H}_{1,1,1,0}+H_{0,1,1}\overline{H}_1+H_0\overline{H}_{1,1,1}+H_{0,1}\overline{H}_{1,1}-2\zeta_3\,\overline{H}_1\,,\\
\mathcal{L}_{1,0,1,0}(z) &= H_{1,0,1,0}+\overline{H}_{0,1,0,1}+H_{1,0,1}\overline{H}_0+H_1\overline{H}_{0,1,0}+H_{1,0}\overline{H}_{0,1}-4\zeta_3\,\overline{H}_1\,,\\
\mathcal{L}_{1,0,1,1}(z) &= H_{1,0,1,1}+\overline{H}_{1,1,0,1}+H_{1,0,1}\overline{H}_1+H_1\overline{H}_{1,1,0}+H_{1,0}\overline{H}_{1,1}+6\zeta_3\,\overline{H}_1\,,\\
\mathcal{L}_{1,1,0,0}(z) &= H_{1,1,0,0}+\overline{H}_{0,0,1,1}+H_{1,1,0}\overline{H}_0+H_1\overline{H}_{0,0,1}+H_{1,1}\overline{H}_{0,0}+2\zeta_3\,\overline{H}_1\,,\\
\mathcal{L}_{1,1,0,1}(z) &= H_{1,1,0,1}+\overline{H}_{1,0,1,1}+H_{1,1,0}\overline{H}_1+H_1\overline{H}_{1,0,1}+H_{1,1}\overline{H}_{1,0}-6\zeta_3\,\overline{H}_1\,,\\
\mathcal{L}_{1,1,1,0}(z) &= H_{1,1,1,0}+\overline{H}_{0,1,1,1}+H_{1,1,1}\overline{H}_0+H_1\overline{H}_{0,1,1}+H_{1,1}\overline{H}_{0,1}+2\zeta_3\,\overline{H}_1\,.
\end{aligned} \tag{3.35}$$

Finally, we remark that the generating series $\mathcal{L}(z)$ provides a convenient way to represent the differential equations (3.15). Together with the y alphabet, it also allows us to write down the differential equations in $\bar{z}$,

$$\frac{\partial}{\partial z}\mathcal{L}(z)=\left(\frac{x_0}{z}+\frac{x_1}{1-z}\right)\mathcal{L}(z) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}}\mathcal{L}(z)=\mathcal{L}(z)\left(\frac{y_0}{\bar{z}}+\frac{y_1}{1-\bar{z}}\right). \tag{3.36}$$

These equations will be particularly useful in Section 5 when we study the multi-Regge limit of the ratio function of the six-point NMHV amplitude.

4 The six-point remainder function in LLA and NLLA

In Section 2, we showed that in MRK the remainder function is fully determined by the coefficient functions $g_n^{(\ell)}(w,w^*)$ and $h_n^{(\ell)}(w,w^*)$ in the logarithmic expansion of its real and imaginary part in eq. (2.7). We further argued that these functions are single-valued in the complex w plane, and suggested that they can be computed explicitly by interpreting the ν-integral in eq. (2.9) as a contour integral and summing the residues. In this section, we describe how knowledge about the space of SVHPLs can be used to facilitate this calculation. In particular, we present results for LLA through ten loops and for NLLA through nine loops.

The main integral we consider is eq. (2.9), which we reproduce here for clarity, rewriting the last factor to take into account eqs. (2.13) and (2.18),

$$\begin{aligned} e^{R+i\pi\delta}|_{\rm MRK} \;=\; & \cos\pi\omega_{ab} + i\,\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\,\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{+\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\,|w|^{2i\nu}\,\Phi_{\rm Reg}(\nu,n) \\ & \times\exp\left[-\omega(\nu,n)\left(\log(1-u_1)+i\pi+\frac{1}{2}\log\frac{|w|^2}{|1+w|^4}\right)\right]\,. \end{aligned} \tag{4.1}$$

The integrand depends on the BFKL eigenvalue and impact factor, which are known through order a^2 and are given in eqs. (2.15), (2.16) and (2.17). These functions can be written as rational functions of ν and n, and polygamma functions (ψ and its derivatives) with arguments $1\pm i\nu+|n|/2$. Recalling that the polygamma functions have poles at the non-positive integers, it is easy to see that all poles are found in the complex ν plane at values $\nu=-i(m+\frac{|n|}{2})$, $m\in\mathbb{N}$, $n\in\mathbb{Z}$. When the integral is performed by summing residues, the result will be of the form,

$$\sum_{m,n} a_{m,n}\,w^{m+n}\,w^{*m}\,. \tag{4.2}$$

Because residues of the polygamma functions are rational numbers, and because polygamma functions evaluate to Euler-Zagier sums for positive integers, the coefficients $a_{m,n}$ are combinations of

1. rational functions in m and n,

2. Euler-Zagier sums of the form $Z_{\vec{\imath}}(m)$, $Z_{\vec{\imath}}(n)$ and $Z_{\vec{\imath}}(m+n)$,

3. $\log|w|$, arising from taking residues at multiple poles.

Identifying $(z,\bar{z})\equiv(-w,-w^*)$, and comparing the double sum (4.2) to the formal series expansion of the HPLs around $z=0$, eq. (3.9), we conclude that the double sums will evaluate to linear combinations of terms of the form $H_{w_1}(w)H_{w_2}(w^*)$. Moreover, as discussed above, this combination should be single-valued. Therefore, based on the discussion in Section 3, we expect $g_n^{(\ell)}(w,w^*)$ and $h_n^{(\ell)}(w,w^*)$ to belong to the space spanned by the SVHPLs.

Furthermore, we know that $g_n^{(\ell)}(w,w^*)$ and $h_n^{(\ell)}(w,w^*)$ are invariant under the action of the $\mathbb{Z}_2\times\mathbb{Z}_2$ transformations of eq. (2.8). In terms of SVHPLs, this symmetry is just an (abelian) subgroup of the larger $\mathbb{Z}_2\times S_3$ symmetry, where the $\mathbb{Z}_2$ is complex conjugation and the S_3 action is given in eq. (3.11). As such, we do not expect an arbitrary linear combination of SVHPLs, but only those that are eigenfunctions with eigenvalue $(+,+)$ of the $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry.

Putting everything together, and taking into account that scattering amplitudes in $\mathcal{N}=4$ SYM are expected to have uniform transcendentality, we are led to conjecture that, to all loop orders, $g_n^{(\ell)}(w,w^*)$ and $h_n^{(\ell)}(w,w^*)$ should be expressible as a linear combination of SVHPLs in $(z,\bar z)=(-w,-w^*)$ of uniform transcendental weight, with eigenvalue $(+,+)$ under the $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry. Inspecting eq. (2.7), the weight should be $2\ell-n-1$ for $g_n^{(\ell)}$ and $2\ell-n-2$ for $h_n^{(\ell)}$. Our conjecture allows us to predict *a priori* the set of functions that can appear at a given loop order, and in practice this set turns out to be rather small. Knowledge of this set of functions can be used to facilitate the evaluation of eq. (4.1). We outline two strategies to achieve this:

1. Evaluate the double sum (4.2) with the summation algorithms of ref. [57]. The result is a complicated expression involving multiple polylogarithms which can be matched to a combination of SVHPLs and zeta values by means of the symbol [20, 21, 22, 23, 24] and coproduct [58, 59, 60].

2. The double sum (4.2) should be equal to the formal series expansion of some linear combination of SVHPLs and zeta values. The unknown coefficients of this combination can be fixed by matching the two expressions term by term.

To see how this works, we calculate the two-loop remainder function in MRK. Expanding eq. (4.1) to two loops, we find,

$$\begin{aligned}a^2R^{(2)}\simeq 2\pi i\,\Bigg\{&a\left[-\frac{1}{2}L_1^+ +\frac{1}{4}\mathcal{I}[1]\right]+\\ &a^2\left[\log(1-u_1)\,\frac{1}{4}\,\mathcal{I}[E_{\nu,n}]+\left(\frac{1}{2}\zeta_2L_1^+ +\frac{1}{4}\,\mathcal{I}[\Phi_{\text{Reg}}^{(1)}(\nu,n)]+\frac{1}{4}L_1^+\,\mathcal{I}[E_{\nu,n}]\right)\right.\\ &\left.+2\pi i\left(\frac{1}{32}\,[L_0^-]^2+\frac{1}{8}\,[L_1^+]^2-\frac{1}{8}\,L_1^+\,\mathcal{I}[1]+\frac{1}{8}\,\mathcal{I}[E_{\nu,n}]\right)\right]\Bigg\}\,,\end{aligned}\tag{4.3}$$

where we have introduced the notation,

$$\mathcal{I}[\mathcal{F}(\nu,n)]=\frac{1}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\,\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{+\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\,|w|^{2i\nu}\,\mathcal{F}(\nu,n)\,.\tag{4.4}$$

Explicit expressions for the functions $L_w^\pm$ for low weights are provided in Appendix A. Equation (4.3) is consistent only if the term of order a vanishes. Indeed this is the case,

$$\begin{aligned}
\mathcal{I}[1] &= \frac{1}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{+\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\,|w|^{2i\nu} \\
&= \log|w|^2 + 2\sum_{n=1}^{\infty}\frac{(-w)^n}{n} + 2\sum_{n=1}^{\infty}\frac{(-w^*)^n}{n} \\
&= \log|w|^2 - 2\log|1+w|^2 \\
&= 2L_1^+\,.
\end{aligned} \tag{4.5}$$

As previously mentioned, we only take half of the residue at $\nu = n = 0$.

Moving on to the terms of order a^2, we refer to eq. (2.7) and extract from eq. (4.3) the expressions for the coefficient functions,

$$\begin{aligned}
g_1^{(2)}(w,w^*) &= \frac{1}{4}\,\mathcal{I}[E_{\nu,n}] \\
g_0^{(2)}(w,w^*) &= \frac{1}{2}\zeta_2 L_1^+ + \frac{1}{4}\,\mathcal{I}[\Phi_{\text{Reg}}^{(1)}(\nu,n)] + \frac{1}{4}L_1^+\,\mathcal{I}[E_{\nu,n}] \\
h_0^{(2)}(w,w^*) &= \frac{1}{32}\,[L_0^-]^2 + \frac{1}{8}\,[L_1^+]^2 - \frac{1}{8}\,L_1^+\,\mathcal{I}[1] + \frac{1}{8}\,\mathcal{I}[E_{\nu,n}]\,.
\end{aligned} \tag{4.6}$$

Note that $h_1^{(2)} = 0$, in accordance with the general expectation that $h_{l-1}^{(l)} = 0$. Proceeding onwards, we have to calculate $\mathcal{I}[E_{\nu,n}]$,

$$\begin{aligned}
\mathcal{I}[E_{\nu,n}] &= \frac{1}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{+\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\,|w|^{2i\nu}\left\{2\gamma_E + \frac{|n|}{2(\nu^2+\frac{n^2}{4})}\right. \\
&\qquad\left. + \psi\left(i\nu+\frac{|n|}{2}\right) + \psi\left(-i\nu+\frac{|n|}{2}\right)\right\} \\
&= \sum_{m=1}^{\infty}\left\{2\frac{|w|^{2m}}{m^2} - 2\frac{(-w)^m+(-w^*)^m}{m^2} + [\log|w|^2 + 2Z_1(m)]\frac{(-w)^m+(-w^*)^m}{m}\right\} \\
&\quad + 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^n}{m(m+n)}\,\left\{w^{m+n}w^{*m} + w^m w^{*m+n}\right\}\,.
\end{aligned} \tag{4.7}$$

The single sum in the first line immediately evaluates to polylogarithms,

$$\begin{aligned}
&\sum_{m=1}^{\infty}\left\{2\frac{|w|^{2m}}{m^2} - 2\frac{(-w)^m+(-w^*)^m}{m^2} + [\log|w|^2 + 2Z_1(m)]\frac{(-w)^m+(-w^*)^m}{m}\right\} \\
&\quad = \sum_{m=1}^{\infty}\left\{2\frac{|w|^{2m}}{m^2} + [\log|w|^2 + 2Z_1(m-1)]\frac{(-w)^m+(-w^*)^m}{m}\right\} \\
&\quad = \log|w|^2\,[H_1(-w) + H_1(-w^*)] + 2H_{0,1}(|w|^2) + 2H_{1,1}(-w) + 2H_{1,1}(-w^*)\,.
\end{aligned} \tag{4.8}$$

Next we transform the double sum into a nested sum by shifting the summation variables by $n = N - m$,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^n}{m(m+n)}\left\{w^{m+n}w^{*m}+w^m w^{*m+n}\right\} = \sum_{N=1}^{\infty}\sum_{m=1}^{N-1}\left\{\frac{(-w)^N(-w^*)^m}{N\,m}+\frac{(-w)^m(-w^*)^N}{N\,m}\right\}$$
$$= \mathrm{Li}_{1,1}(-w,-w^*)+\mathrm{Li}_{1,1}(-w^*,-w)$$
$$= H_1(-w)\,H_1(-w^*) - H_{0,1}(|w|^2)\,, \tag{4.9}$$

where the last step follows from a stuffle identity among multiple polylogarithms [61]. Putting everything together, we obtain

$$\begin{aligned}\mathcal{I}[E_{\nu,n}] &= \log|w|^2\left[H_1(-w)+H_1(-w^*)\right]+2H_{1,1}(-w)+2H_{1,1}(-w^*)+2H_1(-w)\,H_1(-w^*)\\ &= [L_1^+]^2-\frac{1}{4}[L_0^-]^2\,.\end{aligned} \tag{4.10}$$

Referring to eqs. (4.5) and (4.6), we can now write down the results,

$$\begin{aligned}g_1^{(2)}(w,w^*) &= \frac{1}{4}\,[L_1^+]^2-\frac{1}{16}\,[L_0^-]^2\,,\\ h_0^{(2)}(w,w^*) &= 0\,.\end{aligned} \tag{4.11}$$

For higher weights the nested double sums can be more complicated, but they are always of a form that can be performed using the algorithms of ref. [57]. These algorithms will in general produce complicated multiple polylogarithms that, unlike in eq. (4.9), cannot in general be reduced to HPLs by the simple application of stuffle identities. In this case we can use symbols [22, 23, 24] and the coproduct on multiple polylogarithms [58, 59, 60] to perform this reduction.

The above strategy becomes computationally taxing for high weights. For this reason, we also employ an alternative strategy, based on matching series expansions, which is computationally simpler. We demonstrate this method in the computation of $g_0^{(2)}$, for which the only missing ingredient in eq. (4.6) is $\mathcal{I}[\Phi^{(1)}_{\rm Reg}(\nu,n)]$, where $\Phi^{(1)}_{\rm Reg}(\nu,n)$ is defined in eq. (2.17). To proceed, we write the ν-integral as a sum of residues, and truncate the resulting double sum to some finite order,

$$\begin{aligned}\mathcal{I}[\Phi^{(1)}_{\rm Reg}(\nu,n)] = \frac{1}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{+\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\,|w|^{2i\nu}\Bigg\{-\zeta_2-\frac{3}{8}\,\frac{n^2}{(\nu^2+\frac{n^2}{4})^2}\\ -\frac{1}{2}\left(2\gamma_E+\frac{|n|}{2(\nu^2+\frac{n^2}{4})}+\psi\left(i\nu+\frac{|n|}{2}\right)+\psi\left(-i\nu+\frac{|n|}{2}\right)\right)^2\Bigg\}\end{aligned} \tag{4.12}$$

$$
\begin{aligned}
&= -\zeta_2 \log|w|^2 - \left(\log|w|^2\right)|w|^2 - \left(1 + \frac{1}{4}\log|w|^2\right)|w|^4 + \ldots \\
&\quad + (w + w^*)\left[2\zeta_2 + \left(4 - 2\log|w|^2 + \frac{1}{2}\log^2|w|^2\right) + \left(1 + \frac{1}{2}\log|w|^2\right)|w|^2 + \ldots\right] \\
&\quad + (w^2 + w^{*2})\left[-\zeta_2 - \left(\frac{1}{2} + \frac{1}{4}\log^2|w|^2\right) + \left(-1 - \frac{1}{3}\log|w|^2\right)|w|^2 + \ldots\right] \\
&\quad + \ldots .
\end{aligned}
$$

Here we show on separate lines the contributions to the sum from $n = 0$, $n = \pm 1$, and $n = \pm 2$. Next, we construct an ansatz of SVHPLs whose series expansion we attempt to match to the above expression. We expect the result to be a weight-three SVHPL with parity $(+,+)$ under conjugation and inversion. Including zeta values, there are five functions satisfying these criteria, and we can write the ansatz as,

$$
\mathcal{I}[\Phi_{\mathrm{Reg}}^{(1)}(\nu, n)] = c_1\, L_3^+ + c_2\, [L_0^-]^2 L_1^+ + c_3\, [L_1^+]^3 + c_4\, \zeta_2\, L_1^+ + c_5\, \zeta_3\,. \tag{4.13}
$$

Using the series expansions of the constituent HPLs (3.9), it is straightforward to produce the series expansion of this ansatz,

$$
\begin{aligned}
\mathcal{I}[\Phi_{\mathrm{Reg}}^{(1)}(\nu, n)] &= \left(\frac{c_1}{12} + \frac{c_2}{2} + \frac{c_3}{8}\right)\log^3|w|^2 + \frac{1}{2}c_4\zeta_2\log|w|^2 + c_5\,\zeta_3 + 3\,c_3\left(\log|w|^2\right)|w|^2 + \ldots \\
&\quad + (w + w^*)\left[-\zeta_2 c_4 + \left(-c_1 + \frac{1}{2}c_1\, \log|w|^2\right) + \left(-\frac{c_1}{4} - c_2 - \frac{3c_3}{4}\right)\log^2|w|^2 + \ldots\right] \\
&\quad + \ldots .
\end{aligned}
\tag{4.14}
$$

We have only listed the terms necessary to fix the undetermined constants. In practice we generate many more terms than necessary to cross-check the result. Consistency of eqs. (4.12) and (4.14) requires,

$$
c_1 = -4, \quad c_2 = \frac{3}{4}, \quad c_3 = -\frac{1}{3}, \quad c_4 = -2, \quad c_5 = 0\,, \tag{4.15}
$$

which gives,

$$
\mathcal{I}[\Phi_{\mathrm{Reg}}^{(1)}(\nu, n)] = -4\, L_3^+ + \frac{3}{4}\, [L_0^-]^2 L_1^+ - \frac{1}{3}\, [L_1^+]^3 - 2\,\zeta_2\, L_1^+\,. \tag{4.16}
$$

Finally, putting everything together in eq. (4.6),

$$
g_0^{(2)}(w, w^*) \; = -L_3^+ + \frac{1}{6}\left[L_1^+\right]^3 + \frac{1}{8}[L_0^-]^2\, L_1^+\,. \tag{4.17}
$$

This completes the two-loop calculation, and we find agreement with [37, 38]. Moving on to three loops, we can proceed in exactly the same way, and we reproduce the LLA [38] and NLLA results [28, 40] for the imaginary parts of the coefficient functions,

$$
\begin{aligned}
g_2^{(3)}(w,w^*) &= -\frac{1}{8}L_3^+ + \frac{1}{12}\left[L_1^+\right]^3 , \\
g_1^{(3)}(w,w^*) &= \frac{1}{8}L_0^-\, L_{2,1}^- - \frac{5}{8}L_1^+\, L_3^+ + \frac{5}{48}[L_1^+]^4 + \frac{1}{16}[L_0^-]^2\,[L_1^+]^2 - \frac{5}{768}[L_0^-]^4 \\
&\quad - \frac{\pi^2}{12}[L_1^+]^2 + \frac{\pi^2}{48}[L_0^-]^2 + \frac{1}{4}\zeta_3\, L_1^+ .
\end{aligned}
\tag{4.18}
$$

(The result for $g_1^{(3)}$ agrees with that in ref. [28] once the constants are fixed to $c = 0$ and $\gamma' = -9/2$ [40].) The real parts are given by,

$$
\begin{aligned}
h_2^{(3)}(w,w^*) &= 0 , \\
h_1^{(3)}(w,w^*) &= -\frac{1}{8}L_3^+ - \frac{1}{24}[L_1^+]^3 + \frac{1}{32}[L_0^-]^2\, L_1^+ ,
\end{aligned}
\tag{4.19}
$$

in agreement with ref. [38]. Using the fact that

$$
L_1^+ = \frac{1}{2}\log\frac{|w|^2}{|1+w|^4} , \tag{4.20}
$$

it is easy to check that $h_1^{(3)}(w,w^*)$ satisfies eq. (2.19) for $\ell = 3$.

It is straightforward to extend these methods to higher loops. We have produced results for all functions with weight less than or equal to 10, which is equivalent to 10 loops in the LLA, and 9 loops in the NLLA. Using the C++ symbolic computation framework GiNaC [62], which allows for the efficient numerical evaluation of HPLs to high precision [63], we can evaluate these functions numerically. Figures 1 and 2 show the functions plotted on the line segment for which $w = w^*$ and $0 < w < 1$. Here we also show the analytical results through six loops. We provide a separate computer-readable text file, compatible with the `Mathematica` package `HPL` [64, 65], which contains all the expressions through weight 10.

Up to six loops, we find,

$$
\begin{aligned}
g_3^{(4)}(w,w^*) &= \frac{1}{48}\,[L_2^-]^2 + \frac{1}{48}\,[L_0^-]^2\,[L_1^+]^2 + \frac{7}{2304}\,[L_0^-]^4 + \frac{1}{48}\,[L_1^+]^4 - \frac{1}{16}\,L_0^-\,L_{2,1}^- \\
&\quad - \frac{5}{48}\,L_1^+\,L_3^+ - \frac{1}{8}\,L_1^+\,\zeta_3 ,
\end{aligned}
\tag{4.21}
$$

$$
\begin{aligned}
g_2^{(4)}(w,w^*) &= \frac{3}{64}\,[L_0^-]^2\,[L_1^+]^3 + \frac{1}{128}\,L_1^+\,[L_0^-]^4 - \frac{3}{32}\,L_3^+\,[L_0^-]^2 + \frac{1}{8}\,[L_0^-]^2\,\zeta_3 \\
&\quad - \frac{1}{8}\,[L_1^+]^2\,\zeta_3 + \frac{3}{80}\,[L_1^+]^5 - \frac{\pi^2}{24}\,[L_1^+]^3 - \frac{1}{16}\,L_0^-\,L_{2,1}^-\,L_1^+ + \frac{13}{16}\,L_5^+ \\
&\quad + \frac{3}{8}\,L_{3,1,1}^+ + \frac{1}{4}\,L_{2,2,1}^+ - \frac{5}{16}\,L_3^+\,[L_1^+]^2 + \frac{\pi^2}{16}\,L_3^+ ,
\end{aligned}
\tag{4.22}
$$

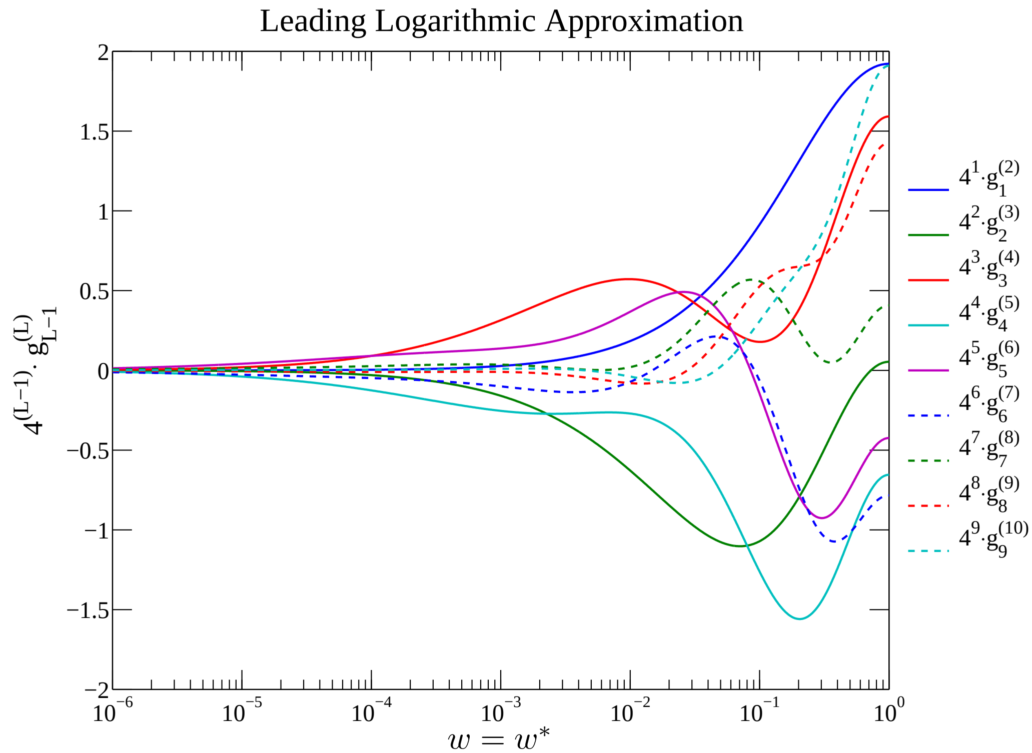

Figure 1: Imaginary parts $g_{\ell-1}^{(\ell)}$ of the MHV remainder function in MRK and LLA through 10 loops, on the line segment with $w = w^*$ running from 0 to 1. The functions have been rescaled by powers of 4 so that they are all roughly the same size.

$$\begin{aligned} g_4^{(5)}(w,w^*) \;=\;& \frac{1}{96}\,[L_0^-]^2\,[L_1^+]^3 + \frac{17}{9216}\,L_1^+\,[L_0^-]^4 - \frac{5}{384}\,L_3^+\,[L_0^-]^2 + \frac{1}{24}\,[L_0^-]^2\,\zeta_3 \\ & - \frac{1}{12}\,[L_1^+]^2\,\zeta_3 + \frac{1}{240}\,[L_1^+]^5 - \frac{1}{24}\,L_0^-\,L_{2,1}^-\,L_1^+ + \frac{43}{384}\,L_5^+ + \frac{1}{8}\,L_{3,1,1}^+ + \frac{1}{12}\,L_{2,2,1}^+ \\ & - \frac{1}{24}\,L_3^+\,[L_1^+]^2\,, \end{aligned} \tag{4.23}$$

$$\begin{aligned} g_3^{(5)}(w,w^*) \;=\;& -\frac{1}{384}\,[L_2^-]^2\,[L_0^-]^2 + \frac{5}{64}\,[L_2^-]^2\,[L_1^+]^2 - \frac{\pi^2}{72}\,[L_2^-]^2 + \frac{1}{384}\,[L_0^-]^4\,[L_1^+]^2 - \frac{7}{48}\,\zeta_3^2 \\ & + \frac{5}{144}\,[L_0^-]^2\,[L_1^+]^4 - \frac{\pi^2}{72}\,[L_0^-]^2\,[L_1^+]^2 - \frac{31}{1152}\,L_{2,1}^-\,[L_0^-]^3 - \frac{11}{384}\,L_1^+\,L_3^+\,[L_0^-]^2 \\ & - \frac{7}{48}\,L_1^+\,[L_0^-]^2\,\zeta_3 + \frac{31}{69120}\,[L_0^-]^6 - \frac{7\pi^2}{3456}\,[L_0^-]^4 + \frac{7}{48}\,[L_{2,1}^-]^2 - \frac{31}{192}\,L_0^-\,L_{2,1}^-\,[L_1^+]^2 \\ & - \frac{65}{576}\,L_3^+\,[L_1^+]^3 - \frac{13}{96}\,[L_1^+]^3\,\zeta_3 + \frac{7}{720}\,[L_1^+]^6 - \frac{\pi^2}{72}\,[L_1^+]^4 + \frac{1}{48}\,[L_3^+]^2 + \frac{5}{96}\,L_4^-\,L_2^- \\ & - \frac{7}{24}\,L_2^-\,L_{2,1,1}^- + \frac{1}{192}\,L_0^-\,L_{4,1}^- + \frac{1}{16}\,L_0^-\,L_{3,2}^- + \frac{\pi^2}{24}\,L_0^-\,L_{2,1}^- + \frac{9}{16}\,L_0^-\,L_{2,1,1,1}^- \\ & + \frac{33}{64}\,L_5^+\,L_1^+ + \frac{5\pi^2}{72}\,L_1^+\,L_3^+ - \frac{7}{48}\,L_1^+\,L_{3,1,1}^+ + \frac{25}{32}\,L_1^+\,\zeta_5 + \frac{\pi^2}{12}\,L_1^+\,\zeta_3 - \frac{5}{32}\,L_3^+\,\zeta_3\,, \end{aligned} \tag{4.24}$$

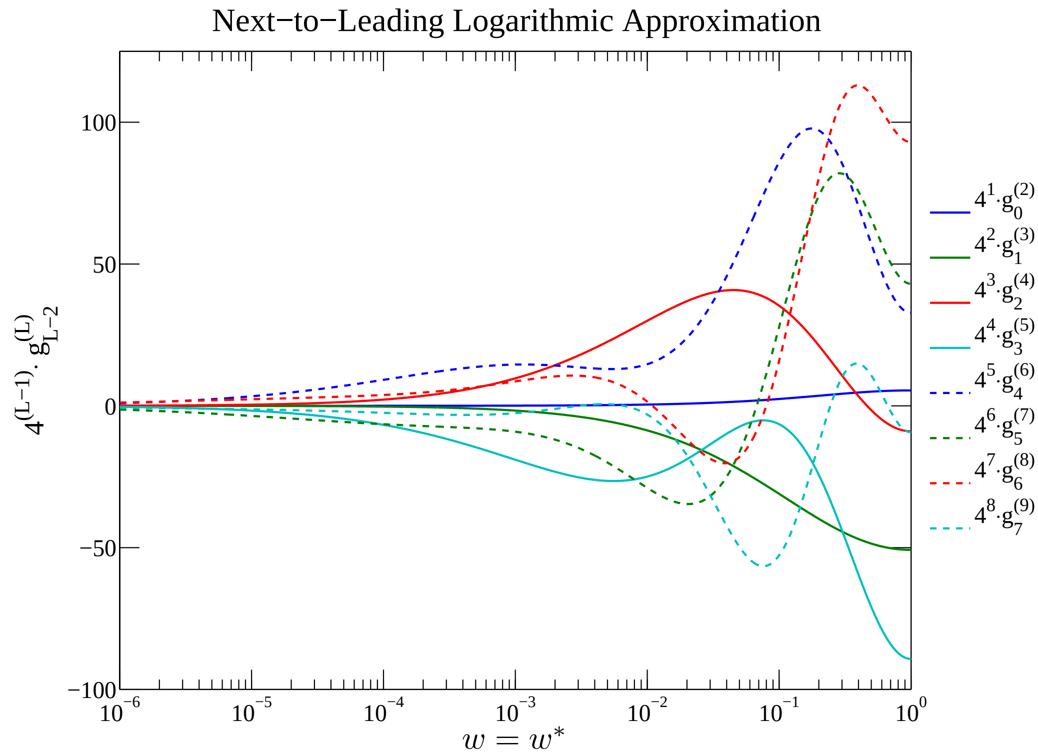

Figure 2: Imaginary parts $g^{(\ell)}_{\ell-2}$ of the MHV remainder function in MRK and NLLA through 9 loops.

$$\begin{aligned}
g_5^{(6)}(w,w^*) \;=\;& \frac{103}{15360}\,[L_2^-]^2\,[L_0^-]^2 - \frac{1}{64}\,[L_2^-]^2\,[L_1^+]^2 + \frac{1}{576}\,[L_0^-]^2\,[L_1^+]^4 + \frac{1}{720}\,[L_0^-]^4\,[L_1^+]^2 \\
&+\frac{29}{9216}\,L_{2,1}^-\,[L_0^-]^3 - \frac{77}{5120}\,L_1^+\,L_3^+\,[L_0^-]^2 + \frac{29}{512}\,L_1^+\,[L_0^-]^2\,\zeta_3 + \frac{73}{1382400}\,[L_0^-]^6 \\
&-\frac{1}{48}\,[L_{2,1}^-]^2 - \frac{1}{192}\,L_0^-\,L_{2,1}^-\,[L_1^+]^2 - \frac{7}{576}\,L_3^+\,[L_1^+]^3 - \frac{1}{32}\,[L_1^+]^3\,\zeta_3 + \frac{1}{1440}\,[L_1^+]^6 \\
&+\frac{43}{3840}\,[L_3^+]^2 - \frac{29}{960}\,L_4^-\,L_2^- + \frac{1}{24}\,L_2^-\,L_{2,1,1}^- - \frac{25}{768}\,L_0^-\,L_{4,1}^- - \frac{3}{128}\,L_0^-\,L_{3,2}^- \\
&-\frac{1}{16}\,L_0^-\,L_{2,1,1,1}^- + \frac{301}{3840}\,L_5^+\,L_1^+ + \frac{7}{48}\,L_1^+\,L_{3,1,1}^+ + \frac{1}{12}\,L_1^+\,L_{2,2,1}^+ - \frac{3}{128}\,L_1^+\,\zeta_5 \\
&+\frac{3}{128}\,L_3^+\,\zeta_3 + \frac{1}{48}\,\zeta_3^2\,,
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
g_4^{(6)}(w,w^*) \;=\;& \frac{5}{1536}\,L_1^+\,[L_2^-]^2\,[L_0^-]^2 + \frac{1}{48}\,[L_2^-]^2\,[L_1^+]^3 - \frac{1}{48}\,[L_2^-]^2\,\zeta_3 - \frac{101}{3072}\,L_3^+\,[L_0^-]^2\,[L_1^+]^2 \\
&+\frac{89}{1536}\,[L_0^-]^2\,[L_1^+]^2\,\zeta_3 + \frac{59}{5760}\,[L_0^-]^2\,[L_1^+]^5 + \frac{85}{18432}\,[L_0^-]^4\,[L_1^+]^3 - \frac{5\pi^2}{576}\,[L_0^-]^2\,[L_1^+]^3 \\
&-\frac{317}{9216}\,L_{2,1}^-\,L_1^+\,[L_0^-]^3 - \frac{43}{768}\,L_5^+\,[L_0^-]^2 + \frac{77}{221184}\,L_1^+\,[L_0^-]^6 - \frac{85\pi^2}{55296}\,L_1^+\,[L_0^-]^4 \\
&+\frac{65}{9216}\,L_3^+\,[L_0^-]^4 + \frac{25\pi^2}{2304}\,L_3^+\,[L_0^-]^2 - \frac{1}{128}\,L_{2,2,1}^+\,[L_0^-]^2 + \frac{1}{768}\,[L_0^-]^4\,\zeta_3
\end{aligned} \tag{4.26}$$

$$
\begin{aligned}
&-\frac{17}{192}\,[L_0^-]^2\,\zeta_5-\frac{5\pi^2}{144}\,[L_0^-]^2\,\zeta_3-\frac{1}{24}\,L_1^+\,[L_{2,1}^-]^2-\frac{3}{64}\,L_0^-\,L_{2,1}^-\,[L_1^+]^3+\frac{205}{768}\,L_5^+\,[L_1^+]^2\\
&-\frac{17}{576}\,L_3^+\,[L_1^+]^4+\frac{5\pi^2}{144}\,L_3^+\,[L_1^+]^2-\frac{1}{48}\,L_{3,1,1}^+\,[L_1^+]^2+\frac{1}{24}\,L_{2,2,1}^+\,[L_1^+]^2-\frac{7}{96}\,[L_1^+]^4\,\zeta_3\\
&+\frac{65}{128}\,[L_1^+]^2\,\zeta_5+\frac{5\pi^2}{72}\,[L_1^+]^2\,\zeta_3+\frac{1}{504}\,[L_1^+]^7-\frac{\pi^2}{288}\,[L_1^+]^5+\frac{7}{192}\,L_1^+\,[L_3^+]^2\\
&-\frac{5}{192}\,L_4^-\,L_2^-\,L_1^++\frac{11}{192}\,L_2^-\,L_0^-\,L_{3,1}^+-\frac{1}{6}\,L_2^-\,L_{2,1,1}^-\,L_1^+-\frac{5}{768}\,L_0^-\,L_{4,1}^-\,L_1^+\\
&-\frac{13}{384}\,L_0^-\,L_{3,2}^-\,L_1^++\frac{5\pi^2}{144}\,L_0^-\,L_{2,1}^-\,L_1^++\frac{23}{384}\,L_0^-\,L_{2,1}^-\,L_3^+-\frac{21}{64}\,L_0^-\,L_{2,1}^-\,\zeta_3\\
&+\frac{3}{16}\,L_0^-\,L_{2,1,1,1}^-\,L_1^+-\frac{215\pi^2}{2304}\,L_5^+-\frac{29}{384}\,L_1^+\,L_3^+\,\zeta_3-\frac{19}{192}\,L_1^+\,\zeta_3^2+\frac{1}{16}\,L_7^+\\
&-\frac{151}{128}\,L_{5,1,1}^+-\frac{3}{32}\,L_{4,1,2}^+-\frac{27}{64}\,L_{4,2,1}^+-\frac{5\pi^2}{48}\,L_{3,1,1}^+-\frac{7}{64}\,L_{3,3,1}^+-\frac{5\pi^2}{72}\,L_{2,2,1}^+\\
&+\frac{13}{4}\,L_{3,1,1,1,1}^++\frac{1}{2}\,L_{2,1,2,1,1}^++\frac{3}{2}\,L_{2,2,1,1,1}^+\,.
\end{aligned}
$$

We present only the imaginary parts, as the real parts are determined by eq. (2.19). However, as a cross-check of our result, we computed the $h_n^{(\ell)}$ explicitly and checked that eq. (2.19) is satisfied. Furthermore, we checked that in the collinear limit $w \to 0$ our results agree with the all-loop prediction for the six-point MHV amplitude in the double-leading-logarithmic (DLL) and next-to-double-leading-logarithmic (NDLL) approximations of ref. [66],

$$
\begin{aligned}
e^{R_{\text{DLLA}}} &= i\pi\,a\,(w+w^*)\,\left[1-I_0\left(2\sqrt{a\log|w|\log(1-u_1)}\right)\right]\,,\\
\mathrm{Re}\left(e^{R_{\text{NDLLA}}}\right) &= \pi^2 a^{3/2}(w+w^*)\,\sqrt{\log|w|}\,\frac{I_1\left(2\sqrt{a\log|w|\log(1-u_1)}\right)}{\sqrt{\log(1-u_1)}}\\
&\quad-\pi^2a^2\,(w+w^*)\,\log|w|\,I_0\left(2\sqrt{a\log|w|\log(1-u_1)}\right)\,,
\end{aligned}
\tag{4.27}
$$

where $I_0(z)$ and $I_1(z)$ denote modified Bessel functions.

Let us conclude this section with an observation: All the results for the six-point remainder function that we computed only involve ordinary ζ values of depth one (ζ_k for some k), despite the fact that multiple ζ values are expected to appear starting from weight eight. In addition, the LLA results only involve odd ζ values – even ζ values never appear.

5 The six-point NMHV amplitude in MRK

So far we have only discussed the multi-Regge limit of the six-point amplitude in an MHV helicity configuration. In this section we extend the discussion to the second independent helicity configuration for six points, the NMHV configuration. We will see that the SVHPLs provide the natural function space for describing this case as well.

The NMHV case was recently analyzed in the LLA [43]. It was shown that the two-loop expression agrees with the limit of the analytic formula for the NMHV amplitude for general kinematics [67], and the three-loop result was also obtained. Here we will extend these results to 10 loops.

Due to helicity conservation along the high-energy line, the only difference between the MHV and NMHV configurations is a flip in helicity of one of the lower energy external gluons (labeled by 4 and 5). Instead of the MHV helicity configuration $(++-++-)$, we consider $(++--+-)$. The tree amplitudes for MHV and NMHV become identical in MRK [43]. In this limit, we can define the NMHV remainder function $R_{\rm NMHV}$ in the same way as in the MHV case (1.1),

$$A_6^{\rm NMHV}|_{\rm MRK} = A_6^{\rm BDS} \times \exp(R_{\rm NMHV})\,. \tag{5.1}$$

Recall the LLA version[4] of eq. (2.9):

$$R_{\rm MHV}^{\rm LLA} = i\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n \int_{-\infty}^{+\infty}\frac{d\nu\, w^{i\nu+n/2}\, w^{*i\nu-n/2}}{(i\nu+\frac{n}{2})(-i\nu+\frac{n}{2})}\,(1-u_1)^{a\,E_{\nu,n}}\,. \tag{5.2}$$

At LLA, the effect of changing the impact factor for emitting gluon 4 with positive helicity to the one for a negative-helicity emission is simply to perform the replacement

$$\frac{1}{-i\nu+\frac{n}{2}} \to -\frac{1}{i\nu+\frac{n}{2}} \tag{5.3}$$

in eq. (5.2), obtaining [43]

$$R_{\rm NMHV}^{\rm LLA} \simeq -\frac{ia}{2}\sum_{n=-\infty}^{\infty}(-1)^n \int_{-\infty}^{+\infty}\frac{d\nu\, w^{i\nu+n/2}\, w^{*i\nu-n/2}}{(i\nu+\frac{n}{2})^2}\,(1-u_1)^{a\,E_{\nu,n}}\,. \tag{5.4}$$

The NMHV ratio function is normally defined in terms of as the ratio of NMHV to MHV super-amplitudes $\mathcal{A}$,

$$\mathcal{P}_{\rm NMHV} = \frac{\mathcal{A}_{\rm NMHV}}{\mathcal{A}_{\rm MHV}}\,. \tag{5.5}$$

However, in MRK, because the tree amplitudes become identical, it suffices to consider the ordinary ratio, which in LLA becomes

$$\mathcal{P}_{\rm NMHV}^{\rm LLA} = \frac{A_{\rm NMHV}^{\rm LLA}}{A_{\rm MHV}^{\rm LLA}} = \exp(R_{\rm NMHV}^{\rm LLA} - R_{\rm MHV}^{\rm LLA})\,. \tag{5.6}$$

Thus eq. (5.4), together with eq. (5.2), is sufficient to generate both the remainder function and the ratio function in LLA.

[4]The distinction between R and $\exp(R)$ is irrelevant at LLA, because the LLA has one fewer logarithm than the loop order, so the square of an LL term has two fewer logarithms and is NLL.

Comparing eq. (5.4) to eq. (2.9), we see that in MRK the MHV and NMHV remainder functions are related by

$$R_{\rm NMHV}^{\rm LLA} = \int dw\, \frac{w^*}{w}\, \frac{\partial}{\partial w^*} R_{\rm MHV}^{\rm LLA}\,. \tag{5.7}$$

It is convenient to write this equation slightly differently. First, define a sequence of single-valued functions $f^{(l)}(w,w^*)$ in analogy with eq. (2.7)[5]

$$R_{\rm NMHV}^{\rm LLA} = 2\pi i \sum_{l=2}^{\infty} a^l \log^{l-1}(1-u_1) \left[\frac{1}{1+w^*} f^{(l)}(w,w^*) + \frac{w^*}{1+w^*} f^{(l)}\Big(\frac{1}{w},\frac{1}{w^*}\Big)\right]. \tag{5.8}$$

Then eq. (5.7) can be used to relate $f^{(l)}(w,w^*)$ to $g_{l-1}^{(l)}(w,w^*)$,

$$\int dw\, \frac{w^*}{w}\, \frac{\partial}{\partial w^*} g_{l-1}^{(l)}(w,w^*) = \frac{1}{1+w^*} f^{(l)}(w,w^*) + \frac{w^*}{1+w^*} f^{(l)}\Big(\frac{1}{w},\frac{1}{w^*}\Big)\,. \tag{5.9}$$

In Section 4 we computed the MHV remainder function in the LLA in the multi-Regge limit up to ten loops. Using these results and eq. (5.9), we can immediately obtain NMHV expressions through ten loops as well. Indeed, $g_{l-1}^{(l)}(w,w^*)$ is a sum of SVHPLs, so the differentiation $\frac{\partial}{\partial w^*}$ can be performed with the aid of eq. (3.36). The result is again a sum of SVHPLs with rational coefficients $1/(1+w^*)$ and $w^*/(1+w^*)$. As such, the differential equations (3.36) also uniquely determine the result of the w-integral as a sum of SVHPLs, up to an undetermined function $F(w^*)$. This function can be at most a constant in order to preserve the single-valuedness condition. It turns out that to respect the vanishing of the remainder function in the collinear limit, $F(w^*)$ must actually be zero.

To see how this works, consider the two loop case. From eq. (4.11),

$$g_1^{(2)}(w,w^*) \; = \frac{1}{4}\,[L_1^+]^2 - \frac{1}{16}\,[L_0^-]^2 = \frac{1}{2}\mathcal{L}_{1,1} + \frac{1}{4}\mathcal{L}_{0,1} + \frac{1}{4}\mathcal{L}_{1,0}. \tag{5.10}$$

Recalling that $(w,w^*)=(-z,-\bar{z})$, first use the second eq. (3.36) to take the w^* derivative, which clips off the last index in the SVHPL, with a different prefactor depending on whether it is a '0' or a '1' (and with corrections due to the y alphabet at higher weights):

$$\begin{aligned} w^* \frac{\partial}{\partial w^*} g_1^{(2)}(w,w^*) &= -\frac{1}{2}\left(\frac{w^*}{1+w^*}\right)\mathcal{L}_1 - \frac{1}{4}\left(\frac{w^*}{1+w^*}\right)\mathcal{L}_0 + \frac{1}{4}\mathcal{L}_1 \\ &= \frac{w^*}{1+w^*}\left[-\frac{1}{4}\,\mathcal{L}_1 - \frac{1}{4}\,\mathcal{L}_0\right] + \frac{1}{1+w^*}\left[\frac{1}{4}\,\mathcal{L}_1\right]\,. \end{aligned} \tag{5.11}$$

Next, use the first eq. (3.36) to perform the w-integration. In practice, this amounts to prepending a '0' to the weight vector of each SVHPL,

$$\begin{aligned} \int dw\, \frac{w^*}{w}\, \frac{\partial}{\partial w^*} g_1^{(2)} &= \frac{w^*}{1+w^*}\left[-\frac{1}{4}\,\mathcal{L}_{0,1} - \frac{1}{4}\,\mathcal{L}_{0,0}\right] + \frac{1}{1+w^*}\left[\frac{1}{4}\,\mathcal{L}_{0,1}\right] \\ &= \frac{1}{1+w^*} f^{(2)}(w,w^*) + \frac{w^*}{1+w^*} f^{(2)}\Big(\frac{1}{w},\frac{1}{w^*}\Big)\,, \end{aligned} \tag{5.12}$$

[5] Ref. [43] defines a similar set of functions, f_l, which are related to ours by $f_2 = -\frac{1}{4} f^{(2)}$, $f_3 = \frac{1}{8} f^{(3)}$, etc.

where

$$
\begin{aligned}
f^{(2)}(w,w^*) &= \frac{1}{4}\,\mathcal{L}_{0,1} \\
&= \frac{1}{4}\,L_2 + \frac{1}{8}\,L_0\,L_1 \\
&= -\frac{1}{4}\left(\log|w|^2\log(1+w^*) - \mathrm{Li}_2(-w) + \mathrm{Li}_2(-w^*)\right) .
\end{aligned}
\tag{5.13}
$$

This result agrees with the one presented in ref. [43]. Furthermore, we can check that the inversion property implicit in eq. (5.12) is satisfied,

$$
\begin{aligned}
f^{(2)}\Big(\frac{1}{w},\frac{1}{w^*}\Big) &= -\frac{1}{4}\left[-\log|w|^2\log\left(1+\frac{1}{w^*}\right) - \mathrm{Li}_2\left(-\frac{1}{w}\right) + \mathrm{Li}_2\left(-\frac{1}{w^*}\right)\right] \\
&= -\frac{1}{4}\left[\frac{1}{2}\log^2|w|^2 - \log|w|^2\log(1+w^*) + \mathrm{Li}_2(-w) - \mathrm{Li}_2(-w^*)\right] \\
&= -\frac{1}{4}\,\mathcal{L}_{0,1} - \frac{1}{4}\,\mathcal{L}_{0,0}\,.
\end{aligned}
\tag{5.14}
$$

Moving on to three loops, we start with the MHV LLA term,

$$
\begin{aligned}
g_2^{(3)}(w,w^*) \; &= -\frac{1}{8}L_3^+ + \frac{1}{12}\left[L_1^+\right]^3 \\
&= \frac{1}{16}\mathcal{L}_{0,0,1} + \frac{1}{8}\mathcal{L}_{0,1,0} + \frac{1}{4}\mathcal{L}_{0,1,1} + \frac{1}{16}\mathcal{L}_{1,0,0} + \frac{1}{4}\mathcal{L}_{1,0,1} + \frac{1}{4}\mathcal{L}_{1,1,0} + \frac{1}{2}\mathcal{L}_{1,1,1}\,.
\end{aligned}
\tag{5.15}
$$

As before, we can take derivatives and integrate using eq. (3.36),

$$
\begin{aligned}
\int dw\,\frac{w^*}{w}\,\frac{\partial}{\partial w^*}g_2^{(3)} = \frac{w^*}{1+w^*}&\left[-\frac{1}{16}\,\mathcal{L}_{0,0,0} - \frac{1}{8}\mathcal{L}_{0,0,1} - \frac{3}{16}\mathcal{L}_{0,1,0} - \frac{1}{4}\mathcal{L}_{0,1,1}\right] \\
+ \frac{1}{1+w^*}&\left[\frac{1}{8}\,\mathcal{L}_{0,0,1} + \frac{1}{16}\,\mathcal{L}_{0,1,0} + \frac{1}{4}\,\mathcal{L}_{0,1,1}\right] ,
\end{aligned}
\tag{5.16}
$$

and we find,

$$
\begin{aligned}
f^{(3)}(w,w^*) &= \frac{1}{8}\,\mathcal{L}_{0,0,1} + \frac{1}{16}\,\mathcal{L}_{0,1,0} + \frac{1}{4}\,\mathcal{L}_{0,1,1} \\
&= \frac{1}{4}\,L_{2,1} + \frac{1}{8}\,L_1\,L_2 + \frac{1}{16}\,L_0\,L_2 + \frac{1}{32}\,L_0^2\,L_1 \\
&= \frac{1}{8}\,\Big[-\frac{1}{2}\log^2|w|^2\log(1+w^*) + \log(-w)\Big(\log^2(1+w^*) - \log^2(1+w)\Big) \\
&\qquad + 2\,\zeta_2\log|1+w|^2 + \frac{1}{2}\log|w|^2\,\Big(\mathrm{Li}_2(-w) - \mathrm{Li}_2(-w^*)\Big) \\
&\qquad - 2\log|1+w|^2\,\mathrm{Li}_2(-w) - 2\,\mathrm{Li}_3(1+w) - 2\,\mathrm{Li}_3(1+w^*) + 4\,\zeta_3\,\Big]\,.
\end{aligned}
\tag{5.17}
$$

The last form agrees with the one given in ref. [43], up to the sign of the second term, which we find must be +1 for the function to be single-valued.

Continuing on to higher loops, we find,

$$\begin{aligned}f^{(4)}(w,w^*) =& -\frac{1}{8}\,L_1\,\zeta_3+\frac{1}{4}\,L_{2,1,1}-\frac{1}{8}\,L_{3,1}+\frac{1}{32}\,L_2^2-\frac{1}{32}\,L_4+\frac{1}{8}\,L_1\,L_{2,1}-\frac{1}{16}\,L_1\,L_3 \\ &-\frac{1}{96}\,L_0\,L_1^3+\frac{1}{96}\,L_0^2\,L_2-\frac{1}{192}\,L_0\,L_3+\frac{1}{256}\,L_0^3\,L_1+\frac{3}{128}\,L_0^2\,L_1^2 \\ &+\frac{1}{16}\,L_0\,L_1\,L_2\,,\end{aligned} \tag{5.18}$$

$$\begin{aligned}f^{(5)}(w,w^*) =& -\frac{1}{96}\,L_2\,\zeta_3-\frac{1}{24}\,L_0\,L_1\,\zeta_3+\frac{1}{4}\,L_{2,1,1,1}-\frac{1}{8}\,L_{2,2,1}+\frac{1}{32}\,L_{4,1}+\frac{1}{48}\,L_{3,2} \\ &+\frac{1}{8}\,L_1\,L_{2,1,1}+\frac{1}{16}\,L_0\,L_{2,1,1}-\frac{1}{16}\,L_1\,L_{3,1}+\frac{1}{32}\,L_1\,L_2^2-\frac{1}{64}\,L_1\,L_4-\frac{1}{96}\,L_0^2\,L_{2,1} \\ &-\frac{1}{96}\,L_1^3\,L_2+\frac{1}{192}\,L_0\,L_2^2-\frac{1}{256}\,L_0\,L_4-\frac{1}{384}\,L_0^2\,L_1^3+\frac{1}{1152}\,L_0^3\,L_2-\frac{1}{1536}\,L_0^2\,L_3 \\ &+\frac{5}{768}\,L_0^3\,L_1^2+\frac{5}{18432}\,L_0^4\,L_1-\frac{7}{192}\,L_0\,L_{3,1}+\frac{1}{16}\,L_0\,L_1\,L_{2,1}-\frac{1}{48}\,L_0\,L_1\,L_3 \\ &+\frac{1}{64}\,L_0\,L_1^2\,L_2+\frac{11}{768}\,L_0^2\,L_1\,L_2-\frac{3}{8}\,L_{3,1,1}\,,\end{aligned} \tag{5.19}$$

$$\begin{aligned}f^{(6)}(w,w^*) =& \frac{1}{4}\,L_{2,1,1,1,1}-\frac{1}{8}\,L_{3,1,1,1}+\frac{1}{12}\,L_{3,2,1}-\frac{1}{32}\,L_{2,1}^2+\frac{1}{48}\,L_{5,1}+\frac{1}{288}\,L_2^3+\frac{1}{384}\,L_3^2 \\ &+\frac{1}{768}\,L_6-\frac{1}{768}\,L_{4,2}+\frac{7}{32}\,L_{4,1,1}+\frac{1}{8}\,L_1\,L_{2,1,1,1}-\frac{1}{16}\,L_1\,L_{3,1,1}+\frac{1}{16}\,L_2\,L_{2,1,1} \\ &+\frac{1}{24}\,L_1\,L_{3,2}+\frac{1}{32}\,L_3\,L_{2,1}-\frac{1}{32}\,L_2\,L_{3,1}+\frac{1}{96}\,L_0^2\,L_{2,1,1}-\frac{1}{96}\,L_1^3\,L_{2,1}+\frac{1}{96}\,L_1^3\,\zeta_3 \\ &-\frac{1}{128}\,L_1^2\,L_2^2-\frac{1}{192}\,L_0\,L_{3,1,1}-\frac{1}{192}\,L_1\,\zeta_5+\frac{1}{192}\,L_1^3\,L_3-\frac{1}{256}\,L_0^2\,L_1^4+\frac{1}{384}\,L_3\,\zeta_3 \\ &-\frac{1}{512}\,L_0\,L_{3,2}-\frac{1}{768}\,L_0\,L_{4,1}+\frac{1}{960}\,L_0\,L_1^5-\frac{1}{2560}\,L_0^2\,L_4+\frac{1}{7680}\,L_0\,L_5 \\ &-\frac{1}{18432}\,L_0^3\,L_3+\frac{1}{73728}\,L_0^5\,L_1+\frac{5}{96}\,L_{2,1}\,\zeta_3+\frac{5}{384}\,L_1\,L_5+\frac{5}{2048}\,L_0^2\,L_2^2 \\ &+\frac{5}{4096}\,L_0^4\,L_1^2+\frac{7}{64}\,L_1\,L_{4,1}+\frac{7}{1536}\,L_0^3\,L_1^3-\frac{11}{1536}\,L_0^2\,L_{3,1}-\frac{11}{1536}\,L_2\,L_4 \\ &+\frac{11}{184320}\,L_0^4\,L_2-\frac{19}{9216}\,L_0^3\,L_{2,1}+\frac{1}{16}\,L_0\,L_1\,L_{2,1,1}-\frac{1}{24}\,L_1\,L_2\,\zeta_3 \\ &-\frac{1}{32}\,L_0\,L_1\,L_{3,1}+\frac{1}{32}\,L_0\,L_1^2\,L_{2,1}-\frac{1}{48}\,L_0\,L_{2,1}\,L_2-\frac{1}{48}\,L_1\,L_3\,L_2 \\ &+\frac{1}{96}\,L_0^2\,L_1^2\,L_2-\frac{1}{192}\,L_0\,L_1^3\,L_2+\frac{1}{384}\,L_0\,L_1\,L_2^2-\frac{3}{256}\,L_0^2\,L_1\,L_{2,1} \\ &-\frac{3}{512}\,L_0^2\,L_1\,\zeta_3-\frac{5}{96}\,L_0\,L_1^2\,\zeta_3-\frac{5}{768}\,L_0\,L_2\,\zeta_3-\frac{11}{1536}\,L_0\,L_1\,L_4 \\ &-\frac{11}{2048}\,L_0^2\,L_1\,L_3-\frac{19}{768}\,L_0\,L_1^2\,L_3+\frac{49}{18432}\,L_0^3\,L_1\,L_2\,.\end{aligned} \tag{5.20}$$

The remaining expressions through 10 loops can be found in computer-readable format in a separate file attached to this article.

6 Single-valued HPLs and Fourier-Mellin transforms

6.1 The multi-Regge limit in (ν, n) space

So far we have only used the machinery of SVHPLs in order to obtain compact analytic expressions for the six-point MHV amplitude in the LL and NLL approximation. However, this was only possible because we knew *a priori* the BFKL eigenvalues and the impact factor to the desired order in perturbation theory. Going beyond NLLA requires higher-order corrections to the BFKL eigenvalues and the impact factor which, by the same logic, can be computed if the corresponding amplitude is known. In other words, if we are given the functions $g_n^{(\ell)}(w, w^*)$ up to some loop order, we can use them to extract the corresponding impact factors and BFKL eigenvalues by transforming the expression from (w, w^*) space back to (ν, n) space. The impact factors and BFKL eigenvalues obtained in this way can then be used to compute the six-point amplitude to any loop order for a given logarithmic accuracy.

In ref. [28] the three-loop six point amplitude was computed up to next-to-next-to-leading logarithmic accuracy (NNLLA),

$$\begin{aligned}
g_0^{(3)}(w,w^*) &= \frac{27}{8}\,L_5^+ + \frac{3}{4}\,L_{3,1,1}^+ - \frac{1}{2}\,L_3^+\,[L_1^+]^2 - \frac{15}{32}\,L_3^+\,[L_0^-]^2 - \frac{1}{8}\,L_1^+\,L_{2,1}^-\,L_0^- \\
&+ \frac{3}{32}\,[L_0^-]^2\,[L_1^+]^3 + \frac{19}{384}\,L_1^+\,[L_0^-]^4 + \frac{3}{8}\,[L_1^+]^2\,\zeta_3 - \frac{5}{32}\,[L_0^-]^2\,\zeta_3 + \frac{\pi^2}{96}\,[L_1^+]^3 \\
&- \frac{\pi^2}{384}\,L_1^+\,[L_0^-]^2 - \frac{3}{4}\,\zeta_5 - \frac{\pi^2}{6}\,\gamma''\left\{L_3^+ - \frac{1}{6}[L_1^+]^3 - \frac{1}{8}[L_0^-]^2\,L_1^+\right\} \\
&+ \frac{1}{4}\,d_1\,\zeta_3\left\{[L_1^+]^2 - \frac{1}{4}[L_0^-]^2\right\} - \frac{\pi^2}{3}d_2\,L_1^+\left\{[L_1^+]^2 - \frac{1}{4}[L_0^-]^2\right\} + \frac{1}{30}\,[L_1^+]^5\,, \\
h_0^{(3)}(w,w^*) &= \frac{3}{16}\,L_1^+\,L_3^+ + \frac{1}{16}\,L_{2,1}^-\,L_0^- - \frac{1}{32}\,[L_1^+]^4 - \frac{1}{32}\,[L_0^-]^2\,[L_1^+]^2 \\
&- \frac{5}{1536}\,[L_0^-]^4 + \frac{1}{8}\,L_1^+\,\zeta_3\,,
\end{aligned} \tag{6.1}$$

where d_1, d_2 and γ'' are some undetermined rational numbers. (To obtain eq. (6.1) from ref. [28] one also needs the value for another constant, $\gamma' = -9/2$, or equivalently $\gamma''' = 0$, which was obtained in ref. [40] using the MRK limit at NLLA.)

These functions can be used to extract the NNLLA correction to the impact factor[6]. Indeed, the NNLL impact factor has already been expressed [40] as an integral over the complex w plane,

$$\Phi_{\text{Reg}}^{(2)}(\nu,n) = (-1)^n\,\left(\nu^2 + \frac{n^2}{4}\right)\int \frac{d^2w}{\pi}\,\rho(w,w^*)\,|w|^{-2i\nu-2}\,\left(\frac{w^*}{w}\right)^{\frac{n}{2}}\,, \tag{6.2}$$

[6]In principle we should expect the amplitude to NNLLA to depend on both the NNLLA impact factor and BFKL eigenvalue. The NNLL BFKL eigenvalue however only enters at four loops, see Section 7.2.

where the kernel $\rho(w,w^*)$ is related to the three-loop amplitude in MRK,

$$\begin{aligned}\rho(w,w^*) &= 2\left[g_0^{(3)}(w,w^*)+\log\frac{|1+w|^2}{|w|}\,g_1^{(3)}(w,w^*)+\left(\log^2\frac{|1+w|^2}{|w|}+\pi^2\right)g_2^{(3)}(w,w^*)\right]\\ &\quad+\log\frac{|1+w|^2}{|w|}\left(\zeta_2\,\log^2\frac{|1+w|^2}{|w|}-\frac{11}{2}\,\zeta_4\right).\end{aligned} \tag{6.3}$$

However, no analytic expression for $\Phi^{(2)}_{\mathrm{Reg}}(\nu,n)$ is yet known. Indeed, an explicit evaluation of the integral (6.2) would require a detailed study of the integrand's branch structure, a task which, if feasible in this case, does not seem particularly amenable to generalization.

Here we propose an alternative to evaluating the integral explicitly. The basic idea is to write down an ansatz for the function in (ν,n) space, and then perform the inverse transform to fix the unknown coefficients. The inverse transform is easily performed using the methods outlined in Section 4, so we are left only with the task of writing down a suitable ansatz. To be precise, consider the inverse Fourier-Mellin transform defined in eq. (4.4). Our goal is to find a set of linearly independent functions $\{\mathcal{F}_i\}$ defined in (ν,n) space such that their transforms $\{\mathcal{I}[\mathcal{F}_i]\}$:

1. are combinations of HPLs of uniform weight,
2. are single-valued in the complex w plane,
3. have a definite parity under $\mathbb{Z}_2\times\mathbb{Z}_2$ transformations in (w,w^*) space,
4. span the whole space of SVHPLs.

Through weight six, we find empirically that this problem has a unique solution, the construction of which we present in the remainder of this section. In particular, we will be led to extend the action of the $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry and the notion of uniform transcendentality to (ν,n) space.

6.2 Symmetries in (ν,n) space

Let us start by analyzing the $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry in (ν,n) space. It is easy to see from eq. (4.4) that

$$\begin{aligned}\mathcal{I}[\mathcal{F}(\nu,n)](w^*,w) &= \mathcal{I}[\mathcal{F}(\nu,-n)](w,w^*)\,,\\ \mathcal{I}[\mathcal{F}(\nu,n)]\left(\frac{1}{w},\frac{1}{w^*}\right) &= \mathcal{I}[\mathcal{F}(-\nu,-n)](w,w^*)\,.\end{aligned} \tag{6.4}$$

In other words, the $\mathbb{Z}_2\times\mathbb{Z}_2$ of conjugation and inversion acts on the (ν,n) space via $[n\leftrightarrow -n]$ and $[\nu\leftrightarrow-\nu,\ n\leftrightarrow -n]$, respectively. Hence, in order that the functions in (w,w^*) space have definite parity under conjugation and inversion, $\mathcal{F}(\nu,n)$ should have definite parity under $n\leftrightarrow -n$ and $\nu\leftrightarrow-\nu$. Our experience shows that the n- and ν-symmetries manifest themselves differently: the ν-symmetry appears as an explicit symmetrization or anti-symmetrization, whereas the n-symmetry requires the introduction of an overall factor of $\mathrm{sgn}(n)$. For example, suppose the target

$(w \leftrightarrow w^*, w \leftrightarrow 1/w)$	$(\nu \leftrightarrow -\nu, n \leftrightarrow -n)$	$\mathcal{F}(\nu, n)$
$(+,+)$	$[+,+]$	$1/2\ [f(\nu,\lvert n\rvert) + f(-\nu,\lvert n\rvert)]$
$(+,-)$	$[-,+]$	$1/2\ [f(\nu,\lvert n\rvert) - f(-\nu,\lvert n\rvert)]$
$(-,+)$	$[-,-]$	$1/2\ \mathrm{sgn}(n)\,[f(\nu,\lvert n\rvert) - f(-\nu,\lvert n\rvert)]$
$(-,-)$	$[+,-]$	$1/2\ \mathrm{sgn}(n)\,[f(\nu,\lvert n\rvert) + f(-\nu,\lvert n\rvert)]$

Table 2: Decomposition of functions in (ν, n) space into eigenfunctions of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action. Note the use of brackets rather than parentheses to denote the parity under (ν, n) transformations.

function in (w, w^*) space is odd under conjugation, and even under inversion. This implies that the function in (ν, n) space must be odd under $n \leftrightarrow -n$ and odd under $\nu \leftrightarrow -\nu$. Such a function will decompose as follows,

$$\mathcal{F}(\nu, n) = \frac{1}{2}\ \mathrm{sgn}(n)\,[f(\nu, |n|) - f(-\nu, |n|)]\ , \tag{6.5}$$

for some suitable function f. See Table 2 for the typical decomposition in all four cases. Furthermore, in the cases we have studied so far, the constituents $f(\nu, |n|)$ can always be expressed as sums of products of single-variable functions with arguments $\pm i\nu + |n|/2$,

$$f(\nu, |n|) = \sum_j c_j \prod_k f_{j,k}(\delta_k i\nu + |n|/2), \tag{6.6}$$

where c_j are constants, $\delta_k \in \{+1, -1\}$, and the $f_{j,k}(z)$ are single-variable functions that we now describe.

6.3 General construction

The functional form of $\mathcal{F}_i(\nu, n)$ can be further restricted by demanding that the integral (4.4) evaluate to a combination of HPLs. To see how, consider closing the ν-contour in the lower half plane and summing residues at poles with $\mathrm{Im}(\nu) < 0$. A necessary condition for the result to yield HPLs is that the residues evaluate exclusively to rational functions and generalized harmonic numbers, e.g., the Euler-Zagier sums defined in eq. (3.10). This condition will clearly be satisfied if the $f_{j,k}(z)$ are purely rational functions of z. Less obviously, it is also satisfied by polygamma functions. Indeed, the polygamma functions evaluate to ordinary (depth one) harmonic numbers at integer values,

$$\psi(1+n) = -\gamma_E + Z_1(n) \quad \text{and} \quad \psi^{(k)}(1+n) = (-1)^{k+1} k!\,(\zeta_{k+1} - Z_{k+1}(n))\,, \tag{6.7}$$

where $\psi^{(1)} = \psi'$, $\psi^{(2)} = \psi''$, etc. Referring to eq. (3.9), we see that all HPLs through weight three can be constructed using ordinary harmonic numbers[7].

[7]Harmonic numbers of depth greater than one do appear at weight three; however, after applying the stuffle algebra relations for Euler-Zagier sums, they all can be rewritten in terms of ordinary harmonic numbers of depth one, namely $Z_{1,1}(k-1) = \frac{1}{2} Z_1(k-1)^2 - \frac{1}{2} Z_2(k-1)$.

We therefore expect the $f_{j,k}(z)$ to be rational functions or polygamma functions through weight three. Starting at weight four, however, ordinary harmonic numbers are insufficient to cover all possible HPLs. Indeed, at weight four, the HPL

$$H_{1,2,1}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k} Z_{2,1}(k-1) \tag{6.8}$$

requires a depth-two sum[8], $Z_{2,1}(k-1)$. A meromorphic function that generates $Z_{2,1}(k-1)$ was presented in ref. [50]. It can be written as a Mellin transform,

$$F_4(N) = \mathbf{M}\left[\left(\frac{\mathrm{Li}_2(x)}{1-x}\right)_+\right](N)\,, \quad N \in \mathbb{C}\,, \tag{6.9}$$

where the Mellin transform $\mathbf{M}$ is defined by

$$\mathbf{M}[(f(x))_+](N) \equiv \int_0^1 dx\,(x^N - 1)\, f(x)\,. \tag{6.10}$$

If N is a positive integer, then $F_4(N)$ evaluates to harmonic numbers of depth two,

$$F_4(N) = Z_{2,1}(N) + Z_3(N) - \zeta_2\, Z_1(N)\,, \quad N \in \mathbb{N}\,. \tag{6.11}$$

Going to higher weight, new harmonic sums will be necessary to construct the full space of HPLs, and, correspondingly, new meromorphic functions will be necessary to give rise to those sums. The analysis of refs. [49, 50, 51] uncovers precisely the functions we need[9]. They are summarized in Appendix B. Through weight five, three new functions are necessary: F_4, F_{6a} and F_7.

There is one final special case that deserves attention. Unlike the other SVHPLs, the pure logarithmic functions $[L_0^-]^k$ diverge as $|w| \to 0$. These functions have special behavior in (ν, n) space as well, requiring a Kronecker delta function:

$$\mathcal{I}[\delta_{n,0}/(i\nu)^k] = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{w}{w^*}\right)^{\frac{n}{2}} \int_{-\infty}^{+\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} |w|^{2i\nu} \frac{\delta_{n,0}}{(i\nu)^k} = \frac{[L_0^-]^{k+1}}{(k+1)!}\,. \tag{6.12}$$

Altogether, we find that the following functions of $z = \pm i\nu + |n|/2$ are sufficient to construct all the remaining SVHPLs through weight five:

$$f_{j,k}(z) \in \left\{1, \frac{1}{z}, \psi(1+z), \psi'(1+z), \psi''(1+z), \psi'''(1+z), F_4(z), F_{6a}(z), F_7(z)\right\}\,. \tag{6.13}$$

However, as we will see, not all combinations of elements in the list (6.13) lead to functions of (w, w^*) that are both single-valued and of definite transcendental weight. Instead we will construct a smaller set of *building blocks* that do have this property.

[8] Another depth-two sum appears in $H_{1,1,2}(x) = \sum_{k=1}^{\infty} \frac{x^k}{k} Z_{1,2}(k-1)$ but the two are related by a stuffle identity, $Z_{2,1}(k-1) + Z_{1,2}(k-1) = Z_2(k-1)\, Z_1(k-1) - Z_3(k-1)$.

[9] Actually, in refs. [49, 50, 51] a more general class of functions is defined. It involves generic HPLs that are singular at $x = -1$ as well as at $x = 0$ and 1. As we never encounter these HPLs in our present context, we do not discuss these functions any further.

6.4 Examples

Let us see how to use the elements in the list (6.13) to construct SVHPLs. The simplest case is $f(\nu,|n|)=1$. Referring to Table 2, only two of the four sectors yield non-zero choices for $\mathcal{F}$. One of these, $\mathcal{F}=\mathrm{sgn}(n)$, produces something proportional to $H_1-\overline{H}_1$, which is not single-valued. This leaves $\mathcal{F}=1$, which should produce a function in the $(+,+)$ sector. Closing the ν-contour in the lower half plane, and summing up the residues at $\nu=-i|n|/2$, we obtain the integral of eq. (4.5),

$$\mathcal{I}[1]=2L_1^+\,, \tag{6.14}$$

indeed a function in the $(+,+)$ sector. Including the special case L_0^- from eq. (6.12), this completes the analysis at weight one.

The next simplest element is $1/z$, yielding $f(\nu,|n|)=1/(i\nu+|n|/2)$. It generates two single-valued functions, one in the $(+,-)$ sector and one in the $(-,-)$ sector (using the (w,w^*) labeling in the first column of Table 2). Symmetrizing as indicated in Table 2, the two functions in (ν,n) space are $\mathcal{F}=-V$ and $\mathcal{F}=N/2$, with the useful shorthands

$$\begin{aligned} V &\equiv -\frac{1}{2}\left[\frac{1}{i\nu+\frac{|n|}{2}}-\frac{1}{-i\nu+\frac{|n|}{2}}\right]=\frac{i\nu}{\nu^2+\frac{|n|^2}{4}}\,,\\ N &\equiv \mathrm{sgn}(n)\left[\frac{1}{i\nu+\frac{|n|}{2}}+\frac{1}{-i\nu+\frac{|n|}{2}}\right]=\frac{n}{\nu^2+\frac{|n|^2}{4}}\,. \end{aligned} \tag{6.15}$$

The transforms of these functions yield two of the four SVHPLs of weight two.

$$\begin{aligned} \mathcal{I}[V] &= -L_0^-\,L_1^+\,,\\ \mathcal{I}[N] &= 4\,L_2^-\,. \end{aligned} \tag{6.16}$$

A third weight-two function is the pure logarithmic function $[L_0^-]^2$, a special case already considered. To find the fourth weight-two function, we turn to the next element in the list (6.13), $\psi(1+z)$. On its own, it does not generate any single-valued functions; however, a particular linear combination of $\{1,1/z,\psi(1+z)\}$ indeed produces such a function. Specifically, $f(\nu,|n|)=2\psi(1+i\nu+|n|/2)+2\gamma_E-1/(i\nu+|n|/2)$ generates the last weight-two SVHPL, which transforms in the $(+,+)$ sector. The function in (ν,n) space is actually the leading-order BFKL eigenvalue, $E_{\nu,n}$,

$$\mathcal{F}=\psi\left(1+i\nu+\frac{|n|}{2}\right)+\psi\left(1-i\nu+\frac{|n|}{2}\right)+2\gamma_E-\frac{\mathrm{sgn}(n)N}{2}=E_{\nu,n}\,, \tag{6.17}$$

and its transform is the last SVHPL of weight two,

$$\mathcal{I}[E_{\nu,n}] = [L_1^+]^2-\frac{1}{4}\,[L_0^-]^2\,. \tag{6.18}$$

The next element in the list (6.13) is $\psi'(1+z)$. Like $\psi(1+z)$, $\psi'(1+z)$ does not by itself generate any single-valued functions; however, there is a particular linear combination that does, and it is given by $f(\nu,|n|) = 2\psi'(1+i\nu+|n|/2) + 1/(i\nu+|n|/2)^2$. Notice that, for the first time, the product in eq. (6.6) extends over more than one term (in this case, $f_{1,1} = f_{1,2} = 1/(i\nu+|n|/2)$, but in general the $f_{j,k}$ will be different). The function in (ν, n) space is,

$$\mathcal{F} = \psi'\left(1 + i\nu + \frac{|n|}{2}\right) - \psi'\left(1 - i\nu + \frac{|n|}{2}\right) - \mathrm{sgn}(n) N V = D_\nu E_{\nu,n}\,, \tag{6.19}$$

where $D_\nu \equiv -i\partial_\nu \equiv -i\,\partial/\partial\nu$. The main observation is that the basis in eq. (6.13) can be modified to consistently generate single-valued functions: $1/z$ is replaced by V and N, ψ is replaced by $E_{\nu,n}$, and $\psi^{(k)}$ is replaced by $D_\nu^k E_{\nu,n}$.

Furthermore, as mentioned previously, the basis at weight four requires a new function $F_4(z)$ that is outside the class of polygamma functions. Like the polygamma functions, $F_4(z)$ does not by itself generate a single-valued function; it too requires additional terms. We denote the resulting basis element by $\tilde{F}_4$. It is related to the function $F_4(z)$ in eq. (6.9) by,

$$\begin{aligned} \tilde{F}_4 \, = \mathrm{sgn}(n) \Bigg\{ & F_4\Big(i\nu + \frac{|n|}{2}\Big) + F_4\Big(-i\nu + \frac{|n|}{2}\Big) - \frac{1}{4} D_\nu^2 E_{\nu,n} - \frac{1}{8} N^2 E_{\nu,n} - \frac{1}{2} V^2 E_{\nu,n} \\ & + \frac{1}{2}\Big(\psi_- + V\Big) D_\nu E_{\nu,n} + \zeta_2 E_{\nu,n} - 4\,\zeta_3 \Bigg\} + N\left\{\frac{1}{2} V \psi_- + \frac{1}{2}\zeta_2\right\}, \end{aligned} \tag{6.20}$$

where

$$\psi_- \equiv \psi\Big(1 + i\nu + \frac{|n|}{2}\Big) - \psi\Big(1 - i\nu + \frac{|n|}{2}\Big). \tag{6.21}$$

Appendix B contains further details about the functions in (ν, n) space, including the basis through weight five and expressions for the building blocks $\tilde{F}_{6a}$ and $\tilde{F}_7$ generated by the functions $F_{6a}(z)$ and $F_7(z)$.

Finally, we describe a heuristic method for assembling the basis in (ν, n) space. The idea is to start with the building blocks,

$$\{1, N, V, E_{\nu,n}, \tilde{F}_4, \tilde{F}_{6a}, \tilde{F}_7\}, \tag{6.22}$$

and piece them together with multiplication and ν-differentiation. These two operations do not always produce independent functions. For example,

$$D_\nu N = 2\,NV \quad \text{and} \quad D_\nu V = \frac{1}{4} N^2 + V^2\,. \tag{6.23}$$

The building blocks have definite parity under $\nu \leftrightarrow -\nu$ and $n \leftrightarrow -n$ which helps determine which combinations appear in which sector. Additionally, we observe that they can be assigned a transcendental weight, which further assists in the classification. The weight in (w, w^*) space is found by calculating the total weight of the constituent building blocks in (ν, n) space, and

	weight	$(\nu \leftrightarrow -\nu, n \leftrightarrow -n)$
1	0	$[+,+]$
D_ν	1	$[-,+]$
V	1	$[-,+]$
N	1	$[+,-]$

	weight	$(\nu \leftrightarrow -\nu, n \leftrightarrow -n)$
$E_{\nu,n}$	1	$[+,+]$
$\tilde{F}_4$	3	$[+,-]$
$\tilde{F}_{6a}$	4	$[-,-]$
$\tilde{F}_7$	4	$[-,+]$

Table 3: Properties of the building blocks for the basis in (ν, n) space.

then adding one (to account for the increase in weight due to the integral transform itself). The relevant properties of the basic building blocks are summarized in Table 3.

As an example, let us consider the function $ND_\nu E_{\nu,n}$. Referring to Table 3, the transcendental weight is $1+1+1=3$ in (ν,n) space, or $3+1=4$ in (w,w^*) space. Under $[\nu \leftrightarrow -\nu, n \leftrightarrow -n]$, N has parity $[+,-]$, D_ν has parity $[-,+]$, and $E_{\nu,n}$ has parity $[+,+]$, so $ND_\nu E_{\nu,n}$ has parity $[-,-]$. We therefore expect this function to transform into a weight four function of (w,w^*), with parity $(-,+)$ under $(w \leftrightarrow w^*, w \leftrightarrow 1/w)$ (see Table 2). Indeed this turns out to be the case. A complete basis through weight four is presented in Table 4.

7 Applications in (ν, n) space: the BFKL eigenvalues and impact factor

7.1 The impact factor at NNLLA

In this section we report results for $g_1^{(4)}$ and $g_0^{(4)}$ and discuss how to transform these functions to (ν,n) space using the basis constructed in the previous section. We then give our results for the new data for the MRK logarithmic expansion: $\Phi_{\rm Reg}^{(2)}$, $\Phi_{\rm Reg}^{(3)}$, and $E_{\nu,n}^{(2)}$.

Before discussing the case of the higher-order corrections to the BFKL eigenvalue and the impact factor, let us review how the known results for $E_{\nu,n}$, $E_{\nu,n}^{(1)}$ and $\Phi_{\rm Reg}^{(1)}$ fit into the framework for (ν,n) space that we have developed in the previous section. First, we have already seen in Section 6 that the LL BFKL eigenvalue is one of our basis elements of weight one in (ν,n) space (See Table 3). Next, we know that the first time the NLL impact factor $\Phi_{\rm Reg}^{(1)}$ appears is in the NLLA of the two-loop amplitude, $g_0^{(2)}(w,w^*)$, which is a pure single-valued function of weight three. Following our analysis from the previous section, it should then be possible to express $\Phi_{\rm Reg}^{(1)}$ as a pure function of weight two in (ν,n) space with the correct symmetries. Indeed, we can easily recast eq. (2.17) in terms of the basis elements shown in Table 3,

$$\Phi_{\rm Reg}^{(1)}(\nu,n) = -\frac{1}{2}E_{\nu,n}^2 - \frac{3}{8}N^2 - \zeta_2\,. \tag{7.1}$$

Similarly, the NLL BFKL eigenvalue can be written as a linear combination of weight three of

weight	$\mathbb{Z}_2 \times \mathbb{Z}_2$	(w, w^*) basis	(ν, n) basis	dimension
1	$(+,+)$	L_1^+	1	1
	$(+,-)$	L_0^-	$\delta_{n,0}$	1
	$(-,+)$	$-$	$-$	0
	$(-,-)$	$-$	$-$	0
2	$(+,+)$	$[L_1^+]^2,\ [L_0^-]^2$	$\delta_{n,0}/(i\nu),\ E_{\nu,n}$	2
	$(+,-)$	$L_0^- L_1^+$	V	1
	$(-,+)$	$-$	$-$	0
	$(-,-)$	L_2^-	N	1
3	$(+,+)$	$[L_1^+]^3,\ [L_0^-]^2 L_1^+,\ L_3^+$	$V^2,\ N^2,\ E_{\nu,n}^2$	3
	$(+,-)$	$[L_0^-]^3,\ L_0^- [L_1^+]^2,\ L_{2,1}^-$	$\delta_{n,0}/(i\nu)^2,\ V E_{\nu,n},\ D_\nu E_{\nu,n}$	3
	$(-,+)$	$L_0^- L_2^-$	VN	1
	$(-,-)$	$L_1^+ L_2^-$	$N E_{\nu,n}$	1
4	$(+,+)$	$[L_0^-]^4,\ [L_1^+]^4,\ [L_0^-]^2 [L_1^+]^2,$ $[L_2^-]^2,\ L_0^- L_{2,1}^-,\ L_1^+ L_3^+$	$\delta_{n,0}/(i\nu)^3,\ E_{\nu,n}^3,\ N^2 E_{\nu,n},$ $V^2 E_{\nu,n},\ V D_\nu E_{\nu,n},\ D_\nu^2 E_{\nu,n}$	6
	$(+,-)$	$L_0^- [L_1^+]^3,\ [L_0^-]^3 L_1^+,\ L_0^- L_3^+,\ L_1^+ L_{2,1}^-$	$V^3,\ N^2 V,\ V E_{\nu,n}^2,\ E_{\nu,n} D_\nu E_{\nu,n}$	4
	$(-,+)$	$L_0^- L_1^+ L_2^-,\ L_{3,1}^+$	$N V E_{\nu,n},\ N D_\nu E_{\nu,n}$	2
	$(-,-)$	$[L_0^-]^2 L_2^-,\ [L_1^+]^2 L_2^-,\ L_4^-,\ L_{2,1,1}^-$	$N^3,\ N V^2,\ N E_{\nu,n}^2,\ \tilde{F}_4$	4

Table 4: Basis of SVHPLs in (w, w^*) and (ν, n) space through weight four. Note that at each weight we can also add the product of zeta values with lower-weight entries.

the basis elements in Table 3,

$$E_{\nu,n}^{(1)} = -\frac{1}{4}\, D_\nu^2 E_{\nu,n} + \frac{1}{2}\, V\, D_\nu E_{\nu,n} - \zeta_2\, E_{\nu,n} - 3\,\zeta_3\,. \tag{7.2}$$

This completes the data for the MRK logarithmic expansion that can be extracted through two loops.

Now we proceed to three loops. By expanding eq. (4.1) to order a^3, we obtain the following relation for the NNLLA correction to the impact factor, $\Phi_{\rm Reg}^{(2)}(\nu, n)$,

$$\begin{aligned} \mathcal{I}\left[\Phi_{\rm Reg}^{(2)}(\nu,n)\right] &= 4\, g_2^{(3)}(w,w^*)\, \left\{[L_1^+]^2 + \pi^2\right\} - 4\, g_1^{(3)}(w,w^*)\, L_1^+ + 4\, g_0^{(3)}(w,w^*) \\ &\quad - 4\pi^2 g_1^{(2)}(w,w^*)\, L_1^+ + \frac{\pi^2}{180} L_1^+ \left\{-45\,[L_0^-]^2 + 120\,[L_1^+]^2 + 22\,\pi^2\right\}\,. \end{aligned} \tag{7.3}$$

This expression is exactly $2\,\rho(w, w^*)$, where ρ was given in eq. (6.3) and in ref. [40]. (The factor of two just has to do with our normalization of the Fourier-Mellin transform.)

To invert eq. (7.3) and obtain $\Phi^{(2)}_{\rm Reg}(\nu,n)$, we begin by observing that the right-hand side is a pure function of weight five in (w,w^*) space. Moreover, it is an eigenfunction with eigenvalue $(+,+)$ under the $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry. Following the analysis of Section 6, and using the results at the end of Appendix B, we are led to make the following ansatz,

$$\begin{aligned}\Phi^{(2)}_{\rm Reg}(\nu,n) &= \alpha_1\,E_{\nu,n}^4+\alpha_2\,N^2E_{\nu,n}^2+\alpha_3\,N^4+\alpha_4\,V^2\,E_{\nu,n}^2+\alpha_5\,N^2V^2+\alpha_6\,V^4\\ &\quad+\alpha_7\,E_{\nu,n}\,V\,D_\nu E_{\nu,n}+\alpha_8\,[D_\nu E_{\nu,n}]^2+\alpha_9\,E_{\nu,n}\,D_\nu^2E_{\nu,n}+\alpha_{10}\,\tilde{F}_4\,N\\ &\quad+\alpha_{11}\,\zeta_2E_{\nu,n}^2+\alpha_{12}\,\zeta_2N^2+\alpha_{13}\,\zeta_2V^2+\alpha_{14}\,\zeta_3E_{\nu,n}+\alpha_{15}\,\zeta_3\,[\delta_{n,0}/(i\nu)]+\alpha_{16}\,\zeta_4\,.\end{aligned}\tag{7.4}$$

The α_i are rational numbers that can be determined by computing the integral transform to (w,w^*) space of eq. (7.4) (see Appendix B) and then matching the result to the right-hand side of eq. (7.3). We find

$$\begin{aligned}\Phi^{(2)}_{\rm Reg}(\nu,n) &= \frac{1}{2}\left[\Phi^{(1)}_{\rm Reg}(\nu,n)\right]^2-E^{(1)}_{\nu,n}\,E_{\nu,n}+\frac{1}{8}\,[D_\nu E_{\nu,n}]^2+\frac{5\pi^2}{16}\,E_{\nu,n}^2-\frac{1}{2}\,\zeta_3\,E_{\nu,n}+\frac{5}{64}\,N^4\\ &\quad+\frac{5}{16}\,N^2\,V^2-\frac{5\pi^2}{64}\,N^2-\frac{\pi^2}{4}\,V^2+\frac{17\pi^4}{360}+d_1\,\zeta_3\,E_{\nu,n}-d_2\,\frac{\pi^2}{6}\,\left[12\,E_{\nu,n}^2+N^2\right]\\ &\quad+\gamma''\,\frac{\pi^2}{6}\,\left[E_{\nu,n}^2-\frac{1}{4}\,N^2\right]\,.\end{aligned}\tag{7.5}$$

Here d_1, d_2 and γ'' are the (not yet determined) rational numbers that appear in eq. (6.1). We emphasize that the expression for $\Phi^{(2)}_{\rm Reg}(\nu,n)$ does not involve the basis element $N\,\tilde{F}_4$ (see eq. (B.52)). That is, $\Phi^{(2)}_{\rm Reg}(\nu,n)$ can be written purely in terms of ψ functions (and their derivatives).

To determine the six-point remainder function in MRK to all loop orders in the NNLL approximation, we must apply some additional information beyond $\Phi^{(2)}_{\rm Reg}(\nu,n)$. In particular, at four loops and higher, the second-order correction to the BFKL eigenvalue, $E^{(2)}_{\nu,n}$, is necessary. In the next section, we will show how to use information from the symbol of the four-loop remainder function to determine $E^{(2)}_{\nu,n}$. We will also derive the next correction to the impact factor, $\Phi^{(3)}_{\rm Reg}(\nu,n)$, which enters the N^{3}LL approximation.

7.2 The four-loop remainder function in the multi-Regge limit

In order to compute the next term in the perturbative expansion of the BFKL eigenvalue and the impact factor, we need the analytic expressions for the four-loop six-point remainder function in the multi-Regge limit. In an independent work, the symbol of the four-loop six-point remainder function has been heavily constrained [52]. In ref. [52] the symbol of $R_6^{(4)}$ is written in the form

$$\mathcal{S}(R_6^{(4)})=\sum_{i=1}^{113}\alpha_i\,S_i\,,\tag{7.6}$$

where α_i are undetermined rational numbers. The S_i denote integrable tensors of weight eight satisfying the first- and final-entry conditions mentioned in the introduction, such that:

1. All entries in the symbol are drawn from the set $\{u_i, 1-u_i, y_i\}_{i=1,2,3}$, where the y_i's are defined in eq. (1.4).
2. The symbol is integrable.
3. The tensor is totally symmetric in u_1, u_2, u_3. Note that under a permutation $u_i \to u_{\sigma(i)}$, $\sigma \in S_3$, the y_i variables transform as $y_i \to 1/y_{\sigma(i)}$.
4. The tensor is invariant under the transformation $y_i \to 1/y_i$.
5. The tensor vanishes in all simple collinear limits.
6. The tensor is in agreement with the prediction coming from the collinear OPE of ref. [25]. We implement this condition on the leading singularity exactly as was done at three loops [28].

In Section 4, we presented analytic expressions for the four-loop remainder function in the LLA and NLLA of MRK. We can use these results to obtain further constraints on the free coefficients α_i appearing in eq. (7.6). In order to achieve this, we first have to understand how to write the symbol (7.6) in MRK. In the following we give very brief account of this procedure.

To begin, recall that the remainder function is non-zero in MRK only after performing the analytic continuation (2.6), $u_1 \to e^{-2\pi i}\,|u_1|$. The function can then be expanded as in eq. (2.7),

$$R_6^{(4)}|_{\mathrm{MRK}} = 2\pi i \sum_{n=0}^{3} \log^n(1-u_1)\,\left[g_n^{(4)}(w,w^*) + 2\pi i\, h_n^{(4)}(w,w^*)\right] . \tag{7.7}$$

The symbols of the imaginary and real parts can be extracted by taking single and double discontinuities,

$$\begin{aligned} 2\pi i \sum_{n=0}^{3} \mathcal{S}\left[\log^n(1-u_1)\, g_n^{(4)}(w,w^*)\right] &= \mathcal{S}(\Delta_{u_1} R_6^{(4)})|_{\mathrm{MRK}} \\ &= -2\pi i \sum_{i=1}^{113} \alpha_i\, \Delta_{u_1}(S_i)|_{\mathrm{MRK}} \\ (2\pi i)^2 \sum_{n=0}^{3} \mathcal{S}\left[\log^n(1-u_1)\, h_n^{(4)}(w,w^*)\right] &= \mathcal{S}(\Delta_{u_1}^2 R_6^{(4)})|_{\mathrm{MRK}} \\ &= (-2\pi i)^2 \sum_{i=1}^{113} \alpha_i\, \Delta_{u_1}^2(S_i)|_{\mathrm{MRK}} , \end{aligned} \tag{7.8}$$

where the discontinuity operator Δ acts on symbols via,

$$\Delta_{u_1}(a_1 \otimes a_2 \otimes \ldots \otimes a_n) = \begin{cases} a_2 \otimes \ldots \otimes a_n , & \text{if } a_1 = u_1 , \\ 0 , & \text{otherwise.} \end{cases} \tag{7.9}$$

$$\Delta_{u_1}^2(a_1 \otimes a_2 \otimes \ldots \otimes a_n) = \begin{cases} \frac{1}{2}\,(a_3 \otimes \ldots \otimes a_n) , & \text{if } a_1 = a_2 = u_1 , \\ 0 , & \text{otherwise.} \end{cases} \tag{7.10}$$

As indicated in eq. (7.8), we need to evaluate the symbols S_i in MRK, which we do by taking the multi-Regge limit of each entry of the symbol. This can be achieved by replacing u_2 and u_3 by the variables x and y, defined in eq. (2.3) (which we then write in terms of w and w^* using eq. (2.4)), while the y_i's are replaced by their limits in MRK [28],

$$y_1 \to 1\,, \qquad y_2 \to \frac{1+w^*}{1+w}\,, \qquad y_3 \to \frac{w^*(1+w)}{w(1+w^*)}\,. \tag{7.11}$$

Finally, we drop all terms in $\Delta^k_{u_1}(S_i)$, $k=1,2$, that have an entry corresponding to u_1, y_1, $1-u_2$ or $1-u_3$, since these quantities approach unity in MRK. In the end, the resulting tensors have entries drawn from the set $\{1-u_1, w, w^*, 1+w, 1+w^*\}$. The $1-u_1$ entries come from factors of $\log(1-u_1)$ and can be shuffled out, so that we can write eq. (7.8) as,

$$\begin{aligned}
\sum_{n=0}^{3} \mathcal{S}\left[\log^n(1-u_1)\right] ⧢ \mathcal{S}\left[g_n^{(4)}(w,w^*))\right] &= \sum_{i=1}^{113}\sum_{n=0}^{7} \alpha_i\, \mathcal{S}\left[\log^n(1-u_1)\right] ⧢ G_{i,n}\\
\sum_{n=0}^{3} \mathcal{S}\left[\log^n(1-u_1)\right] ⧢ \mathcal{S}\left[h_n^{(4)}(w,w^*))\right] &= \sum_{i=1}^{113}\sum_{n=0}^{6} \alpha_i\, \mathcal{S}\left[\log^n(1-u_1)\right] ⧢ H_{i,n}\,,
\end{aligned} \tag{7.12}$$

for some suitable tensors $G_{i,n}$ of weight $(7-n)$ and $H_{i,n}$ of weight $(6-n)$. The sums on the right-hand side of eq. (7.12) turn out to extend past $n=3$. Because the sums on the left-hand side do not, we immediately obtain homogeneous constraints on the α_i for the cases $n=4,5,6,7$. Furthermore, since the quantities on the left-hand side of eq. (7.12) are known for $n=3$ and $n=2$, we can use this information to further constrain the α_i. Finally, there is a consistency condition which relates the real and imaginary parts,

$$\begin{aligned}
h_1^{(4)}(w,w^*) &= g_2^{(4)}(w,w^*) + \frac{\pi^2}{12}\, g_1^{(2)}(w,w^*)\, L_1^+ - \frac{1}{2}\, g_1^{(3)}(w,w^*)\, L_1^+ - g_1^{(2)}(w,w^*)\, g_0^{(2)}(w,w^*)\,,\\
h_0^{(4)}(w,w^*) &= \frac{1}{2}\, g_1^{(4)}(w,w^*) + \pi^2\, g_3^{(4)}(w,w^*) - \pi^2\, g_2^{(3)}(w,w^*)\, L_1^+ - \frac{1}{2}\, g_0^{(3)}(w,w^*)\, L_1^+\\
&\quad + \frac{\pi^2}{2}\, g_1^{(2)}(w,w^*)\, [L_1^+]^2 + \frac{\pi^2}{12}\, g_0^{(2)}(w,w^*)\, L_1^+ + \frac{\pi^2}{64}\, [L_0^-]^2\, [L_1^+]^2 - \frac{\pi^2}{1536}\, [L_0^-]^4\\
&\quad + \frac{3}{640}\pi^4\, [L_0^-]^2 - \frac{5}{96}\pi^2\, [L_1^+]^4 - \frac{3}{160}\pi^4\, [L_1^+]^2 - \frac{1}{2}\, [g_0^{(2)}(w,w^*)]^2\,.
\end{aligned} \tag{7.13}$$

In total, these constraints allow us to fix all of the coefficients α_i that survive in the multi-Regge limit, except for a single parameter which we will refer to as a_0.

The results of the above analysis are expressions for the symbols of the functions $g_1^{(4)}$ and $g_0^{(4)}$. We would like to use this information to calculate new terms in the perturbative expansions of the BFKL eigenvalue $\omega(\nu,n)$ and the MHV impact facator $\Phi_{\rm Reg}(\nu,n)$. For this purpose, we actually need the functions $g_1^{(4)}$ and $g_0^{(4)}$, and not just their symbols. Thankfully, using our knowledge of the space of SVHPLs, it is easy to integrate these symbols. We can constrain the beyond-the-symbol ambiguities by demanding that the function vanish in the collinear limit $(w,w^*)\to 0$, and that it be invariant under conjugation and inversion of the w variables. Putting everything together, we find the following expressions for $g_1^{(4)}$ and $g_0^{(4)}$,

$$
\begin{aligned}
g_1^{(4)}(w,w^*) \;=\;& \frac{3}{128}\,[L_2^-]^2\,[L_0^-]^2 - \frac{3}{32}\,[L_2^-]^2\,[L_1^+]^2 + \frac{19}{384}\,[L_0^-]^2\,[L_1^+]^4 + \frac{73}{1536}\,[L_0^-]^4\,[L_1^+]^2 \\
&+ \frac{1}{96}\,L_{2,1}^-\,[L_0^-]^3 - \frac{29}{64}\,L_1^+\,L_3^+\,[L_0^-]^2 - \frac{11}{30720}\,[L_0^-]^6 - \frac{1}{8}\,[L_{2,1}^-]^2 - \frac{17}{48}\,L_3^+\,[L_1^+]^3 \\
&+ \frac{23}{12}\,[L_1^+]^3\,\zeta_3 + \frac{11}{480}\,[L_1^+]^6 + \frac{5}{32}\,[L_3^+]^2 - \frac{1}{4}\,L_4^-\,L_2^- + \frac{1}{4}\,L_2^-\,L_{2,1,1}^- + \frac{1}{4}\,L_0^-\,L_{4,1}^- \\
&- \frac{3}{4}\,L_0^-\,L_{2,1,1,1}^- + \frac{19}{8}\,L_5^+\,L_1^+ + \frac{5}{4}\,L_1^+\,L_{3,1,1}^+ + \frac{1}{2}\,L_1^+\,L_{2,2,1}^+ - \frac{3}{2}\,L_1^+\,\zeta_5 + \frac{1}{8}\,\zeta_3^2 \\
&+ a_0\Bigg\{\frac{1027}{2}\,[L_2^-]^2\,[L_0^-]^2 + \frac{417}{8}\,[L_0^-]^2\,[L_1^+]^4 + \frac{431}{24}\,[L_0^-]^4\,[L_1^+]^2 + \frac{3155}{48}\,L_{2,1}^-\,[L_0^-]^3 \\
&\qquad - \frac{1581}{16}\,L_1^+\,L_3^+\,[L_0^-]^2 + \frac{9823}{1152}\,[L_0^-]^6 - \frac{871}{4}\,L_0^-\,L_{2,1}^-\,[L_1^+]^2 - \frac{709}{4}\,L_3^+\,[L_1^+]^3 \\
&\qquad + \frac{2223}{2}\,L_5^+\,L_1^+ - 157\,[L_2^-]^2\,[L_1^+]^2 - 256\,[L_{2,1}^-]^2 + 1593\,[L_1^+]^3\,\zeta_3 \\
&\qquad + 681\,[L_3^+]^2 - 1606\,L_4^-\,L_2^- + 512\,L_2^-\,L_{2,1,1}^- - 3371\,L_0^-\,L_{4,1}^- \\
&\qquad - 1730\,L_0^-\,L_{3,2}^- - 299\,L_0^-\,L_{2,1,1,1}^- + 2127\,L_1^+\,L_{3,1,1}^+ + 744\,L_1^+\,L_{2,2,1}^+ \\
&\qquad + 5489\,L_1^+\,\zeta_5 + 256\,\zeta_3^2\Bigg\} + a_1\,\pi^2\,g_1^{(3)}(w,w^*) + a_2\,\pi^2\,g_3^{(4)}(w,w^*) \\
&+ a_3\,\pi^2\,[g_1^{(2)}(w,w^*)]^2 + a_4\,\pi^2\,h_2^{(4)}(w,w^*) + a_5\,\pi^2\,h_0^{(3)}(w,w^*) \\
&+ a_6\,\pi^4\,g_1^{(2)}(w,w^*) + a_7\,\zeta_3\,g_0^{(2)}(w,w^*) + a_8\,\zeta_3\,g_2^{(3)}(w,w^*)\,.
\end{aligned}
\tag{7.14}
$$

$$
\begin{aligned}
g_0^{(4)}(w,w^*) \;=\;& \frac{5}{64}\,L_1^+\,[L_2^-]^2\,[L_0^-]^2 - \frac{1}{16}\,[L_2^-]^2\,[L_1^+]^3 - \frac{21}{64}\,L_3^+\,[L_0^-]^2\,[L_1^+]^2 + \frac{7}{144}\,[L_0^-]^4\,[L_1^+]^3 \\
&+ \frac{9}{320}\,[L_0^-]^2\,[L_1^+]^5 - \frac{7}{192}\,L_{2,1}^-\,L_1^+\,[L_0^-]^3 + \frac{129}{64}\,L_5^+\,[L_0^-]^2 + \frac{1007}{46080}\,L_1^+\,[L_0^-]^6 \\
&- \frac{5}{24}\,L_3^+\,[L_0^-]^4 + \frac{3}{32}\,L_{3,1,1}^+\,[L_0^-]^2 - \frac{1}{16}\,L_{2,2,1}^+\,[L_0^-]^2 + \frac{7}{16}\,[L_0^-]^2\,\zeta_5 \\
&- \frac{1}{16}\,L_0^-\,L_{2,1}^-\,[L_1^+]^3 + \frac{25}{16}\,L_5^+\,[L_1^+]^2 - \frac{7}{48}\,L_3^+\,[L_1^+]^4 + \frac{7}{8}\,L_{3,1,1}^+\,[L_1^+]^2 \\
&+ \frac{25}{12}\,[L_1^+]^4\,\zeta_3 + \frac{1}{210}\,[L_1^+]^7 - \frac{1}{4}\,L_4^-\,L_2^-\,L_1^+ - \frac{5}{16}\,L_2^-\,L_0^-\,L_{3,1}^+ + \frac{1}{4}\,L_2^-\,L_{2,1,1}^-\,L_1^+ \\
&+ \frac{1}{4}\,L_0^-\,L_{4,1}^-\,L_1^+ - \frac{1}{8}\,L_0^-\,L_{2,1}^-\,L_3^+ - \frac{1}{4}\,L_0^-\,L_{2,1,1,1}^-\,L_1^+ + \frac{3}{2}\,L_1^+\,\zeta_3^2 - \frac{125}{8}\,L_7^+ \\
&+ \frac{1}{2}\,L_{4,1,2}^+ + \frac{11}{4}\,L_{4,2,1}^+ + \frac{3}{4}\,L_{3,3,1}^+ - \frac{1}{2}\,L_{2,1,2,1,1}^+ - \frac{3}{2}\,L_{2,2,1,1,1}^+ + \frac{25}{4}\,\zeta_7 + 5\,L_{5,1,1}^+
\end{aligned}
\tag{7.15}
$$

$$
\begin{aligned}
&- 4\, L^{+}_{3,1,1,1,1} + \frac{1}{4}\, L^{+}_{2,2,1}\, [L^{+}_{1}]^2 + a_0 \Bigg\{ - \frac{1309}{4}\, L^{+}_{1}\, [L^{-}_{2}]^2\, [L^{-}_{0}]^2 \\
&\quad - \frac{8535}{4}\, L^{+}_{3}\, [L^{-}_{0}]^2\, [L^{+}_{1}]^2 + \frac{235}{4}\, [L^{-}_{0}]^2\, [L^{+}_{1}]^5 + \frac{4617}{16}\, [L^{-}_{0}]^4\, [L^{+}_{1}]^3 \\
&\quad - \frac{32027}{24}\, L^{-}_{2,1}\, L^{+}_{1}\, [L^{-}_{0}]^3 - \frac{11415}{8}\, L^{+}_{5}\, [L^{-}_{0}]^2 - \frac{310}{9}\, L^{+}_{1}\, [L^{-}_{0}]^6 \\
&\quad + \frac{15225}{64}\, L^{+}_{3}\, [L^{-}_{0}]^4 + \frac{24279}{4}\, L^{+}_{3,1,1}\, [L^{-}_{0}]^2 - \frac{823}{2}\, L^{-}_{0}\, L^{-}_{2,1}\, [L^{+}_{1}]^3 \\
&\quad + \frac{2235}{2}\, L^{+}_{5}\, [L^{+}_{1}]^2 - \frac{365}{4}\, L^{+}_{3}\, [L^{+}_{1}]^4 + 205\, [L^{-}_{2}]^2\, [L^{+}_{1}]^3 + 1911\, L^{+}_{3}\, [L^{-}_{2}]^2 \\
&\quad + 2130\, L^{+}_{2,2,1}\, [L^{-}_{0}]^2 - 2623\, [L^{-}_{0}]^2\, \zeta_5 + 992\, L^{+}_{1}\, [L^{-}_{2,1}]^2 + 63\, L^{+}_{3,1,1}\, [L^{+}_{1}]^2 \\
&\quad - 288\, L^{+}_{2,2,1}\, [L^{+}_{1}]^2 + 2396\, [L^{+}_{1}]^4\, \zeta_3 + 1830\, L^{+}_{1}\, [L^{+}_{3}]^2 - 1612\, L^{-}_{4}\, L^{-}_{2}\, L^{+}_{1} \\
&\quad + 1344\, L^{-}_{2}\, L^{-}_{0}\, L^{+}_{3,1} - 520\, L^{-}_{2}\, L^{-}_{2,1,1}\, L^{+}_{1} + 11839\, L^{-}_{0}\, L^{-}_{4,1}\, L^{+}_{1} \\
&\quad + 4330\, L^{-}_{0}\, L^{-}_{3,2}\, L^{+}_{1} + 3780\, L^{-}_{0}\, L^{-}_{2,1}\, L^{+}_{3} + 562\, L^{-}_{0}\, L^{-}_{2,1,1,1}\, L^{+}_{1} \\
&\quad + 3556\, L^{+}_{1}\, \zeta_3^2 + 2256\, L^{+}_{7} - 164778\, L^{+}_{5,1,1} - 33216\, L^{+}_{4,1,2} - 89088\, L^{+}_{4,2,1} \\
&\quad - 33912\, L^{+}_{3,3,1} - 12048\, L^{+}_{3,2,2} - 17820\, L^{+}_{3,1,1,1,1} - 2928\, L^{+}_{2,1,2,1,1} \\
&\quad - 8784\, L^{+}_{2,2,1,1,1} - 23796\, \zeta_7 \Bigg\} + b_1\, \zeta_2\, [L^{-}_{2}]^2 L^{+}_{1} + b_2\, \zeta_2\, [L^{-}_{0}]^2 L^{+}_{1}\, g^{(2)}_{1}(w,w^*) \\
&+ b_3\, \zeta_2\, g^{(2)}_{1}(w,w^*)\, g^{(3)}_{2}(w,w^*) + b_4\, \zeta_2\, g^{(2)}_{0}(w,w^*)\, g^{(2)}_{1}(w,w^*) + b_5\, \zeta_2\, h^{(4)}_{1}(w,w^*) \\
&+ b_6\, \zeta_2\, h^{(5)}_{3}(w,w^*) + b_7\, \zeta_2\, g^{(3)}_{0}(w,w^*) + b_8\, \zeta_2\, g^{(4)}_{2}(w,w^*) + b_9\, \zeta_2\, g^{(5)}_{4}(w,w^*) \\
&+ b_{10}\, \zeta_3\, h^{(4)}_{2}(w,w^*) + b_{11}\, \zeta_3\, h^{(3)}_{0}(w,w^*) + b_{12}\, \zeta_3\, [g^{(2)}_{1}(w,w^*)]^2 \\
&+ b_{13}\, \zeta_3\, g^{(4)}_{3}(w,w^*) + b_{14}\, \zeta_3\, g^{(3)}_{1}(w,w^*) + b_{15}\, \zeta_4\, g^{(3)}_{2}(w,w^*) + b_{16}\, \zeta_4\, g^{(2)}_{0}(w,w^*) \\
&+ b_{17}\, \zeta_3\, \zeta_2\, g^{(2)}_{1}(w,w^*) + b_{18}\, \zeta_5\, g^{(2)}_{1}(w,w^*)\,.
\end{aligned}
$$

In these expressions, a_i for $i = 0, \ldots, 8$, and b_j for $j = 1, \ldots, 18$, denote undetermined rational numbers. The one symbol-level parameter, a_0, enters both $g_1^{(4)}$ and $g_0^{(4)}$. We observe that a_0 enters these formulae in a complicated way, and that there is no nonzero value of a_0 that simplifies the associated large rational numbers. We therefore suspect that $a_0 = 0$, although we currently have no proof. The remaining parameters account for beyond-the-symbol ambiguities. We will see in the next section that one of these parameters, b_1, is not independent of the others.

7.3 Analytic results for the NNLL correction to the BFKL eigenvalue and the N^{3}LL correction to the impact factor

Having at our disposal analytic expressions for the four-loop remainder function at NNLLA and N^{3}LLA, we use these results to extract the BFKL eigenvalue and the impact factors to the same accuracy in perturbation theory. We proceed as in Section 7.1, i.e., we use our knowledge of the space of SVHPLs and the corresponding functions in (ν, n) space to find a function whose inverse

Fourier-Mellin transform reproduces the four-loop results we have derived.

Let us start with the computation of the BFKL eigenvalue at NNLLA. Expanding eq.(4.1) to order a^4, we can extract the following relation,

$$\begin{aligned}\mathcal{I}\left[E_{\nu,n}^{(2)}\right] &= 12\,\left\{[L_1^+]^2+\pi^2\right\} g_3^{(4)}(w,w^*) - 8\,L_1^+\,g_2^{(4)}(w,w^*) + 4\,g_1^{(4)}(w,w^*)\\ &\quad - 8\,L_1^+\,\pi^2\,g_2^{(3)}(w,w^*) + 2\,\pi^2\,g_1^{(2)}(w,w^*)\,[L_1^+]^2\\ &\quad - \mathcal{I}\left[E_{\nu,n}^{(1)}\,\Phi_{\text{Reg}}^{(1)}(\nu,n)\right] - \mathcal{I}\left[E_{\nu,n}\,\Phi_{\text{Reg}}^{(2)}(\nu,n)\right]\,.\end{aligned} \tag{7.16}$$

The right-hand side of eq. (7.16) is completely known, up to some rational numbers mostly parametrizing our ignorance of beyond-the-symbol terms in the three- and four-loop coefficient functions at NNLLA. It can be written exclusively in terms of SVHPLs of weight six with eigenvalue $(+,+)$ under $\mathbb{Z}_2\times\mathbb{Z}_2$ transformations. The results of Section 6 then allow us to write down an ansatz for the NNLLA correction to the BFKL eigenvalue, similar to the ansatz (7.4) we made for the NNLLA correction to the impact factor, but at higher weight. More precisely, we assume that we can write $E_{\nu,n}^{(2)} = \sum_i \alpha_i\,P_i$, where α_i denote rational numbers and P_i runs through all possible monomials of weight five with the correct symmetry properties that we can construct out of the building blocks given in eq. (6.22), i.e.,

$$P_i \in \left\{E_{\nu,n}^5, \zeta_2\,V\,D_\nu E_{\nu,n}, E_{\nu,n}\,N\,\tilde{F}_4, \zeta_5, \ldots\right\}\,. \tag{7.17}$$

The rational coefficients α_i can then be fixed by inserting our ansatz into eq. (7.16) and performing the inverse Fourier-Mellin transform to (w,w^*) space. We find that there is a unique solution for the α_i, and the result for the NNLLA correction to the BFKL eigenvalue then takes the form,

$$\begin{aligned}E_{\nu,n}^{(2)} &= -E_{\nu,n}^{(1)}\,\Phi_{\text{Reg}}^{(1)}(\nu,n) - E_{\nu,n}\,\Phi_{\text{Reg}}^{(2)}(\nu,n) + \frac{3}{8}\,D_\nu^2 E_{\nu,n}\,E_{\nu,n}^2 + \frac{3}{32}\,N^2\,D_\nu^2 E_{\nu,n} + \frac{1}{8}\,V^2\,D_\nu^2 E_{\nu,n}\\ &\quad - \frac{1}{8}\,V\,D_\nu^3 E_{\nu,n} + \frac{1}{48}\,D_\nu^4 E_{\nu,n} + \frac{\pi^2}{12}\,D_\nu^2 E_{\nu,n} - \frac{3}{4}\,D_\nu E_{\nu,n}\,V\,E_{\nu,n}^2 - \frac{5}{16}\,D_\nu E_{\nu,n}\,N^2\,V\\ &\quad - \frac{\pi^2}{4}\,D_\nu E_{\nu,n}\,V + \frac{1}{8}\,E_{\nu,n}\,[D_\nu E_{\nu,n}]^2 + \frac{3}{16}\,N^2\,E_{\nu,n}^3 + \frac{61}{4}\,E_{\nu,n}^2\,\zeta_3 + \frac{1}{8}\,E_{\nu,n}^5 + \frac{5\pi^2}{6}\,E_{\nu,n}^3\\ &\quad + \frac{19}{128}\,E_{\nu,n}\,N^4 + \frac{5}{16}\,E_{\nu,n}\,N^2\,V^2 + \frac{3\pi^2}{16}\,E_{\nu,n}\,N^2 + \frac{\pi^2}{4}\,E_{\nu,n}\,V^2 + \frac{35}{16}\,N^2\,\zeta_3 + \frac{1}{2}\,V^2\,\zeta_3\\ &\quad + \frac{11\pi^2}{6}\,\zeta_3 + 10\,\zeta_5 + a_0\,\mathcal{E}_5 + \sum_{i=1}^{5} a_i\,\zeta_2\,\mathcal{E}_{3,i} + a_6\,\zeta_4\,\mathcal{E}_2 + \sum_{i=7}^{8} a_i\,\zeta_3\,\mathcal{E}_{1,i}\,,\end{aligned} \tag{7.18}$$

where the quantities $\mathcal{E}_{3,i}$, $\mathcal{E}_2$, and $\mathcal{E}_{1,i}$ capture the beyond-the-symbol ambiguities in $g_1^{(4)}$, and $\mathcal{E}_5$

corresponds to the one symbol-level ambiguity. They are given by,

$$
\begin{aligned}
\mathcal{E}_5 &= \frac{124}{3}\,N^2\,D_\nu^2 E_{\nu,n} + \frac{1210}{3}\,V^2\,D_\nu^2 E_{\nu,n} - \frac{35}{3}\,V\,D_\nu^3 E_{\nu,n} - \frac{31}{6}\,D_\nu^4 E_{\nu,n} - \frac{151}{2}\,D_\nu E_{\nu,n}\,N^2\,V \\
&\quad + \frac{124}{3}\,N^2\,E_{\nu,n}^3 - \frac{140}{3}\,V^2\,E_{\nu,n}^3 - \frac{31}{2}\,E_{\nu,n}\,N^4 + \frac{10903}{12}\,N^2\,\zeta_3 + \frac{13960}{3}\,V^2\,\zeta_3 \\
&\quad - 62\,D_\nu^2 E_{\nu,n}\,E_{\nu,n}^2 + 70\,D_\nu E_{\nu,n}\,V\,E_{\nu,n}^2 - 760\,D_\nu E_{\nu,n}\,V^3 + 248\,E_{\nu,n}\,[D_\nu E_{\nu,n}]^2 \\
&\quad + 7431\,E_{\nu,n}^2\,\zeta_3 - 97\,E_{\nu,n}\,N^2\,V^2 + 16072\,\zeta_5\,,
\end{aligned}
\tag{7.19}
$$

$$\mathcal{E}_{3,1} = -\frac{3}{4}\,E_{\nu,n}\,N^2 - D_\nu^2 E_{\nu,n} + 5\,E_{\nu,n}^3 + 6\,E_{\nu,n}\,V^2 - 2\,E_{\nu,n}\,\pi^2 + 8\,\zeta_3\,, \tag{7.20}$$

$$\mathcal{E}_{3,2} = E_{\nu,n}^3\,, \tag{7.21}$$

$$\mathcal{E}_{3,3} = \frac{3}{4}\,E_{\nu,n}\,N^2 - 3\,D_\nu E_{\nu,n}\,V + 3\,E_{\nu,n}^3 + 12\,\zeta_3\,, \tag{7.22}$$

$$\mathcal{E}_{3,4} = -\frac{1}{8}\,D_\nu^2 E_{\nu,n} + \frac{9}{4}\,D_\nu E_{\nu,n}\,V - \frac{3}{4}\,E_{\nu,n}\,N^2 - \frac{3}{2}\,E_{\nu,n}\,V^2 - \frac{25}{2}\,\zeta_3 - 2\,E_{\nu,n}^3\,, \tag{7.23}$$

$$\mathcal{E}_{3,5} = \frac{3}{8}\,E_{\nu,n}\,N^2 - \frac{3}{2}\,E_{\nu,n}^3\,, \tag{7.24}$$

$$\mathcal{E}_2 = 90\,E_{\nu,n}\,, \tag{7.25}$$

$$\mathcal{E}_{1,7} = E_{\nu,n}^2 - \frac{1}{4}\,N^2\,, \tag{7.26}$$

$$\mathcal{E}_{1,8} = \frac{1}{2}\,E_{\nu,n}^2\,. \tag{7.27}$$

We observe that the most complicated piece is $\mathcal{E}_5$. It would be absent if our conjecture that $a_0 = 0$ is correct. Some further comments are in order about eq. (7.18):

1. In ref. [40] it was argued, based on earlier work [68, 69, 70, 71], that the BFKL eigenvalue should vanish as $(\nu, n) \to 0$ to all orders in perturbation theory, i.e., $\omega(0,0) = 0$. While this statement depends on how one approaches the limit, the most natural way seems to set the discrete variable n to 0 before taking the limit $\nu \to 0$. Indeed in this limit $E_{\nu,n}$ and $E_{\nu,n}^{(1)}$ vanish. However, we find that $E_{\nu,n}^{(2)}$ does not vanish in this limit, but rather it approaches a constant,
$$\lim_{\nu\to 0} E_{\nu,0}^{(2)} = -\frac{1}{2}\pi^2\,\zeta_3\,. \tag{7.28}$$
We stress that the limit is independent of any of the undetermined constants that parameterize the beyond-the-symbol terms in the three- and four-loop coefficients. While we have confidence in our result for $E_{\nu,n}^{(2)}$ given our assumptions (such as the vanishing of $g_n^{(\ell)}$ and $h_n^{(\ell)}$ as $w \to 0$), we have so far no explanation for this observation.

2. While the (ν, n)-space basis constructed in Section 6 involves the new functions $\tilde{F}_4$, $\tilde{F}_{6a}$ and $\tilde{F}_7$, we find that $E_{\nu,n}^{(2)}$ is free of these functions and can be expressed entirely in terms of ψ functions and rational functions of ν and n. Moreover, the ψ functions arise only in the form of the LLA BFKL eigenvalue and its derivative with respect to ν. We are therefore

led to conjecture that, to all loop orders, the BFKL eigenvalue and the impact factor can be expressed as linear combinations of uniform weight of monomials that are even in both ν and n and constructed exclusively out of multiple ζ values[10] and the quantities N, V, $E_{\nu,n}$ and D_ν defined in Section 6.

We now move on and and extract the impact factor at N^{3}LLA from the four-loop amplitude at the same logarithmic accuracy. Equation (4.1) at order a^4 yields the following relation for the impact factor at N^{3}LLA,

$$\begin{aligned}\mathcal{I}\left[\Phi^{(3)}_{\rm Reg}(\nu,n)\right] &= -4\,\left\{[L_1^+]^3+3\,L_1^+\,\pi^2\right\}\,g_3^{(4)}(w,w^*)+4\left\{[L_1^+]^2+\pi^2\right\}\,g_2^{(4)}(w,w^*)\\&-4\,L_1^+\,g_1^{(4)}(w,w^*)+4\,g_0^{(4)}(w,w^*)+8\,\pi^2\,g_2^{(3)}(w,w^*)\,[L_1^+]^2\\&-4\,L_1^+\,\pi^2\,g_1^{(3)}(w,w^*)-2\,\pi^2\left\{[L_1^+]^3-\frac{\pi^2}{3}\,L_1^+\right\}\,g_1^{(2)}(w,w^*)\\&+2\,\pi^2\,g_0^{(2)}(w,w^*)\,[L_1^+]^2+\frac{\pi^4}{8}\,L_1^+\,[L_0^-]^2-\frac{\pi^4}{3}\,[L_1^+]^3-\frac{73\pi^6}{1260}\,L_1^+-2\,L_1^+\,\zeta_3^2\,.\end{aligned}\tag{7.29}$$

In order to determine $\Phi^{(3)}_{\rm Reg}(\nu,n)$, we proceed in the same way as we did for $E^{(2)}_{\nu,n}$, i.e., we write down an ansatz for $\Phi^{(3)}_{\rm Reg}(\nu,n)$ that has the correct transcendentality and symmetry properties and fix the free coefficients by requiring the inverse Fourier-Mellin transform of the ansatz to match the right-hand side of eq. (7.29). Building upon our conjecture that the impact factor can be expressed purely in terms of ψ functions and rational functions of ν and n, we construct a restricted ansatz[11] that is a linear combination just of monomials of ζ values and N, V, D_ν and $E_{\nu,n}$. Just like in the case of $E^{(2)}_{\nu,n}$, we find that there is a unique solution for the coefficients in the ansatz, thus giving further support to our conjecture. Furthermore, we are forced along the way to fix one of the beyond-the-symbol parameters appearing in $g_0^{(4)}$,

$$b_1=-\frac{15}{8}\,a_1-\frac{3}{16}\,a_2-\frac{3}{32}\,a_4+\frac{9}{16}\,a_5+\frac{1}{64}\,b_3+\frac{1}{8}\,b_4-\frac{3}{16}\,b_5-\frac{1}{32}\,b_6+\frac{1}{4}\,b_7+\frac{3}{32}\,b_8+\frac{3}{16}\,.\tag{7.30}$$

The final result for the impact factor at N^{3}LLA then takes the form,

$$\begin{aligned}\Phi^{(3)}_{\rm Reg}(\nu,n) &= \frac{1}{3}\left[\Phi^{(1)}_{\rm Reg}(\nu,n)\right]^3-E^{(2)}_{\nu,n}\,E_{\nu,n}-\Phi^{(2)}_{\rm Reg}(\nu,n)\,E^2_{\nu,n}-\frac{1}{24}\,[D^2_\nu E_{\nu,n}]^2\\&+\frac{1}{4}\,D_\nu E_{\nu,n}\,V\,D^2_\nu E_{\nu,n}-\frac{1}{24}\,D_\nu E_{\nu,n}\,D^3_\nu E_{\nu,n}+\frac{1}{8}\,D^2_\nu E_{\nu,n}\,E^3_{\nu,n}-\frac{3}{32}\,E_{\nu,n}\,N^2\,D^2_\nu E_{\nu,n}\\&-\frac{37\pi^2}{96}\,E_{\nu,n}\,D^2_\nu E_{\nu,n}-\frac{1}{24}\,D^2_\nu E_{\nu,n}\,\zeta_3-\frac{1}{4}\,D_\nu E_{\nu,n}\,V\,E^3_{\nu,n}+\frac{3}{16}\,D_\nu E_{\nu,n}\,E_{\nu,n}\,N^2\,V\\&+\frac{11\pi^2}{24}\,D_\nu E_{\nu,n}\,E_{\nu,n}\,V+\frac{9}{4}\,D_\nu E_{\nu,n}\,V\,\zeta_3+\frac{1}{16}\,[D_\nu E_{\nu,n}]^2\,E^2_{\nu,n}-\frac{3}{64}\,N^2\,[D_\nu E_{\nu,n}]^2\end{aligned}\tag{7.31}$$

[10]Note that we can not exclude the appearance of multiple ζ values at higher weights, as multiple ζ values are reducible to ordinary ζ values until weight eight.

[11]We have constructed the full basis of functions in (ν,n) space through weight six and the explicit map to (w,w^*) functions of weight seven. It is therefore not necessary for us to restrict our ansatz in this way. It is, however, sufficient, and computationally simpler to do so.

$$
\begin{aligned}
&-\frac{1}{8}\,V^2\,[D_\nu E_{\nu,n}]^2+\frac{3\pi^2}{32}\,[D_\nu E_{\nu,n}]^2+\frac{37}{256}\,N^4\,E_{\nu,n}^2+\frac{5}{32}\,N^2\,V^2\,E_{\nu,n}^2 \\
&-\frac{23\pi^2}{128}\,N^2\,E_{\nu,n}^2-\frac{21\pi^2}{32}\,V^2\,E_{\nu,n}^2+\frac{161}{12}\,E_{\nu,n}^3\,\zeta_3+\frac{7}{48}\,E_{\nu,n}^6+\frac{\pi^2}{3}\,E_{\nu,n}^4-\frac{\pi^4}{72}\,E_{\nu,n}^2 \\
&+\frac{7}{16}\,E_{\nu,n}\,N^2\,\zeta_3-\frac{13\pi^2}{2}\,E_{\nu,n}\,\zeta_3-\frac{45}{1024}\,N^6-\frac{41}{128}\,N^4\,V^2+\frac{5\pi^2}{512}\,N^4-\frac{3}{16}\,N^2\,V^4 \\
&-\frac{5\pi^2}{128}\,N^2\,V^2+\frac{\pi^4}{24}\,N^2+\frac{\pi^4}{8}\,V^2+\frac{5}{2}\,\zeta_3^2-\frac{311\pi^6}{11340}+3\,E_{\nu,n}\,V^2\,\zeta_3+10\,E_{\nu,n}\,\zeta_5 \\
&+\frac{15}{64}\,N^2\,E_{\nu,n}^4+a_0\,\mathcal{P}_6+\sum_{i=1}^{5}a_i\,\zeta_2\,\mathcal{P}_{a,4,i}+a_6\,\zeta_4\,\mathcal{P}_{a,2}+\sum_{i=7}^{8}a_i\,\zeta_3\,\mathcal{P}_{a,3,i} \\
&+\sum_{i=2}^{9}b_i\,\zeta_2\,\mathcal{P}_{b,4,i}+\sum_{i=10}^{14}b_i\,\zeta_3\,\mathcal{P}_{b,3,i}+\sum_{i=15}^{16}b_i\,\zeta_4\,\mathcal{P}_{b,2,i}+b_{17}\,\zeta_2\zeta_3\,\mathcal{P}_{b,1,1}+b_{18}\,\zeta_5\,\mathcal{P}_{b,1,2}\,,
\end{aligned}
\tag{7.32}
$$

where $\mathcal{P}_{i,j,\ldots}$ parametrize the beyond-the-symbol terms in the four-loop coefficient functions, and $\mathcal{P}_6$ parameterizes the one symbol-level ambiguity,

$$
\begin{aligned}
\mathcal{P}_6 \;=\;& \frac{105}{2}\,[D_\nu^2 E_{\nu,n}]^2-\frac{152}{3}\,E_{\nu,n}\,N^2\,D_\nu^2 E_{\nu,n}-\frac{2690}{3}\,E_{\nu,n}\,V^2\,D_\nu^2 E_{\nu,n}+\frac{595}{3}\,E_{\nu,n}\,V\,D_\nu^3 E_{\nu,n} \\
&-\frac{7}{6}\,E_{\nu,n}\,D_\nu^4 E_{\nu,n}-\frac{10455}{2}\,D_\nu^2 E_{\nu,n}\,\zeta_3+\frac{249}{8}\,N^2\,[D_\nu E_{\nu,n}]^2+\frac{2655}{2}\,V^2\,[D_\nu E_{\nu,n}]^2 \\
&+\frac{103}{16}\,N^4\,E_{\nu,n}^2+\frac{317}{4}\,N^2\,V^2\,E_{\nu,n}^2+\frac{197}{24}\,N^2\,E_{\nu,n}^4+\frac{515}{6}\,V^2\,E_{\nu,n}^4+\frac{61793}{6}\,E_{\nu,n}\,N^2\,\zeta_3 \\
&+\frac{13777}{3}\,E_{\nu,n}\,V^2\,\zeta_3+\frac{111}{128}\,N^6+\frac{345}{32}\,N^4\,V^2-385\,D_\nu E_{\nu,n}\,V\,D_\nu^2 E_{\nu,n}-30\,D_\nu E_{\nu,n}\,D_\nu^3 E_{\nu,n} \\
&+16\,D_\nu^2 E_{\nu,n}\,E_{\nu,n}^3-420\,D_\nu E_{\nu,n}\,V\,E_{\nu,n}^3+7\,D_\nu E_{\nu,n}\,E_{\nu,n}\,N^2\,V-760\,D_\nu E_{\nu,n}\,E_{\nu,n}\,V^3 \\
&-22606\,D_\nu E_{\nu,n}\,V\,\zeta_3-34\,[D_\nu E_{\nu,n}]^2\,E_{\nu,n}^2+1140\,V^4\,E_{\nu,n}^2+15231\,E_{\nu,n}^3\,\zeta_3+6548\,E_{\nu,n}\,\zeta_5 \\
&+46992\,\zeta_3^2\,,
\end{aligned}
\tag{7.33}
$$

$$
\begin{aligned}
\mathcal{P}_{a,4,1} \;=\;& \frac{5}{8}\,E_{\nu,n}\,D_\nu^2 E_{\nu,n}-\frac{3}{2}\,D_\nu E_{\nu,n}\,E_{\nu,n}\,V+\frac{33}{8}\,[D_\nu E_{\nu,n}]^2-\frac{183}{32}\,N^2\,E_{\nu,n}^2 \\
&-\frac{129}{8}\,V^2\,E_{\nu,n}^2-\frac{5}{4}\,E_{\nu,n}^4+\frac{3}{128}\,N^4+\frac{171}{32}\,N^2\,V^2+\frac{\pi^2}{4}\,N^2+\pi^2\,E_{\nu,n}^2-68\,E_{\nu,n}\,\zeta_3\,,
\end{aligned}
\tag{7.34}
$$

$$
\begin{aligned}
\mathcal{P}_{a,4,2} \;=\;& -\frac{3}{16}\,E_{\nu,n}\,D_\nu^2 E_{\nu,n}+\frac{3}{4}\,D_\nu E_{\nu,n}\,E_{\nu,n}\,V+\frac{7}{16}\,[D_\nu E_{\nu,n}]^2-\frac{51}{64}\,N^2\,E_{\nu,n}^2 \\
&-\frac{33}{16}\,V^2\,E_{\nu,n}^2-\frac{1}{4}\,E_{\nu,n}^4-\frac{7}{256}\,N^4+\frac{19}{64}\,N^2\,V^2-12\,E_{\nu,n}\,\zeta_3\,,
\end{aligned}
\tag{7.35}
$$

$$
\begin{aligned}
\mathcal{P}_{a,4,3} \;=\;& -\frac{3}{2}\,E_{\nu,n}\,D_\nu^2 E_{\nu,n}+\frac{9}{4}\,[D_\nu E_{\nu,n}]^2-\frac{3}{2}\,N^2\,E_{\nu,n}^2-\frac{9}{2}\,V^2\,E_{\nu,n}^2-\frac{3}{4}\,E_{\nu,n}^4-\frac{9}{64}\,N^4 \\
&+\frac{9}{8}\,N^2\,V^2+6\,D_\nu E_{\nu,n}\,E_{\nu,n}\,V-48\,E_{\nu,n}\,\zeta_3\,,
\end{aligned}
\tag{7.36}
$$

$$\begin{aligned}
\mathcal{P}_{a,4,4} &= \frac{49}{32}\,E_{\nu,n}\,D_\nu^2 E_{\nu,n} - \frac{27}{8}\,D_\nu E_{\nu,n}\,E_{\nu,n}\,V - \frac{45}{32}\,[D_\nu E_{\nu,n}]^2 + \frac{117}{128}\,N^2\,E_{\nu,n}^2 \\
&\quad + \frac{111}{32}\,V^2\,E_{\nu,n}^2 + \frac{1}{2}\,E_{\nu,n}^4 + \frac{73}{2}\,E_{\nu,n}\,\zeta_3 + \frac{69}{512}\,N^4 - \frac{21}{128}\,N^2\,V^2\,, \qquad (7.37)\\
\mathcal{P}_{a,4,5} &= -\frac{3}{16}\,E_{\nu,n}\,D_\nu^2 E_{\nu,n} - \frac{3}{4}\,D_\nu E_{\nu,n}\,E_{\nu,n}\,V - \frac{15}{16}\,[D_\nu E_{\nu,n}]^2 + \frac{105}{64}\,N^2\,E_{\nu,n}^2 \\
&\quad + \frac{63}{16}\,V^2\,E_{\nu,n}^2 + \frac{3}{8}\,E_{\nu,n}^4 + \frac{3}{256}\,N^4 - \frac{69}{64}\,N^2\,V^2 + 18\,E_{\nu,n}\,\zeta_3\,, \qquad (7.38)\\
\mathcal{P}_{a,2} &= -\frac{45}{4}\,N^2 - 45\,E_{\nu,n}^2\,, \qquad (7.39)\\
\mathcal{P}_{a,3,7} &= \frac{1}{6}\,D_\nu^2 E_{\nu,n} - \frac{1}{3}\,E_{\nu,n}^3 - \frac{4}{3}\,\zeta_3 - E_{\nu,n}\,V^2\,, \qquad (7.40)\\
\mathcal{P}_{a,3,8} &= -\frac{1}{24}\,D_\nu^2 E_{\nu,n} + \frac{1}{4}\,D_\nu E_{\nu,n}\,V - \frac{1}{6}\,E_{\nu,n}^3 - \frac{1}{8}\,E_{\nu,n}\,N^2 - \frac{1}{2}\,E_{\nu,n}\,V^2 - \frac{13}{6}\,\zeta_3\,, \qquad (7.41)\\
\mathcal{P}_{b,4,2} &= \frac{3}{4}\,N^2\,E_{\nu,n}^2 + \frac{3}{16}\,N^4 + \frac{21}{4}\,N^2\,V^2 + 3\,E_{\nu,n}\,D_\nu^2 E_{\nu,n} + 12\,D_\nu E_{\nu,n}\,E_{\nu,n}\,V \\
&\quad + 3\,[D_\nu E_{\nu,n}]^2 + 9\,V^2\,E_{\nu,n}^2\,, \qquad (7.42)\\
\mathcal{P}_{b,4,3} &= \frac{7}{192}\,E_{\nu,n}\,D_\nu^2 E_{\nu,n} - \frac{7}{16}\,D_\nu E_{\nu,n}\,E_{\nu,n}\,V - \frac{9}{64}\,[D_\nu E_{\nu,n}]^2 + \frac{33}{256}\,N^2\,E_{\nu,n}^2 \\
&\quad + \frac{19}{64}\,V^2\,E_{\nu,n}^2 + \frac{5}{24}\,E_{\nu,n}^4 + \frac{37}{12}\,E_{\nu,n}\,\zeta_3 + \frac{9}{1024}\,N^4 - \frac{1}{256}\,N^2\,V^2\,, \qquad (7.43)\\
\mathcal{P}_{b,4,4} &= -\frac{5}{24}\,E_{\nu,n}\,D_\nu^2 E_{\nu,n} - \frac{1}{2}\,D_\nu E_{\nu,n}\,E_{\nu,n}\,V - \frac{3}{8}\,[D_\nu E_{\nu,n}]^2 + \frac{9}{32}\,N^2\,E_{\nu,n}^2 \\
&\quad + \frac{7}{8}\,V^2\,E_{\nu,n}^2 + \frac{5}{12}\,E_{\nu,n}^4 + \frac{14}{3}\,E_{\nu,n}\,\zeta_3 + \frac{3}{128}\,N^4 + \frac{11}{32}\,N^2\,V^2\,, \qquad (7.44)\\
\mathcal{P}_{b,4,5} &= \frac{3}{16}\,E_{\nu,n}\,D_\nu^2 E_{\nu,n} + \frac{1}{2}\,D_\nu E_{\nu,n}\,E_{\nu,n}\,V + \frac{1}{2}\,[D_\nu E_{\nu,n}]^2 - \frac{31}{64}\,N^2\,E_{\nu,n}^2 \\
&\quad - \frac{27}{16}\,V^2\,E_{\nu,n}^2 - \frac{9}{16}\,E_{\nu,n}^4 + \frac{\pi^2}{8}\,E_{\nu,n}^2 - \frac{1}{128}\,N^4 - \frac{3}{64}\,N^2\,V^2 + \frac{\pi^2}{32}\,N^2 - 8\,E_{\nu,n}\,\zeta_3\,, \qquad (7.45)\\
\mathcal{P}_{b,4,6} &= -\frac{5}{96}\,E_{\nu,n}\,D_\nu^2 E_{\nu,n} + \frac{1}{2}\,D_\nu E_{\nu,n}\,E_{\nu,n}\,V + \frac{17}{96}\,[D_\nu E_{\nu,n}]^2 - \frac{25}{128}\,N^2\,E_{\nu,n}^2 \\
&\quad - \frac{15}{32}\,V^2\,E_{\nu,n}^2 - \frac{11}{48}\,E_{\nu,n}^4 - \frac{49}{12}\,E_{\nu,n}\,\zeta_3 - \frac{17}{1536}\,N^4 + \frac{11}{384}\,N^2\,V^2\,, \qquad (7.46)\\
\mathcal{P}_{b,4,7} &= \Phi^{(2)}_{\mathrm{Reg}}(\nu,n) - \frac{2}{3}\,E_{\nu,n}\,D_\nu^2 E_{\nu,n} - \frac{3}{8}\,[D_\nu E_{\nu,n}]^2 + \frac{1}{4}\,N^2\,E_{\nu,n}^2 + \frac{7}{4}\,V^2\,E_{\nu,n}^2 \\
&\quad - \frac{\pi^2}{2}\,E_{\nu,n}^2 + \frac{1}{3}\,E_{\nu,n}\,\zeta_3 - \frac{5}{128}\,N^4 - \frac{7}{8}\,N^2\,V^2 + \frac{\pi^2}{48}\,N^2 + \frac{\pi^2}{4}\,V^2 - \frac{11\pi^4}{180} \\
&\quad + \frac{5}{24}\,E_{\nu,n}^4 + D_\nu E_{\nu,n}\,E_{\nu,n}\,V\,, \qquad (7.47)\\
\mathcal{P}_{b,4,8} &= -\frac{5}{32}\,E_{\nu,n}\,D_\nu^2 E_{\nu,n} + \frac{1}{8}\,D_\nu E_{\nu,n}\,E_{\nu,n}\,V - \frac{7}{32}\,[D_\nu E_{\nu,n}]^2 + \frac{27}{128}\,N^2\,E_{\nu,n}^2 \\
&\quad + \frac{33}{32}\,V^2\,E_{\nu,n}^2 + \frac{3}{8}\,E_{\nu,n}^4 - \frac{\pi^2}{4}\,E_{\nu,n}^2 + \frac{7}{512}\,N^4 - \frac{19}{128}\,N^2\,V^2 + 3\,E_{\nu,n}\,\zeta_3\,, \qquad (7.48)
\end{aligned}$$

$$\mathcal{P}_{b,4,9} = \frac{1}{24}\, E_{\nu,n}^4 \,, \tag{7.49}$$

$$\mathcal{P}_{b,3,10} = -\frac{1}{48}\, D_\nu^2 E_{\nu,n} + \frac{3}{8}\, D_\nu E_{\nu,n}\, V - \frac{1}{3}\, E_{\nu,n}^3 - \frac{1}{8}\, E_{\nu,n}\, N^2 - \frac{1}{4}\, E_{\nu,n}\, V^2 - \frac{25}{12}\, \zeta_3 \,, \tag{7.50}$$

$$\mathcal{P}_{b,3,11} = \frac{1}{16}\, E_{\nu,n}\, N^2 - \frac{1}{4}\, E_{\nu,n}^3 \,, \tag{7.51}$$

$$\mathcal{P}_{b,3,12} = -\frac{1}{2}\, D_\nu E_{\nu,n}\, V + \frac{1}{2}\, E_{\nu,n}^3 + \frac{1}{8}\, E_{\nu,n}\, N^2 + 2\, \zeta_3 \,, \tag{7.52}$$

$$\mathcal{P}_{b,3,13} = \frac{1}{6}\, E_{\nu,n}^3 \,, \tag{7.53}$$

$$\mathcal{P}_{b,3,14} = -\frac{1}{6}\, D_\nu^2 E_{\nu,n} + \frac{5}{6}\, E_{\nu,n}^3 - \frac{1}{8}\, E_{\nu,n}\, N^2 - \frac{\pi^2}{3}\, E_{\nu,n} + \frac{4}{3}\, \zeta_3 + E_{\nu,n}\, V^2 \,, \tag{7.54}$$

$$\mathcal{P}_{b,2,15} = \frac{1}{2}\, E_{\nu,n}^2 \,, \tag{7.55}$$

$$\mathcal{P}_{b,2,16} = E_{\nu,n}^2 - \frac{1}{4}\, N^2 \,, \tag{7.56}$$

$$\mathcal{P}_{b,1,1} = E_{\nu,n} \,, \tag{7.57}$$

$$\mathcal{P}_{b,1,2} = E_{\nu,n} \,. \tag{7.58}$$

Again, the undetermined function at symbol level, $\mathcal{P}_6$, is the most complicated term, but it would be absent if $a_0 = 0$.

Finally, we remark that the $\nu \to 0$ behavior of $\Phi_{\text{Reg}}^{(\ell)}(\nu, n)$ is nonvanishing, and even singular for $\ell = 2$ and 3. Taking the limit after setting $n = 0$, as in the case of $E_{\nu,n}^{(2)}$, we find that the constant term is given in terms of the cusp anomalous dimension,

$$\lim_{\nu\to 0} \Phi_{\text{Reg}}^{(1)}(\nu, 0) \sim \frac{\gamma_K^{(2)}}{4} + \mathcal{O}(\nu^4) \,, \tag{7.59}$$

$$\lim_{\nu\to 0} \Phi_{\text{Reg}}^{(2)}(\nu, 0) \sim \frac{\pi^2}{4\,\nu^2} + \frac{\gamma_K^{(3)}}{4} + \mathcal{O}(\nu^2) \,, \tag{7.60}$$

$$\lim_{\nu\to 0} \Phi_{\text{Reg}}^{(3)}(\nu, 0) \sim -\frac{\pi^4}{8\,\nu^2} + \frac{\gamma_K^{(4)}}{4} + \mathcal{O}(\nu^2) \,. \tag{7.61}$$

This fact is presumably related to the appearance of $\gamma_K(a)$ in the factors ω_{ab} and δ, which carry logarithmic dependence on $|w|$ as $w \to 0$. It may play a role in understanding the failure of $E_{\nu,0}^{(2)}$ to vanish as $\nu \to 0$ in eq. (7.28).

8 Conclusions and Outlook

In this article we exposed the structure of the multi-Regge limit of six-gluon scattering in planar $\mathcal{N} = 4$ super-Yang-Mills theory in terms of the single-valued harmonic polylogarithms introduced by Brown. Given the finite basis of such functions, it is extremely simple to determine any quantity that is defined by a power series expansion around the origin of the (w, w^*) plane. Two examples which we could evaluate with no ambiguity are the LL and NLL terms in the multi-Regge limit of the MHV amplitude. We could carry this exercise out through transcendental weight 10, and we presented the analytic formulae explicitly through six loops in Section 4. The NMHV amplitudes also fit into the same mathematical framework, as we saw in Section 5: An integro-differential operator that generates the NMHV LLA terms from the MHV LLA ones [43] has a very natural action on the SVHPLs, making it simple to generate NMHV LLA results to high order as well. A clear avenue for future investigation utilizing the SVHPLs is the NMHV six-point amplitude at next-to-leading-logarithm and beyond.

A second thrust of this article was to understand the Fourier-Mellin transform from (w, w^*) to (ν, n) variables. In practice, we constructed this map in the reverse direction: We built an ansatz out of various elements: harmonic sums and specific rational combinations of ν and n. We then implemented the inverse Fourier-Mellin transform as a truncated sum, or power series around the origin of the (w, w^*) plane, and matched to the basis of SVHPLs. We thereby identified specific combinations of the elements as building blocks from which to generate the full set of SVHPL Fourier-Mellin transforms. We have executed this procedure completely through weight six in the (ν, n) space, corresponding to weight seven in the (w, w^*) space. In generalizing the procedure to yet higher weight, we expect the procedure to be much the same. Beginning with a linear combination of weight $(p-2)$ HPLs in a single variable x, perform a Mellin transformation to produce weight $(p-1)$ harmonic sums such as ψ, F_4, F_{6a}, etc. For suitable combinations of these elements, the inverse Fourier-Mellin transform will generate weight p SVHPLs in the complex conjugate pair (w, w^*). The step of determining which combinations of elements correspond to the SVHPLs was carried out empirically in this paper. It would be interesting to investigate further the mathematical properties of these building blocks.

Using our understanding of the Fourier-Mellin transform, we could explicitly evaluate the NNLL MHV impact factor $\Phi_{\rm Reg}^{(2)}(\nu, n)$ which derives from a knowledge of the three-loop remainder function in the MRK limit [28, 40]. We then went on to four loops, using a computation of the four-loop symbol [52] in conjunction with additional constraints from the multi-Regge limit to determine the MRK symbol up to one free parameter a_0 (which we suspect is zero). We matched this symbol to the symbols of the SVHPLs in order to determine the complete four-loop remainder function in MRK, up to a number of beyond-the-symbol constants. This data, in particular $g_1^{(4)}$ and $g_0^{(4)}$ then led to the NNLL BFKL eigenvalue $E_{\nu,n}^{(2)}$ and N^{3}LL impact factor $\Phi_{\rm Reg}^{(3)}(\nu, n)$. These quantities also contain the various beyond-the-symbol constants. Clearly the higher-loop NNLL MRK terms can be determined just as we did at LL and NLL, using the master formula (2.9) and the SVHPL basis. However, it would also be worthwhile to understand what constraint can fix a_0, and the host of beyond-the-symbol constants, since they will afflict all of these terms. This task may require backing away somewhat from the multi-Regge limit, or utilizing coproduct information in some way.

We also remind the reader that we found that the NNLL BFKL eigenvalue $E_{\nu,n}^{(2)}$ does not vanish as $\nu \to 0$, taking the limit after setting $n = 0$. This behavior is in contrast to what happens in the LL and NLL case. It also goes against the expectations in ref. [40], and thus calls for further study.

Although the structure of QCD amplitudes in the multi-Regge limit is more complicated than those of planar $\mathcal{N}=4$ super-Yang-Mills theory, one can still hope that the understanding of the Fourier-Mellin (ν, n) space that we have developed here may prove useful in the QCD context.

Finally, we remark that the SVHPLs are very likely to be applicable to another current problem in $\mathcal{N}=4$ super-Yang-Mills theory, namely the determination of correlation functions for four off-shell operators. Conformal invariance implies that these quantities depend on two separate cross ratios. The natural arguments of the polylogarithms that appear at low loop order, after a change of variables from the original cross ratios, are again a complex pair (w, w^*) (or $(z, \bar{z})$). The same single-valued conditions apply here as well. For example, the one-loop off-shell box integral that enters the correlation function is proportional to $L_2^-(z,\bar{z})/(z-\bar{z})$. We expect that the SVHPL framework will allow great progress to be made in this arena, just as it has to the study of the multi-Regge limit.

Acknowledgments

We thank Vittorio Del Duca for useful discussions, and Johannes Blümlein and Francis Brown for helpful comments on the manuscript. This research was supported by the US Department of Energy under contract DE–AC02–76SF00515 and by the ERC grant “IterQCD”.

A Single-valued harmonic polylogarithms

A.1 Expression of the $L^{\pm}$ functions in terms of ordinary HPLs

In this appendix we present the expressions for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ eigenfunctions $L_w^{\pm}(z)$ defined in eq. (3.19) as linear combinations of ordinary HPLs of the form $H_{w_1}(z)\, H_{w_2}(\bar{z})$ up to weight 5. All expressions up to weight 6 are attached as ancillary files in computer-readable format. We give results only for the Lyndon words, as all other cases can be reduced to the latter. In the following, we use the condensed notation (3.27) for the HPL arguments z and $\bar{z}$ to improve the readability of the formulas.

A.2 Lyndon words of weight 1

$$L_0^- \;=\; H_0 + \overline{H}_0 = \log|w|^2\,, \tag{A.1}$$

$$L_1^+ \;=\; H_1 + \overline{H}_1 + \frac{1}{2}H_0 + \frac{1}{2}\overline{H}_0 = -\log|1+w|^2 + \frac{1}{2}\log|w|^2\,, \tag{A.2}$$

A.3 Lyndon words of weight 2

$$\begin{aligned} L_2^- &= \frac{1}{4}\Big[-2\,H_{1,0} + 2\,\overline{H}_{1,0} + 2\,H_0\,\overline{H}_1 - 2\,\overline{H}_0\,H_1 + 2\,H_2 - 2\,\overline{H}_2 \Big] \\ &= \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\,\log|z|^2(\log(1-z) - \log(1-\bar{z}))\,, \end{aligned} \tag{A.3}$$

A.4 Lyndon words of weight 3

$$
\begin{aligned}
L_3^+ &= \frac{1}{4}\Big[2\,H_0\,\overline{H}_{0,0} + 2\,H_0\,\overline{H}_{1,0} + 2\,\overline{H}_0\,H_{0,0} + 2\,\overline{H}_0\,H_{1,0} + 2\,H_1\,\overline{H}_{0,0} + 2\,\overline{H}_1\,H_{0,0} \\
&\quad +2\,H_{0,0,0} + 2\,H_{1,0,0} + 2\,\overline{H}_{0,0,0} + 2\,\overline{H}_{1,0,0} + 2\,H_3 + 2\,\overline{H}_3\Big] \\
&= \mathrm{Li}_3(z) + \mathrm{Li}_3(\bar{z}) - \frac{1}{2}\log|z|^2\Big[\mathrm{Li}_2(\bar{z}) + \mathrm{Li}_2(z)\Big] \\
&\quad -\frac{1}{4}\log^2|z|^2\Big[\log(1-z) + \log(1-\bar{z})\Big] + \frac{1}{12}\log^3|z|^2\,,
\end{aligned} \tag{A.4}
$$

$$
\begin{aligned}
L_{2,1}^- &= \frac{1}{4}\Big[H_0\,\overline{H}_{1,0} + \overline{H}_0\,H_{1,0} + H_1\,\overline{H}_{0,0} + \overline{H}_1\,H_{0,0} + 2\,H_0\,\overline{H}_{0,0} + 2\,H_0\,\overline{H}_{1,1} \\
&\quad +2\,\overline{H}_0\,H_{0,0} + 2\,\overline{H}_0\,H_{1,1} + H_{1,0,0} + 2\,H_{0,0,0} + 2\,H_{2,0} + 2\,H_{2,1} + 2\,H_{1,1,0} \\
&\quad +\overline{H}_{1,0,0} + 2\,\overline{H}_{0,0,0} + 2\,\overline{H}_{2,0} + 2\,\overline{H}_{2,1} + 2\,\overline{H}_{1,1,0} + 2\,H_0\,\overline{H}_2 + 2\,\overline{H}_0\,H_2 \\
&\quad +2\,H_1\,\overline{H}_2 + 2\,\overline{H}_1\,H_2 + H_3 + \overline{H}_3 - 4\,\zeta_3\Big] \\
&= -\mathrm{Li}_3(1-z) - \mathrm{Li}_3(1-\bar{z}) - \frac{1}{2}\Big[\mathrm{Li}_3(z) + \mathrm{Li}_3(\bar{z})\Big] + \frac{1}{4}\log|z|^2\Big[\mathrm{Li}_2(z) + \mathrm{Li}_2(\bar{z})\Big] \\
&\quad -\frac{1}{2}\log|1-z|^2\Big[\mathrm{Li}_2(z) + \mathrm{Li}_2(\bar{z})\Big] - \frac{1}{8}\log|z|^2\log|1-z|^2 + \frac{1}{12}\log^3|z|^2 \\
&\quad -\frac{1}{4}\log\frac{z}{\bar{z}}\Big[\log^2(1-z) - \log^2(1-\bar{z})\Big] + \zeta_2\,\log|1-z|^2 + \zeta_3\,,
\end{aligned} \tag{A.5}
$$

A.5 Lyndon words of weight 4

$$
\begin{aligned}
L_{3,1}^+ &= \frac{1}{4}\Big[H_0\,\overline{H}_{2,0} + H_0\,\overline{H}_{1,0,0} - \overline{H}_0\,H_{2,0} - \overline{H}_0\,H_{1,0,0} - H_1\,\overline{H}_{0,0,0} + \overline{H}_1\,H_{0,0,0} \\
&\quad +H_{0,0}\,\overline{H}_2 + H_{0,0}\,\overline{H}_{1,0} - \overline{H}_{0,0}\,H_2 - \overline{H}_{0,0}\,H_{1,0} + 2\,H_0\,\overline{H}_{1,1,0} - 2\,\overline{H}_0\,H_{1,1,0} \\
&\quad +2\,H_{0,0}\,\overline{H}_{1,1} - 2\,\overline{H}_{0,0}\,H_{1,1} + H_{3,0} - H_{2,0,0} - H_{1,0,0,0} + 2\,H_{3,1} - 2\,H_{1,1,0,0} \\
&\quad +\overline{H}_{2,0,0} + \overline{H}_{1,0,0,0} - \overline{H}_{3,0} - 2\,\overline{H}_{3,1} + 2\,\overline{H}_{1,1,0,0} - H_0\,\overline{H}_3 + \overline{H}_0\,H_3 - 2\,H_1\,\overline{H}_3 \\
&\quad +2\,\overline{H}_1\,H_3 + 4\,H_1\,\zeta_3 + H_4 - 4\,\overline{H}_1\,\zeta_3 - \overline{H}_4\Big]\,,
\end{aligned} \tag{A.6}
$$

$$
\begin{aligned}
L_4^- &= \frac{1}{4}\Big[2\,H_0\,\overline{H}_{1,0,0} - 2\,\overline{H}_0\,H_{1,0,0} - 2\,H_1\,\overline{H}_{0,0,0} + 2\,\overline{H}_1\,H_{0,0,0} + 2\,H_{0,0}\,\overline{H}_{1,0} \\
&\quad -2\,\overline{H}_{0,0}\,H_{1,0} - 2\,H_{1,0,0,0} + 2\,\overline{H}_{1,0,0,0} + 2\,H_4 - 2\,\overline{H}_4\Big]\,,
\end{aligned} \tag{A.7}
$$

$$
\begin{aligned}
L^-_{2,1,1} = \frac{1}{4}\Big[& H_0\,\overline{H}_{1,0,0} + H_0\,\overline{H}_{1,2} + H_0\,\overline{H}_{1,1,0} - \overline{H}_0\,H_{1,0,0} - \overline{H}_0\,H_{1,2} - \overline{H}_0\,H_{1,1,0} \\
& -H_1\,\overline{H}_{0,0,0} - H_1\,\overline{H}_{2,0} + \overline{H}_1\,H_{0,0,0} + \overline{H}_1\,H_{2,0} + H_{0,0}\,\overline{H}_{1,0} + H_{0,0}\,\overline{H}_{1,1} \\
& -\overline{H}_{0,0}\,H_{1,0} - \overline{H}_{0,0}\,H_{1,1} + H_2\,\overline{H}_{1,0} - \overline{H}_2\,H_{1,0} + 2\,H_0\,\overline{H}_{1,1,1} - 2\,\overline{H}_0\,H_{1,1,1} \\
& -2\,H_1\,\overline{H}_{2,1} + 2\,\overline{H}_1\,H_{2,1} + 2\,H_2\,\overline{H}_{1,1} - 2\,\overline{H}_2\,H_{1,1} + H_{3,1} + H_{2,2} \\
& -H_{1,0,0,0} - H_{1,2,0} - H_{1,1,0,0} + 2\,H_{2,1,1} - 2\,H_{1,1,1,0} + \overline{H}_{1,0,0,0} + \overline{H}_{1,2,0} + \overline{H}_{1,1,0,0} \\
& -\overline{H}_{3,1} - \overline{H}_{2,2} - 2\,\overline{H}_{2,1,1} + 2\,\overline{H}_{1,1,1,0} - H_1\,\overline{H}_3 + \overline{H}_1\,H_3 + 2\,H_1\,\zeta_3 + H_4 \\
& -2\,\overline{H}_1\,\zeta_3 - \overline{H}_4 \Big],
\end{aligned}
\tag{A.8}
$$

A.6 Lyndon words of weight 5

$$
\begin{aligned}
L^+_5 = \frac{1}{4}\Big[& 2\,H_0\,\overline{H}_{0,0,0,0} + 2\,H_0\,\overline{H}_{1,0,0,0} + 2\,\overline{H}_0\,H_{0,0,0,0} + 2\,\overline{H}_0\,H_{1,0,0,0} + 2\,H_1\,\overline{H}_{0,0,0,0} \\
& +2\,\overline{H}_1\,H_{0,0,0,0} + 2\,H_{0,0}\,\overline{H}_{0,0,0} + 2\,H_{0,0}\,\overline{H}_{1,0,0} + 2\,\overline{H}_{0,0}\,H_{0,0,0} + 2\,\overline{H}_{0,0}\,H_{1,0,0} \\
& +2\,H_{1,0}\,\overline{H}_{0,0,0} + 2\,\overline{H}_{1,0}\,H_{0,0,0} + 2\,H_{0,0,0,0,0} + 2\,H_{1,0,0,0,0} + 2\,\overline{H}_{0,0,0,0,0} + 2\,\overline{H}_{1,0,0,0,0} \\
& +2\,H_5 + 2\,\overline{H}_5 \Big],
\end{aligned}
\tag{A.9}
$$

$$
\begin{aligned}
L^+_{3,1,1} = \frac{1}{4}\Big[& H_5 + \overline{H}_5 + H_{4,0} + \overline{H}_{4,0} + H_{4,1} + \overline{H}_{4,1} + H_{3,2} + \overline{H}_{3,2} + H_{3,1,0} + \overline{H}_{3,1,0} \\
& +H_{2,0,0,0} + \overline{H}_{2,0,0,0} + H_{2,1,0,0} + \overline{H}_{2,1,0,0} + H_{1,0,0,0,0} + \overline{H}_{1,0,0,0,0} + H_{1,2,0,0} \\
& +\overline{H}_{1,2,0,0} + H_{1,1,0,0,0} + \overline{H}_{1,1,0,0,0} + 2\,H_{0,0,0,0,0} + 2\,\overline{H}_{0,0,0,0,0} + 2\,H_{3,0,0} + 2\,\overline{H}_{3,0,0} \\
& +2\,H_{3,1,1} + 2\,\overline{H}_{3,1,1} + 2\,H_{1,1,1,0,0} + 2\,\overline{H}_{1,1,1,0,0} + 4\,\zeta_5 + H_0\,\overline{H}_4 + H_0\,\overline{H}_{3,1} + H_0\,\overline{H}_{2,0,0} \\
& +H_0\,\overline{H}_{2,1,0} + H_0\,\overline{H}_{1,0,0,0} + H_0\,\overline{H}_{1,2,0} + H_0\,\overline{H}_{1,1,0,0} + \overline{H}_0\,H_4 + \overline{H}_0\,H_{3,1} + \overline{H}_0\,H_{2,0,0} \\
& +\overline{H}_0\,H_{2,1,0} + \overline{H}_0\,H_{1,0,0,0} + \overline{H}_0\,H_{1,2,0} + \overline{H}_0\,H_{1,1,0,0} + H_1\,\overline{H}_{0,0,0,0} + H_1\,\overline{H}_4 + H_1\,\overline{H}_{3,0} \\
& +\overline{H}_1\,H_{0,0,0,0} + \overline{H}_1\,H_4 + \overline{H}_1\,H_{3,0} + H_{0,0}\,\overline{H}_{2,0} + H_{0,0}\,\overline{H}_{2,1} + H_{0,0}\,\overline{H}_{1,0,0} + H_{0,0}\,\overline{H}_{1,2} \\
& +H_{0,0}\,\overline{H}_{1,1,0} + \overline{H}_{0,0}\,H_{2,0} + \overline{H}_{0,0}\,H_{2,1} + \overline{H}_{0,0}\,H_{1,0,0} + \overline{H}_{0,0}\,H_{1,2} + \overline{H}_{0,0}\,H_{1,1,0} \\
& +H_2\,\overline{H}_{0,0,0} + H_2\,\overline{H}_3 + \overline{H}_2\,H_{0,0,0} + \overline{H}_2\,H_3 + H_{1,0}\,\overline{H}_{0,0,0} + H_{1,0}\,\overline{H}_3 + \overline{H}_{1,0}\,H_{0,0,0} \\
& +\overline{H}_{1,0}\,H_3 + H_{1,1}\,\overline{H}_{0,0,0} + \overline{H}_{1,1}\,H_{0,0,0} + 2\,H_0\,\overline{H}_{0,0,0,0} + 2\,H_0\,\overline{H}_{3,0} + 2\,H_0\,\overline{H}_{1,1,1,0} \\
& +2\,\overline{H}_0\,H_{0,0,0,0} + 2\,\overline{H}_0\,H_{3,0} + 2\,\overline{H}_0\,H_{1,1,1,0} + 2\,H_1\,\overline{H}_{3,1} + 2\,\overline{H}_1\,H_{3,1} + 2\,H_{0,0}\,\overline{H}_{0,0,0} \\
& +2\,H_{0,0}\,\overline{H}_3 + 2\,H_{0,0}\,\overline{H}_{1,1,1} + 2\,\overline{H}_{0,0}\,H_{0,0,0} + 2\,\overline{H}_{0,0}\,H_3 + 2\,\overline{H}_{0,0}\,H_{1,1,1} - 2\,H_2\,\zeta_3 \\
& -2\,\overline{H}_2\,\zeta_3 - 2\,H_{1,0}\,\zeta_3 - 2\,\overline{H}_{1,0}\,\zeta_3 + 2\,H_{1,1}\,\overline{H}_3 + 2\,\overline{H}_{1,1}\,H_3 - 4\,H_{1,1}\,\zeta_3 - 4\,\overline{H}_{1,1}\,\zeta_3
\end{aligned}
\tag{A.10}
$$

$$
-2\,H_0\,\overline{H}_1\,\zeta_3 - 2\,\overline{H}_0\,H_1\,\zeta_3\Bigg],
$$

$$
\begin{aligned}
L^{+}_{2,2,1} \;=\; \frac{1}{4}\Bigg[& H_5 + \overline{H}_5 + H_{4,1} + \overline{H}_{4,1} + H_{2,3} + \overline{H}_{2,3} + H_{1,0,0,0,0} + \overline{H}_{1,0,0,0,0} + H_{1,3,0} \\
& +\overline{H}_{1,3,0} + H_{1,1,0,0,0} + \overline{H}_{1,1,0,0,0} + 2\,H_{0,0,0,0,0} + 2\,\overline{H}_{0,0,0,0,0} + 2\,H_{4,0} + 2\,\overline{H}_{4,0} \\
& +2\,H_{2,0,0,0} + 2\,\overline{H}_{2,0,0,0} + 2\,H_{2,2,0} + 2\,\overline{H}_{2,2,0} + 2\,H_{2,2,1} + 2\,\overline{H}_{2,2,1} + 2\,H_{1,1,2,0} \\
& +2\,\overline{H}_{1,1,2,0} - 6\,\zeta_5 + H_0\,\overline{H}_{1,0,0,0} + H_0\,\overline{H}_{1,3} + H_0\,\overline{H}_{1,1,0,0} + \overline{H}_0\,H_{1,0,0,0} + \overline{H}_0\,H_{1,3} \\
& +\overline{H}_0\,H_{1,1,0,0} + H_1\,\overline{H}_{0,0,0,0} + H_1\,\overline{H}_4 + H_1\,\overline{H}_{2,0,0} + \overline{H}_1\,H_{0,0,0,0} + \overline{H}_1\,H_4 + \overline{H}_1\,H_{2,0,0} \\
& +H_{0,0}\,\overline{H}_{1,0,0} + H_{0,0}\,\overline{H}_{1,1,0} + \overline{H}_{0,0}\,H_{1,0,0} + \overline{H}_{0,0}\,H_{1,1,0} + H_2\,\overline{H}_{1,0,0} + \overline{H}_2\,H_{1,0,0} \\
& +H_{1,0}\,\overline{H}_{0,0,0} + H_{1,0}\,\overline{H}_{2,0} + \overline{H}_{1,0}\,H_{0,0,0} + \overline{H}_{1,0}\,H_{2,0} + H_{1,1}\,\overline{H}_{0,0,0} + \overline{H}_{1,1}\,H_{0,0,0} \\
& +2\,H_0\,\overline{H}_{0,0,0,0} + 2\,H_0\,\overline{H}_4 + 2\,H_0\,\overline{H}_{2,0,0} + 2\,H_0\,\overline{H}_{2,2} + 2\,H_0\,\overline{H}_{1,1,2} + 2\,\overline{H}_0\,H_{0,0,0,0} \\
& +2\,\overline{H}_0\,H_4 + 2\,\overline{H}_0\,H_{2,0,0} + 2\,\overline{H}_0\,H_{2,2} + 2\,\overline{H}_0\,H_{1,1,2} + 2\,H_1\,\overline{H}_{2,2} + 2\,\overline{H}_1\,H_{2,2} \\
& +2\,H_{0,0}\,\overline{H}_{0,0,0} + 2\,H_{0,0}\,\overline{H}_{2,0} + 2\,\overline{H}_{0,0}\,H_{0,0,0} + 2\,\overline{H}_{0,0}\,H_{2,0} + 2\,H_2\,\overline{H}_{0,0,0} + 2\,H_2\,\overline{H}_{2,0} \\
& +2\,H_2\,\overline{H}_{1,1,0} + 2\,H_2\,\zeta_3 + 2\,\overline{H}_2\,H_{0,0,0} + 2\,\overline{H}_2\,H_{2,0} + 2\,\overline{H}_2\,H_{1,1,0} + 2\,\overline{H}_2\,\zeta_3 \\
& +2\,H_{1,1}\,\overline{H}_{2,0} + 2\,\overline{H}_{1,1}\,H_{2,0} - 4\,H_{0,0}\,\zeta_3 - 4\,\overline{H}_{0,0}\,\zeta_3 + 4\,H_{1,0}\,\zeta_3 + 4\,\overline{H}_{1,0}\,\zeta_3 \\
& +8\,H_{1,1}\,\zeta_3 + 8\,\overline{H}_{1,1}\,\zeta_3 - 4\,H_0\,\overline{H}_0\,\zeta_3 + 4\,H_0\,\overline{H}_1\,\zeta_3 + 4\,\overline{H}_0\,H_1\,\zeta_3\Bigg],
\end{aligned} \tag{A.11}
$$

$$
\begin{aligned}
L^{-}_{4,1} \;=\; \frac{1}{4}\Bigg[& H_0\,\overline{H}_{2,0,0} + H_0\,\overline{H}_{1,0,0,0} + \overline{H}_0\,H_{2,0,0} + \overline{H}_0\,H_{1,0,0,0} + H_1\,\overline{H}_{0,0,0,0} + \overline{H}_1\,H_{0,0,0,0} \\
& +H_{0,0}\,\overline{H}_{2,0} + H_{0,0}\,\overline{H}_{1,0,0} + \overline{H}_{0,0}\,H_{2,0} + \overline{H}_{0,0}\,H_{1,0,0} + H_2\,\overline{H}_{0,0,0} + \overline{H}_2\,H_{0,0,0} \\
& +H_{1,0}\,\overline{H}_{0,0,0} + \overline{H}_{1,0}\,H_{0,0,0} + 2\,H_0\,\overline{H}_{0,0,0,0} + 2\,H_0\,\overline{H}_{1,1,0,0} + 2\,\overline{H}_0\,H_{0,0,0,0} \\
& +2\,\overline{H}_0\,H_{1,1,0,0} + 2\,H_{0,0}\,\overline{H}_{0,0,0} + 2\,H_{0,0}\,\overline{H}_{1,1,0} + 2\,\overline{H}_{0,0}\,H_{0,0,0} + 2\,\overline{H}_{0,0}\,H_{1,1,0} \\
& +2\,H_{1,1}\,\overline{H}_{0,0,0} + 2\,\overline{H}_{1,1}\,H_{0,0,0} - 4\,H_{0,0}\,\zeta_3 - 4\,H_{1,0}\,\zeta_3 + H_{4,0} + H_{2,0,0,0} \\
& +H_{1,0,0,0,0} + 2\,H_{0,0,0,0,0} + 2\,H_{4,1} + 2\,H_{1,1,0,0,0} - 4\,\overline{H}_{0,0}\,\zeta_3 - 4\,\overline{H}_{1,0}\,\zeta_3 + \overline{H}_{4,0} \\
& +\overline{H}_{2,0,0,0} + \overline{H}_{1,0,0,0,0} + 2\,\overline{H}_{0,0,0,0,0} + 2\,\overline{H}_{4,1} + 2\,\overline{H}_{1,1,0,0,0} - 4\,H_0\,\overline{H}_0\,\zeta_3 \\
& -4\,H_0\,\overline{H}_1\,\zeta_3 - 4\,\overline{H}_0\,H_1\,\zeta_3 + H_0\,\overline{H}_4 + \overline{H}_0\,H_4 + 2\,H_1\,\overline{H}_4 + 2\,\overline{H}_1\,H_4 \\
& +H_5 + \overline{H}_5 - 4\,\zeta_5\Bigg],
\end{aligned} \tag{A.12}
$$

$$
\begin{aligned}
L^{-}_{3,2} \;=\; \frac{1}{4}\Bigg[& H_0\,\overline{H}_{1,0,0,0} + \overline{H}_0\,H_{1,0,0,0} + H_1\,\overline{H}_{0,0,0,0} + \overline{H}_1\,H_{0,0,0,0} + H_{0,0}\,\overline{H}_{1,0,0} \\
& +\overline{H}_{0,0}\,H_{1,0,0} + H_{1,0}\,\overline{H}_{0,0,0} + \overline{H}_{1,0}\,H_{0,0,0} + 2\,H_0\,\overline{H}_{0,0,0,0} + 2\,H_0\,\overline{H}_{3,0} + 2\,H_0\,\overline{H}_{1,2,0} \\
& +2\,\overline{H}_0\,H_{0,0,0,0} + 2\,\overline{H}_0\,H_{3,0} + 2\,\overline{H}_0\,H_{1,2,0} + 2\,H_1\,\overline{H}_{3,0} + 2\,\overline{H}_1\,H_{3,0} + 2\,H_{0,0}\,\overline{H}_{0,0,0}
\end{aligned} \tag{A.13}
$$

$$+2\,H_{0,0}\,\overline{H}_{3} + 2\,H_{0,0}\,\overline{H}_{1,2} + 2\,\overline{H}_{0,0}\,H_{0,0,0} + 2\,\overline{H}_{0,0}\,H_{3} + 2\,\overline{H}_{0,0}\,H_{1,2} + 2\,H_{1,0}\,\overline{H}_{3}$$
$$+2\,\overline{H}_{1,0}\,H_{3} + 8\,H_{0,0}\,\zeta_3 + 8\,H_{1,0}\,\zeta_3 + H_{1,0,0,0,0} + 2\,H_{0,0,0,0,0} + 2\,H_{3,0,0} + 2\,H_{3,2}$$
$$+2\,H_{1,2,0,0} + 8\,\overline{H}_{0,0}\,\zeta_3 + 8\,\overline{H}_{1,0}\,\zeta_3 + \overline{H}_{1,0,0,0,0} + 2\,\overline{H}_{0,0,0,0,0} + 2\,\overline{H}_{3,0,0} + 2\,\overline{H}_{3,2}$$
$$\left.+2\,\overline{H}_{1,2,0,0} + 8\,H_0\,\overline{H}_0\,\zeta_3 + 8\,H_0\,\overline{H}_1\,\zeta_3 + 8\,\overline{H}_0\,H_1\,\zeta_3 + H_5 + \overline{H}_5 + 16\,\zeta_5\right],$$

$$L^{-}_{2,1,1,1} = \frac{1}{4}\left[H_5 + \overline{H}_5 + H_{4,1} + \overline{H}_{4,1} + H_{3,2} + \overline{H}_{3,2} + H_{3,1,1} + \overline{H}_{3,1,1} + H_{2,3} + \overline{H}_{2,3}\right. \tag{A.14}$$
$$+H_{2,2,1} + \overline{H}_{2,2,1} + H_{2,1,2} + \overline{H}_{2,1,2} + H_{1,0,0,0,0} + \overline{H}_{1,0,0,0,0} + H_{1,3,0} + \overline{H}_{1,3,0}$$
$$+H_{1,2,0,0} + \overline{H}_{1,2,0,0} + H_{1,2,1,0} + \overline{H}_{1,2,1,0} + H_{1,1,0,0,0} + \overline{H}_{1,1,0,0,0} + H_{1,1,2,0} + \overline{H}_{1,1,2,0}$$
$$+H_{1,1,1,0,0} + \overline{H}_{1,1,1,0,0} + 2\,H_{0,0,0,0,0} + 2\,\overline{H}_{0,0,0,0,0} + 2\,H_{4,0} + 2\,\overline{H}_{4,0} + 2\,H_{3,0,0}$$
$$+2\,\overline{H}_{3,0,0} + 2\,H_{3,1,0} + 2\,\overline{H}_{3,1,0} + 2\,H_{2,0,0,0} + 2\,\overline{H}_{2,0,0,0} + 2\,H_{2,2,0} + 2\,\overline{H}_{2,2,0}$$
$$+2\,H_{2,1,0,0} + 2\,\overline{H}_{2,1,0,0} + 2\,H_{2,1,1,0} + 2\,\overline{H}_{2,1,1,0} + 2\,H_{2,1,1,1} + 2\,\overline{H}_{2,1,1,1} + 2\,H_{1,1,1,1,0}$$
$$+2\,\overline{H}_{1,1,1,1,0} - 4\,\zeta_5 + H_0\,\overline{H}_{1,0,0,0} + H_0\,\overline{H}_{1,3} + H_0\,\overline{H}_{1,2,0} + H_0\,\overline{H}_{1,2,1} + H_0\,\overline{H}_{1,1,0,0}$$
$$+H_0\,\overline{H}_{1,1,2} + H_0\,\overline{H}_{1,1,1,0} + \overline{H}_0\,H_{1,0,0,0} + \overline{H}_0\,H_{1,3} + \overline{H}_0\,H_{1,2,0} + \overline{H}_0\,H_{1,2,1}$$
$$+\overline{H}_0\,H_{1,1,0,0} + \overline{H}_0\,H_{1,1,2} + \overline{H}_0\,H_{1,1,1,0} + H_1\,\overline{H}_{0,0,0,0} + H_1\,\overline{H}_4 + H_1\,\overline{H}_{3,0} + H_1\,\overline{H}_{3,1}$$
$$+H_1\,\overline{H}_{2,0,0} + H_1\,\overline{H}_{2,2} + H_1\,\overline{H}_{2,1,0} + \overline{H}_1\,H_{0,0,0,0} + \overline{H}_1\,H_4 + \overline{H}_1\,H_{3,0} + \overline{H}_1\,H_{3,1}$$
$$+\overline{H}_1\,H_{2,0,0} + \overline{H}_1\,H_{2,2} + \overline{H}_1\,H_{2,1,0} + H_{0,0}\,\overline{H}_{1,0,0} + H_{0,0}\,\overline{H}_{1,2} + H_{0,0}\,\overline{H}_{1,1,0}$$
$$+H_{0,0}\,\overline{H}_{1,1,1} + \overline{H}_{0,0}\,H_{1,0,0} + \overline{H}_{0,0}\,H_{1,2} + \overline{H}_{0,0}\,H_{1,1,0} + \overline{H}_{0,0}\,H_{1,1,1} + H_2\,\overline{H}_{1,0,0}$$
$$+H_2\,\overline{H}_{1,2} + H_2\,\overline{H}_{1,1,0} + \overline{H}_2\,H_{1,0,0} + \overline{H}_2\,H_{1,2} + \overline{H}_2\,H_{1,1,0} + H_{1,0}\,\overline{H}_{0,0,0} + H_{1,0}\,\overline{H}_3$$
$$+H_{1,0}\,\overline{H}_{2,0} + H_{1,0}\,\overline{H}_{2,1} + \overline{H}_{1,0}\,H_{0,0,0} + \overline{H}_{1,0}\,H_3 + \overline{H}_{1,0}\,H_{2,0} + \overline{H}_{1,0}\,H_{2,1} + H_{1,1}\,\overline{H}_{0,0,0}$$
$$+H_{1,1}\,\overline{H}_3 + H_{1,1}\,\overline{H}_{2,0} + \overline{H}_{1,1}\,H_{0,0,0} + \overline{H}_{1,1}\,H_3 + \overline{H}_{1,1}\,H_{2,0} + 2\,H_0\,\overline{H}_{0,0,0,0} + 2\,H_0\,\overline{H}_4$$
$$+2\,H_0\,\overline{H}_{3,0} + 2\,H_0\,\overline{H}_{3,1} + 2\,H_0\,\overline{H}_{2,0,0} + 2\,H_0\,\overline{H}_{2,2} + 2\,H_0\,\overline{H}_{2,1,0} + 2\,H_0\,\overline{H}_{2,1,1}$$
$$+2\,H_0\,\overline{H}_{1,1,1,1} + 2\,\overline{H}_0\,H_{0,0,0,0} + 2\,\overline{H}_0\,H_4 + 2\,\overline{H}_0\,H_{3,0} + 2\,\overline{H}_0\,H_{3,1} + 2\,\overline{H}_0\,H_{2,0,0}$$
$$+2\,\overline{H}_0\,H_{2,2} + 2\,\overline{H}_0\,H_{2,1,0} + 2\,\overline{H}_0\,H_{2,1,1} + 2\,\overline{H}_0\,H_{1,1,1,1} + 2\,H_1\,\overline{H}_{2,1,1} + 2\,\overline{H}_1\,H_{2,1,1}$$
$$+2\,H_{0,0}\,\overline{H}_{0,0,0} + 2\,H_{0,0}\,\overline{H}_3 + 2\,H_{0,0}\,\overline{H}_{2,0} + 2\,H_{0,0}\,\overline{H}_{2,1} + 2\,\overline{H}_{0,0}\,H_{0,0,0} + 2\,\overline{H}_{0,0}\,H_3$$
$$+2\,\overline{H}_{0,0}\,H_{2,0} + 2\,\overline{H}_{0,0}\,H_{2,1} + 2\,H_2\,\overline{H}_{0,0,0} + 2\,H_2\,\overline{H}_3 + 2\,H_2\,\overline{H}_{2,0} + 2\,H_2\,\overline{H}_{2,1}$$
$$+2\,H_2\,\overline{H}_{1,1,1} + 2\,\overline{H}_2\,H_{0,0,0} + 2\,\overline{H}_2\,H_3 + 2\,\overline{H}_2\,H_{2,0} + 2\,\overline{H}_2\,H_{2,1} + 2\,\overline{H}_2\,H_{1,1,1}$$
$$\left.+2\,H_{1,1}\,\overline{H}_{2,1} - 2\,H_{1,1}\,\zeta_3 + 2\,\overline{H}_{1,1}\,H_{2,1} - 2\,\overline{H}_{1,1}\,\zeta_3\right].$$

A.7 Expression of Brown's SVHPLs in terms of the $L^{\pm}$ functions

In this appendix we present the expression of Brown's SVHPLs corresponding to Lyndon words in terms of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ eigenfunctions $L_w^{\pm}(z)$.

$$
\begin{aligned}
L_0 &= L_0^- \,, \\
L_1 &= L_1^+ - \frac{1}{2}\, L_0^- \,, \\
L_2 &= L_2^- \,, \\
L_3 &= L_3^+ - \frac{1}{12}\, [L_0^-]^3 \,, \\
L_{2,1} &= -\frac{1}{4}\, L_1^+\, [L_0^-]^2 + \frac{1}{2}\, L_3^+ + L_{2,1}^- + \zeta_3 \,, \\
L_4 &= L_4^- \,, \\
L_{3,1} &= -\frac{1}{4}\, L_2^-\, [L_0^-]^2 + L_4^- + L_{3,1}^+ \,, \\
L_{2,1,1} &= -\frac{1}{4}\, L_2^-\, L_0^-\, L_1^+ + \frac{1}{2}\, L_{3,1}^+ + L_{2,1,1}^- \,, \\
L_5 &= L_5^+ - \frac{1}{240}\, [L_0^-]^5 \,, \\
L_{4,1} &= \frac{1}{48}\, L_1^+\, [L_0^-]^4 - \frac{1}{4}\, L_3^+\, [L_0^-]^2 + \frac{1}{2}\, [L_0^-]^2\, \zeta_3 + \frac{3}{2}\, L_5^+ + L_{4,1}^- + \zeta_5 \,, \\
L_{3,2} &= -\frac{1}{16}\, L_1^+\, [L_0^-]^4 + \frac{1}{2}\, L_3^+\, [L_0^-]^2 - \frac{7}{2}\, L_5^+ - [L_0^-]^2\, \zeta_3 + L_{3,2}^- - 4\,\zeta_5 \,, \\
L_{3,1,1} &= \frac{1}{16}\, [L_0^-]^3\, [L_1^+]^2 - \frac{1}{4}\, L_{2,1}^-\, [L_0^-]^2 + \frac{7}{960}\, [L_0^-]^5 - \frac{1}{4}\, L_0^-\, L_1^+\, L_3^+ + \frac{1}{2}\, L_0^-\, L_1^+\, \zeta_3 \\
&\quad + L_{4,1}^- + L_{3,1,1}^+ \,, \\
L_{2,2,1} &= -\frac{3}{16}\, [L_0^-]^3\, [L_1^+]^2 + \frac{1}{2}\, L_{2,1}^-\, [L_0^-]^2 - \frac{13}{960}\, [L_0^-]^5 + \frac{3}{4}\, L_0^-\, L_1^+\, L_3^+ - \frac{1}{2}\, L_0^-\, L_1^+\, \zeta_3 \\
&\quad - \frac{7}{2}\, L_{4,1}^- - \frac{1}{2}\, L_{3,2}^- + L_{2,2,1}^+ \,, \\
L_{2,1,1,1} &= \frac{1}{48}\, [L_0^-]^2\, [L_1^+]^3 - \frac{1}{192}\, L_1^+\, [L_0^-]^4 + \frac{1}{16}\, L_3^+\, [L_0^-]^2 - \frac{1}{8}\, [L_0^-]^2\, \zeta_3 - \frac{1}{4}\, L_0^-\, L_{2,1}^-\, L_1^+ \\
&\quad - \frac{1}{4}\, L_5^+ + \frac{1}{2}\, L_{3,1,1}^+ + \frac{1}{2}\, \zeta_5 + L_{2,1,1,1}^- \,, \\
L_6 &= L_6^- \,, \\
L_{5,1} &= -\frac{1}{4}\, L_4^-\, [L_0^-]^2 + \frac{1}{48}\, L_2^-\, [L_0^-]^4 + 2\, L_6^- + L_{5,1}^+ \,, \\
L_{4,2} &= \frac{3}{4}\, L_4^-\, [L_0^-]^2 - \frac{1}{12}\, L_2^-\, [L_0^-]^4 - \frac{11}{2}\, L_6^- + L_{4,2}^+ \,,
\end{aligned}
\qquad (A.15)
$$

$$
\begin{aligned}
L_{4,1,1} &= \frac{1}{16}\, L_2^-\, L_1^+\, [L_0^-]^3 - \frac{1}{4}\, L_{3,1}^+\, [L_0^-]^2 - \frac{1}{4}\, L_4^-\, L_0^-\, L_1^+ + \frac{1}{2}\, L_2^-\, L_0^-\, \zeta_3 + \frac{3}{2}\, L_{5,1}^+ + L_{4,1,1}^- \,, \\
L_{3,2,1} &= -\frac{3}{16}\, L_2^-\, L_1^+\, [L_0^-]^3 + \frac{1}{2}\, L_{3,1}^+\, [L_0^-]^2 + \frac{3}{4}\, L_4^-\, L_0^-\, L_1^+ - \frac{1}{2}\, L_2^-\, L_0^-\, \zeta_3 - \frac{7}{2}\, L_{5,1}^+ + L_{3,2,1}^- \,, \\
L_{3,1,2} &= -\frac{1}{4}\, L_2^-\, L_0^-\, L_3^+ - \frac{3}{2}\, L_2^-\, L_0^-\, \zeta_3 + L_{3,1,2}^- + 3\, L_{5,1}^+ + L_{4,2}^+ \,, \\
L_{3,1,1,1} &= \frac{1}{16}\, L_2^-\, [L_0^-]^2\, [L_1^+]^2 + \frac{1}{4}\, L_4^-\, [L_0^-]^2 - \frac{5}{192}\, L_2^-\, [L_0^-]^4 - \frac{1}{4}\, L_{2,1,1}^-\, [L_0^-]^2 - \frac{1}{4}\, L_0^-\, L_1^+\, L_{3,1}^+ \\
&\quad - L_6^- + L_{4,1,1}^- + L_{3,1,1,1}^+ \,, \\
L_{2,2,1,1} &= -\frac{1}{4}\, L_2^-\, [L_0^-]^2\, [L_1^+]^2 - \frac{3}{4}\, L_4^-\, [L_0^-]^2 + \frac{1}{12}\, L_2^-\, [L_0^-]^4 + \frac{3}{4}\, L_{2,1,1}^-\, [L_0^-]^2 + \frac{11}{4}\, L_6^- \\
&\quad + \frac{1}{4}\, L_2^-\, L_1^+\, L_3^+ - \frac{1}{2}\, L_2^-\, L_1^+\, \zeta_3 + \frac{3}{4}\, L_0^-\, L_1^+\, L_{3,1}^+ - \frac{1}{2}\, L_{3,1,2}^- - 5\, L_{4,1,1}^- - L_{3,2,1}^- + L_{2,2,1,1}^+ \,, \\
L_{2,1,1,1,1} &= -\frac{5}{192}\, L_2^-\, L_1^+\, [L_0^-]^3 + \frac{1}{16}\, L_{3,1}^+\, [L_0^-]^2 + \frac{1}{48}\, L_2^-\, L_0^-\, [L_1^+]^3 + \frac{1}{8}\, L_4^-\, L_0^-\, L_1^+ - \frac{1}{4}\, L_2^-\, L_0^-\, \zeta_3 \\
&\quad - \frac{1}{4}\, L_0^-\, L_{2,1,1}^-\, L_1^+ - \frac{1}{4}\, L_{5,1}^+ + \frac{1}{2}\, L_{3,1,1,1}^+ + L_{2,1,1,1,1}^- \,.
\end{aligned}
\tag{A.16}
$$

B Analytic continuation of harmonic sums

In this section we review the analytic continuation of multiple harmonic sums and the structural relations between them, as presented by Blümlein [50]. Multiple harmonic sums are defined by,

$$
S_{a_1,\cdots,a_n}(N) = \sum_{k_1=1}^{N} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} \frac{\mathrm{sgn}(a_1)^{k_1}}{k_1^{|a_1|}} \cdots \frac{\mathrm{sgn}(a_n)^{k_n}}{k_n^{|a_n|}} \,, \tag{B.1}
$$

where the a_k are positive or negative integers, and N is a positive integer. For the cases in which we are interested, they are similar to the Euler-Zagier sums (3.10), except that the summation range differs slightly. They are related to Mellin transforms of real functions or distributions $f(x)$,

$$
S_{a_1,\ldots,a_n}(N) = \int_0^1 dx\; x^N\, f_{a_1,\ldots,a_n} = \mathbf{M}[f_{a_1,\ldots,a_n}(x)](N) \,. \tag{B.2}
$$

Typically $f(x)$ are HPLs weighted by factors of $1/(1 \pm x)$. To avoid singularities at $x = 1$, it is often useful to consider the +-distribution,

$$
\mathbf{M}[(f(x))_+](N) = \int_0^1 dx\; \left(x^N - 1\right)\, f(x) \,. \tag{B.3}
$$

The weight $|w|$ of the harmonic sum is given by $|w| = \sum_{k=1}^{n} |a_k|$. The number of harmonic sums of weight w is equal to $2 \cdot 3^{|w|-1}$, but not all of them are independent. For example, they obey shuffle relations [72]. It is natural to ask whether these are the only relations they satisfy. In fact, it is known that in the special case $N \to \infty$, in which the sums reduce to multiple zeta values, many new relations emerge [73, 74, 61, 75]. In ref. [50], an analytic continuation of the harmonic sums was considered. It is defined by the integral respresentation, eq. (B.2), where N is allowed to take complex values. This allows for two new operations—differentiation and evaluation at fractional arguments—which generate new structural relations among the harmonic sums.

In the present work, harmonic sums with negative indices do not appear, so we will assume that $a_k > 0$. This assumption provides a considerable simplification. The derivative relations allow for the extraction of logarithmic factors,

$$\mathbf{M}[\log^l(x) f(x)](N) = \frac{d^l}{dN^l}\mathbf{M}[f(x)](N)\,, \tag{B.4}$$

which explains why the derivatives of the building blocks in Section 6 generate SVHPLs. In ref. [50], all available relations are imposed, and the following are the irreducible functions through weight five:

<u>weight 1</u>

$$S_1(N) = \psi(N+1) + \gamma_E = \mathbf{M}\left[\left(\frac{1}{x-1}\right)_+\right](N) \tag{B.5}$$

<u>weight 3</u>

$$F_4(N) = \mathbf{M}\left[\left(\frac{\mathrm{Li}_2(x)}{1-x}\right)_+\right](N) \tag{B.6}$$

<u>weight 4</u>

$$\begin{aligned} F_{6a}(N) &= \mathbf{M}\left[\left(\frac{\mathrm{Li}_3(x)}{1-x}\right)_+\right](N) \\ F_7(N) &= \mathbf{M}\left[\left(\frac{S_{1,2}(x)}{x-1}\right)_+\right](N) \end{aligned} \tag{B.7}$$

<u>weight 5</u>

$$\begin{aligned} F_9(N) &= \mathbf{M}\left[\left(\frac{\mathrm{Li}_4(x)}{x-1}\right)_+\right](N) \\ F_{11}(N) &= \mathbf{M}\left[\left(\frac{S_{2,2}(x)}{x-1}\right)_+\right](N) \\ F_{13}(N) &= \mathbf{M}\left[\left(\frac{\mathrm{Li}_2^2(x)}{x-1}\right)_+\right](N) \\ F_{17}(N) &= \mathbf{M}\left[\left(\frac{S_{1,3}(x)}{x-1}\right)_+\right](N) \end{aligned} \tag{B.8}$$

There are no irreducible basis functions of weight two. These functions are meromorphic with poles at the negative integers. To use these functions in the integral transform (4.4), we need the expansions near the poles. Actually, we only need the expansions around zero, since the expansions around any integer can be obtained from them using the recursion relations of ref. [50],

$$
\begin{aligned}
\psi^{(n)}(1+z) &= \psi^{(n)}(z) + (-1)^n \frac{n!}{z^{n+1}} \\
F_4(z) &= F_4(z-1) - \frac{1}{z}\left[\zeta_2 - \frac{S_1(z)}{z}\right] \\
F_{6a}(z) &= F_{6a}(z-1) - \frac{\zeta_3}{z} + \frac{1}{z^2}\left[\zeta_2 - \frac{S_1(z)}{z}\right] \\
F_7(z) &= F_7(z-1) + \frac{\zeta_3}{z} - \frac{1}{2z^2}\left[S_1^2(z) + S_2(z)\right] \\
F_9(z) &= F_9(z-1) + \frac{\zeta_4}{z} - \frac{\zeta_3}{z^2} + \frac{\zeta_2}{z^3} - \frac{1}{z^4} S_1(z) \\
F_{11}(z) &= F_{11}(z-1) + \frac{\zeta_4}{4z} - \frac{\zeta_3}{z^2} + \frac{1}{2z^3}\left[S_1^2(z) + S_2(z)\right] \\
F_{13}(z) &= F_{13}(z-1) + \frac{\zeta_2^2}{z} - \frac{4\zeta_3}{z^2} - \frac{2\zeta_2}{z^2} S_1(z) + \frac{2S_{2,1}(z)}{z^2} + \frac{2}{z^3}\left[S_1^2(z) + S_2(z)\right] \\
F_{17}(z) &= F_{17}(z-1) + \frac{\zeta_4}{z} - \frac{1}{6z^2}\left[S_1^3(z) + 3S_1(z)S_2(z) + 2S_3(z)\right] .
\end{aligned}
\tag{B.9}
$$

The expansions around zero can be obtained from the integral reprsesentations. We find that, for $\delta \to 0$, the expansions can all be expressed simply in terms of multiple zeta values,

$$
\begin{aligned}
S_1(\delta) &= -\sum_{n=1}^{\infty} (-\delta)^n \zeta_{n+1} , \\
F_4(\delta) &= \sum_{n=1}^{\infty} (-\delta)^n \zeta_{n+1,2} , \\
F_{6a}(\delta) &= \sum_{n=1}^{\infty} (-\delta)^n \zeta_{n+1,3} , \\
F_7(\delta) &= -\sum_{n=1}^{\infty} (-\delta)^n \zeta_{n+1,2,1} , \\
F_9(\delta) &= -\sum_{n=1}^{\infty} (-\delta)^n \zeta_{n+1,4} , \\
F_{11}(\delta) &= -\sum_{n=1}^{\infty} (-\delta)^n \zeta_{n+1,3,1} , \\
F_{13}(\delta) &= -\sum_{n=1}^{\infty} (-\delta)^n \left(2\zeta_{n+1,2,2} + 4\zeta_{n+1,3,1}\right) , \\
F_{17}(\delta) &= -\sum_{n=1}^{\infty} (-\delta)^n \zeta_{n+1,2,1,1} .
\end{aligned}
\tag{B.10}
$$

These single-variable functions can be assembled to form two-variable functions of ν and n, such that their inverse Mellin-Fourier transforms produce sums of SVHPLs. This construction is not unique, because other building blocks could be added. We choose to define the two-variable functions as,

$$\begin{aligned}
\tilde{F}_4 \;&= \operatorname{sgn}(n)\left\{F_4\Big(i\nu+\frac{|n|}{2}\Big)+F_4\Big(-i\nu+\frac{|n|}{2}\Big)-\frac{1}{4}D_\nu^2 E_{\nu,n}-\frac{1}{8}N^2 E_{\nu,n}-\frac{1}{2}V^2 E_{\nu,n}\right.\\
&\qquad\left.+\frac{1}{2}\Big(\psi_-+V\Big)D_\nu E_{\nu,n}+\zeta_2 E_{\nu,n}-4\,\zeta_3\right\}+N\left\{\frac{1}{2}V\psi_-+\frac{1}{2}\zeta_2\right\},\\
\tilde{F}_{6a} \;&= \operatorname{sgn}(n)\left\{F_{6a}\Big(i\nu+\frac{|n|}{2}\Big)-F_{6a}\Big(-i\nu+\frac{|n|}{2}\Big)-\frac{1}{12}D_\nu^3 E_{\nu,n}-\frac{3}{8}N^2\,V E_{\nu,n}-\frac{1}{2}V^3 E_{\nu,n}\right.\\
&\qquad\left.+\frac{1}{4}\Big(\psi_-+V\Big)D_\nu^2 E_{\nu,n}+\zeta_2\,D_\nu E_{\nu,n}+\zeta_3\,\psi_-\right\}+N\left\{\frac{1}{16}\left(N^2+12\,V^2\right)\psi_-+\zeta_2 V\right\},\\
\tilde{F}_7 \;&= F_7\Big(i\nu+\frac{|n|}{2}\Big)-F_7\Big(-i\nu+\frac{|n|}{2}\Big)-\frac{1}{2}\tilde{F}_{6a}+\frac{1}{2}V\tilde{F}_4-\left[\frac{1}{8}(\psi_-)^2-\frac{1}{4}\psi'_+ +\frac{1}{2}\zeta_2\right]D_\nu E_{\nu,n}\\
&\quad+\left[\frac{1}{2}\tilde{F}_4+\frac{1}{16}N^2 E_{\nu,n}+\frac{1}{4}V^2 E_{\nu,n}-\frac{1}{4}V\,D_\nu E_{\nu,n}+\frac{1}{8}D_\nu^2 E_{\nu,n}-\zeta_3\right]\psi_- +5\,V\zeta_3\\
&\quad+\operatorname{sgn}(n)N\left\{-\frac{1}{8}V\,E_{\nu,n}^2-\frac{1}{2}V^3-\frac{3}{32}V N^2-\left[\frac{1}{8}(\psi_-)^2-\frac{1}{4}\psi'_+ +\frac{1}{2}\zeta_2\right]V\right\},
\end{aligned} \tag{B.11}$$

where

$$\begin{aligned}
\psi_- &\equiv \psi\Big(1+i\nu+\frac{|n|}{2}\Big)-\psi\Big(1-i\nu+\frac{|n|}{2}\Big),\\
\psi'_+ &\equiv \psi'\Big(1+i\nu+\frac{|n|}{2}\Big)+\psi'\Big(1-i\nu+\frac{|n|}{2}\Big).
\end{aligned} \tag{B.12}$$

B.1 The basis in (ν, n) space in terms of single-valued HPLs

In this appendix we present the analytic expressions for the basis of $\mathbb{Z}_2\times\mathbb{Z}_2$ eigenfunctions in (ν, n) space in terms of single-valued HPLs in (w, w^*) space up to weight five. The $\mathbb{Z}_2\times\mathbb{Z}_2$ acts on (w, w^*) space via inversion, while it acts on (ν, n) space via via $[n\leftrightarrow -n, \nu\leftrightarrow -\nu]$ and $[\nu\leftrightarrow -\nu]$. The eigenvalue under $\mathbb{Z}_2\times\mathbb{Z}_2$ in (w, w^*) space will be referred to as *parity*.

Basis of weight 1 with parity $(+,+)$:

$$\mathcal{I}\,[1] \;= 2\,L_1^+\,. \tag{B.13}$$

Basis of weight 1 with parity $(+,-)$:

$$\mathcal{I}\,[\delta_{0,n}] \;= L_0^-\,. \tag{B.14}$$

Basis of weight 2 with parity $(+,+)$:

$$\mathcal{I}\left[E_{\nu,n}\right] \;=\; [L_1^+]^2 - \frac{1}{4}\,[L_0^-]^2\,, \tag{B.15}$$

$$\mathcal{I}\left[\delta_{0,n}/(i\nu)\right] \;=\; \frac{1}{2}\,[L_0^-]^2\,. \tag{B.16}$$

Basis of weight 2 with parity $(+,-)$:

$$\mathcal{I}\left[V\right] \,= -L_0^-\, L_1^+\,. \tag{B.17}$$

Basis of weight 2 with parity $(-,-)$:

$$\mathcal{I}\left[N\right] \,= 4\,L_2^-\,. \tag{B.18}$$

Basis of weight 3 with parity $(+,+)$:

$$\mathcal{I}\left[E_{\nu,n}^2\right] \;=\; \frac{2}{3}\,[L_1^+]^3 - L_3^+\,, \tag{B.19}$$

$$\mathcal{I}\left[N^2\right] \;=\; 12\,L_3^+ - 2\,L_1^+\,[L_0^-]^2\,, \tag{B.20}$$

$$\mathcal{I}\left[V^2\right] \;=\; \frac{1}{2}\,L_1^+\,[L_0^-]^2 - L_3^+\,. \tag{B.21}$$

Basis of weight 3 with parity $(+,-)$:

$$\mathcal{I}\left[V\,E_{\nu,n}\right] \;=\; \frac{1}{6}\,[L_0^-]^3 - 2\,L_{2,1}^-\,, \tag{B.22}$$

$$\mathcal{I}\left[D_\nu E_{\nu,n}\right] \;=\; -\frac{1}{12}\,[L_0^-]^3 - L_0^-\,[L_1^+]^2 + 4\,L_{2,1}^-\,, \tag{B.23}$$

$$\mathcal{I}\left[\delta_{0,n}/(i\nu)^2\right] \;=\; \frac{1}{6}\,[L_0^-]^3\,. \tag{B.24}$$

Basis of weight 3 with parity $(-,+)$:

$$\mathcal{I}\left[N\,V\right] = -L_2^-\,L_0^-\,. \tag{B.25}$$

Basis of weight 3 with parity $(-,-)$:

$$\mathcal{I}\left[N\,E_{\nu,n}\right] \,= 2\,L_2^-\,L_1^+\,. \tag{B.26}$$

Basis of weight 4 with parity $(+,+)$:

$$
\begin{aligned}
\mathcal{I}\left[E^3_{\nu,n}\right] &= \frac{1}{2}\,[L_2^-]^2 + \frac{1}{2}\,[L_0^-]^2\,[L_1^+]^2 + \frac{7}{96}\,[L_0^-]^4 + \frac{1}{2}\,[L_1^+]^4 - \frac{3}{2}\,L_0^-\,L_{2,1}^- \\
&\quad - \frac{5}{2}\,L_1^+\,L_3^+ - 3\,L_1^+\,\zeta_3\,, && \text{(B.27)}\\
\mathcal{I}\left[N^2\,E_{\nu,n}\right] &= \frac{1}{12}\,[L_0^-]^4 + 2\,[L_2^-]^2 - 2\,L_0^-\,L_{2,1}^- + 2\,L_1^+\,L_3^+ - 4\,L_1^+\,\zeta_3\,, && \text{(B.28)}\\
\mathcal{I}\left[V^2\,E_{\nu,n}\right] &= -\frac{1}{2}\,[L_2^-]^2 - \frac{1}{4}\,[L_0^-]^2\,[L_1^+]^2 - \frac{1}{12}\,[L_0^-]^4 + \frac{3}{2}\,L_0^-\,L_{2,1}^- + \frac{1}{2}\,L_1^+\,L_3^+ \\
&\quad - L_1^+\,\zeta_3\,, && \text{(B.29)}\\
\mathcal{I}\left[V\,D_\nu E_{\nu,n}\right] &= \frac{3}{4}\,[L_0^-]^2\,[L_1^+]^2 + \frac{1}{16}\,[L_0^-]^4 + [L_2^-]^2 - 2\,L_0^-\,L_{2,1}^- - 2\,L_1^+\,L_3^+ + 4\,L_1^+\,\zeta_3\,, && \text{(B.30)}\\
\mathcal{I}\left[D_\nu^2 E_{\nu,n}\right] &= -\frac{1}{2}\,[L_0^-]^2\,[L_1^+]^2 - \frac{1}{24}\,[L_0^-]^4 - 2\,[L_2^-]^2 + 4\,L_1^+\,L_3^+ - 8\,L_1^+\,\zeta_3\,, && \text{(B.31)}\\
\mathcal{I}\left[\delta_{0,n}/(i\nu)^3\right] &= \frac{1}{24}\,[L_0^-]^4\,. && \text{(B.32)}
\end{aligned}
$$

Basis of weight 4 with parity $(+,-)$:

$$
\begin{aligned}
\mathcal{I}\left[V\,E^2_{\nu,n}\right] &= \frac{1}{8}\,L_1^+\,[L_0^-]^3 + \frac{1}{6}\,L_0^-\,[L_1^+]^3 - L_0^-\,\zeta_3 - 2\,L_{2,1}^-\,L_1^+\,, && \text{(B.33)}\\
\mathcal{I}\left[N^2\,V\right] &= \frac{1}{3}\,L_1^+\,[L_0^-]^3 - 2\,L_0^-\,L_3^+\,, && \text{(B.34)}\\
\mathcal{I}\left[V^3\right] &= \frac{1}{2}\,L_0^-\,L_3^+ - \frac{1}{6}\,L_1^+\,[L_0^-]^3\,, && \text{(B.35)}\\
\mathcal{I}\left[E_{\nu,n}\,D_\nu E_{\nu,n}\right] &= -\frac{1}{8}\,L_1^+\,[L_0^-]^3 - \frac{1}{2}\,L_0^-\,[L_1^+]^3 + \frac{1}{2}\,L_0^-\,L_3^+ + L_0^-\,\zeta_3 + 2\,L_{2,1}^-\,L_1^+\,, && \text{(B.36)}
\end{aligned}
$$

Basis of weight 4 with parity $(-,+)$:

$$
\begin{aligned}
\mathcal{I}\left[N\,V\,E_{\nu,n}\right] &= -2\,L_{3,1}^+\,, && \text{(B.37)}\\
\mathcal{I}\left[N\,D_\nu E_{\nu,n}\right] &= 8\,L_{3,1}^+ - 2\,L_2^-\,L_0^-\,L_1^+\,. && \text{(B.38)}
\end{aligned}
$$

Basis of weight 4 with parity $(-,-)$:

$$
\begin{aligned}
\mathcal{I}\left[\tilde{F}_4\right] &= -\frac{1}{4}\,L_2^-\,[L_0^-]^2 + L_2^-\,[L_1^+]^2 + 4\,L_4^- - 6\,L_{2,1,1}^-\,, && \text{(B.39)}\\
\mathcal{I}\left[N\,E^2_{\nu,n}\right] &= \frac{1}{2}\,L_2^-\,[L_0^-]^2 - 6\,L_4^- + 8\,L_{2,1,1}^-\,, && \text{(B.40)}\\
\mathcal{I}\left[N^3\right] &= 40\,L_4^- - 6\,L_2^-\,[L_0^-]^2\,, && \text{(B.41)}\\
\mathcal{I}\left[N\,V^2\right] &= \frac{1}{2}\,L_2^-\,[L_0^-]^2 - 2\,L_4^-\,. && \text{(B.42)}
\end{aligned}
$$

Basis of weight 5 with parity $(+,+)$:

$$\mathcal{I}\left[E_{\nu,n}^4\right] = \frac{17}{96}\,L_1^+\,[L_0^-]^4 - \frac{5}{4}\,L_3^+\,[L_0^-]^2 + \frac{2}{5}\,[L_1^+]^5 + \frac{43}{4}\,L_5^+ + [L_0^-]^2\,[L_1^+]^3 + 4\,[L_0^-]^2\,\zeta_3 - 4\,L_3^+\,[L_1^+]^2 - 8\,[L_1^+]^2\,\zeta_3 - 4\,L_0^-\,L_{2,1}^-\,L_1^+ + 12\,L_{3,1,1}^+ + 8\,L_{2,2,1}^+\,, \tag{B.43}$$

$$\mathcal{I}\left[N^2\,E_{\nu,n}^2\right] = \frac{1}{3}\,[L_0^-]^2\,[L_1^+]^3 - \frac{1}{24}\,L_1^+\,[L_0^-]^4 + 4\,L_1^+\,[L_2^-]^2 + 3\,L_3^+\,[L_0^-]^2 - 8\,[L_0^-]^2\,\zeta_3 - 25\,L_5^+ - 24\,L_{3,1,1}^+ - 16\,L_{2,2,1}^+\,, \tag{B.44}$$

$$\mathcal{I}\left[N^4\right] = \frac{13}{6}\,L_1^+\,[L_0^-]^4 - 20\,L_3^+\,[L_0^-]^2 + 140\,L_5^+\,, \tag{B.45}$$

$$\mathcal{I}\left[V^2\,E_{\nu,n}^2\right] = -\frac{1}{12}\,[L_0^-]^2\,[L_1^+]^3 - \frac{13}{96}\,L_1^+\,[L_0^-]^4 + \frac{1}{4}\,L_3^+\,[L_0^-]^2 - \frac{1}{4}\,L_5^+ - L_1^+\,[L_2^-]^2 + 2\,[L_0^-]^2\,\zeta_3 + 10\,L_{3,1,1}^+ + 4\,L_{2,2,1}^+ - 4\,\zeta_5\,, \tag{B.46}$$

$$\mathcal{I}\left[N^2\,V^2\right] = -\frac{1}{8}\,L_1^+\,[L_0^-]^4 + L_3^+\,[L_0^-]^2 - 5\,L_5^+\,, \tag{B.47}$$

$$\mathcal{I}\left[V^4\right] = \frac{5}{96}\,L_1^+\,[L_0^-]^4 - \frac{1}{4}\,L_3^+\,[L_0^-]^2 + \frac{3}{4}\,L_5^+\,, \tag{B.48}$$

$$\mathcal{I}\left[V\,E_{\nu,n}\,D_\nu E_{\nu,n}\right] = \frac{7}{48}\,L_1^+\,[L_0^-]^4 - \frac{3}{4}\,L_3^+\,[L_0^-]^2 - \frac{3}{2}\,[L_0^-]^2\,\zeta_3 + \frac{7}{2}\,L_5^+ + L_1^+\,[L_2^-]^2 + L_0^-\,L_{2,1}^-\,L_1^+ - 12\,L_{3,1,1}^+ - 4\,L_{2,2,1}^+ + 6\,\zeta_5\,, \tag{B.49}$$

$$\mathcal{I}\left[[D_\nu E_{\nu,n}]^2\right] = \frac{3}{2}\,[L_0^-]^2\,[L_1^+]^3 - \frac{1}{3}\,L_1^+\,[L_0^-]^4 - 2\,L_1^+\,[L_2^-]^2 + 2\,L_3^+\,[L_0^-]^2 + 2\,[L_0^-]^2\,\zeta_3 - 4\,L_3^+\,[L_1^+]^2 + 8\,[L_1^+]^2\,\zeta_3 - 8\,L_0^-\,L_{2,1}^-\,L_1^+ - 9\,L_5^+ + 48\,L_{3,1,1}^+ + 16\,L_{2,2,1}^+ - 24\,\zeta_5\,, \tag{B.50}$$

$$\mathcal{I}\left[E_{\nu,n}\,D_\nu^2 E_{\nu,n}\right] = \frac{1}{6}\,L_1^+\,[L_0^-]^4 - [L_0^-]^2\,[L_1^+]^3 - L_3^+\,[L_0^-]^2 + 4\,L_3^+\,[L_1^+]^2 - 8\,[L_1^+]^2\,\zeta_3 + 4\,L_0^-\,L_{2,1}^-\,L_1^+ + 2\,L_5^+ - 24\,L_{3,1,1}^+ - 8\,L_{2,2,1}^+ + 12\,\zeta_5 \tag{B.51}$$

$$\mathcal{I}\left[N\,\tilde{F}_4\right] = \frac{1}{12}\,L_1^+\,[L_0^-]^4 - \frac{7}{4}\,L_3^+\,[L_0^-]^2 + \frac{7}{2}\,[L_0^-]^2\,\zeta_3 - L_1^+\,[L_2^-]^2 - L_0^-\,L_{2,1}^-\,L_1^+ + 15\,L_5^+ + 12\,L_{3,1,1}^+ + 8\,L_{2,2,1}^+\,. \tag{B.52}$$

Basis of weight 5 with parity $(+,-)$:

$$\mathcal{I}\left[\tilde{F}_7\right] = \frac{5}{8}\,L_0^-\,[L_2^-]^2 - \frac{11}{48}\,[L_0^-]^3\,[L_1^+]^2 + \frac{1}{4}\,L_{2,1}^-\,[L_0^-]^2 + \frac{59}{3840}\,[L_0^-]^5 + \frac{5}{48}\,L_0^-\,[L_1^+]^4 + \frac{3}{2}\,L_0^-\,L_1^+\,L_3^+ - \frac{7}{2}\,L_{3,2}^- - L_{2,1}^-\,[L_1^+]^2 - 8\,L_0^-\,L_1^+\,\zeta_3 - 10\,L_{4,1}^- + 7\,L_{2,1,1,1}^-\,, \tag{B.53}$$

$$\mathcal{I}\left[V\,E_{\nu,n}^3\right] = \frac{1}{2}\,L_0^-\,[L_2^-]^2 + \frac{3}{16}\,[L_0^-]^3\,[L_1^+]^2 + \frac{3}{4}\,L_{2,1}^-\,[L_0^-]^2 - \frac{1}{192}\,[L_0^-]^5 - \frac{1}{4}\,L_0^-\,L_1^+\,L_3^+ + \frac{9}{2}\,L_0^-\,L_1^+\,\zeta_3 - \frac{9}{2}\,L_{3,2}^- - 6\,L_{4,1}^- - 12\,L_{2,1,1,1}^-\,, \tag{B.54}$$

$$\mathcal{I}\left[N^2\, V\, E_{\nu,n}\right] \;=\; -\frac{1}{4}\,[L_0^-]^3\,[L_1^+]^2 - \frac{1}{48}\,[L_0^-]^5 + L_{2,1}^-\,[L_0^-]^2 + L_0^-\,L_1^+\,L_3^+ - 2\,L_0^-\,L_1^+\,\zeta_3 - 8\,L_{4,1}^- - 2\,L_{3,2}^-\,, \tag{B.55}$$

$$\mathcal{I}\left[V^3\, E_{\nu,n}\right] \;=\; \frac{3}{16}\,[L_0^-]^3\,[L_1^+]^2 - \frac{3}{4}\,L_{2,1}^-\,[L_0^-]^2 + \frac{23}{960}\,[L_0^-]^5 - \frac{3}{4}\,L_0^-\,L_1^+\,L_3^+ + \frac{3}{2}\,L_0^-\,L_1^+\,\zeta_3 + \frac{3}{2}\,L_{3,2}^- + 4\,L_{4,1}^-\,, \tag{B.56}$$

$$\mathcal{I}\left[E_{\nu,n}^2\, D_\nu E_{\nu,n}\right] \;=\; -\frac{1}{2}\,L_0^-\,[L_2^-]^2 - \frac{7}{24}\,[L_0^-]^3\,[L_1^+]^2 - \frac{1}{48}\,[L_0^-]^5 - \frac{1}{6}\,L_0^-\,[L_1^+]^4 + L_0^-\,L_1^+\,L_3^+ - 2\,L_0^-\,L_1^+\,\zeta_3 + 4\,L_{4,1}^- + 3\,L_{3,2}^- + 8\,L_{2,1,1,1}^-\,, \tag{B.57}$$

$$\mathcal{I}\left[N^2\, D_\nu E_{\nu,n}\right] \;=\; \frac{3}{2}\,[L_0^-]^3\,[L_1^+]^2 + \frac{1}{24}\,[L_0^-]^5 - 2\,L_0^-\,[L_2^-]^2 - 4\,L_{2,1}^-\,[L_0^-]^2 - 8\,L_0^-\,L_1^+\,L_3^+ + 16\,L_0^-\,L_1^+\,\zeta_3 + 48\,L_{4,1}^- + 12\,L_{3,2}^-\,, \tag{B.58}$$

$$\mathcal{I}\left[V^2\, D_\nu E_{\nu,n}\right] \;=\; \frac{1}{2}\,L_0^-\,[L_2^-]^2 - \frac{3}{8}\,[L_0^-]^3\,[L_1^+]^2 - \frac{1}{480}\,[L_0^-]^5 + L_{2,1}^-\,[L_0^-]^2 + 2\,L_0^-\,L_1^+\,L_3^+ - 4\,L_0^-\,L_1^+\,\zeta_3 - 12\,L_{4,1}^- - 5\,L_{3,2}^-\,, \tag{B.59}$$

$$\mathcal{I}\left[V\, D_\nu^2 E_{\nu,n}\right] \;=\; -\frac{1}{15}\,[L_0^-]^5 - 2\,L_0^-\,[L_2^-]^2 - 2\,L_0^-\,L_1^+\,L_3^+ + 4\,L_0^-\,L_1^+\,\zeta_3 + 24\,L_{4,1}^- + 12\,L_{3,2}^-\,, \tag{B.60}$$

$$\mathcal{I}\left[D_\nu^3 E_{\nu,n}\right] \;=\; \frac{1}{2}\,[L_0^-]^3\,[L_1^+]^2 + \frac{7}{40}\,[L_0^-]^5 + 6\,L_0^-\,[L_2^-]^2 - 48\,L_{4,1}^- - 24\,L_{3,2}^-\,, \tag{B.61}$$

$$\mathcal{I}\left[\delta_{0,n}/(i\nu)^4\right] \;=\; \frac{1}{120}\,[L_0^-]^5\,. \tag{B.62}$$

Basis of weight 5 with parity $(-,+)$:

$$\mathcal{I}\left[\tilde{F}_{6a}\right] \;=\; \frac{1}{12}\,L_2^-\,[L_0^-]^3 - L_4^-\,L_0^- + L_2^-\,L_{2,1}^- - L_1^+\,L_{3,1}^+\,, \tag{B.63}$$

$$\mathcal{I}\left[V\,\tilde{F}_4\right] \;=\; \frac{1}{48}\,L_2^-\,[L_0^-]^3 - \frac{1}{2}\,L_2^-\,L_0^-\,[L_1^+]^2 - \frac{3}{4}\,L_4^-\,L_0^- - L_2^-\,L_{2,1}^- + 3\,L_0^-\,L_{2,1,1}^- + L_1^+\,L_{3,1}^+\,, \tag{B.64}$$

$$\mathcal{I}\left[N\,V\,E_{\nu,n}^2\right] \;=\; -\frac{1}{48}\,L_2^-\,[L_0^-]^3 + \frac{1}{2}\,L_2^-\,L_0^-\,[L_1^+]^2 + \frac{3}{4}\,L_4^-\,L_0^- - 2\,L_0^-\,L_{2,1,1}^- - 2\,L_1^+\,L_{3,1}^+\,, \tag{B.65}$$

$$\mathcal{I}\left[N^3\,V\right] \;=\; \frac{3}{4}\,L_2^-\,[L_0^-]^3 - 5\,L_4^-\,L_0^-\,, \tag{B.66}$$

$$\mathcal{I}\left[N\,V^3\right] \;=\; \frac{3}{4}\,L_4^-\,L_0^- - \frac{7}{48}\,L_2^-\,[L_0^-]^3\,, \tag{B.67}$$

$$\mathcal{I}\left[N\,E_{\nu,n}\,D_\nu E_{\nu,n}\right] \;=\; -\frac{5}{24}\,L_2^-\,[L_0^-]^3 + \frac{3}{2}\,L_4^-\,L_0^- - L_2^-\,L_0^-\,[L_1^+]^2 + 4\,L_1^+\,L_{3,1}^+\,. \tag{B.68}$$

Basis of weight 5 with parity $(-,-)$:

$$\mathcal{I}\left[N\,E^3_{\nu,n}\right] = \frac{5}{8}\,L_2^-\,L_1^+\,[L_0^-]^2 - \frac{15}{2}\,L_4^-\,L_1^+ - \frac{1}{2}\,L_2^-\,L_3^+ - L_2^-\,[L_1^+]^3 + 12\,L_{2,1,1}^-\,L_1^+\,, \tag{B.69}$$

$$\mathcal{I}\left[E_{\nu,n}\,N^3\right] = -\frac{1}{2}\,L_2^-\,L_1^+\,[L_0^-]^2 + 2\,L_4^-\,L_1^+ + 6\,L_2^-\,L_3^+ - 16\,L_2^-\,\zeta_3 - 4\,L_0^-\,L_{3,1}^+\,, \tag{B.70}$$

$$\mathcal{I}\left[E_{\nu,n}\,N\,V^2\right] = -\frac{1}{8}\,L_2^-\,L_1^+\,[L_0^-]^2 + \frac{1}{2}\,L_4^-\,L_1^+ - \frac{1}{2}\,L_2^-\,L_3^+ + L_0^-\,L_{3,1}^+\,, \tag{B.71}$$

$$\mathcal{I}\left[N\,V\,D_\nu E_{\nu,n}\right] = \frac{3}{4}\,L_2^-\,L_1^+\,[L_0^-]^2 - 3\,L_4^-\,L_1^+ + L_2^-\,L_3^+ + 4\,L_2^-\,\zeta_3 - 2\,L_0^-\,L_{3,1}^+\,, \tag{B.72}$$

$$\mathcal{I}\left[N\,D_\nu^2 E_{\nu,n}\right] = -L_2^-\,L_1^+\,[L_0^-]^2 + 12\,L_4^-\,L_1^+ - 4\,L_2^-\,L_3^+ - 16\,L_2^-\,\zeta_3\,, \tag{B.73}$$

$$\mathcal{I}\left[E_{\nu,n}\,\tilde{F}_4\right] = \frac{1}{8}\,L_2^-\,L_1^+\,[L_0^-]^2 + \frac{2}{3}\,L_2^-\,[L_1^+]^3 + \frac{1}{2}\,L_4^-\,L_1^+ + \frac{1}{2}\,L_2^-\,L_3^+ - \frac{1}{2}\,L_0^-\,L_{3,1}^+ - 2\,L_2^-\,\zeta_3 - 4\,L_{2,1,1}^-\,L_1^+\,. \tag{B.74}$$

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