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Non-perturbative String Theory from Water Waves

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Abstract

We use a combination of a 't Hooft limit and numerical methods to find non–perturbative solutions of exactly solvable string theories, showing that perturbative solutions in different asymptotic regimes are connected by smooth interpolating functions. Our earlier perturbative work showed that a large class of minimal string theories arise as special limits of a Painlevé IV hierarchy of string equations that can be derived by a similarity reduction of the dispersive water wave hierarchy of differential equations. The hierarchy of string equations contains new perturbative solutions, some of which were conjectured to be the type IIA and IIB string theories coupled to $(4, 4k-2)$ superconformal minimal models of type (A, D). Our present paper shows that these new theories have smooth non-perturbative extensions. We also find evidence for putative new string theories that were not apparent in the perturbative analysis.

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1 Introduction

In the quest to better understand non–perturbative phenomena in string theory, and to find clues as to the full nature of M–theory [1–3], rich solvable examples are of considerable value. The type 0 minimal string theories (formulated in refs. [4–8] and refs. [9, 10], and recognized as type 0 strings in ref. [11]) have highly tractable non–perturbative physics, and despite being exactly solvable contain a rich set of phenomena such as holography and open–closed dualities. Like the original minimal strings [12–14], they have bosonic spacetime physics, although they have worldsheet fermions, and may be thought of as two–dimensional supergravity coupled to $\hat{c} < 1$ superconformal minimal models, with a diagonal GSO projection.

The physics of the minimal strings can be succinctly formulated in terms of associated non–linear ordinary differential equations, known as the string equations. They furnish asymptotic expansions for the free energy once the boundary conditions are fixed. These expansions take the form of a string world–sheet expansion. Typically (for the cases of the type 0A and 0B systems coupled to the $(2, 4k)$ superconformal minimal models) there are two perturbative regimes in which such an expansion can be developed. One is interpreted as purely closed string backgrounds with integer, N, units of R–R flux, while the other has both open and closed string sectors with N D–branes in the background[1].

Remarkably, these models have a non–perturbative completion (first discovered in refs. [4–8] in the context of the type 0A systems) that connects these two asymptotic regimes, furnishing an example of a so–called "geometric transition" between open and closed string backgrounds. While in the case of type 0A, complete (numerical) solutions of the exact string equations for any N can be found, the connectedness of the type 0B case has been argued for on the basis of a large N, 't Hooft–like, limit where the interpolating solution can be found using algebraic techniques [11]. This has more than just shades of the AdS/CFT correspondence [17–19], where in the prototype, the open string physics of N D3–branes can be rewritten in terms of that of closed strings in $\mathrm{AdS}_5 \times S^5$ with N units of R–R flux. In the present context we have a precise analogue of this important example. There is of course a 't Hooft large N limit there too, connecting supergravity to large N Yang–Mills. That we have here not just the analogue of the large N limit but also knowledge of how to go beyond, may ultimately prove instructive.

In ref. [20], we investigated a new system called the Dispersive Water Wave (DWW)

[1]Interestingly, in the associated non–linear system — the generalized KdV system for type 0A — the N D–branes or flux units correspond to special soliton solutions, as shown in refs. [15, 16].

hierarchy [21–25], which we argued should yield new string equations *via* a similarity reduction. These string equations turn out to be a Painlevé IV hierarchy of equations. We found that both the type 0A and 0B string theories were found to be naturally embedded within this framework. In addition to this new non–perturbative connection, we found further that the two string theories are merely special points in a much larger tapestry of possibilities that also appear to be string theories. This is somewhat suggestive of an M–theory, now for minimal strings, which is exactly solvable. It clearly deserves further study.

Among the other special points which suggested themselves (using perturbation theory) were some that we conjectured to be the type IIA and IIB string theories coupled to the $(4, 4k-2)$ superconformal minimal models. Much of the analysis carried out to support this identification was using perturbative techniques.

The subject of the present paper is to continue the study of this rich system into the non–perturbative regime. We argue that the conjectured type II theories have non–perturbative completions by showing that the corresponding perturbative expansions match onto each other smoothly. This is accomplished using a combination of analytical and numerical techniques similar to those used for the type 0 theories coupled to the $(2, 4k)$ superconformal minimal models.

The outline of the paper is as follows: In sections 2 and 3, we review essential aspects of the well–known type 0 theories and provide a summary of results about the DWW system that will be needed later. We reproduce the analytic ('t Hooft limit) technique used for the type 0B string theory, as in ref. [11], in section 4 to keep the presentation self–contained. We then reformulate it in a manner that allows us to extend the technique to our DWW system. We apply this strategy to our system in Section 5 for the first two even DWW flows, which contain the type 0 and conjectured type II theories of interest and find that the type II theories possess smooth non–perturbative solutions. In addition, we find evidence for the existence of new non–perturbative solutions with novel asymptotics. In section 6, we consider the first (and simplest) flow of the DWW system. For this case, we are able to strengthen evidence for the existence of these new non–perturbative solutions with exact numerical results. We end with a brief discussion in section 7. An appendix details the expansions arising from the DWW system for the first few flows along with some of their essential properties.

2 Type 0 strings: A brief review

We begin with a brief review of the type 0 string theories coupled to the $(2, 4k)$ superconformal minimal models of type (A, A). Those familiar with these models can proceed directly to the next section.

2.1 Type 0A

Type 0A string theory coupled to the $(2, 4k)$ superconformal minimal models (first derived and studied in refs. [5–8] and identified as type 0A in ref. [11]) is described by the string equation

$$w\mathcal{R}^2 - \frac{1}{2}\mathcal{R}\mathcal{R}'' + \frac{1}{4}\mathcal{R}'^2 = \nu^2\Gamma^2 \quad , \tag{1}$$

where, for a particular model, $w(z)$ is a real function of the real variable z, a prime denotes $\nu\partial/\partial z$, and Γ and ν are real constants. The quantity $\mathcal{R}$ is a function of $w(z)$ and its z–derivatives. In general $w(z)$ additionally depends on couplings t_k associated with flowing between the various models. Then we have

$$\mathcal{R} = \sum_{k=0}^{\infty} \Big(k + \frac{1}{2}\Big) t_k P_k \quad , \tag{2}$$

where the $P_k[w]$ are polynomials in $w(z)$ and its z–derivatives, called the Gel'fand – Dikii polynomials [26]. They are related by a recursion relation (defining a recursion operator R_2)

$$P'_{k+1} = \frac{1}{4}P'''_k - wP'_k - \frac{1}{2}w'P_k \equiv R_2 \circ P'_k \ , \tag{3}$$

and fixed by the value of the constant P_0 and the requirement that the rest vanish for vanishing w. The first four are:

$$\begin{aligned} & P_0 = \frac{1}{2} \ , \quad P_1 = -\frac{1}{4}w \ , \quad P_2 = \frac{1}{16}(3w^2 - w'') \ , \\ \text{and} \qquad & P_3 = -\frac{1}{64}(10w^3 - 10ww'' - 5(w')^2 + w''') \ . \end{aligned} \tag{4}$$

The kth model is chosen by setting all the t_j to zero except $t_0 \equiv z$ and

$$t_k = \frac{(-4)^{k+1}(k!)^2}{(2k+1)!} \ . \tag{5}$$

This number is chosen so that the coefficient of w^k in $\mathcal{R}$ is set to -1.[2] The flows between various models are captured by the integrable KdV hierarchy [27, 28].

[2] This gives $w = z^{1/k} + \ldots$ as $z \to +\infty$. If we had instead chosen $t_0 = -z$, we would have chosen the coefficient of w^k to be unity.

The function $w(z)$ defines the partition function $Z = \exp(-F)$ of the string theory *via*

$$w(z) = 2\nu^2 \frac{\partial^2 F}{\partial \mu^2}\Big|_{\mu=z} \quad , \tag{6}$$

where μ is the coefficient of the lowest dimension operator in the world–sheet theory.

The asymptotic expansions of the string equations for the first two cases are:

$k = 1$

$$\begin{aligned} w(z) &= z + \frac{\nu\Gamma}{z^{1/2}} - \frac{\nu^2\Gamma^2}{2z^2} + \frac{5}{32}\frac{\nu^3}{z^{7/2}}\Gamma\left(4\Gamma^2+1\right) + \cdots \quad (z \to \infty) \\ w(z) &= 0 + \frac{\nu^2(4\Gamma^2-1)}{4z^2} + \frac{\nu^4}{8}\frac{(4\Gamma^2-1)(4\Gamma^2-9)}{z^5} + \cdots \quad (z \to -\infty) \end{aligned} \tag{7}$$

$k = 2$

$$\begin{aligned} w(z) &= z^{1/2} + \frac{\nu\Gamma}{2z^{3/4}} - \frac{1}{24}\frac{\nu^2}{z^2}\left(6\Gamma^2+1\right) + \cdots \quad (z \to \infty) \\ w(z) &= (4\Gamma^2-1)\left(\frac{\nu^2}{4z^2} + \frac{1}{32}\frac{\nu^6}{z^7}(4\Gamma^2-9)(4\Gamma^2-25) + \cdots\right) \quad (z \to -\infty) \end{aligned} \tag{8}$$

The solutions for various k for $z > 0$ can be numerically and analytically shown to match onto the solution for $z < 0$, providing a unique [6–8] non–perturbative completion of the theory. (See figure (1) for an example of a solution found using numerical methods.)

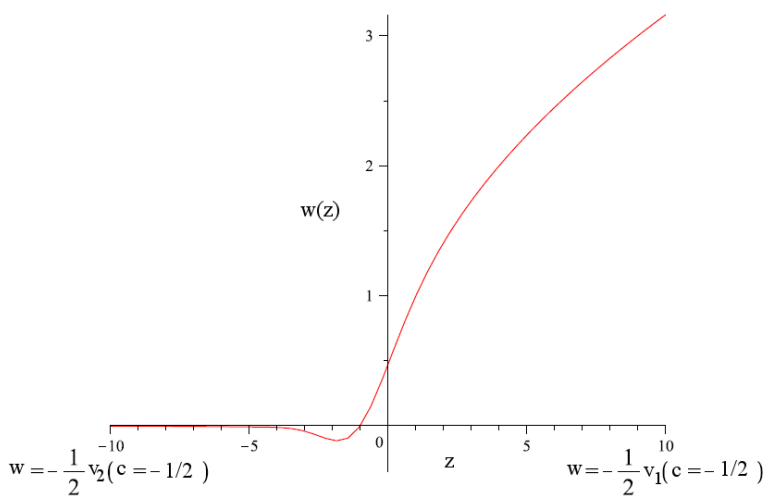

Figure 1: A plot of the $k = 2$ type 0A solution showing how the perturbative regimes at large $|z|$ are smoothly connected. Section 3 discusses a function $v(x)$ ($x \propto z$), which has a number of different classes of behaviour distinguished by choice of boundary condition. The type 0A theory has class $v_1(z)$ in the $+z$ perturbative regime and class $v_2(z)$ in the $-z$ perturbative regime. Here we have set $\nu = 1$ and $\Gamma = 0$.

In the $\mu \to +\infty$ regime, Γ represents [8, 11] the number of background ZZ D–branes [29] in the model, while in the $\mu \to -\infty$ regime, Γ represents the number of units of R–R flux in the background [11].

2.2 Type 0B strings

Type 0B string theory coupled to the $(2, 4k)$ superconformal minimal models [11] is described by the following string equations [9, 10]:

$$\sum_{l=0}^{\infty} t_l (l+1) R_l = 0 \ , \qquad \sum_{l=0}^{\infty} t_l (l+1) H_l + \nu q = 0 \ , \tag{9}$$

where the R_l and H_l are polynomials of functions $r(x)$ and $\omega(x)$ (and their derivatives), and ν and q are real constants.

The function $\widetilde{w}(x) = r^2/4$ defines the partition function of the theory *via*

$$\widetilde{w}(x) = \frac{r^2}{4} = \nu^2 \frac{d^2 F}{dx^2}\Big|_{\mu=x} \quad . \tag{10}$$

where μ is the coefficient of the lowest dimension operator in the world–sheet theory. The nth model is chosen by setting all t_l to zero except $t_0 \sim x$ and t_n. These models have an interpretation as type 0B strings coupled to the $(2, 2n)$ superconformal minimal models only for even $n = 2k$. The flows between various models are captured by the integrable Zakharov–Shabat (ZS) hierarchy [30].

The asymptotic expansions of the string equations (9) for the first two even $n = 2k$ are:

$n = 2 \ \ (k = 1)$

$$\begin{aligned} \widetilde{w}(x) &= \frac{x}{4} + \left(q^2 - \frac{1}{4}\right)\left[\frac{\nu^2}{2x^2} + \left(q^2 - \frac{9}{4}\right)\left(\frac{-2\nu^4}{x^5} + \cdots\right)\right] \ , \quad (x \to \infty) \\ \widetilde{w}(x) &= \frac{\nu q \sqrt{2}}{4|x|^{1/2}} - \frac{\nu^2 q^2}{4|x|^2} + \frac{\nu^3}{|x|^{7/2}} \frac{5\sqrt{2}}{64} q \left(1 + 4q^2\right) + \cdots \quad (x \to -\infty) \end{aligned} \tag{11}$$

$n = 4 \ \ (k = 2)$

$$\begin{aligned} \widetilde{w}(x) &= \frac{\sqrt{x}}{4} + \frac{\nu^2}{144 x^2}\left(64q^2 - 15\right) + \cdots \ ; \quad (x \to \infty) \\ \widetilde{w}(x) &= \frac{\sqrt{|x|}}{2\sqrt{14}} + \frac{\nu}{2|x|^{3/4}} \frac{q}{\sqrt{3} \cdot 7^{1/4}} + \cdots \quad (x \to -\infty) \end{aligned} \tag{12}$$

In the $\mu \to -\infty$ regime, q represents the number of background ZZ D–branes in the model, while in the $\mu \to \infty$ regime it counts the number of units of R–R flux in the

background [11]. The structure of solutions with increasing n is particularly rich [11]; the $n = 4$ expansion for $\mu \to -\infty$ shown above breaks a $\mathbb{Z}_2$ symmetry due to the presence of R–R fields in the dual string theory.

Unlike the 0A case, the solutions for $x > 0$ have not been shown numerically to match onto those for $x < 0$ so far. The highly non-linear nature of the string equations makes it difficult to numerically obtain smooth solutions connecting the two regimes for the 0B case. Nevertheless, as argued in ref. [11] and reviewed in section 4, these theories can be argued to be non-perturbatively complete in a particular ('t Hooft) limit.

For the $n = 2$ ($k = 1$) model[3], the full non–perturbative solution is known since it can be mapped directly to the solution known for the $k = 1$ type 0A case *via* the Morris map [4, 31]. The string equation for the 0A theory becomes the string equation for the 0B theory once one identifies Γ with q, but with the sign of x reversed. Analogues of the Morris map for higher n are not known.

3 The DWW system: A brief review

The DWW system introduced in ref. [20] leads to a Painlevé IV hierarchy of string equations:

$$-\frac{1}{2}\mathcal{L}_x + \frac{1}{2}u\mathcal{L} + \mathcal{K} = \nu c \tag{13}$$

$$\left(-v + \frac{1}{4}u^2 + \frac{1}{2}u_x\right)\mathcal{L}^2 - \frac{1}{2}\mathcal{L}\mathcal{L}_{xx} + \frac{1}{4}\mathcal{L}_x^2 = \nu^2\Gamma^2 \quad , \tag{14}$$

where c and Γ are real constants and ν plays the same role as for the type 0 theories (note that here and in the rest of the paper, for any function $G(x)$, G_x will denote $\nu\,\partial G/\partial x$).

The functions $u(x)$ and $v(x)$ are generalizations of the two functions $r(x)$ and $\omega(x)$ used to describe the 0B theory. The polynomials $\mathcal{L}[u, v]$ and $\mathcal{K}[u, v]$ are defined by

$$\begin{pmatrix} \mathcal{L} \\ \mathcal{K} \end{pmatrix} = \sum_{n=0}^{\infty} \frac{1}{2}(n+1)t_n \begin{pmatrix} L_n[u,v] \\ K_n[u,v] \end{pmatrix} \quad , \tag{15}$$

where $L_n[u, v]$ and $K_n[u, v]$ are polynomials in $u(x)$, $v(x)$ and their derivatives, similar to the polynomials R_n and H_n for the 0B theory. They satisfy the recursion relation

$$\begin{pmatrix} L_{n+1}[u,v] \\ K_{n+1}[u,v] \end{pmatrix} = R \begin{pmatrix} L_n[u,v] \\ K_n[u,v] \end{pmatrix} \quad .$$

[3]The central charge of the $\mathcal{N} = 1$ (p, q) super–conformal minimal models is given by $\hat{c} = 1 - \frac{2(p-q)^2}{pq}$. For $n = 2$, the central charge of the $(2, 4)$ superconformal minimal model is thus zero and we simply have the pure world–sheet supergravity sector.

where R is the recursion operator of the DWW hierarchy, given by

$$R \equiv \frac{1}{2} \begin{pmatrix} \partial_x u \partial_x^{-1} - \partial_x & 2 \\ 2v + v_x \partial_x^{-1} & u + \partial_x \end{pmatrix} \quad . \tag{16}$$

The first few L_n and K_n are as follows:

$$\begin{aligned} L_0 &= 2; \quad K_0 = 0; \\ L_1 &= u; \quad K_1 = v; \\ L_2 &= \frac{1}{2}u^2 + v - \frac{1}{2}u_x; \quad K_2 = uv + \frac{1}{2}v_x; \end{aligned} \tag{17}$$

The nth model is chosen by setting all t_i equal to zero except for $t_0 = x$ and t_n which is chosen to be a numerical factor to fix the normalization. The parameter t_n can be replaced by

$$g_n \equiv \frac{1}{\frac{1}{2}(n+1)t_n} \quad , \tag{18}$$

in order to make direct contact with recent literature which discusses this system in a much different (mathematical) context [25].

Analytic expansion solutions to the string equations (13) and (14) for all n can be organized into five main classes on the basis of the ν^0 (or g_s^{-2}) behavior of $u(x)$ and $v(x)$, as follows,

$$\begin{aligned} &\text{Class 1:} \quad u_1 \sim 0 \ , \qquad v_1 \sim x^{2/n} \ ; \\ &\text{Class 2:} \quad u_2 \sim 0 \ , \qquad v_2 \sim 0 \ ; \\ &\text{Class 3:} \quad u_3 \sim x^{1/n} \ , \quad v_3 \sim 0 \ ; \\ &\text{Class 4:} \quad u_4 \sim x^{1/n} \ , \quad v_4 \sim x^{2/n}, \quad u_4^2/v_4 \sim 1/4 \ ; \\ &\text{Class 5:} \quad u_5 \sim x^{1/n} \ , \quad v_5 \sim x^{2/n}, \quad u_5^2/v_5 \sim a \neq 1/4 \ . \end{aligned} \tag{19}$$

They follow interesting patterns[4] with increasing n and have been explored in detail in ref. [20]. Representative solutions for the first few n have been provided in the Appendix for ease of reference. In addition to the type 0 theories reviewed earlier, these solutions encode new string theories, some of which were conjectured to be type II string theories coupled to super–conformal minimal models as reviewed below.

The type 0 theories reviewed earlier can be recovered completely from this system of equations by applying appropriate constraints [20]. Setting $u(x) = 0$ and $\mathcal{L} = 0$ in (13)

[4]For example, the expansions in Class 1 appear only for even $n = 2k$, while the expansions in Class 5 are the analogues of the $\mathbb{Z}_2$ symmetry-breaking solutions of the 0B theories.

and (14) results in the type 0A and type 0B theories respectively[5]. Consistency of the two equations then requires that $c = -1/2$ for type 0A and $\Gamma = 0$ for type 0B, while the free parameter counts the branes and fluxes in the respective theories. The function $v(x)$ (after appropriate redefinition) encodes the free energies of the respective theories, according to equations (6) and (10).

One can also recover solutions corresponding to the type 0 theories by setting $c = -1/2$ or $\Gamma = 0$ in appropriately chosen expansions from the five classes listed above. Expansions in Classes 1 and 2 (labeled $v_1(x)$ and $v_2(x)$) reduce to those of the type 0A theory (see (7) and (8)) once we set $c = -1/2$. This helps fix the directions of the various expansions by requiring that $v(x)$ be real, with the result that some expansions are real and others complex for a given n. Further details can be found in ref. [20].

A full solution for $v(x)$ is constructed by specifying its behavior in the two asymptotic regimes, as seen for the $k = 2$ 0A theory in figure (1). This 0A solution is obtained from the DWW expansions, $v_1(x)$ and $v_2(x)$, by setting $c = -1/2$. Using this scheme, the above expansions can be organized in the form of a square as shown in figure (2). The four corners of the square correspond to four different string theories. The top corners represent the known type 0 strings coupled to the $(2, 4k)$ (A, A) superminimal models. The bottom corners represent two new theories that were conjectured in ref. [20] to be type IIA and IIB string theories coupled to the $(4, 4k-2)$ (A, D) minimal models (for even k only). At each corner, one parameter from (c, Γ) is frozen to the value indicated on the figure, while the other parameter counts the branes/fluxes in the appropriate regime. The horizontal sides of the square correspond to expansions in the $+x$ direction, while the vertical sides represent expansions in the $-x$ direction. For all n, there are two special points on the vertical sides corresponding to $c = \pm\Gamma$, where the expansions v_2 and v_3 identically vanish.

It must be noted that the square fully exists only for even $n = 0 \mod 4$. For $n = 2 \mod 4$, the expansions labeled v_4 become complex as $+x$ expansions and the expansions labeled v_1 do not appear for odd n.

The new theories were conjectured to be type II strings coupled to super–minimal matter by matching the genus zero contributions from the asymptotic expansions (the terms appearing at order g_s^0 in the free energy) with the continuum calculation of their one–loop partition functions. In the absence of non–perturbative constraints (like $u = 0$ for the 0A theory) leading to the conjectured type II theories, the corresponding values of $c = 0$ and $\Gamma^2 = 1/4$ in figure 2, were obtained by systematic analysis of the properties of the genus zero

[5]One also needs to make the identification $t_{2n}^{DWW} = \frac{1}{4} t_n^{\mathrm{KdV}} = \frac{(-1)^{n+1} 4^n (n!)^2}{(2n+1)!} \Rightarrow g_{2n} = 2 \frac{(-1)^{n+1} (2n)!}{4^n (n!)^2}$.

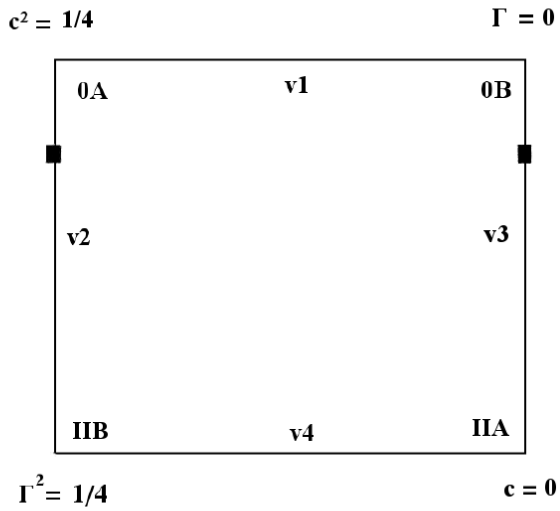

Figure 2: A family of string theories, forming a square. See text for details.

contributions [20].

However, pairing perturbative expansions in this manner does not guarantee the existence of a full non-perturbative solution with the desired properties. A type of 't Hooft limit will allow us to argue for the existence of such non-perturbative solutions for the type II theories. We demonstrate how this limit works for the two parameter DWW system in section (5). Before that, we review the 't Hooft limit for the known type 0 theories.

4 Expansion Matching for type 0 theories : The known

The theories encoded by the string equations described so far can be non–perturbatively completed by matching the perturbative solutions for $x \to \infty$ on to those for $x \to -\infty$. Numerical solutions have been found for the string equations for the 0A models that smoothly interpolate between the two asymptotic regimes as discussed earlier. See, e.g., figure (1). The string equations for the 0B models, however, are more difficult to analyze numerically and attempts to find smooth numerical solutions similar to the 0A ones have been unsuccessful thus far[6]. This difficulty is inherited by the DWW string equations presented in section 3.

It was shown in ref. [11] that the asymptotic expansions for the 0B theories (for example, those in equations (11) and (12)) match on to each other in a 't Hooft limit, with

[6]The only exception is $n = 2$ ($k = 1$) 0B theory, where the numerical $k = 1$ type 0A solution can be converted into it *via* the Morris map, as mentioned in section 2.2.

the brane/flux counting parameter (labeled q for 0B and Γ for 0A) taken to be large. This was then taken to suggest that the theories are non–perturbatively complete even when the parameter is finite. We review this limit in some detail, before analyzing our system in the same limit in section 5.

4.1 The 0B theory in a 't Hooft limit

4.1.1 The $n = 2$ theory

The string equation for the simplest 0B theory with $n = 2$ is

$$\nu^2 \frac{\partial^2 r}{\partial x^2} - \frac{1}{2} r^3 + \frac{1}{2} x r + \nu^2 \frac{q^2}{r^3} = 0 \quad , \tag{20}$$

obtained by using the explicit forms of the polynomials R_2 and H_2 and eliminating $\omega(x)$ between the two equations (9).

Consider the limit $q \to \infty$, $x \to \pm\infty$ with $t = (\nu q)^{-2/3} x$ fixed. This is our 't Hooft limit with $t^{-3/2}$ being the 't Hooft coupling[7]. Defining $s = (\nu q)^{-1/3} r$, the string equation (20) becomes

$$\frac{2}{q^2} s^3 \partial_t^2 s - s^6 + t s^4 + 2 = 0 \quad . \tag{21}$$

In the large q limit, the first term containing the derivative can be neglected resulting in an algebraic equation for $f(t) = s^2$,

$$f^3 - t f^2 = 2 \quad . \tag{22}$$

For generic t this equation has 3 solutions, only *one* of which is real for all t (an easy check is to consider $t \approx 0$). This solution reads

$$f(t) = \frac{1}{3} \left[t + (t^3 + 27 - 3\sqrt{81 + 6t^3})^{1/3} + (t^3 + 27 + 3\sqrt{81 + 6t^3})^{1/3} \right] \quad . \tag{23}$$

For $t > -\frac{3}{2^{1/3}}$, the arguments of the square roots are positive and $f(t)$ is real. Since there are opposite signs in front of the square roots in the above expression, the half-integer powers of t cancel when one expands $f(t)$ for $t \to \infty$.

For $t < -\frac{3}{2^{1/3}}$ the arguments of the square roots are negative and the second and third terms in the solution equation are complex, but $f(t)$ is real since all the imaginary contributions cancel out.

[7] With the identification $g_s = \frac{\nu}{x^{3/2}}$ and q as counting the number of units of R–R flux, this is recognized as the usual 't Hooft coupling $\lambda \sim g_s N$ familiar from higher-dimensional string theories. Taking $x \to \pm\infty$ amounts to taking the limit $g_s \to 0$.

The expansions for $f(t)$ for large negative and positive t are

$$\begin{aligned} f(t) &= \frac{\sqrt{2}}{t^{1/2}} - \frac{1}{t^2} + \frac{5\sqrt{2}}{4t^{7/2}} - \frac{4}{t^5} + \cdots , \quad (t \to -\infty) \\ f(t) &= t + \frac{2}{t^2} - \frac{8}{t^5} + \frac{56}{t^8} + \cdots \quad (t \to \infty) \end{aligned} \tag{24}$$

These expansions reproduce the coefficients of the highest powers of q in the asymptotic expansions to the string equation (11) after remembering that $\widetilde{w} = f/4$. One smooth function (23) captures the limiting behavior of large q and connects[8] the two asymptotic regions smoothly.

This limit has been interpreted in [11] as a 't Hooft limit where only spherical topologies with boundaries (D–branes) or fluxes survive. For $q = 0$, the 0B theory exhibits the Gross–Witten phase transition at $x = 0$ (namely $F'' = x/4$ for $x > 0$ and $F'' = 0$ for $x < 0$). This transition can be smoothed out by either the genus expansion with $q = 0$ or by the expansion in the 't Hooft parameter for large q.

4.1.2 The $n = 4$ theory

A similar analysis shows that the asymptotic expansions for the $n = 4$ theory (12) match smoothly onto each other. The analysis is somewhat complicated because $\omega(x)$ cannot be eliminated in favor of $r(x)$, unlike the $n = 2$ case and one has to use the asymptotic expansions for both $r(x)$ and $\omega(x)$. The expansions for $\omega(x)$ are [11]:

$$\begin{aligned} \omega(x) &= -\frac{2}{3}\frac{q}{x} + \frac{2}{3}\frac{q}{x^{7/2}}\left(\frac{80}{27}q^2 - \frac{5}{4}\right) + \cdots \quad (x \to \infty) \\ \omega(x) &= -\frac{\sqrt{3}}{2}\left(\frac{2}{7}|x|\right)^{1/4} + \frac{2}{3}\frac{q}{|x|} - \frac{5}{96\sqrt{3}}\left(\frac{7}{2}\right)^{1/4}\frac{3+32q^2}{|x|^{9/4}} + \cdots \quad (x \to -\infty) \end{aligned} \tag{25}$$

In the $q \to \infty$ limit with $\omega = (\nu q)^{1/5} h$, $x = (\nu q)^{4/5} t$ and $r^2 = (\nu q)^{2/5} f$, the derivative terms in the string equation for this case are suppressed, resulting in

$$\begin{aligned} \frac{3}{8}f^2 - 3fh^2 + h^4 - \frac{3}{8}t &= 0 \quad , \\ fh\left(-\frac{3}{2}f + 2h^2\right) &= 1 \quad . \end{aligned} \tag{26}$$

[8] One can also formulate an equivalent argument as in ref. [11] by starting with the action $S \sim \int dx \left[\frac{1}{2}r'^2 + \frac{1}{8}(r^2 - x)^2 + \frac{1}{2}\frac{q^2}{r^2}\right] = \int dx \left[\frac{1}{2}r'^2 + V(r^2)\right]$ whose equation of motion is (20). Since this action is bounded below, it is clear that a solution to this equation will exist. The original matrix model integral dual to the theory is well defined and convergent, so one expects a finite and real answer for the free energy F. It is then natural to expect that equation (20) will have a unique real and smooth solution.

The second equation, on solving for f, gives

$$f_\pm = \frac{2}{3}h^2 \pm \sqrt{\left(\frac{2}{3}h^2\right)^2 - \frac{2}{3h}} \quad . \tag{27}$$

From equation (25), it can be seen that h is negative, so that only the solution f_+ can be chosen[9] for real f.

Substituting f_+ into the first equation gives, after some rearrangement:

$$-12 - 864h^5 + 448h^{10} - 36ht - 96h^6t - 27h^2t^2 = 0 \quad . \tag{28}$$

Defining $y = ht$ reduces the above equation to a quadratic in h^5,

$$-(864 + 96y)h^5 + 448h^{10} - (36y + 27y^2 + 12) = 0 \quad . \tag{29}$$

Solving for h^5 as a function of y

$$h^5_\pm = \frac{54 + 6y \pm 5\sqrt{3}\sqrt{28 + 3(y+2)^2}}{56} \quad , \tag{30}$$

will allow a solution for $t = y/h$. The solution h^5_+ is always positive and non–zero as a function of y. To get the asymptotics we desire, we focus on h^5_-, which is negative and zero only for $y_0 = -2/3$. Using $t = y/h_-$ it is easy to see that as $y \to y_0$, $t \to \infty$ and and as $y \to \infty$, $t \to -\infty$. As t changes continuously, we will always lie on the h_- branch since the two branches never cross. The expansion for h_- as a function of t can then be worked out [11] to be

$$\begin{aligned} h_- &= -\frac{\sqrt{3}}{2}(2|t|/7)^{1/4} + \frac{2}{3|t|} - \frac{5(7/2)^{1/4}}{3\sqrt{3}|t|^{9/4}} + \cdots \quad (t \to -\infty) \\ h_- &= -\frac{2}{3t} + \frac{160}{81t^{7/2}} + \cdots \quad (t \to \infty) \end{aligned} \tag{31}$$

The coefficients in these expansions agree with the leading powers for large q in the expansions of $\omega(x)$ in equation (25) and can again be interpreted as a limit in which only spherical topologies with boundaries or fluxes survive.. This shows that an appropriately chosen solution of equation (26) interpolates between the two asymptotic regimes in the limit of large q.

This 't Hooft limit removes the derivative terms from the string equations, resulting in algebraic equations that are simpler to analyze. It can then be sensibly thought of as *algebraic*

[9]Note the presence of more than one solution for f. Such multiple solutions will be prominent in our more general analysis in the next section.

limit. Including the derivatives gives terms *sub–leading* in powers of q and presumably does not introduce any singularities that would destroy the smooth interpolation. The sub–leading powers of q likely smooth out any sharp transitions (as we saw for the Gross–Witten phase transition above) from large negative x to large positive x.

4.2 The 0A theory in a 't Hooft limit

It is interesting to work out the 't Hooft limit of the type 0A theories (with string equations (1)). Since, as already discussed, solutions of the full equations have been obtained numerically, it is instructive to compare the 't Hooft limit (*i.e.,* algebraic) results to the numerical results. (Note that while the last section's type 0B results were a review, the type 0A analysis is presented here for the first time.)

4.2.1 The $k = 1$ theory

The string equation for this theory is,

$$w\,(w-z)^2 - \frac{1}{2}\nu^2 \frac{\partial^2 w}{\partial z^2}\,(w-z) + \frac{1}{4}\nu^2 \left(\frac{\partial w}{\partial z} - 1\right)^2 = \nu^2\Gamma^2 \quad .$$

Consider the limit $\Gamma \to \infty$, $z \to \pm\infty$ with $\rho = (\nu\Gamma)^{-2/3} z$ fixed. This is the 't Hooft limit with $\rho^{-3/2}$ being the 't Hooft coupling. Defining $w = (\nu\Gamma)^{-2/3} g$, the above string equation becomes

$$g(g-\rho)^2 - \frac{1}{2\Gamma^2}(g-\rho)\partial_\rho^2 g + \frac{1}{4\Gamma^2}(\partial_\rho g - 1)^2 = 1 \quad . \tag{32}$$

In the large Γ limit, the derivative terms can be neglected to give the simple algebraic equation

$$g(g-\rho)^2 = 1 \ . \tag{33}$$

As for the type 0B case of the last section, the solutions to this cubic equation[10] can be expanded as a Taylor series for $\rho \to \pm\infty$ and seen to reproduce the coefficients of the highest powers of Γ in the asymptotic expansions (7). One smooth function essentially connects the two asymptotic regions, as in the 0B theory.

Figure (3) shows a comparison between the solution to equation (33) and the solution to the exact string equations (32) obtained by numerical methods, both with $\Gamma = 1$. The two solutions deviate slightly from each other in the interior, and asymptotically, they match. This can be seen in figure (4), which plots the difference between the exact and algebraic

[10] One can also use the Morris map referred to at the end of section (2.2) to obtain this equation directly from the corresponding 0B algebraic equation (22).

solutions, which goes to zero for large $|x|$. Physics sub–leading in powers of Γ contribute to the exact solution, while only the highest powers of Γ contribute to the algebraic solution. The inclusion of the derivative terms does not introduce any singularities.

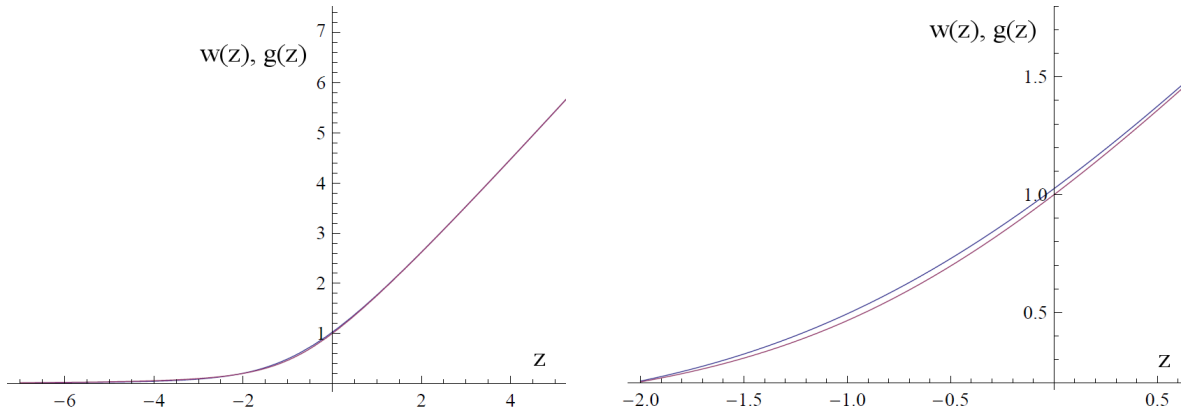

Figure 3: Comparison of the solution (in the 't Hooft limit) to equation (33) and the numerical solution to the full string equations (32) obtained by numerical methods, with $\Gamma = 1$. The curve which is uppermost at *e.g.*, $x = -1.0$ represents the solution of the exact equation. The plot on the right shows the same curves within a smaller domain for better resolution.

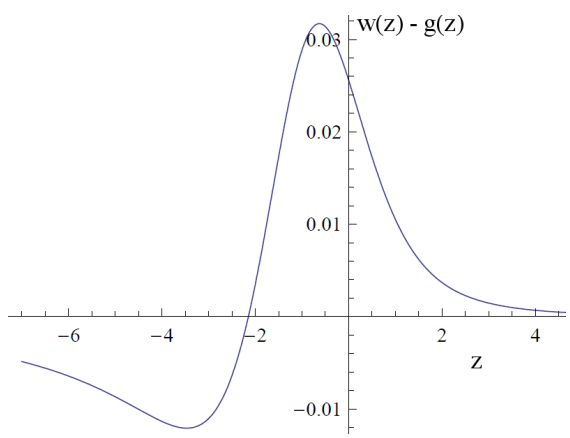

Figure 4: The difference between the algebraic and full numerical solutions for the $k = 1$ 0A theory. For large $|x|$, the difference goes to zero.

4.2.2 The $k = 2$ theory

The string equation for the $k = 2$ 0A theory has $\mathcal{R} = \frac{w''}{3} - w^2 - z$. In the large $\Gamma \to \infty$ and $z \to \pm\infty$ limit, with $\rho = (\nu\Gamma)^{-5/4} z$ held fixed, the derivative terms drop out. Defining $g = (\nu\Gamma)^{-5/4} w$ results in the algebraic equation

$$g(g^2 - z)^2 = 1 \ . \tag{34}$$

The solutions to this equation reproduce the coefficients of the highest powers of Γ in the asymptotic expansions (8).

Figure (5) shows a comparison between the solution to equation (34) and the solution to the exact $k = 2$ 0A string equation obtained by numerical methods. The difference between the two solutions is plotted in figure 6 and can be seen to approach zero for large $|x|$. The deviation for finite x and asymptotic matching is evident, and can be explained by the additional contributions coming from sub–leading powers of Γ in the exact solution.

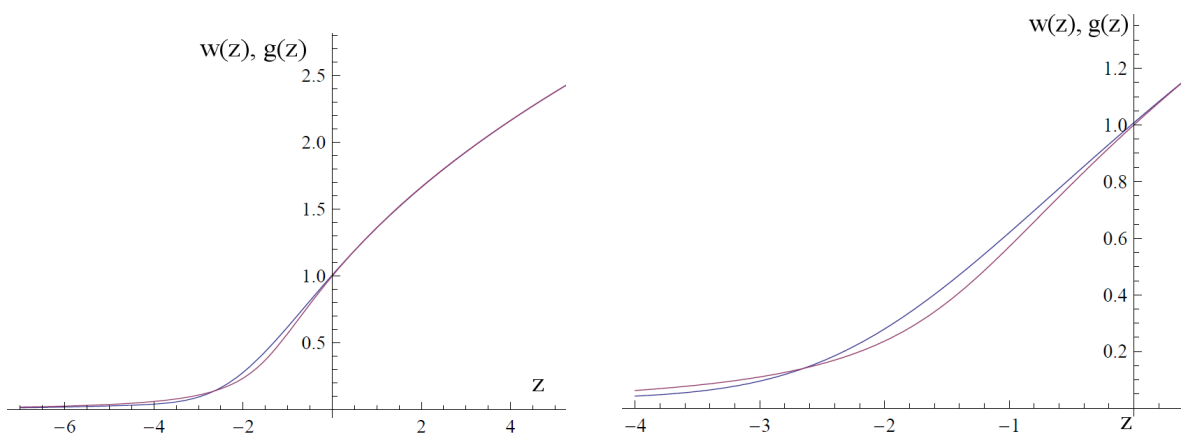

Figure 5: Comparison of the solution (in the 't Hooft limit) to equation (34) and the solution to the exact string equation obtained by numerical methods, with $\Gamma = 1$. The curve which is uppermost at *e.g.*, $x = -2$ represents the solution of the exact equation. The plot on the right shows the same curves within a smaller domain for better resolution.

4.3 A modified 't Hooft limit

A modification of the above limit (one that does not affect the physics) will allow us to apply it to the Painlevé IV hierarchy of string equations. The need for such a modification will be explained in the next section.

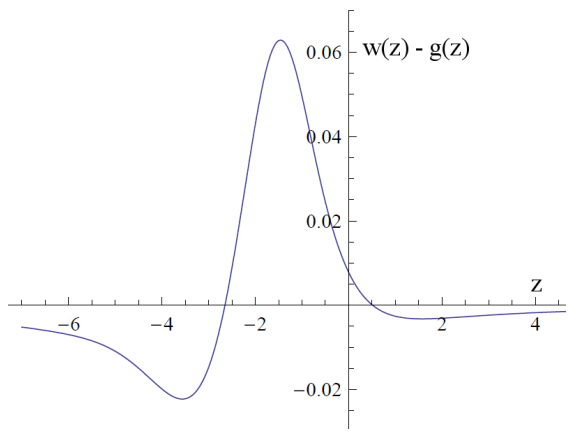

Figure 6: The difference between the algebraic and full numerical solutions for the $k = 2$ 0A theory. For large $|x|$, the difference goes to zero.

Instead of rescaling variables in the 0B (0A) theory to eliminate q (Γ), one can remove the derivatives by first rescaling

$$q \to \frac{q}{\nu} \quad , \tag{35}$$

and then taking the limit

$$\nu \to 0 \quad , \tag{36}$$

in that order, without affecting the physics. Note that the rescaled q is large, so this is still a large q limit. The limit $\nu \to 0$ amounts to taking $g_s \to 0$, as is clear from the definition of g_s ($g_s = \frac{\nu}{\mu^{3/2}}$ for $n = 2$ and $g_s = \frac{\nu}{\mu^{5/4}}$ for $n = 4$). This modified limit therefore extracts the physics same as the 't Hooft limit of the previous subsections.

In this modified limit, the following equations are obtained for the type 0B theories for $n = 2$,

$$r^6 - xr^4 = 2q^2 \quad , \tag{37}$$

and for $n = 4$,

$$\begin{aligned} \frac{3}{8}r^5 - 3r^3\omega^2 + r\omega^4 - \frac{3}{8}xr &= 0, \\ \frac{3}{2}r^4\omega - 2r^2\omega^3 &= -q \quad . \end{aligned} \tag{38}$$

These are the same as equations (22) and (26) with $r^2 \sim f$, $\omega \sim h$ and $x \sim t$, but with the rescaled parameter q explicitly present.

For the 0A theory, after rescaling $\Gamma \to \frac{\Gamma}{\nu}$, the following equations are obtained for $k = 1$ and $k = 2$, respectively,

$$w(w-z)^2 = \Gamma^2, \qquad w(w^2-z)^2 = \Gamma^2 \quad . \tag{39}$$

In fact, for general k, the type 0A string equation (1) effectively reduces to

$$w\widetilde{\mathcal{R}}^2 = \Gamma^2 \quad , \tag{40}$$

where $\widetilde{\mathcal{R}} = w^k - z$ is obtained from $\mathcal{R}$ after dropping all the derivatives of $w(z)$.

5 Expansion Matching for DWW: The unknown

5.1 't Hooft limit for DWW : Two parameters

The string equations of the DWW hierarchy (reviewed in section 3) are quite general, encoding the string equations of type 0A and of type 0B as special cases. This generality, however, comes at a price: all but the simplest cases are too complicated to be solved using standard numerical methods. This complicates our task of performing a non–perturbative analysis of the string theories proposed in ref. [20], which were based on strictly perturbative considerations. To facilitate this analysis, we examine the DWW string equations in a (modified) 't Hooft limit. The result of this study is considerable evidence that the conjectured type II theories possess smooth solutions connecting the perturbative expansions. This analysis also uncovers new non-perturbative solutions, absent from the earlier perturbative analysis, which we suspect may encode new unidentified string theories.

In section 4 we described how to take the 't Hooft limit of the type 0A and type 0B theories. Implicit in these methods was the fact that the type 0 theories each depend on a single parameter, which counts branes or fluxes. The full DWW equations, in contrast, have two free parameters, so it is not immediately clear how to implement the 't Hooft limit in this more general case.

The perturbative analysis of ref. [20] suggests that, in addition to the type 0 theories, the full DWW string equations also describe type II theories whose branes and fluxes are counted by a single parameter. This motivates our assumption that *new theories are described by a one–dimensional subspace of the parameters c and Γ*. We further restrict our study to subspaces defined by *linear* combinations of c and Γ. This is the simplest possibility and although there could be interesting one-dimensional subspaces corresponding to non-trivial curves in (c, Γ) parameter space, we will not consider them here.

Given this assumption and the scaling prescription described in section 4.3, we can utilize the 't Hooft limit to analyze the string equations (13) and (14). For given η and ξ, let $c + \eta\Gamma = \xi$ be the constraint defining the one-dimensional parameter subspace of a new theory under consideration. Since the constraint is (by assumption) satisfied by the solutions of interest, we can impose the constraint on the DWW equations themselves. We are left with a single free parameter which we can recale as in section 4.3 to take the 't Hooft limit. In this way we obtain algebraic equations which are simple to analyze.

Unfortunately, in the search for new theories, we do not *a priori* know the values of η and ξ, so the above method is not directly applicable. Nevertheless, we can proceed by adopting a slightly different perspective. Note that the constraint $c + \eta\Gamma = \xi$ implies that the free parameter can be taken to be c or Γ, or any linear combination of them (as long as it is independent of the constraint). Rescaling the free parameter by $1/\nu$ is therefore equivalent to rescaling $c \to c/\nu$ and $\Gamma \to \Gamma/\nu$. This procedure is independent of the details of the original constraint, so it can be implemented without actually knowing the values of η and ξ. The result is a set of algebraic equations which depends on c and Γ.

Moreover, the solutions to these equations can be used to deduce information about the original constraint. To see this, note that since ξ is a constant, after taking the limit the constraint takes the form $c + \eta\Gamma = 0$. Therefore, the original constraint is satisfied by any solution of the algebraic equations which satisfies $c + \eta\Gamma = 0$. Turning this around, if a particular solution to the algebraic equations only exists when $c + \eta\Gamma = 0$, then this is a strong indication that it descends from the special solution to the full equations subject to $c + \eta\Gamma = \xi$. That is to say, it is an algebraic approximation to the full solutions of some new string theory. Note that ξ is not determined from this approach, but η is.

5.2 't Hooft limit for DWW : Strategy

To obtain the algebraic equations, we begin with the DWW string equations (13) and (14),

$$\begin{aligned} -\frac{1}{2}\mathcal{L}_x + \frac{1}{2}u\mathcal{L} + \mathcal{K} &= \nu c \\ \left(-v + \frac{1}{4}u^2 + \frac{1}{2}u_x\right)\mathcal{L}^2 - \frac{1}{2}\mathcal{L}\mathcal{L}_{xx} + \frac{1}{4}\mathcal{L}_x^2 &= \nu^2\Gamma^2 \quad . \end{aligned}$$

In the modified 't Hooft limit discussed above,

$$\Gamma \to \frac{\Gamma}{\nu} \quad , \quad c \to \frac{c}{\nu} \quad , \quad \nu \to 0 \quad , \tag{41}$$

the string equations simplify to

$$\frac{1}{2}u\tilde{\mathcal{L}} + \tilde{\mathcal{K}} = c \ , \qquad \text{and} \qquad \left(-v + \frac{1}{4}u^2\right)\tilde{\mathcal{L}}^2 = \Gamma^2 \ , \tag{42}$$

where $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{K}}$ are the polynomials in u and v obtained from $\mathcal{L}$ and $\mathcal{K}$ after dropping all the derivative terms. Before analyzing these equations in full generality, let us discuss a few special cases to illustrate our methods.

1. $u = 0$ (type 0A)

   The restriction to type 0A requires n to be even, in which case $\widetilde{\mathcal{K}} \propto u$. Thus, setting $u = 0$ in first equation of (42) implies $c = 0$. This constraint can be written as $c + \eta\Gamma = 0$ with $\eta = 0$. As described above, this constraint lifts to $c + \eta\Gamma = c = \xi$ at the level of the full equations. If we hadn't known the parameter constraint of the 0A theory ($c = -\frac{1}{2}$), the algebraic equations alone would indicate that $c = \xi$, a constant.

   Alternatively one could start by looking for solutions to the equations (42) with $c = 0$. A subset of smooth solutions, identified by their boundary behavior, will correspond to asymptotic expansions in Classes 1 and 2 with c fixed (recall that these correspond to 0A). The boundary behavior of the smooth solutions is matched with the behavior of expansions in Classes 1 and 2, which are the asymptotic expansions of the type 0A theory.

2. $\widetilde{\mathcal{L}} = 0$ (type 0B)

   Setting $\widetilde{\mathcal{L}} = 0$ forces $\Gamma = 0$ and the first equation gives $\widetilde{\mathcal{K}} = c$. For even n, these reduce to the 0B algebraic equations after redefining the variables appropriately. The constraint $\Gamma = 0$ can be rewritten as $c + \eta\Gamma = 0$ with $\eta = \infty$ and lifts to a constraint on the full equations which takes the form $\Gamma = \xi$, a constant. The true parameter constraint for the full $0B$ theory is $\Gamma = 0$; the algebraic equations alone do not determine the precise value of ξ.

   As in the 0A case above, one could look for smooth solutions to the algebraic equations subject to $\Gamma = 0$. If they obey the correct boundary conditions (expansion Classes 1 and 3 (or 5)) one could identify these as 0B solutions.

Our strategy to demonstrate the existence of smooth non–perturbative solutions to the type 0 and type II theories and identify potentially new theories can be summarized as follows. Take the limit (41) of the full DWW equations to obtain a set of algebraic equations.

Search for solutions to these equations subject to one of three constraints[11]:

$$c = 0 \ , \quad \Gamma = 0 \ , \quad \text{or} \quad c \pm \Gamma = 0 \ . \tag{43}$$

By matching the asymptotic behavior of these solutions onto the various expansions, we identify the type 0 and type II string theories. We also find new solutions with asymptotics different from those of the type 0 or type II theories. We speculate that these solutions correspond to some new unknown string theories. As emphasized above, the precise parameter constraints of these new theories cannot be fully determined from the algebraic equations.

5.3 DWW $n = 2$ in a 't Hooft limit

The string equations for $n = 2$ reduce in the algebraic limit (41) to

$$\begin{aligned} u\left(v + \frac{1}{2}u^2 + x\right) + 2uv &= 2c \quad , \\ (uv - c)^2 - v\left(v + \frac{1}{2}u^2 + x\right)^2 &= \Gamma^2 \quad . \end{aligned} \tag{44}$$

They produce a total of nine asymptotic expansions which fall into 4 classes (see Appendix A). As the expansions within each class are related by various $\mathbb{Z}_2$ symmetries [20], we label each expansion only by a subscript specifying its class. The solutions corresponding to the three constraints in parameter space are described below.

1. $c = 0$

   A plot of solutions to the algebraic equations (44) with $c = 0$ and $\Gamma = 2$ is shown in figure (7). The plot on the right shows two smooth solutions that connect regions[12] of large $-x$ and $+x$. In our conventions, these are $v_2|v_1$ and $v_3|v_4$ solutions[13]. The plot on the left shows the solutions which do not join negative asymptotics to positive asymptotics, a feature we expect to find in any solutions describing new string theories. For this reason, these types of solutions will not be our primary interest.

   The $v_2|v_1$ solutions correspond to the 0A theory (with $k = 1$) coupled to pure supergravity. It is unclear if the $v_3|v_4$ solutions correspond to a consistent new theory. Based on the square of theories in figure (2), it is tempting to conclude that this theory might

[11] It turns out that more general linear combinations are relevant only for $n \geq 4$, and for simplicity we exclude such cases from the present analysis.

[12] The algebraic solutions are represented by the solid black curves, while the asymptotic expansions are represented by the dashed curves. We adopt this convention in the rest of the paper.

[13] We label a solution with $-x$ asymptotics v_L and $+x$ asymptotics v_R by $v_L|v_R$.

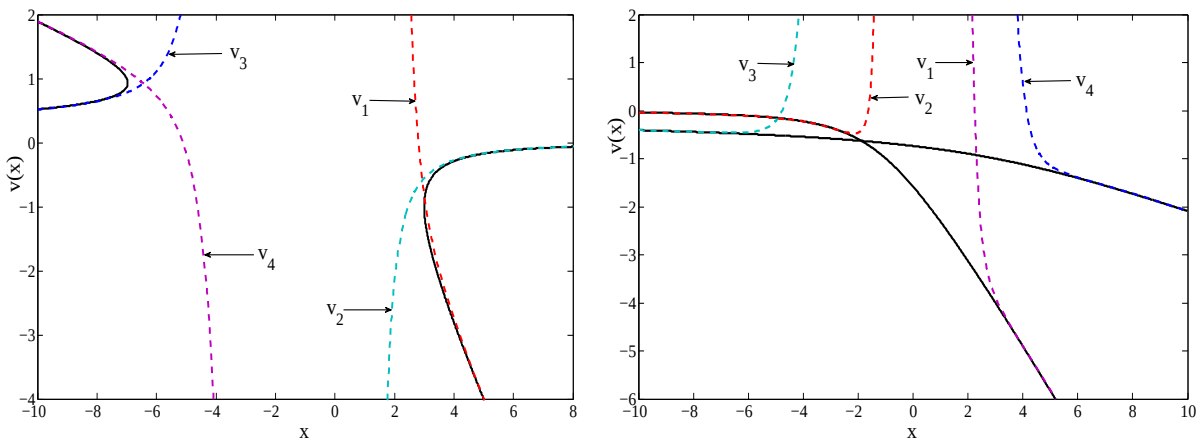

Figure 7: $n = 2$ algebraic solutions with $c = 0$ and $\Gamma = 2$.

be the type IIA string theory coupled to the $(4, 2)$ (A, D) superconformal minimal model. However, as mentioned in the discussion below figure (2), the v_4 expansion is complex as a $+x$ expansion for $n = 2 \mod 4$. For the particular case of $n = 2$ and $c = 0$, v_4 happens to be real, but this is an exception. We find it unlikely that the corresponding solution to the full equations exists and encodes a consistent theory, but we leave this question for future investigation.

2. $\Gamma = 0$

   Figure (8) shows the algebraic solutions for $\Gamma = 0$ and $c = 1.5$. The right figure shows two smooth solutions, $v_3|v_1$ and $v_4|v_2$, connecting the negative and positive regions.

   The $v_3|v_1$ solutions are algebraic approximations to the full solutions of the 0B theory (with $n = 2$) coupled to pure supergravity. Analogous to the previous case with $c = 0$, it is unclear if the $v_4|v_2$ solutions descend from a consistent new theory. It is tempting to conclude that these are type IIB string theories coupled to the $(4, 2)$ (A, D) superminimal model, but this is likely incorrect because v_4 is complex as a $+x$ expansion for $n = 2 \mod 4$.

3. $c = \pm\Gamma$

   We consider $c = \Gamma$. The case of $c = -\Gamma$ is completely analogous. Figure (9) shows solutions for $c = 2$ and $\Gamma = 2$. The solutions in the right figure are $v_4|v_1$ and $v_3|v_1$ and connect the negative region smoothly to the positive region.

   The fixed value of $c \mp \Gamma$ cannot be deduced from the algebraic solutions.

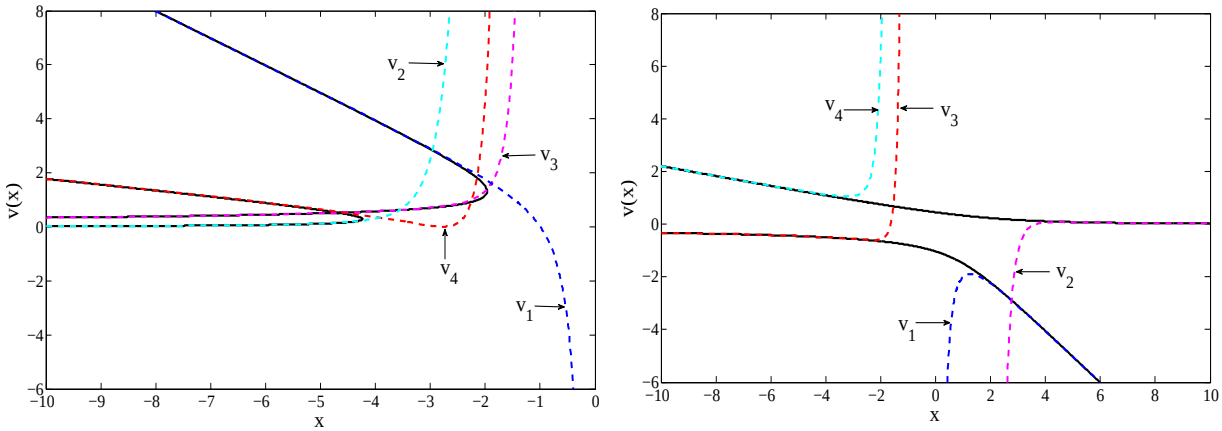

Figure 8: $n = 2$ algebraic solutions with $\Gamma = 0$ and $c = 1.5$.

5.4 DWW $n = 4$ in a 't Hooft limit

The string equations in the 't Hooft limit, after setting $g_4 = -3/4$, are too long to be reproduced here. The polynomials $\mathcal{L}$ and $\mathcal{K}$ reduce to,

$$\begin{aligned} \widetilde{\mathcal{L}} &= \frac{3}{4}v^2 + \frac{3}{2}vu^2 + \frac{1}{8}u^4 - \frac{3}{4}x \quad , \\ \widetilde{\mathcal{K}} &= \frac{3}{2}uv^2 + \frac{1}{2}u^3 v \quad , \end{aligned} \tag{45}$$

from which the corresponding algebraic string equations can be easily obtained using (42). There are a total of twenty–five expansions [20], falling into five classes. Solutions to the algebraic string equations with the three constraints are shown below.

1. $c = 0$

   Figure (10) shows solutions with $c = 0$ and $\Gamma = 2$. There are three solutions that interpolate between the negative region and the positive region, $v_2|v_1$, $v_5|v_3$ and $v_5|v_4$.

   The $v_2|v_1$ solutions represent type 0A coupled to the $(2, 8)$ (A, A) superminimal model. The $v_5|v_4$ solutions fit with our conjectured type IIA string theory coupled to the $(4, 6)$ (A, D) superminimal model and are analogues of the $\mathbb{Z}_2$ symmetry–breaking solutions of the 0B theory.

   The $v_5|v_3$ solutions are new. On setting $c = 0$ in the *full* expansions, the powers of Γ in both correspond to a parameter counting branes. An underlying string theory with

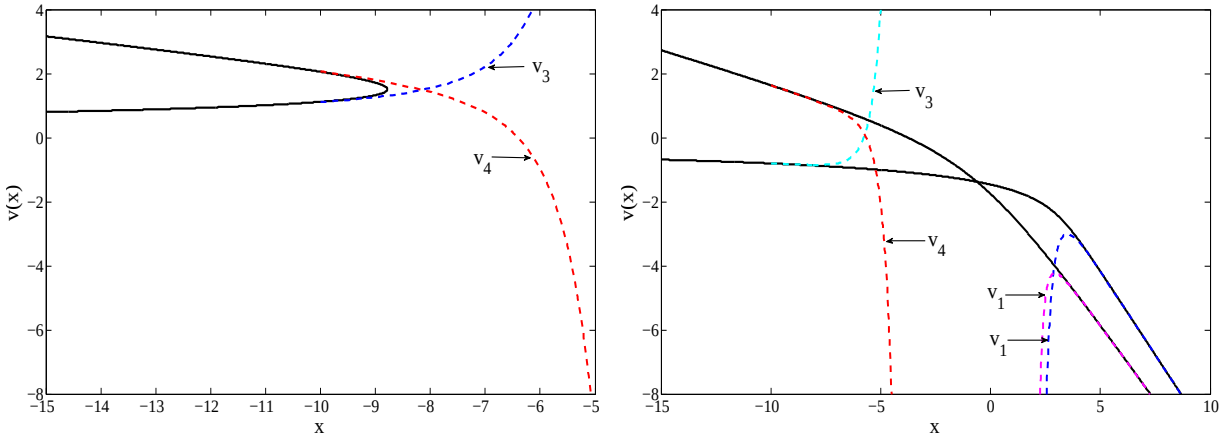

Figure 9: $n = 2$ algebraic solutions with $c = 2$ and $\Gamma = 2$.

these solutions, if it exists, would have branes in both asymptotic regimes[14]. Further work is needed to conclude if such an underlying theory exists.

2. $\Gamma = 0$

   Figure (11) shows solutions with $\Gamma = 0$ and $c = 1$. The solutions which interpolate between $-x$ and $+x$ are $v_5|v_1$, $v_5|v_3$ and $v_2|v_4$. The $v_5|v_1$ solutions correspond to the $\mathbb{Z}_2$ symmetry–breaking solutions of type 0B string theory coupled to the $(2, 8)$ superminimal model, while the $v_2|v_4$ solutions fit with our conjectured type IIB string theory coupled to the $(4, 6)$ (A, D) superminimal models.

   The $v_5|v_3$ solutions with $\Gamma = 0$ are new, similar to those above with $c = 0$. Further work is required to understand the existence and nature of such theories.

3. $c = \pm\Gamma$

   Again we consider $c = \Gamma$ since the case of $c = -\Gamma$ is completely analogous. Figure (12) plots the solutions for $c = 1$ and $\Gamma = 1$. The four smooth solutions which interpolate between $-x$ and $+x$ are two $v_5|v_1$ solutions, one $v_5|v_4$ solution and one $v_5|v_3$ solution. (The two $v_5|v_1$ solutions can be obtained from one another by applying $\mathbb{Z}_2$ symmetries on the signs of the parameters). All of these solutions are new, and it is possible that they correspond to well-defined underlying string theories, but we leave any definitive

[14]It is possible that such brane–brane solutions could be summed up to form rational solutions. This would be analogous to the rational solutions of the type 0A string equations that were considered in a string theory context in ref. [32]. The rational solutions have v_2 type expansions (for $c = -1/2$) in both asymptotic directions for x.

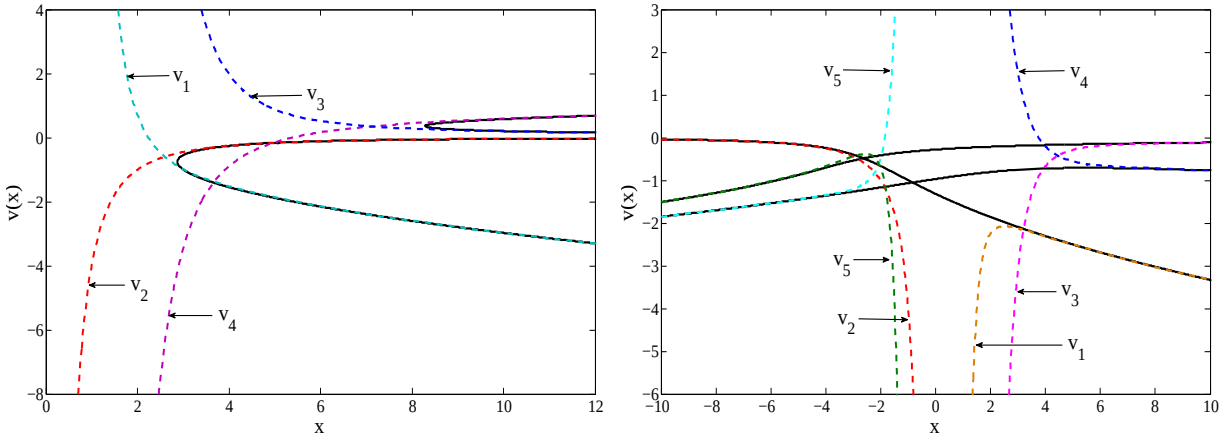

Figure 10: $n = 4$ algebraic solutions with $c = 0$ and $\Gamma = 2$.

claims to future work.

6 DWW $n = 1$: 't Hooft limit and Numerical Results

The string equations for $n = 1$ are,

$$2v - u_x + u^2 + g_1 x u = 2\nu g_1 (c + \frac{1}{2}) \quad , \tag{46}$$
$$\left(-v + \frac{1}{4}u^2 + \frac{1}{2}u_x\right)(u + g_1 x)^2 - \frac{1}{2}u_{xx}(u + g_1 x) + \frac{1}{4}\left(u_x + \nu g_1\right)^2 = \nu^2 g_1^2 \Gamma^2 \quad ,$$

where we have used the relation $g_1 = \frac{1}{t_1}$. These equations are simple enough to allow for complete numerical solutions, which will help us classify new non-perturbatively complete solutions.

We will also examine these equations in the limit (41), under which they reduce to

$$\begin{aligned} u^2 + 2v - 2ux &= -4c \quad , \\ v^2 + 2v\left(u^2 - 4ux + 4x^2 - 4c\right) &= 4(c^2 - \Gamma^2) \quad , \end{aligned} \tag{47}$$

where we have used $g_1 = -2$. These equations admit four asymptotic expansions, which we

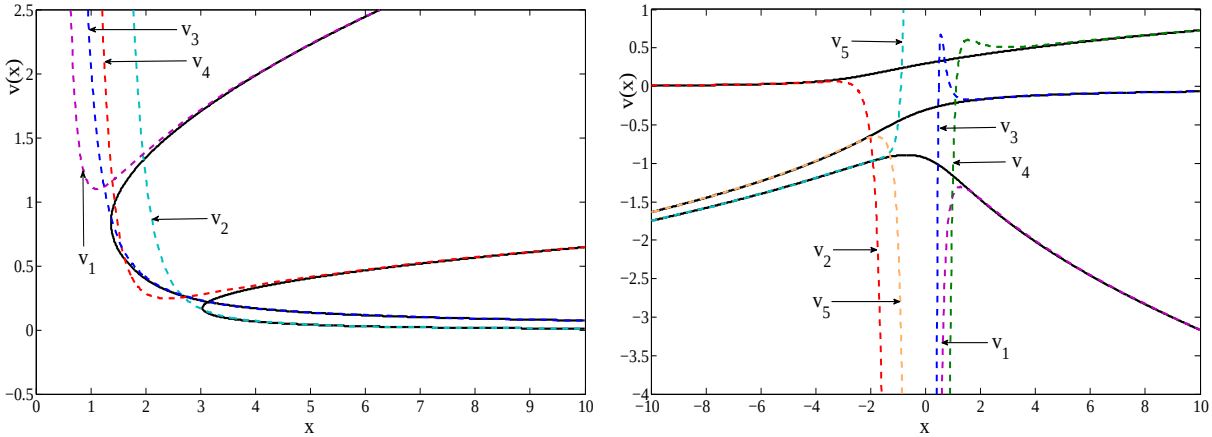

Figure 11: $n = 4$ algebraic solutions with $\Gamma = 0$ and $c = 1$.

label as:

$$\begin{aligned} v_2 &\sim \frac{\nu^2}{x^2}\left(c^2 - \Gamma^2\right) \quad , \\ v_{3a} &\sim g_1 \nu \left(c + \Gamma\right) \quad , \\ v_{3b} &\sim g_1 \nu \left(c - \Gamma\right) \quad , \\ v_4 &\sim \frac{1}{9} g_1^2 x^2 \quad . \end{aligned} \tag{48}$$

The solutions with the three different parameter constraints are presented below.

1. $c = 0$

   A plot of solutions to the algebraic equations (47) with $c = 0$ is shown in figure (13). There are two smooth solutions interpolating between $+x$ and $-x$, labeled $v_3|v_2$ in our convention.

   As outlined in the Appendix (A.4), v_2 with $c = 0$ can be interpreted as a flux expansion (for odd n) in Γ, while v_3 with $c = 0$ is a brane expansion in Γ. These interpretations are possible strictly for $c = 0$; a finite value of c results in powers of g_s that do not allow v_2 to be interpreted as a flux expansion. So a string theory corresponding to this solution (if it exists) should require $c = 0$ in the exact string equations.

2. $\Gamma = 0$

   The plots for this case are shown in figure (14). There are three smooth solutions, two of which are $v_4|v_2$ and $v_2|v_4$ solutions. The third solution is parallel to the x–axis

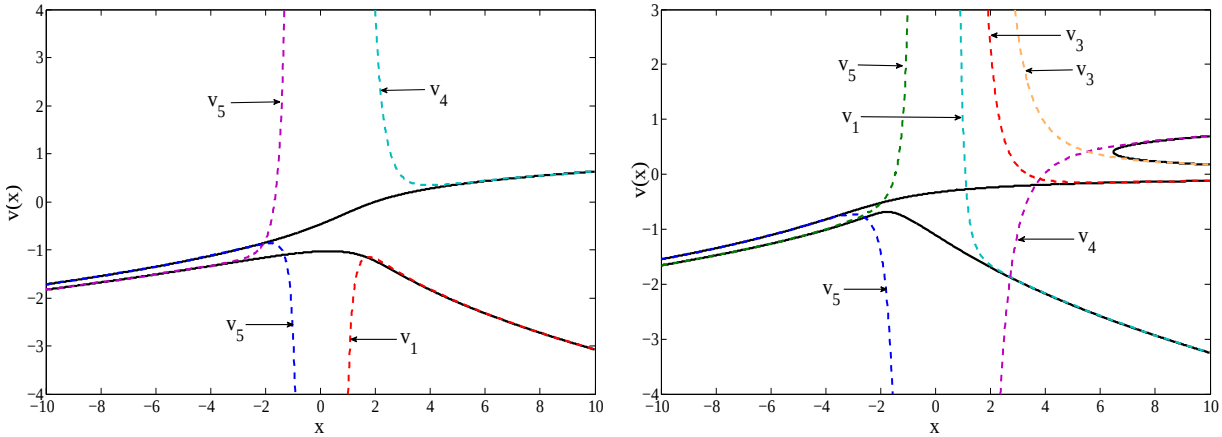

Figure 12: $n = 4$ algebraic solutions with $c = 1$ and $\Gamma = 1$.

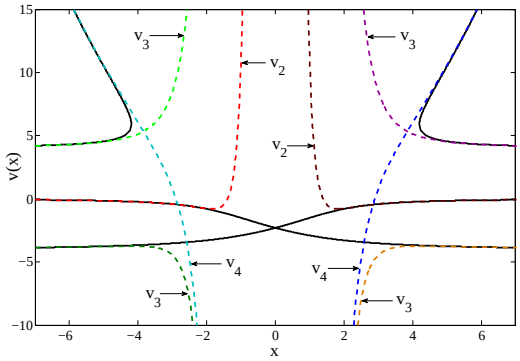

Figure 13: $n = 1$ algebraic solution with $c = 0$ and $\Gamma = -2$.

and exactly equals v_{3a} and v_{3b}. It is a 0B solution, the 'topological point' of the 0B theory [11] with $v = \nu g_1 c$.

The $v_4|v_2$ and $v_2|v_4$ solutions are new. For any value of Γ, v_2 and v_4 have powers of c and g_s consistent with a parameter counting branes. This suggests that a string theory underlying such solutions (if it exists) should have branes in *both* asymptotic regimes.

Interestingly, algebraic $v_4|v_2$ solutions do not exist for $c > 0$. Nevertheless, we have demonstrated the existence of numerical solutions to the full equations in this and other cases. See figure (15). These solutions[15] were obtained using the `bvp4c` algorithm in

[15]Note the development of a bump in the interior of the solution as c decreases. There are a number of qualitative features of this family of solutions that are akin to those seen in studies of the type 0A case in refs. [15, 16], when examining the case of $\Gamma \to -1$.

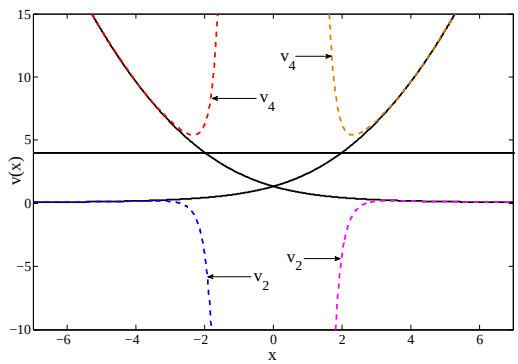

Figure 14: $n = 1$ algebraic solution with $c = -2$ and $\Gamma = 0$.

MATLAB. One lesson learned is that the failure to find algebraic solutions in a t 'Hooft limit is not a guarantee that smooth solutions to the full equations do not exist.

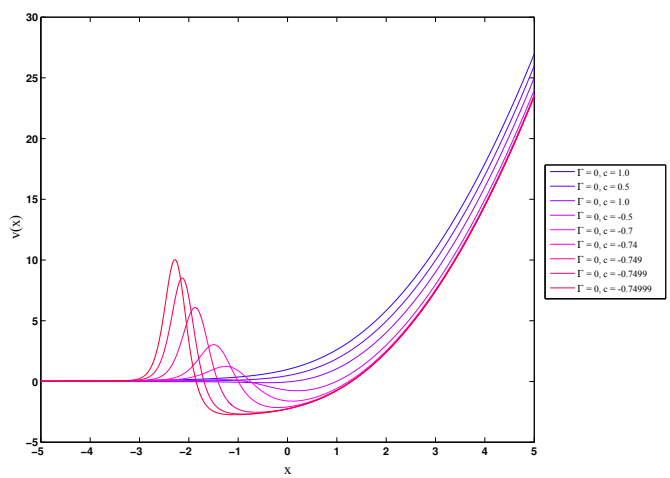

Figure 15: $n = 1$ solution to the full string equations (46) with $\Gamma = 0$.

3. $c = \pm\Gamma$

   Figure (16) shows algebraic solutions where $c = 2$ and $\Gamma = 2$. These are $v_4|v_3$ and $v_3|v_4$ solutions in our notation.

   The constraint $c \pm \Gamma = 0$ on the algebraic equations lifts to $c \pm \Gamma = \xi$ on the full solutions. The constant ξ cannot be determined from algebraic solutions alone. In

this case, however, a thorough numerical analysis of the full equations is possible and it shows that $\xi = -1/2$. Examples of these numerical solutions are displayed in figure (17).

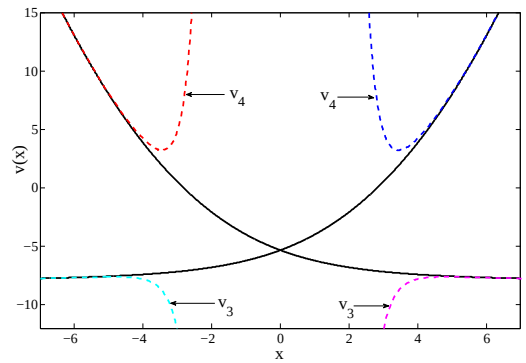

Figure 16: $n = 1$ algebraic solutions with $c = 2$ and $\Gamma = \pm 2$.

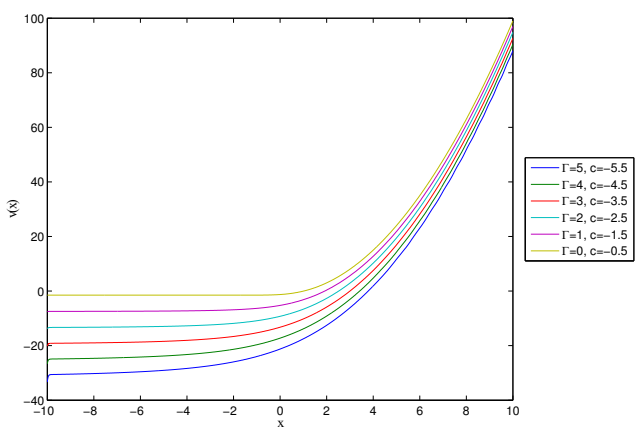

Figure 17: $n = 1$ solution to the full string equations (46) with $c + \Gamma = -1/2$.

The numerical solution strongly suggests that such new solutions are not just an artifact of the 't Hooft limit and should be taken seriously as new examples of non–perturbatively complete solutions. Such solutions were not apparent in the perturbative analysis of ref. [20] and lead us to believe that the modified 't Hooft limit presented here is a good way of unearthing new non–perturbative solutions.

7 Discussion

We have presented a modification of the 't Hooft limit, first used by the authors of ref. [11], to argue for the existence of new non–perturbative solutions to string equations that are difficult to obtain numerically. We have analyzed the Painlevé IV hierarchy of string equations introduced in ref. [20] in this limit, showing that examples of the conjectured type II string theories coupled to the $(4, 4k-2)$ superconformal minimal models have well–defined non–perturbative solutions. As in the case of type 0A, higher k solutions are likely to exist as a consequence of these [7], since the underlying integrable flow structure should evolve lower k solutions into higher k ones.

Although this limit results only in the highest powers of the brane/flux parameter surviving (so that we have only spherical topologies with boundaries or fluxes), it is likely that smooth solutions exist for the full string equations too. The higher genus terms in the free energy are obtained by including the derivative corrections to the string equations and from the lesson of type 0A (where we can compare to numerical solutions of the full equations — see section 4.3) it seems that they do not introduce any singular behaviour. This can presumably be checked through further numerical work for the 0B and conjectured type II theories.

We have uncovered a number of clear examples of new non–perturbative solutions which also seem to be string–like. It would be interesting to determine if these indeed correspond to new string theories. This could presumably be checked using perturbative techniques of the sort that we used in ref. [20] to identify the type II theories. We have demonstrated, in at least the $n = 1$ case, that such new solutions are not simply an artifact of the 't Hooft limit, and that smooth solutions to the full equations can be obtained numerically obeying the same parameter constraints. It would be interesting to find new solutions of this type numerically for higher n.

Acknowledgements

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A The DWW expansions

We list the asymptotic expansions for $n = 1$, $n = 2$ and $n = 4$ here. For each n, there are multiple expansions within each class that can be related to each other by various $\mathbb{Z}_2$

symmetries. We list these explicitly for $n = 1$. The symmetries for $n = 2$ and $n = 4$ are similar to those for $n = 1$ and can be found in ref. [20].

We also summarize the conditions under which a given expansion can be thought of as encoding ZZ–branes or fluxes.

A.1 $n = 1$

There are three classes of expansions for $v(x)$ (expansions in Class 1 appear only for even n),

$$\begin{aligned} v_2 &= \frac{\nu^2}{x^2}(c^2-\Gamma^2)\left(1-\frac{\nu}{g_1x^2}6c+\frac{\nu^2}{g_1^2x^4}(45c^2-5\Gamma^2+5)-\cdots\right) \quad , \\ v_3 &= \nu g_1(c-\Gamma)\left(1-\frac{\nu}{g_1x^2}2\Gamma-\frac{\nu^2}{g_1^2x^4}6\Gamma(c-3\Gamma)-\cdots\right) \quad , \\ v_4 &= \frac{1}{9}g_1^2x^2+\nu\frac{2g_1c}{3}-\frac{\nu^2}{x^2}\frac{1}{3}(3c^2+9\Gamma^2-1)+\frac{\nu^3}{g_1x^4}6c(c^2-9\Gamma^2)-\cdots \quad . \end{aligned} \tag{49}$$

Integrating $v(x)$ twice, one can obtain the free energy for a genus expansion of a string theory which allows us to identify the string coupling to be $g_s = \nu/x^2$.

A.1.1 Symmetries for $n = 1$

The other expansions within each class can be obtained by the following symmetry operation:

$$f_1 : \Gamma \to -\Gamma \quad .$$

Since v_2 and v_4 contain only even powers of Γ, the are invariant under this map; however, $f_1 \circ v_3 \neq v_3$. Thus there are two expansions in the v_3 class, and, together with v_2 and v_4, these comprise four total $n = 1$ expansions.

A.2 $n = 2$

In this case, there are four relevant classes of expansions

$$\begin{aligned} v_1 &= -g_2x-\frac{\nu g_2^{1/2}}{x^{1/2}}\Gamma+\frac{\nu^2}{x^2}\frac{1}{8}\left(-4c^2+4\Gamma^2+1\right)+\cdots \quad , \\ v_2 &= \frac{\nu^2}{x^2}\left(c^2-\Gamma^2\right)\left(1-\frac{2\nu^2}{g_2x^3}(5c^2-\Gamma^2+1)+\cdots\right) \quad , \\ v_3 &= \frac{g_2^{1/2}\nu}{x^{1/2}}(c-\Gamma)\left(\frac{i}{\sqrt{2}}+\frac{\nu}{g_2^{1/2}x^{3/2}}\frac{1}{4}(c-5\Gamma)-\cdots\right) \quad , \\ v_4 &= -\frac{g_2x}{5}+\frac{\nu g_2^{1/2}}{x^{1/2}}\frac{ic}{\sqrt{5}}-\frac{\nu^2}{x^2}\frac{1}{4}(2c^2+10\Gamma^2-1)-\cdots \quad . \end{aligned} \tag{50}$$

Here, we see that the string coupling is $g_s = \nu/x^{\frac{3}{2}}$. There are *nine* distinct expansions related to each other by various discrete symmetries as for $n = 1$.

A.3 $n = 4$

The following five classes of expansions are obtained:

$$
\begin{aligned}
v_1(x) &= -\frac{2i}{\sqrt{3}}\sqrt{g_4}\sqrt{x} - \frac{\nu g_4^{1/4}}{x^{3/4}}\frac{(1+i)\Gamma}{2\cdot 3^{1/4}} - \frac{\nu^2}{x^2}\frac{1}{48}\left(12c^2 - 12\Gamma^2 - 5\right) + \cdots \quad , \\
v_2(x) &= -\frac{\nu^2}{x^2}(\Gamma^2 - c^2)\left(1 - \frac{3\nu^4}{2g_4x^5}(21c^4 - 14c^2\Gamma^2 + \Gamma^4 + 35c^2 - 5\Gamma^2 + 4)\right) + \cdots \quad , \\
v_3(x) &= -\frac{g_4^{1/4}\nu}{x^{3/4}}(c - \Gamma)\left(\frac{1}{2^{3/4}(1+i)} + \frac{\nu}{g_4^{1/4}x^{5/4}}\frac{7\Gamma - 3c}{8} + \cdots\right) \quad , \\
v_4(x) &= -\frac{2i}{3\sqrt{7}}\sqrt{g_4}\sqrt{x} - \frac{g_4^{1/4}\nu}{x^{3/4}}\frac{c}{\sqrt{3}\cdot 7^{1/4}(1+i)} - \frac{\nu^2}{x^2}\frac{1}{24}\left(6c^2 + 54\Gamma^2 - 5\right) + \cdots \quad , \\
v_5(x) &= -2\sqrt{\frac{2}{21}}\sqrt{g_4}\sqrt{x} - \frac{g_4^{1/4}\nu}{x^{3/4}}\frac{21^{1/4}}{2^{5/4}}\left(-\frac{c}{\sqrt{7}} + \frac{\Gamma}{\sqrt{3}}\right) - \frac{\nu^2}{x^2}\frac{1}{48}\left(12c^2 - 12\Gamma^2 - 5\right) + \cdots \quad .
\end{aligned}
\tag{51}
$$

Here, the string coupling is $g_s = \nu/x^{\frac{5}{4}}$. There are a total of *twenty five* expansions related by various $\mathbb{Z}_2$ symmetries. These have been explicitly listed in [20].

A.4 Brane/flux interpretation

Recall the interpretation given to the parameter Γ of the 0A theory (and also to q of the 0B theory). In the $\mu \to +\infty$ regime, Γ represents the number of background ZZ D–branes in the model, with a factor of Γ for each boundary in the worldsheet expansion. Since an orientable surface with odd (even) Euler characteristic must contain an odd (even) number of boundaries, Γ must be raised to an odd (even) power if g_s is. In addition, the power of Γ must be less than or equal to the power of g_s. In the $\mu \to -\infty$ regime, Γ represents the number of units of R–R flux in the background, with $g_s^2\Gamma^2$ appearing when there is an insertion of pure R–R flux. So in this case both Γ and g_s should appear with even powers.

In applying these observations to the DWW expansions above, we immediately notice the remarkable fact that the various expansions have powers of the parameters which somehow allow for interpretations as counting branes or fluxes. This is by no means guaranteed, and indeed its occurrence was one of our main motivations for in–depth study of the system. The presence of two parameters, however, leads to a few subtleties. For example, in some

expansions an interpretation in terms of branes is only possible if one of the two parameters is set to zero. We summarize these below.

A.4.1 Class 1

- v_1 contains powers of Γ consistent with a parameter counting branes. This remains true for any value of c.

- v_1 contains powers of c consistent with a parameter counting fluxes. However, for arbitrary Γ, g_s appears with odd powers, inconsistent with our requirements for a description of fluxes, as mentioned above. This problem is avoided if we set $\Gamma = 0$ since this forces the odd powers of g_s to vanish.

A.4.2 Class 2

- For even n, v_2 contains powers of Γ and g_s consistent with a parameter counting fluxes. This is true for all values of c. For odd n, the powers of Γ are still consistent with the flux interpretation, but there are odd powers of g_s which are inconsistent with fluxes. These odd powers can be removed by setting $c = 0$.

- For even n, v_2 contains powers of c and g_s consistent with a parameter counting fluxes. This is true for all values of Γ. For odd n, the powers of c are consistent with a parameter counting branes. In this interpretation, there are no contributions from surfaces with only one boundary.

A.4.3 Class 3

- v_3 contains powers of Γ consistent with a parameter counting branes. This is true only for $c = 0$.

- v_3 contains powers of c consistent with a parameter counting branes. This is true only for $\Gamma = 0$.

- We notice that associating one boundary to each factor of c *and* Γ also produces a consistent worldsheet expansion. This encourages us to speculate these expansions might capture c and Γ simultaneously counting branes.

A.4.4 Class 4

- v_4 contains powers of c consistent with a parameter counting branes. This remains true for any value of Γ.
- v_4 contains powers of Γ consistent with a parameter counting fluxes. However, for arbitrary c, g_s appears with odd powers, inconsistent with our requirements for a description of fluxes, as mentioned above. This problem is avoided if we set $c = 0$ since this forces the odd powers of g_s to vanish.

A.4.5 Class 5

- v_5 contains powers of Γ consistent with a parameter counting branes. This is true only for $c = 0$.
- v_5 contains powers of c consistent with a parameter counting branes. This is true only for $\Gamma = 0$.
- As for v_3, we notice that associating one boundary to each factor of c *and* Γ also produces a consistent worldsheet expansion. This encourages us to speculate these expansions might capture c and Γ simultaneously counting branes.
- These expansions do not exist for $n < 3$.

References

[1] C. M. Hull and P. K. Townsend, "Unity of superstring dualities," *Nucl. Phys.* **B438** (1995) 109–137, `arXiv:hep-th/9410167`.

[2] P. K. Townsend, "The eleven-dimensional supermembrane revisited," *Phys. Lett.* **B350** (1995) 184–187, `arXiv:hep-th/9501068`.

[3] E. Witten, "String theory dynamics in various dimensions," *Nucl. Phys.* **B443** (1995) 85–126, `arXiv:hep-th/9503124`.

[4] T. R. Morris, "2-D quantum gravity, multicritical matter and complex matrices,". FERMILAB-PUB-90-136-T.

[5] S. Dalley, C. V. Johnson, and T. R. Morris, "Multicritical complex matrix models and nonperturbative 2- D quantum gravity," *Nucl. Phys.* **B368** (1992) 625–654.

[6] S. Dalley, C. V. Johnson, and T. R. Morris, "Nonperturbative two-dimensional quantum gravity," *Nucl. Phys.* **B368** (1992) 655–670.

[7] C. V. Johnson, T. R. Morris, and A. Watterstam, "Global KdV flows and stable 2-D quantum gravity," *Phys. Lett.* **B291** (1992) 11–18, `arXiv:hep-th/9205056`.

[8] S. Dalley, C. V. Johnson, T. R. Morris, and A. Watterstam, "Unitary matrix models and 2-D quantum gravity," *Mod. Phys. Lett.* **A7** (1992) 2753–2762, `arXiv:hep-th/9206060`.

[9] C. Crnkovic, M. R. Douglas, and G. W. Moore, "Physical solutions for unitary matrix models," *Nucl. Phys.* **B360** (1991) 507–523.

[10] T. J. Hollowood, L. Miramontes, A. Pasquinucci, and C. Nappi, "Hermitian vs. Anti-Hermitian 1-matrix models and their hierarchies," *Nucl. Phys.* **B373** (1992) 247–280, `arXiv:hep-th/9109046`.

[11] I. R. Klebanov, J. M. Maldacena, and N. Seiberg, "Unitary and complex matrix models as 1-d type 0 strings," *Commun. Math. Phys.* **252** (2004) 275–323, `arXiv:hep-th/0309168`.

[12] D. J. Gross and A. A. Migdal, "Nonperturbative Two-Dimensional Quantum Gravity," *Phys. Rev. Lett.* **64** (1990) 127.

[13] E. Brezin and V. A. Kazakov, "Exactly solvable field theories of closed strings," *Phys. Lett.* **B236** (1990) 144–150.

[14] M. R. Douglas and S. H. Shenker, "Strings in less than one-dimension," *Nucl. Phys.* **B335** (1990) 635.

[15] J. E. Carlisle, C. V. Johnson, and J. S. Pennington, "Baecklund transformations, D-branes, and fluxes in minimal type 0 strings," *J. Phys.* **A40** (2007) 12451–12462, `arXiv:hep-th/0501006`.

[16] J. E. Carlisle, C. V. Johnson, and J. S. Pennington, "D-branes and fluxes in supersymmetric quantum mechanics," `arXiv:hep-th/0511002`.

[17] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231–252, `arXiv:hep-th/9711200`.

[18] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," *Phys. Lett.* **B428** (1998) 105–114, `arXiv:hep-th/9802109`.

[19] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, `arXiv:hep-th/9802150`.

[20] R. Iyer, C. V. Johnson, and J. S. Pennington, "String Theory and Water Waves," `arXiv:1002.1120 [hep-th]`.

[21] D. Kaup, "Finding eigenvalue problems for solving nonlinear evolution equations," *Prog. Theor. Phys.* **54** (1975) 72–78.

[22] D. Kaup, "A higher-order water-wave equation and the method for solving it," *Prog. Theor. Phys.* **54** (1975) 396–408.

[23] L. Broer, "Approximate equations for long water waves," *Appl. Sci. Res.* **31** (1975) 377–395.

[24] B. Kuperschmidt, "Mathematics of dispersive water waves," *Comm. Math. Phys.* **99** (1985) 51–73.

[25] P. R. Gordoa, N. Joshi, and A. Pickering, "On a generalized $2+1$ dispersive water wave hierarchy," *Publ. Res. Inst. Math. Sci.* **37** (2001) no. 3, 327–347.

[26] I. M. Gelfand and L. A. Dikii, "Asymptotic behavior of the resolvent of Sturm-Liouville equations and the algebra of the Korteweg-De Vries equations," *Russ. Math. Surveys* **30** (1975) 77–113.

[27] M. R. Douglas, "Strings in less than one dimension and the generalized KdV hierarchies," *Phys. Lett.* **B238** (1990) 176.

[28] T. Banks, M. R. Douglas, N. Seiberg, and S. H. Shenker, "Microscopic and macroscopic loops in non-perturbative two- dimensional gravity," *Phys. Lett.* **B238** (1990) 279.

[29] A. B. Zamolodchikov and A. B. Zamolodchikov, "Liouville field theory on a pseudosphere," `arXiv:hep-th/0101152`.

[30] V. E. Zakharov and A. B. Shabat, "Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II," *Funct. Anal. Appl.* **13** (1979) 166–174.

[31] T. R. Morris, "Multicritical matter from complex matrices," *Class. Quant. Grav.* **9** (1992) 1873–1881.

[32] C. V. Johnson, "String theory without branes," `arXiv:hep-th/0610223`.