# FINAL STATE HADRONS IN INCLUSIVE ELECTROPRODUCTION PROCESSES* 

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#### Abstract

Inclusive deep inelastic electroproduction processes, in which one (or more) hadrons in the final state observed in coincidence with the electron, is investigated using Regge-Mueller language, via the inclusive virtual photoproduction processes. The hadron distributions and generalized scaling laws are obtained, assuming that Pomeron is the dominant Regge trajectory, it is a simple pole and therefore factorizes at high energies. The contribution of the isospin carrying secondary trajectories cause charge asymmetries in the central and current fragmentation regions, and these increase with $Q^{2}$, for fixed total energy. Average multiplicity grows logarithmically with energy, as in the hadronic case. However, in the Bjorken limit, the major contribution comes from the current fragmentation region, contrary to the hadronic case, and the contribution of central region scales, and increases with increasing $1 / \omega=\bar{\omega}$. Finally we show that the transverse momentum of the produced particles is limited, and this limit depends on the density of the particles in the phase space, or rate of increase of average multiplicity with $\ln s$, and $\omega$. All the predictions are consistent with the data.


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## I. INTRODUCTION

The scaling laws proposed by Bjorken [1] for the highly inelastic electron scattering processes are consistent with all the experimental data [2], and this phenomena is quite well established and understood at present [3]. In purely hadronic production processes another scaling law of rather different character had been proposed quite a while ago [4]. The interest in this hadronic scaling law has been reviewed by Feynman [5] and Yang [5] quite recently. This scaling law is also consistent with the present experimental data [5].

Recently there has been increasing theoretical [6] and experimental [7] interest in deep inelastic lepton scattering processes in which one or more of the final hadrons is detected in coincidence with the scattered lepton (though the interest on these processes was shadowed by the very exciting discoveries of heavy narrow vector mesons in the $\mathrm{e}^{+} \mathrm{e}^{-}$annihilation channel [10]). The most recent experimental results of SLAC, Cornell and DESY are reported at the 1975 SLAC conference [9]. The data seem to be offering no big surprises.

On the theoretical side [8] the first attempts have been focused on the possibility that the invariant distributions will exhibit some sort of, similar to that of Bjorken and Feynman and Yang, scaling laws. The exclusive subsets of these processes have been investigated by T. D. Lee [8] using an $\mathrm{SU}_{2}$ phenomenological Lagrangian technique. The inclusive processes have been analyzed by Drell and Yan [8], and Landshoff and Polkinghorne [8], and Feynman [3], using different forms of the parton mpdel. Stack [8] investigated the same problem by using the free-field light-cone commutators, and assuming the dominance of light-cone singularity for certain semiconnected diagrams. Then attempts have been made by the present author [11] and the others [11], to study the problem, by adapting the formalism developed by Mueller [12] for purely hadronic production processes.

This approach based on very simple and plausible assumption that processes involving highly virtual photons are also dominated by Regge trajectory exchanges, Pomeron being the highest, at high energies. The price paid for not having any fictitious entities like partons or quarks, is that the predictions in this approach are not as detailed as that of a parton model [3]. This work is a corrected and updated version of the second reference in [11], which was the first thorough attempt on the problem $\dagger \dagger$. All the predictions are in perfect agreement with the new available data [9].

A final work; although I will use always the terminology electroproduction, everything I say applies to muon production also (the terminology leptoproduction does not seem to be popular somehow).

## II. DEEP INELASTIC ELECTROPRODUCTION

We shall start with a short review of deep inelastic electron-proton scattering from the Regge point of view for the sake of completeness. Treating the electromagnetic interaction to lowest order, the process is represented by the Feynman diagram given in Fig. 1. By applying Feynman rules, the differential cross section for fixed incident electron energy and fixed scattering angle is given by

$$
\begin{equation*}
\frac{\pi}{\mathrm{EE}^{\dagger}} \frac{\mathrm{d} \sigma}{\mathrm{~d} \Omega \mathrm{dE}^{\mathrm{t}}}=\frac{\mathrm{d} \sigma}{\mathrm{~d} Q^{2} \mathrm{~d} \nu}=\frac{4 \pi \alpha^{2}}{\mathrm{Q}^{4}} \frac{\mathrm{E}^{\prime}}{\mathrm{E}}\left[\mathrm{~W}_{2}\left(\mathrm{Q}^{2}, \nu\right) \cos ^{2} \frac{\theta}{2}+2 \mathrm{~W}_{1}\left(\mathrm{Q}^{2}, \nu\right) \sin ^{2} \frac{\theta}{2}\right] \tag{2.1}
\end{equation*}
$$

where $E\left(E^{\prime}\right)$ is the laboratory energy of the incident (final) electron, $\theta$ is the lab scattering angle, m is the nucleon mass, $\alpha$ is the fine structure constant, and q is the four-momentum transferred to the proton by the photon; we have also defined $Q^{2}=-q^{2}$. The invariant structure functions are defined by

$$
\begin{align*}
\mathrm{W}_{\mu \nu}(\mathrm{q}, \mathrm{p}) & \left.=\sum_{\text {spins }} j^{4} \mathrm{x}^{4} \mathrm{e}^{\mathrm{iq} \cdot \mathrm{x}}<\mathrm{p}\left|\left[\mathrm{~J}_{\mu}(\mathrm{x}), \mathrm{J}_{\nu}(0)\right]\right| \mathrm{p}\right\rangle \\
& =-(\Lambda)_{\mu \nu} \mathrm{W}_{1}\left(\mathrm{Q}^{2}, \nu\right)+\frac{1}{\mathrm{~m}^{2}}(\Lambda \cdot \mathrm{p})_{\mu}(\Lambda \cdot \mathrm{p})_{\nu} \mathrm{W}_{2}\left(\mathrm{Q}^{2}, \nu\right) \tag{2.2}
\end{align*}
$$

where $\Lambda_{\mu \nu}=\mathrm{g}_{\mu \nu}-\frac{\mathrm{q}_{\mu} \mathrm{q}_{\nu}}{\mathrm{q}^{2}}$. Both $\mathrm{W}_{1}$ and $\mathrm{W}_{2}$ are antisymmetric under $\nu \rightarrow-\nu$ :

$$
\mathrm{W}_{\mathrm{i}}\left(\mathrm{Q}^{2},-\nu\right)=-\mathrm{W}_{\mathrm{i}}\left(\mathrm{Q}^{2}, \nu\right)
$$

The total absorption cross section for longitudinal and transverse virtual photons, $\sigma^{L, T}$, is defined by

$$
\begin{equation*}
\text { (Flux) } \sigma^{\mathrm{T}, \mathrm{~L}}=4 \pi^{2} \alpha \epsilon_{\mu}^{* \mathrm{~T}, \mathrm{~L}} \mathrm{~W}^{\mu \nu} \epsilon_{\nu}^{\mathrm{T}, \mathrm{~L}} \tag{2.3}
\end{equation*}
$$

where $\epsilon_{\mu}^{\mathrm{T}, \mathrm{L}}$ are the polarization vectors for virtual photons which satisfy the gauge condition $q \cdot \epsilon=0$. In a frame in which $q=\left(q_{0}, 0,0, q_{3}\right)$ we choose them as

$$
\begin{align*}
\epsilon_{\mu}^{\mathrm{L}} & =\frac{1}{\mathrm{Q}}\left(\mathrm{q}_{3}, 0,0, \mathrm{q}_{0}\right) \\
\epsilon_{\mu}^{\mathrm{T} \pm} & =\frac{1}{\sqrt{2}}(0,1, \pm \mathrm{i}, 0) \tag{2.4}
\end{align*}
$$

Substituting in Eq. (2.7) we get

$$
\begin{align*}
& \text { (Flux) } \sigma_{\mathrm{T}}=4 \pi^{2} \alpha \mathrm{~W}_{1}\left(\mathrm{Q}^{2}, \nu\right) \\
& \text { (Flux) } \sigma_{\mathrm{L}}=4 \pi^{2} \alpha\left[\left(1+\frac{\nu^{2}}{\mathrm{Q}^{2}}\right) \mathrm{W}_{2}\left(\mathrm{Q}^{2}, \nu\right)-\mathrm{W}_{1}\left(\mathrm{Q}^{2}, \nu\right)\right] . \tag{2.5}
\end{align*}
$$

By dimensional arguments, $\sigma_{\mathrm{L}} \sim \mathrm{Q}^{2}$ as $\mathrm{Q}^{2} \rightarrow 0$, and $\sigma_{\mathrm{T}}\left(\mathrm{Q}^{2}=0, \nu\right)$ is the total photoabsorption cross section for real photons of energy $\nu$.

It is suggestive to write the differential cross section in terms of $\sigma_{\mathrm{T}}$ and $\sigma_{\mathrm{L}}$ as defined above:

$$
\begin{equation*}
\frac{\mathrm{d} \sigma}{\mathrm{~d} \Omega \mathrm{dE}}{ }^{\mathbf{1}}=\frac{\mathrm{e}^{2}}{2 \pi^{2}} \frac{\sqrt{\nu^{2}+\mathrm{Q}^{2}}}{\mathrm{Q}^{2}} \frac{\mathrm{E}^{\prime}}{\mathrm{E}} \frac{\sigma_{\mathrm{T}}}{1-\epsilon}[1+\epsilon \mathrm{R}] \tag{2.6}
\end{equation*}
$$

where

$$
\begin{equation*}
\epsilon=\left[1+2 \tan ^{2} \frac{\theta}{2}\left(1+\frac{\nu^{2}}{Q^{2}}\right)\right]^{-1} \tag{2.7}
\end{equation*}
$$

is called the polarization parameter [13] and often is small, and $\mathrm{R}=\sigma_{\mathrm{L}} / \sigma_{\mathrm{T}}$.
Bjorken [1] argued that the limits of $\mathrm{W}_{1}$ and $\nu \mathrm{W}_{2}$ exist as both $\nu$ and $\mathrm{Q}^{2}$ tend to infinity with their ratio fixed:

$$
\begin{array}{r}
\mathrm{W}_{1}\left(\mathrm{Q}^{2}, \nu\right) \rightarrow \mathrm{F}_{1}(\omega) \\
\frac{1}{\mathrm{~m}} \nu \mathrm{~W}_{2}\left(\mathrm{Q}^{2}, \nu\right) \rightarrow \mathrm{F}_{2}(\omega) \tag{2.8}
\end{array}
$$

here $\omega=Q^{2} / 2 \mathrm{~m} \nu$. Present experimental data [2] is in agreement with this proposal for $Q^{2}>0.5(\mathrm{GeV} / \mathrm{c})^{2}$, and we will take this scaling phenomenon, Bjorken scaling, as an experimental fact. Furthermore experiments indicate that the ratio $R$ is very small [2]:

$$
R=0.14 \pm 0.067 .
$$

Now consider the forward elastic scattering of a massive photon of space like momentum $q$ by a physical nucleon of momentum $p$; so $q^{2}=-Q^{2}<0, p^{2}=m^{2}$. The amplitude averaged over nucleon spins has the form,

$$
\begin{align*}
\mathrm{T}_{\mu \nu}\left(\mathrm{Q}^{2}, \nu\right) & \left.=\mathrm{i} \int \mathrm{~d}^{4} \mathrm{x} \mathrm{e}^{\mathrm{iq} \cdot \mathrm{x}}<\mathrm{p}\left|\mathrm{~T}\left[\mathrm{~J}_{\mu}^{\mathrm{em}}(\mathrm{x}) \mathrm{J}_{\nu}^{\mathrm{em}}(0)\right]\right| \mathrm{p}\right\rangle \\
& =-(\Lambda)_{\mu \nu} \mathrm{T}_{1}\left(\mathrm{Q}^{2}, \nu\right)+\frac{1}{\mathrm{~m}^{2}}(\Lambda \cdot \mathrm{p})_{\mu}(\Lambda \cdot \mathrm{p})_{\nu} \mathrm{T}_{2}\left(\mathrm{Q}^{2}, \nu\right) \tag{2.9}
\end{align*}
$$

The structure functions $W_{1,2}$ are related to the absorptive parts of $T_{1}$ and $T_{2}$ :

$$
\begin{equation*}
\mathrm{W}_{\mathrm{i}}\left(\mathrm{Q}^{2}, \nu\right)=\frac{1}{2 \pi} \operatorname{Im} \mathrm{~T}_{\mathrm{i}}\left(\mathrm{Q}^{2}, \nu\right) . \tag{2.10}
\end{equation*}
$$

It can be shown that $[14] \mathrm{T}_{1}\left(Q^{2}, \nu\right)$ is a helicity nonflip amplitude for the t-channel process $\gamma+\gamma \rightarrow \mathrm{N}+\overline{\mathrm{N}}$. So we may decompose it into partial waves according to

$$
\begin{equation*}
T_{1}\left(Q^{2}, \nu\right)=\sum_{J=0}^{\infty}(2 J+1) t_{1}\left(Q^{2}, J\right)\left[P_{J}\left(\cos \theta_{\mathrm{t}}\right)+P_{J}\left(-\cos \theta_{\mathrm{t}}\right)\right] \tag{2.11}
\end{equation*}
$$

appropriately analytically continued from the region $t>4 m_{N}^{2}, q^{2}>0$ to the Compton region required for our problem, namely $\mathrm{t}=0, \mathrm{q}^{2}<0$ with $\nu=\mathrm{p} \cdot \mathrm{q}$ physical (Fig. 2). Here $\theta_{t}$ is the scattering angle in the center of mass frame of the $t-$ channel, and for small $t$, is given by

$$
\cos \theta_{\mathrm{t}} \sim \frac{\nu}{\mathrm{q}}
$$

where $q \equiv \sqrt{q}^{2}$. For later convenience we shall also define $Q^{2}=-q^{2}$. Now let us look at the behavior of $\cos \theta_{t}$ at various limits:
(a) Regge limit $=\nu=p \cdot q / m \rightarrow \infty$, while $q^{2}$ is large and fixed.

Obviously $\cos \theta_{t} \rightarrow \infty$.
(b) Bjorken limit $=\nu, \mathrm{q}^{2} \rightarrow \infty$ while $\omega=\frac{\mathrm{Q}^{2}}{2 \mathrm{~m} \nu}$ fixed. In this case $\cos \theta_{t} \sim-q / 2 m \omega \rightarrow \infty$. So the conditions for a Regge expansion, to be more precise a crossed channel SO (3) expansion, are satisfied in the Bjorken scaling region. The essential point is that the $\nu$-dependence in the Bjorken limit is exhibited by the Regge representation.

Dropping the background integral in the Sommerfield-Regge expansion, we get

$$
\begin{equation*}
\mathrm{T}_{1}\left(\mathrm{Q}^{2}, \nu\right) \sim \sum_{\mathrm{n}}\left(2 \alpha_{\pi}+1\right) \frac{\bar{\beta}_{\mathrm{n}}\left(\mathrm{Q}^{2}, \alpha_{\mathrm{n}}\right)}{\sin \pi \alpha_{\mathrm{n}}}\left[\mathrm{P}_{\alpha_{\mathrm{n}}}\left(\cos \theta_{\mathrm{t}}\right)+\mathrm{P}_{\alpha_{\mathrm{n}}}\left(-\cos \theta_{\mathrm{t}}\right)\right] \tag{2.11a}
\end{equation*}
$$

To leading order, $\mathrm{P}_{\alpha_{\mathrm{n}}}\left(\cos \theta_{\mathrm{t}}\right) \sim\left(\cos \theta_{\mathrm{t}}\right)^{\alpha} \mathrm{n}_{\mathrm{n}}$ as $\cos \theta_{\mathrm{t}} \rightarrow \infty$, so we get (Fig. 2)

$$
\begin{equation*}
\mathrm{W}_{1}\left(\mathrm{Q}^{2}, \nu\right)=\frac{1}{2 \pi} \operatorname{Im~}_{1}\left(\mathrm{Q}^{2}, \nu\right)=\sum_{\mathrm{n}} \beta_{\gamma \gamma}^{\alpha} \mathrm{n}^{\left(\mathrm{Q}^{2}\right) \beta_{\mathrm{PP}}}{ }_{\mathrm{n}}^{\alpha}(0)\left(\frac{\nu}{\mathrm{q}}\right)^{\alpha} \mathrm{n}^{(0)} \tag{2.11b}
\end{equation*}
$$

where we have assumed factorization of Regge residues and absorbed the irrelevant factors into the $\beta$ 's. Obviously the validity of this argument requires $Q^{2}$ to be large. For small $Q^{2}$ this is rather dubious, for the neglected terms would be comparable to the nonleading Regge contributions of the Regge expansion. There is also the question of the region of validity of the expansion (2.11a). Since the sum extends over all leading trajectories, by the use of duality arguments, one may hope that (2.11a) is a good representation of the amplitude $T_{1}\left(Q^{2}, \nu\right)$ even for moderate values of $\nu$, and not just the asymptotic region.

Note that we can pass to the Bjorken region from the Regge region, by keeping $\nu$ large and fixed and let $q^{2} \rightarrow \infty$. We observe that the simplest way to obtain the scaling laws (2.8) from the Regge expansion (2.11b) is to have [15]

$$
\begin{equation*}
\left.{ }_{\beta}^{\alpha}{ }_{\gamma \gamma} \mathrm{n}^{2}\right)_{\mathrm{Q} \rightarrow \infty} \simeq \frac{\beta_{\gamma \gamma}^{\alpha} \mathrm{n}_{(0)}}{\alpha_{\mathrm{Q}}(0)} \tag{2.12}
\end{equation*}
$$

for the photon-photon-Reggeon vertex for large $Q^{2}$. Then we see that, if we believe in Bjorken scaling, then, in some loose sense, the photon-Reggeon coupling strength decreases as $\mathrm{Q} \rightarrow \infty$. This simple observation is very interesting and suggestive in the light of lately popular "asymptotically free field theories" [16], which claim that Bjorken scaling is a consequence of asymptotic freedom. Equation (2.12) is the crucial prescription we shall extensively use in the rest of the paper.

Using (2.12) we get

$$
\mathrm{W}_{1}\left(\mathrm{Q}^{2}, \nu\right) \overrightarrow{\mathrm{Bj}} \mathrm{~F}_{1}(\omega) \simeq \sum_{\mathrm{n}}{\beta_{\gamma \gamma}}_{\alpha_{\mathrm{n}}}(0) \beta_{\mathrm{pp}}^{\alpha}{ }_{\mathrm{n}}(0)\left(-\frac{\nu}{\mathrm{q}^{2}}\right)^{\alpha}{ }^{\mathrm{n}^{(0)}}
$$

i.e.,

$$
\begin{equation*}
\mathrm{F}_{1}(\omega) \sim \sum_{\mathrm{n}} \beta_{\gamma \gamma}^{\alpha_{\mathrm{n}}}(0) \beta_{\mathrm{pp}}{ }_{\mathrm{n}}(0)\left(\frac{1}{2 \mathrm{~m} \omega}\right)^{\alpha_{\mathrm{n}}(0)} \tag{2.13}
\end{equation*}
$$

Similar arguments apply to $\mathrm{W}_{2}\left(\mathrm{Q}^{2}, \nu\right)$ and we get a similar result

$$
\begin{equation*}
\frac{1}{\mathrm{~m}} \nu \mathrm{~W}_{2}\left(\mathrm{Q}^{2}, \nu\right) \underset{\mathrm{Bj}}{\mathrm{~F}_{2}(\omega) \simeq \sum_{\mathrm{n}} \beta_{\gamma \gamma}^{\alpha}(0){ }_{\mathrm{n}}{ }_{\mathrm{pp}}(0)\left(\frac{1}{2 \mathrm{~m} \omega}\right)^{\alpha} .} \tag{2.14}
\end{equation*}
$$

Now if we take only the leading (Pomeron) trajectory, we get

$$
\begin{equation*}
\mathrm{F}_{1}(\omega) \underset{\mathrm{Bj}}{\sim} \frac{\alpha}{\omega} \tag{2.15a}
\end{equation*}
$$

$$
\begin{equation*}
\mathrm{F}_{2}(\omega) \underset{\mathrm{Bj}}{\sim} 2 \mathrm{a} \tag{2.15b}
\end{equation*}
$$

in accord with the relation $\sigma_{\mathrm{L}} / \sigma_{\mathrm{T}} \rightarrow 0$. A warning remark is appropriate here. Actually, the point $\alpha(0)=1$ is a nonsense point for $\mathrm{W}_{2}$, and the Pomeranchuk trajectory decouples unless a fixed pole is present which restores its contributions. If we include the secondary Regge trajectory ( $\mathrm{P}^{\mathrm{\prime}}$ ), we get

$$
\begin{align*}
& F_{1}(\omega) \underset{\operatorname{Bj}}{ } \frac{a}{\omega}+\frac{b}{\sqrt{\omega}}  \tag{2.16}\\
& F_{2}(\omega) \underset{B_{j}}{\sim} 2 a+2 b \sqrt{\omega}
\end{align*}
$$

Notice that $F_{2}$ goes to a constant from above, as $\frac{1}{\omega}=\frac{2 p \cdot q}{Q^{2}} \rightarrow \infty$ (Regge limit).

## III. INCLUSIVE ELECTROPRODUCTION

## III. 1 Definitions and Kinematics

We shall consider the process in which an electron scatters from a hadron (nucleon) and one of the hadrons in the final state, in addition to the scattered electron, is detected:

$$
e(\ell)+h(p) \rightarrow e^{\prime}\left(\ell^{\prime}\right)+h^{\prime}(k)+H\left(p_{H}\right)
$$

Treating the electromagnetic interaction to lowest order the process is represented by the Feynman diagram given in Fig. 3, where $E\left(E^{\prime}\right)$ is the lab energy of the incident (final) electron. Applying Feynman rules the differential cross section for fixed incident electron energy, fixed electron scattering angle, and fixed hadron scattering angles ( $\theta$ and $\phi$ ), and summing over all else, is given by

$$
\begin{equation*}
\text { (Flux) } \frac{\mathrm{d} \sigma}{\frac{\mathrm{~d}^{3} \ell^{\prime}}{2 \mathrm{E}^{\mathrm{t}}} \frac{\mathrm{~d}^{3} \mathrm{k}}{2 \mathrm{k}_{0}}}=2 \alpha^{2} \frac{1}{\mathrm{q}^{4}} \ell_{\mu \nu} \hat{\mathrm{w}}^{\mu \nu} \tag{3.1}
\end{equation*}
$$

where, the masses of target and detected hadron are denoted by m and $\mu$.
Simple Dirac algebra gives

$$
\begin{align*}
& \ell_{\mu \nu} \sim 2\left[\ell_{\mu} \ell_{\nu}^{\prime}+\ell_{\nu} \ell_{\mu}^{\prime}-\mathrm{g}_{\mu \nu} \ell \cdot \ell^{\prime}\right] \quad \text { with } \mathrm{m}_{\mathrm{e}}^{2} \sim 0 \\
& \mathrm{~W}_{\mu \nu}= \left.\frac{1}{2} \sum_{\text {spins }} \sum_{\mathrm{II}}\langle\mathrm{p}| \mathrm{J}_{\mu}^{+}(0)\left|\mathrm{H}\left(\mathrm{p}_{\mathrm{H}}\right), \mathrm{k}\right\rangle\left\langle\mathrm{k}, \mathrm{H}\left(\mathrm{p}_{\mathrm{H}}\right)\right| \mathrm{J}_{\nu}(0) \right\rvert\,> \\
& \times(2 \pi)^{3} \delta^{4}\left(\mathrm{q}+\mathrm{p}-\mathrm{k}-\mathrm{p}_{\mathrm{H}}\right) \\
&= \sum_{\text {spins }} \int \frac{\mathrm{d}^{4} \mathrm{x}}{4 \pi} \mathrm{e}^{\mathrm{iq} \cdot \mathrm{x}}\langle\mathrm{p}| J_{\mu}^{+}(\mathrm{x})|\mathrm{k}\rangle\langle\mathrm{k}| J_{\nu}(0)|\mathrm{p}\rangle \tag{3.2}
\end{align*}
$$

where $J_{\mu}$ is the hadron electromagnetic current, $\mid p>$ is a one-proton state and $\mid \mathrm{k}, \mathrm{H}>$ is a state of the one hadron being detected plus all possible others with
quantum numbers summarized by $H$. Here $\sum_{\mathrm{H}}$ means sum over all intermediate states $H$, which is connected to the proton by $\mathrm{J}^{\mathrm{em}}$, and integrate over their invariant phase. Our metric, normalization of states, etc. are the same as in Ref. [17]. We denote the average over the initial spin by $\bar{\Sigma}$.

Lorentz invariance tells us that $\hat{\mathrm{W}}_{\mu \nu}$ must be a second rank tensor. Because of the average over spins we have only the vectors $q_{\mu}, p_{\mu}, k_{\mu}$, and the tensor $g_{\mu \nu}$ at our disposal. Tensors of the type $\epsilon_{\mu \nu \alpha \beta} q^{\alpha}{ }^{p}{ }^{\beta}$ or $\epsilon_{\mu \nu \alpha \beta} q^{\alpha} k^{\beta}$ are excluded because $W_{\mu \nu}$ has positive parity (current operator is a polar vector under spatial reflections). Furthermore, the electromagnetic current is conserved, i.e.,

$$
\mathrm{q}_{\mu} \hat{\mathrm{W}}^{\mu \nu}=\hat{\mathrm{W}}^{\mu \nu} \mathrm{q}_{\nu}=0
$$

Therefore the most general form for the symmetric tensor $\hat{W}_{\mu \nu}$ is

$$
\begin{aligned}
\hat{\mathrm{W}}_{\mu \nu}= & -(\Lambda)_{\mu \nu} \hat{\mathrm{W}}_{1}+\frac{1}{\mathrm{~m}^{2}}(\Lambda \cdot \mathrm{p})_{\mu}(\Lambda \cdot \mathrm{p})_{\nu} \hat{\mathrm{W}}_{2}+\frac{1}{\mu^{2}}(\Lambda \cdot \mathrm{k})_{\mu}(\Lambda \cdot \mathrm{k})_{\nu} \hat{\mathrm{W}}_{3} \\
& +\frac{1}{2 \mathrm{~m} \mu}\left[(\Lambda \cdot \mathrm{p})_{\mu}^{(\Lambda \cdot \mathrm{k})_{\nu}}+(\Lambda \cdot \mathrm{k})_{\mu}(\Lambda \cdot \mathrm{p})_{\nu}\right] \hat{\mathrm{W}}_{4}
\end{aligned}
$$

with

$$
\Lambda_{\mu \nu}=\mathrm{g}_{\mu \nu}-\frac{\mathrm{q}_{\mu} \mathrm{q}_{\nu}}{\mathrm{q}^{2}}
$$

Note that because of the conservation of the leptonic current, and a special choice of the gauge $\left(\epsilon_{\mu} q^{\mu}=0\right)$ the relevant form of the tensor $\hat{W}_{\mu \nu}$ in the inclusive virtual photoproduction processes is

$$
\begin{equation*}
\hat{\mathrm{W}}_{\mu \nu}=-\mathrm{g}_{\mu \nu} \hat{\mathrm{W}}_{1}+\frac{1}{\mathrm{~m}^{2}} \mathrm{p}_{\mu} \mathrm{p}_{\nu} \hat{\mathrm{W}}_{2}+\frac{1}{\mu^{2}} \mathrm{k}_{\mu} \mathrm{k}_{\nu} \hat{\mathrm{W}}_{3}+\frac{1}{2 \mathrm{~m} \mu}\left[\mathrm{p}_{\mu} \mathrm{k}_{\nu}+\mathrm{p}_{\nu} \mathrm{k}_{\mu}\right] \hat{\mathrm{W}}_{4} \tag{3.3}
\end{equation*}
$$

For the process $\gamma+\mathrm{p} \rightarrow \mathrm{h}+$ anything, we need $3 \times 4-10+2=4$ invariant independent variables, and for the process $e+p \rightarrow e^{p}+h+$ anything, we need
$3 \times 5-10+1=6$ independent invariant variables to express the cross sections. But because of the single photon exchange dominance, hadronic and leptonic parts factor in the differential cross section as depicted in (3.1). Because of this fact, which is known as the locality of the lepton vertex, the hadronic tensor $\mathrm{W}_{\mu \nu}$, cannot depend on the $\ell_{\mu}$ and $\ell_{\mu}^{\prime}$, except insofar as the $\mathrm{p}, \mathrm{k}$ and $\mathrm{p}_{\mathrm{H}}$ are tied to $\ell$ and $\ell^{\prime}$ by the overall conservation laws. In virtue of these laws, $q=\ell-\ell^{\prime}$ is effectively a hadronic four-vector. So $\hat{W}_{\mu \nu}$ can depend on its components. But $\left(\ell+\ell^{\prime}\right)_{\mu}=\tilde{\mathrm{q}}_{\mu}$ is not a hadronic four-vector and $\mathrm{W}_{\mu \nu}$ cannot depend on its components to the extent that the latter are unconstrained. There are two such constraints:
a. The magnitude of $\tilde{q}_{\mu}$ is related to the magnitude of $\mathrm{q}_{\mu}$ :

$$
\tilde{\mathrm{q}}^{2}=\left(\ell+\ell^{\prime}\right)^{2}=4 \mathrm{~m}_{\mathrm{e}}^{2}-\mathrm{q}^{2}
$$

b. Projection of $\tilde{\mathrm{q}}_{\mu}$ on $\mathrm{q}_{\mu}$ is not free either

$$
\mathrm{q}_{\mu} \tilde{\mathrm{q}}^{\mu}=\ell^{2}-\ell^{2}{ }^{2}=0
$$

This implies that six variables, which describe the whole process can be chosen in such a way, that $W_{\mu \nu}$ is independent of two of them. The dependence of $d \sigma$, on these two variables must therefore be explicitly contained in the lepton factor $\ell_{\mu \nu}$. In the lab frame (rest frame of the target) we shall define the direction of $q_{\mu}$ as 0 z -axis. We shall take the 0 x -axis, $\quad$ in the plane spanned by the vectors $\vec{\ell}$ and $\vec{l}^{\prime}$. Then the azimuthal angle of the detected hadron, measured from the 0x-axis is going to be the angle between the planes (or their normals) defined $\vec{\ell}$ and $\overrightarrow{\ell^{\prime}}$ and $\overrightarrow{\mathrm{q}}$ and $\overrightarrow{\mathrm{k}}$.

We shall choose the four invariant variables to describe the hadronic tensor $\mathrm{W}_{\mu \nu}$ as follows:

$$
Q^{2}=-q^{2}>0
$$

$$
\begin{aligned}
\mu \nu^{\prime} & =\mathrm{k} \cdot \mathrm{q} \\
\mathrm{~m} \kappa & =\mathrm{p} \cdot \mathrm{k} \\
\tan \phi & =\mathrm{k}_{\mathrm{y}} / \mathrm{k}_{\mathrm{x}}
\end{aligned}
$$

Two new variables are $\nu^{\prime}$, the energy loss of the electrons in the rest frame of the detected hadron, and $\kappa$, the energy of the detected hadron is the lab frame. Other sets of variables which are equally well suited for describing the hadronic part of the $d \sigma$ are $\left(Q^{2}, t, M^{2}, \phi\right)$ or $\left(Q^{2},|\overrightarrow{\mathrm{k}}|, \cos \theta, \phi\right)$ where

$$
\begin{align*}
\mathrm{t} & =(\mathrm{p}-\mathrm{k})^{2}=\mathrm{m}^{2}+\mu^{2}-2 \mathrm{~m} \kappa  \tag{3.4}\\
\mathrm{M}^{2} & =(\mathrm{p}+\mathrm{q}-\mathrm{k})^{2}=\mathrm{m}^{2}+\mu^{2}+2 \mathrm{~m} \nu-2 \mu \nu^{\prime}-2 \mathrm{~m} \kappa-\mathrm{Q}^{2}
\end{align*}
$$

Here $\theta$ is the polar angle of the detected hadron, measured from the photon direction (Fig. 4). The invariant phase space volume element, in terms of these different sets, is given by

$$
\begin{align*}
\frac{\mathrm{d}^{3} \mathrm{k}}{2 \mathrm{k}_{0}} & \left.=\frac{|\overrightarrow{\mathrm{k}}|^{2}}{2 \mathrm{k}_{0}} \mathrm{~d} \right\rvert\, \overrightarrow{\mathrm{k} \mid} \mathrm{d} \cos \theta \mathrm{~d} \phi \\
& =\frac{\mu}{2\left(\nu^{2}+\mathrm{Q}^{2}\right)^{1 / 2}} \mathrm{~d} \nu^{\prime} \mathrm{d} \kappa \mathrm{~d} \phi  \tag{3.5}\\
& =\frac{1}{8 \mathrm{~m}\left(\nu^{2}+\mathrm{Q}^{2}\right)^{1 / 2}} \mathrm{dt} \mathrm{dM}
\end{align*}
$$

The old and new invariant variables $\left(\mathrm{Q}^{2}, \nu\right)$ and $\left(\nu^{\prime}, \kappa\right)$ satisfy the kinematical constraints

$$
\begin{equation*}
0<\omega \equiv \frac{\mathrm{Q}^{2}}{2 \mathrm{~m} \nu}<1, \quad 2 \mu \nu^{\prime}+2 \mathrm{~m} \kappa<2 \mathrm{~m} \nu(1-\omega) \tag{3.6}
\end{equation*}
$$

which are derived from the positivity conditions

$$
\begin{equation*}
\mathrm{s}=(\mathrm{q}+\mathrm{p})^{2} \geq \mathrm{m}^{2} \quad \text { and } \quad \mathrm{M}^{2}=(\mathrm{q}+\mathrm{p}-\mathrm{k})^{2}>0 \tag{3.7}
\end{equation*}
$$

in the Bjorken limit.

As stated above, summing over the final spins, and averaging over the initial spins (unpolarized lepton scattering over unpolarized proton target) the number of independent helicity amplitudes is four. The differential cross section in terms of these functions is
$\frac{\mathrm{d} \sigma}{\mathrm{d} \Omega^{\prime} \mathrm{dE}^{\prime} \frac{\mathrm{d}^{3} \mathrm{k}}{2 \mathrm{k}_{0}}}=\frac{\alpha^{2}}{4 \pi} \frac{\mathrm{E}^{\prime}}{2 \mathrm{mE}} \frac{1}{\mathrm{Q}^{2}(1-\epsilon)}\left[\mathrm{W}^{++}+\epsilon \mathrm{W}^{\mathrm{OO}}-\epsilon \mathrm{W}^{+-} \cos 2 \phi-2 \sqrt{\epsilon+\epsilon^{2}}\left(\operatorname{Re} \mathrm{~W}^{+\mathrm{o}}\right) \cos \phi\right]$
Here $\epsilon$ is the polarization parameter defined in (2.7) and $W^{a b}$ are

$$
\mathrm{w}^{\mathrm{ab}}=\epsilon_{\mu}^{\mathrm{a}^{*}} \epsilon_{\nu}^{\mathrm{b}} \hat{\mathrm{w}}^{\mu \nu} ; \quad \mathrm{a}, \mathrm{~b}=0, \pm
$$

where ( $\pm$ ) stands for two transverse polarizations and (0) for the longitudinal polarization as defined in (2.4). The lengthy relations between $W^{a b}$ and the $\hat{W}_{i}$ are given in Appendix 2.

As is obvious from (3.8), if we integrate over the azimuthal angle $\phi$, the last two terms vanish, and we are left with two structure functions, $\mathrm{W}^{++}$and $W^{o o}$. Let us do this more systematically by integrating the hadronic tenor $W^{\mu \nu}$ over the azimuthal angle $\phi$.

$$
\begin{align*}
\hat{\mathrm{W}}^{\mu \nu}\left(\mathrm{Q}^{2}, \nu^{\prime}, \kappa\right)= & \int \frac{\mathrm{d}^{3} \mathrm{k}}{2 \mathrm{k}_{0}} \delta\left(\nu^{\prime}-\frac{\mathrm{k} \cdot \mathrm{q}}{\mu}\right) \delta\left(\kappa-\frac{\mathrm{p} \cdot \mathrm{k}}{\mathrm{~m}}\right) \hat{\mathrm{w}}^{\mu \nu} \\
= & \int \frac{\mathrm{d}^{3} \mathrm{k}}{2 \mathrm{k}_{0}} \delta\left(\nu^{\prime}-\frac{\mathrm{k} \cdot \mathrm{q}}{\mu}\right) \delta\left(\kappa-\frac{\mathrm{p} \cdot \mathrm{k}}{\mathrm{~m}}\right) \times \\
& \left.\times \int \frac{\mathrm{d}^{4} \mathrm{x}}{4 \pi} \mathrm{e}^{\mathrm{iq} \cdot \mathrm{x}}<\mathrm{p}\left|\mathrm{~J}_{\mu}^{+}(\mathrm{x})\right| \mathrm{k}\right\rangle\langle\mathrm{k}| \mathrm{J}_{\nu}(0)|\mathrm{p}\rangle \tag{3.8}
\end{align*}
$$

Now $\mathrm{W}^{\mu \nu}$ can be written in terms of two structure functions as:

$$
\begin{equation*}
\mathrm{W}^{\mu \nu}=-(\Lambda)_{\mu \nu} \mathscr{W}_{1}\left(\mathrm{Q}^{2}, \nu, \nu^{\mathrm{p}}, \kappa\right)+\frac{1}{\mathrm{~m}^{2}}(\Lambda \cdot \mathrm{p})_{\mu}(\Lambda \cdot \mathrm{p})_{\nu} \mathscr{\mathscr { V }}_{2}\left(\mathrm{Q}^{2}, \nu, \nu^{\imath}, \kappa\right) \tag{3.9}
\end{equation*}
$$

and the differential cross section takes the familiar form

$$
\begin{equation*}
\left\langle\frac{\mathrm{d} \sigma}{\mathrm{~d} \Omega^{\prime} \mathrm{dE} \cdot \frac{\mathrm{~d}^{3} \mathrm{k}}{2 \mathrm{k}_{0}}}\right\rangle=\frac{4 \alpha^{2}}{\mathrm{Q}^{4}} \mathrm{E}^{2}\left[\mathscr{T}_{2} \cos ^{2} \frac{\theta}{2}+2 \mathscr{\mathscr { W } _ { 1 } \operatorname { s i n } ^ { 2 } \frac { \theta } { 2 } ]}\right. \tag{3.10}
\end{equation*}
$$

where " $\rangle$ " stands for the azimuthal integration. The differential cross section for virtual inclusive photoproduction is defined in the usual way:

$$
\begin{equation*}
\text { (Flux) } \frac{\mathrm{d} \sigma}{\frac{\mathrm{~d}^{3} \mathrm{k}}{2 \mathrm{k}_{0}}}=4 \pi^{2} \alpha \epsilon_{\mu}^{*} \epsilon_{\nu} \hat{\mathrm{w}}^{\mu \nu} \tag{3.11}
\end{equation*}
$$

Conventionally the flux factor for the virtual photon is defined as the flux factor for a real photon with the same initial invariant mass $(q+p)^{2}$ as

$$
(\text { Flux })=\mathrm{q} \cdot \mathrm{p}-\frac{1}{2} \mathrm{Q}^{2} \underset{\mathrm{bj}}{\sim} \mathrm{~m} \nu(1-\omega)
$$

Sometimes we shall define the flux factor as we do for hadron beams, whenever it is convenient as, (Flux) $=\left[(q \cdot p)^{2}+m^{2} Q^{2}\right]^{1 / 2}$. A simple calculation gives

$$
\begin{align*}
\epsilon_{\mu}^{*} \mathrm{~T} \mathrm{~W}^{\mu \nu} \epsilon_{\nu}^{\mathrm{T}}= & \hat{\mathrm{W}}_{1}+\frac{1}{\mu^{2}}\left|\epsilon^{\mathrm{T}} \cdot \mathrm{k}\right|^{2} \hat{\mathrm{~W}}_{3} \\
\epsilon_{\mu}^{*} \mathrm{~L} \mathrm{~W}^{\mu \nu} \epsilon_{\nu}^{\mathrm{L}}= & -\hat{\mathrm{W}}_{1}+\frac{\nu^{2}+\mathrm{Q}^{2}}{\mathrm{Q}^{2}} \hat{\mathrm{~W}}_{2}+\frac{1}{\mu^{2}}\left|\epsilon^{\mathrm{L}} \cdot \mathrm{k}\right|^{2} \hat{\mathrm{~W}}_{3}  \tag{3.13}\\
& +\frac{1}{\mathrm{~m} \mu} \operatorname{Re}\left(\epsilon^{*} \mathrm{~L} \cdot \mathrm{p}\right)\left(\epsilon^{\mathrm{L}} \cdot \mathrm{k}\right) \hat{\mathrm{W}}_{4}
\end{align*}
$$

After integrating over the azimuthal angle we get

$$
\begin{aligned}
& \epsilon_{\mu}^{*} \mathrm{~T}_{\mathscr{V}}^{\mu \nu} ; \epsilon_{\nu}^{\mathrm{T}}=\mathscr{W}_{1} \\
& \epsilon_{\mu}^{*} \mathrm{~L}_{\mathscr{V}}^{\mu \nu} \epsilon_{\nu}^{\mathrm{L}}=-\mathscr{V}_{1}+\left(1+\frac{\nu^{2}}{\mathrm{Q}^{2}}\right) \mathscr{\mathscr { V }}_{2}
\end{aligned}
$$

Substituting these in (3.11) we get

$$
\begin{align*}
\left\langle\frac{\mathrm{d} \sigma^{\mathrm{T}}}{\mathrm{~d}^{3} \mathrm{k} / 2 \mathrm{k}_{0}}\right\rangle= & \frac{8 \pi^{2} \alpha}{2 \mathrm{~m} \nu-\mathrm{Q}^{2}} \times \frac{2}{\mu} \sqrt{\nu^{2}+\mathrm{Q}^{2}} \mathscr{W}_{1}\left(\mathrm{Q}^{2}, \nu, \nu^{\prime}, \kappa\right) \\
\left\langle\frac{\mathrm{d} \sigma^{\mathrm{L}}}{\mathrm{~d}^{3} \mathrm{k} / 2 \mathrm{k}_{0}}\right\rangle= & \frac{8 \pi^{2} \alpha}{2 \mathrm{~m} \nu-\mathrm{Q}^{2}} \times \frac{2}{\mu} \sqrt{\nu^{2}+\mathrm{Q}^{2}}\left[-\mathscr{W}_{1}\left(\mathrm{Q}^{2}, \nu, \nu^{\dagger}, \kappa\right)\right. \\
& \left.+\left(1+\frac{\nu^{2}}{\mathrm{Q}^{2}}\right) \mathscr{W}_{2}\left(\mathrm{Q}^{2}, \nu, \nu^{\prime}, \kappa\right)\right] \tag{3.14}
\end{align*}
$$

Or in terms of the normalized inclusive distributions, which is defined as the ratio of the Lorentz invariant distribution to the total cross section (to get rid of all the irrelevant factors), we get

$$
\begin{equation*}
\left.\rho\left(\mathrm{Q}^{2}, \nu, \nu^{\prime}, \kappa\right)=\frac{1}{\sigma_{\text {tot }}} / \frac{\mathrm{d} \sigma}{\left\langle\mathrm{~d}^{3} \mathrm{k} / 2 \mathrm{k}_{0}\right.}\right\rangle=\frac{\mu}{2 \sqrt{\nu^{2}+Q^{2}}} \frac{\epsilon_{\mu}^{*} \epsilon_{\nu} \mathscr{H}^{\mu \nu}}{\epsilon_{\mu}^{*} \epsilon_{\nu} \mathrm{W}^{\mu \nu}} \tag{3.15}
\end{equation*}
$$

Explicitly, in terms of the structure functions

$$
\begin{align*}
& \rho^{\mathrm{T}}=\frac{2}{\mu} \sqrt{\nu^{2}+\mathrm{Q}^{2}} \frac{\mathscr{V}_{1}}{\mathrm{~W}_{1}} \\
& \rho^{\mathrm{T}+\mathrm{L}}=\frac{2}{\mu} \sqrt{\nu^{2}+\mathrm{Q}^{2}}\left[\frac{\frac{1}{\mathrm{~m}} \nu \mathscr{N}_{2}}{\frac{1}{\mathrm{~m}} \nu \mathrm{~W}_{2}}\right] \tag{3.16}
\end{align*}
$$

$\rho^{\mathrm{T}+\mathrm{L}}$ is written in this form, because we know that it is $\frac{1}{\mathrm{~m}} \nu \mathrm{~W}_{2}$ which scales in the Bjorken limit. Noting that $\sqrt{\nu^{2}+\mathrm{Q}^{2}} \simeq \nu$ in the Bjorken limit, we get

$$
\begin{align*}
& \rho^{\mathrm{T}} \underset{\mathrm{Bj}}{\rightarrow} \frac{\frac{1}{\mathrm{~m}} \nu \mathscr{N}_{1}}{\mathrm{~F}_{1}(\omega)} \\
& \rho^{\mathrm{T}+\mathrm{L}} \underset{\mathrm{Bj}}{\vec{~}} \frac{\left[\frac{1}{\mathrm{~m} \mu} \nu^{2} \mathscr{N}_{2}\right]}{\mathrm{F}_{2}(\omega)} \tag{3.17}
\end{align*}
$$

From (3.17) it is obvious that for the inclusive cross sections (integrated over $\phi$ ) there is one more power of $\nu$, multiplying the scaling structure functions, compared to the ordinary electroproduction structure functions. Of course everything we did above is true if the virtual photon vertex factors out. We shall see that this is not true in photon-end of the phase space. And it is impossible to make such simple predictions. For $Q^{2} \rightarrow 0$ we can relate inclusive electroproduction to the inclusive photoproduction as follows (see Appendix 1):

$$
\begin{equation*}
\left\langle\frac{\mathrm{d} v}{\mathrm{~d} \mathrm{Q}^{2} \mathrm{~d} \nu \mathrm{~d} \nu^{\prime} \mathrm{d} \kappa}\right\rangle_{\mathrm{eh} \rightarrow \mathrm{e}^{\prime} \mathrm{h} \cdot \mathrm{H}} \mathrm{Q}^{2} \rightarrow 0 .\left(\frac{2 \alpha}{\pi}\right)\left(\frac{\mathrm{m}}{\mu}\right)\left(\frac{\epsilon^{\prime}}{\epsilon}\right) \times \frac{1}{\mathrm{Q}^{2}}\left(\frac{\mathrm{~d} v}{\mathrm{~d} \nu^{\prime} \mathrm{d} \kappa}\right)_{\gamma h \rightarrow h^{\prime} \mathrm{H}} \tag{3.18}
\end{equation*}
$$

## III. 2 Variables

In discussing purely hadronic inclusive processes it proved to be useful to parametrize the particle momenta in terms of rapidity variables. Therefore we shall use the same parametrization here also:

$$
\begin{align*}
& \mathrm{p}=\mathrm{m}\left(\cosh \mathrm{y}_{2}, 0,0, \sinh \mathrm{y}_{2}\right) \\
& \mathrm{q}=\mathrm{Q}\left(\sinh \mathrm{y}_{1}, 0,0, \cosh \mathrm{y}_{1}\right)  \tag{3.19}\\
& \mathrm{k}=\mu_{\perp}\left(\cosh \mathrm{y}, \frac{\mathrm{k}_{\perp}}{\mu_{\perp}}, \sinh \mathrm{y}\right)
\end{align*}
$$

where $k_{\perp}$ is the transverse momentum of the produced particle, with $p$ and $q$ taken to be colinear in the z -direction; and $\mu_{\perp}=\left(\mu^{2}+\mathrm{k}_{\perp}^{2}\right)^{1 / 2}$ is the transverse mass. The rapidity $y_{i}$ specifies the longitudinal Lorentz transformation that relates the lab frame to the rest frame of the ith hadron. For a space-like photon, rapidity parameter is the Lorentz boost which relates the lab frame (or whichever frame this labeling is done) to the frame where photon has only a space-component and no energy. This frame is called Breit (or Brick-wall) frame and its usefulness in processes involving highly space-like photons was
advertized by Feynman in Ref. [3]. The invariants $\nu=\frac{\mathrm{p} \cdot \mathrm{q}}{\mathrm{m}}, \nu^{\prime}=\frac{\mathrm{k} \cdot \mathrm{q}}{\mu}$, and $\kappa=\frac{\mathrm{k} \cdot \mathrm{p}}{\mathrm{m}}$ which we have chosen to represent the inclusive electroproduction can be expressed in terms of these rapidity variables:

$$
\begin{align*}
& \nu=\frac{1}{\mathrm{~m}} \mathrm{p} \cdot \mathrm{q}=\mathrm{Q} \sinh \left(\mathrm{y}_{1}-\mathrm{y}_{2}\right) \\
& \nu^{\prime}=\frac{1}{\mu} \mathrm{k} \cdot \mathrm{q}=\frac{\mu_{\perp}}{\mu} \mathrm{Q} \sinh \left(\mathrm{y}_{1}-\mathrm{y}\right)  \tag{3.20}\\
& \kappa=\frac{1}{\mathrm{~m}} \mathrm{k} \cdot \mathrm{q}=\mu_{\perp} \cosh \left(\mathrm{y}-\mathrm{y}_{2}\right)
\end{align*}
$$

The longitudinal and transverse momenta, $\mathrm{k}_{3}$ and $\mathrm{k}_{1}$, of the detected hadron in the c.m. frame of the initial photon-hadron system are related to the rapidity variable as follows

$$
\begin{equation*}
\mathrm{y}=\ln \left(\frac{\mathrm{k}_{0}+\mathrm{k}_{3}}{\mu_{\perp}}\right)=\frac{1}{2} \ln \left(\frac{\mathrm{k}_{0}+\mathrm{k}_{3}}{\mathrm{k}_{0}-\mathrm{k}_{3}}\right) \tag{3.21}
\end{equation*}
$$

We put all longitudinally moving frames on an equal footing by the use of rapidity variables, since they are all related by a simple shift of the scale. That is a longitudinal Lorentz transformation, characterized by $\beta=\tanh u$, merely changes $y$ to $y^{\prime}=y+u$.

Let us now find the absolute kinematical limits imposed on $y$, by the energymomentum conservation. In the Regge region (which is a subregion of Bjorken region)

$$
\begin{equation*}
M^{2}=(p+q-k)^{2} \simeq-Q^{2}+2 p \cdot q-2 q \cdot k-2 p \cdot k \geq 0 \tag{3.22}
\end{equation*}
$$

From (3:20) we get in the lab frame

$$
\begin{aligned}
& \nu=\mathrm{Q} \operatorname{Sh} \mathrm{Y} \underset{\mathrm{Bj}}{\simeq} \frac{1}{2} \mathrm{Q} \mathrm{e}^{\mathrm{Y}} \\
& \nu^{\prime}=\frac{\mu_{\perp}}{\mu} \mathrm{Q} \operatorname{Sh}(\mathrm{Y}-\mathrm{y}) \\
& \kappa=\mu_{\perp} \operatorname{Sh} \mathrm{y}
\end{aligned}
$$

Since it is only $\kappa$, which does not involve Y , we can neglect $2 \mathrm{~m} \kappa$ compared to other-terms in (3.22), for the minimum value of $y$, and obtain

$$
2 \mathrm{~m} \nu(1-\omega)>2 \mu \nu^{\prime}
$$

Substituting in the value $2 \mu \nu^{\prime}$ for $\mathrm{y} \simeq \mathrm{y}_{\text {min }}$

$$
2 \mu \nu^{\prime} \sim \mu_{\perp} \mathrm{Q} \mathrm{e}^{\mathrm{Y}-\mathrm{Y}_{\min }}
$$

we finally obtain

$$
\begin{equation*}
\mathrm{y}_{\min } \gtrsim \ln \left(\frac{\mu_{\perp}}{\mu}\right)+\ln \left(\frac{\mu}{\mathrm{m}} \frac{1}{1-\omega}\right) \tag{3.23a}
\end{equation*}
$$

Note that this lower limit is particularly simple for nucleon production:

$$
y_{\min } \simeq \ln \left(\frac{1}{1-\omega}\right) \simeq \omega
$$

for small $\omega$ (which of course means Regge limit). Again from (3.20) we see that for $\mathrm{y} \simeq \mathrm{y}_{\text {max }}, Y-\mathrm{y}_{\max }$ is small; but because of the factor Q in the front $\nu^{\prime}$ should not have been negligible compared to the other terms. Substituting in

$$
\begin{aligned}
& 2 \mu \nu^{\prime}=2 \mu_{\perp} \mathrm{Q} \operatorname{Sh}\left(\mathrm{Y}-\mathrm{y}_{\max }\right) \\
& 2 \mathrm{~m} \kappa=\mathrm{m} \mu_{\perp} \mathrm{e}^{\mathrm{y}_{\max }}
\end{aligned}
$$

we obtain

$$
2 \mathrm{~m} \nu(1-\omega)>2 \mu_{\perp} \nu \mathrm{e}^{-\mathrm{y}_{\max }}+\mathrm{m} \mu_{\perp} \mathrm{e}^{\mathrm{y}_{\max }}
$$

or

$$
\mathrm{e}^{\mathrm{y}_{\max }}+\frac{2 \nu}{\mathrm{~m}} \mathrm{e}^{-\mathrm{y}_{\max }} \lesssim \frac{2 \nu}{\mu_{\perp}}(1-\omega)
$$

A real careful study of this inequality, gives

$$
\begin{equation*}
\mathrm{y}_{\max } \lesssim \ln \left(\frac{\mathrm{s}}{\mathrm{~m} \mu_{\perp}}\right)=\ln \left(\frac{\mathrm{m}}{\mu_{\perp}}\right)+\ln \left(\frac{\mathrm{s}}{\mathrm{~m}^{2}}\right) \tag{3.23b}
\end{equation*}
$$

Again notice that for nucleon production, this upper limit gets very simple

$$
\mathrm{y}_{\max } \lesssim \ln \left(\frac{\mathrm{s}}{\mathrm{~m}^{2}}\right)
$$

So in summary, the absolute kinematical limits on y are

$$
\begin{equation*}
\ln \left(\frac{\mu_{\perp}}{\mathrm{m}}\right)+\ln \left(\frac{1}{1-\omega}\right) \lesssim \mathrm{y} \lesssim \ln \left(\frac{\mathrm{~m}}{\mu_{\perp}}\right)+\ln \left(\frac{\mathrm{s}}{\mathrm{~m}^{2}}\right) \tag{3.24}
\end{equation*}
$$

And for nucleon production, and for $\omega \simeq 0$,

$$
\omega \lesssim \mathrm{y} \lesssim \ln \left(\frac{\mathrm{~s}}{\mathrm{~m}^{2}}\right)
$$

One interesting thing about this results is that lower limit depends on $\omega$. In the deep Bjorken limit this dependence gets more pronounced. Another distinctive feature is that phase space (of the detected hadron) is not fixed by the rapidities of the target or projectile; in other words it is not $y_{2} \leq y \leq y_{1}$. Calculating the difference

$$
\begin{equation*}
\Delta \mathrm{y}=\mathrm{y}_{\max }-\mathrm{y}_{1}=\ln (1-\omega)+\ln \left(\frac{\mathrm{Q}}{\mu_{\perp}}\right) \tag{3.25}
\end{equation*}
$$

we see that, for $Q^{2} \sim \mu_{\perp}^{2}$, this quantity is negative and small (because $Q^{2} \sim \mu_{\perp}^{2}$ means $\omega \simeq 0$ ). As $Q^{2}$ gets larger the second term gets large; but fortunately, the first term also gets large and negative. Therefore we may think that, since the large $\mu_{\perp}$ production is severely suppressed [9,25] (see Section $V$ for a detailed study), $\Delta y$ is always finite, and never gets large.

## III. 3 Generalized Optical Theorem

In purely hadronic inclusive reactions Mueller [12] indicated that the inclusive cross section is a piece of the discontinuity of the forward three-to-three amplitude. Stapp's [12] formal proof does go through for virtual particles also. Therefore we are going to use the generalized optical theorem in our case as
well, to relate the virtual photo inclusive cross section to the discontinuity of the forward three-to-three amplitude $\gamma \mathrm{hh}^{\prime} \rightarrow \gamma \mathrm{hF}^{\prime}$ (Fig. 5).

Denoting this discontinuity by A(q, p, h), optical theorem states

$$
\begin{equation*}
\text { (Flux) } \frac{d \sigma^{T, L}}{d^{3} k / 2 k_{0}}=A^{T, L}(q, p, k) \tag{3.26}
\end{equation*}
$$

where

$$
\begin{equation*}
\left.A^{\mathrm{T}, \mathrm{~L}}(\mathrm{q}, \mathrm{p}, \mathrm{k})=\epsilon_{\mu}^{* \mathrm{~T}, \mathrm{~L}} \epsilon_{\nu}^{\mathrm{T}, \mathrm{~L}}\left\{\sum \int \mathrm{~d}^{4} \mathrm{x} \mathrm{e}^{\mathrm{iq} \cdot \mathrm{x}}<\mathrm{p},-\mathrm{k}\left|\mathrm{~J}_{\mu}^{+}(\mathrm{x}) \mathrm{J}_{\nu}(0)\right| \mathrm{p},-\mathrm{k}\right\rangle\right\} \tag{3.26a}
\end{equation*}
$$

and

$$
\begin{equation*}
\text { (Flux) } \frac{\mathrm{d} \sigma^{\mathrm{T}, \mathrm{~L}}}{\mathrm{~d}^{3} \mathrm{k} / 2 \mathrm{k}_{0}}=4 \pi^{2} \alpha \epsilon_{\mu}^{*} \mathrm{~T}_{\nu}^{\mathrm{T}, \mathrm{~L}} \epsilon_{\nu}^{\mathrm{T}, \mathrm{~L}}\left\{\sum \int \frac{\mathrm{~d}^{4} \mathrm{x}}{4 \pi} \mathrm{e}^{\mathrm{iq} \cdot \mathrm{x}}\langle\mathrm{p}| \mathrm{J}_{\mu}^{+}(\mathrm{x})|\mathrm{k}\rangle\langle\mathrm{k}| \mathrm{J}_{\nu}(0)|\mathrm{p}\rangle\right\} \tag{3.26b}
\end{equation*}
$$

This is not at least implausible, since the virtual photon at hand is spacelike, whereas it is known that the normal threshold branch points and cuts exist only in the right half $q^{2}$ plane. On the other hand, in the production of $\mu^{+} \mu^{-}$ pairs by time-like photons, these singularities may be relevant and the use of Mueller's results would require further justification.

## IV. $0(1,2)$ EXPANSIONS

If his famous work Mueller [12] expanded the absorptive part A into $0(1,2)$ harmonics to investigate the scaling properties in the purely hadronic inclusive reactions. In doing so his main purpose was to connect the scaling properties of particle production (such as pionization and limiting fragmentation), and average multiplicity with Regge singularities familiar from two-body reactions. He showed that an appropriate single $0(1,2)$ expansion gives limiting fragmentation [5] (slow particles in the rest frame of the one of the initial particles are the fragments of that particular particle), and the appropriate double $0(1,2)$ expansion gives pionization (pionization products are those pions which maintain a finite momentum in the c.m. frame of the initial particles as the energy of the initial particles become very large).

Following Mueller, in getting our generalized scaling laws, we shall use $0(1,2)$ expansions, though our kinematic configuration again has sufficient symmetry to render an $0(1,3)$ expansion natural. If we use the $0(1,3)$ group, our expansion parameters would be the rapidity variables which we defined earlier. Both in the Regge limit and Bjorken limit $y_{1}-y_{2}$ becomes very large. There are, as in the purely hadronic case, three distinct type of regions available in a single particle spectrum:
a) $\mathrm{y}_{1}-\mathrm{y}$ large, $\mathrm{y}-\mathrm{y}_{2}$ finite $(\simeq \mathscr{O}(1))=$ target fragmentation region
b) Both $y_{1}-\mathrm{y}$ and $\mathrm{y}-\mathrm{y}_{2}$ large $=$ central region
c) $\dot{\mathrm{y}}-\mathrm{y}_{2}$ large, $\mathrm{y}_{1}-\mathrm{y}$ finite $(\simeq \mathscr{O}(1))=$ current fragmentation region.

Despite the fact that we made these definitions in exact analogy to the purely hadronic production processes, it is not at all obvious whether a hadron detected in any of the above regions should show the distinct features of the one detected in purely hadronic production. Because of the short range correlation assumption
we make in the rapidity space, we shouldn't be surprised to see that a hadron detected in the region-a) above is slow in the target rest frame; but there is no a priori reason why the particles detected in the region-b) should be slow in the barycentric frame, or why the particles detected in the region-c) should be slow in the Breit frame.

Since $A$ is an invariant function of $p \cdot q, k \cdot q$, and $p \cdot k$ (also $Q^{2}$ of course) we may choose any coordinate frame which is convenient for our $0(1,2)$ parametrization. It turns out that the most convenient frame is one in which the produced particle is at rest. We shall make the following $0(1,2)$ parametrization in this particular frame

$$
\begin{align*}
& \mathrm{k}=\mu(1,0,0,0)=\left(\mathrm{k}_{0}, \mathrm{k}_{\mathrm{x}}, \mathrm{k}_{\mathrm{y}}, \mathrm{k}_{\mathrm{z}}\right) \\
& \mathrm{q}=\mathrm{Q}\left(\sinh \xi_{1}, \cosh \xi_{1} \cos \phi, \cosh \xi_{1} \sin \phi, 0\right)  \tag{4.1}\\
& \mathrm{p}=\mathrm{m}\left(\cosh \xi_{2},-\sinh \xi_{2}, 0,0\right)
\end{align*}
$$

where

$$
0<\xi_{1}, \quad \xi_{2} \leq \infty, \quad-\pi \leq \phi \leq \pi
$$

In terms of these, the three invariant variables become

$$
\begin{align*}
& \mathrm{p} \cdot \mathrm{q}=\mathrm{m} \mathrm{Q}\left(\sinh \xi_{1} \cosh \xi_{2}+\cosh \xi_{1} \sinh \xi_{2} \cos \phi\right) \\
& \mathrm{q} \cdot \mathrm{k}=\mu \mathrm{Q} \sinh \xi_{1}  \tag{4.2}\\
& \mathrm{p} \cdot \mathrm{k}=\mathrm{m} \mu \cosh \xi_{2} .
\end{align*}
$$

Considered as a function of the independent variables $\xi_{1}, \xi_{2}$, and $\phi$, A can be expanded in $0(1,2)$ harmonics 18 (neglecting the discrete series)

$$
\begin{equation*}
A\left(Q^{2}, \xi_{2}, \phi, \xi_{1}\right)=\sum_{\mathrm{m}} \int_{-\frac{1}{2}-\mathrm{i} \infty}^{-\frac{1}{2}+\mathrm{i} \infty} \int \mathrm{~d} \Lambda_{2} \mathrm{~d} \Lambda_{1} A_{\mathrm{m}}^{\left.\Lambda_{2} \Lambda^{\Lambda}{ }_{(Q}{ }^{2}\right) \mathrm{d}_{0 \mathrm{~m}}^{\Lambda_{2}}\left(\xi_{2}\right) \mathrm{e}^{-\mathrm{im} \phi}{ }_{\mathrm{d}}^{\mathrm{m} 0}{ }^{\Lambda_{1}}\left(\xi_{1}\right), ~} \tag{4.3}
\end{equation*}
$$

when $\xi_{1}\left(\xi_{2}\right)$ becomes large the behavior of (4.3) is governed by the leading singularities in $\Lambda_{1}\left(\Lambda_{2}\right)$. When both $\xi_{1}$ and $\xi_{2}$ become large the asymptotic behavior is governed by the leading singularities $\alpha_{1}$ and $\alpha_{2}$ in $\Lambda_{1}$ and $\Lambda_{2}$.

Before starting the study of different asymptotic regions let us give the relation between the rapidity variables and the $0(1,2)$ parameters

$$
\begin{align*}
& \mathrm{p} \cdot \mathrm{q}=\mathrm{m} \mathrm{Q} \sinh \left(\mathrm{y}_{1}-\mathrm{y}_{2}\right)=\mathrm{m} \mathrm{Q}\left(\sinh \xi_{1} \cosh \xi_{2}+\cosh \xi_{1} \sinh \xi_{2} \cos \phi\right) \\
& \mathrm{k} \cdot \mathrm{q}=\mu_{\perp} \mathrm{Q} \sinh \left(\mathrm{y}_{1}-\mathrm{y}\right)=\mu \mathrm{Q} \sinh \xi_{1}  \tag{4.4}\\
& \mathrm{k} \cdot \mathrm{p}=\mathrm{m} \mu_{\perp} \cosh \left(\mathrm{y}-\mathrm{y}_{2}\right)=\mathrm{m} \mu \cosh \xi_{2} .
\end{align*}
$$

## IV. 1 Target Fragmentation

From (4.3) we see that when $y_{1}-\mathrm{y}$ is large (photon and the produced hadron is well separated in the rapidity plane) $\xi_{1}$ is also large, and $\xi_{2}$ is finitc because $y-y_{2}$ is (as long as $k_{1}$ is small, which seems to be the case from the experiments). The $0(1,2)$ analysis of $\xi_{1}$, for large $\xi_{1}$, yields (Fig. 6)

$$
\begin{equation*}
\mathrm{A}\left(\mathrm{Q}^{2}, \xi_{2}, \phi, \xi_{1}\right) \underset{\xi_{1} \rightarrow \infty}{\simeq}\left(\cosh \xi_{1}\right)^{\alpha} \beta_{t}\left(\mathrm{Q}^{2}, \xi_{2}, \phi\right) \tag{4.5}
\end{equation*}
$$

where $\alpha$ is the leading singularity in $\Lambda_{1}$ of $A_{m}^{\Lambda_{1}}$, which we assumed to be a simple pole. Assuming that only the Pomeranchuk pole dominates we have $\alpha=1$. Also assuming that the leading singularity is a simple pole probably means its residue factorizes in the usual sense:

$$
\begin{aligned}
& \beta_{\mathrm{h}}\left(\mathrm{Q}^{2}, \xi_{2}, \phi\right)=\beta_{\mathrm{ph}}^{\mathscr{P}}\left(\phi, \xi_{2}\right) \beta_{\gamma \gamma} \mathscr{P}^{\mathcal{P}}\left(\mathrm{Q}^{2}\right) \\
& \simeq \beta_{\mathrm{ph}}^{\mathscr{P}}\left(\phi, \xi_{2}\right) \frac{\beta_{\gamma \gamma}^{\mathscr{P}}}{\mathrm{Q}} \\
& \mathrm{Q}^{2} \rightarrow \infty
\end{aligned}
$$

where we used our prescription (2.12) for the photon-photon-Reggeon coupling in the last step. Substituting these in (4.5) we get

$$
\begin{equation*}
\text { (Flux) } \frac{\mathrm{d} \sigma}{\mathrm{~d}^{3} \mathrm{k} / 2 \mathrm{k}_{0}} \underset{\xi_{1} \rightarrow \infty}{\simeq}\left(\frac{\nu^{\prime}}{\mathrm{Q}}\right) \beta_{\mathrm{ph}}^{\mathscr{P}}\left(\phi, \xi_{2}\right) \frac{\beta^{\mathscr{P}}}{\frac{\gamma \gamma}{\mathrm{Q}}} \tag{4.6}
\end{equation*}
$$

Going back to the invariant variables $\nu, \nu^{\prime}, \kappa$, we get

$$
\begin{aligned}
\cosh \xi_{1} & \sim \frac{\mathrm{k} \cdot \mathrm{q}}{\mu \mathrm{Q}}=\frac{\nu^{\prime}}{\mathrm{Q}} \\
\cosh \xi_{2} & =\frac{\mathrm{k} \cdot \mathrm{p}}{\mathrm{~m} \mu}=\frac{\mathrm{m}^{2}+\mu^{2}-\mathrm{t}}{2 \mathrm{~m} \mu} \\
\cos \phi & =\left[1-\frac{\kappa^{2}}{\mu^{2}}\right]^{-1 / 2}\left[\frac{\nu}{\nu^{\prime}}-\frac{\kappa}{\mu}\right]
\end{aligned}
$$

Large $\xi_{1}$ means large $\nu^{\prime} / Q$, and finiteness of $\xi_{2}$ and $\phi$ means the finiteness of t and $\nu^{\prime} / \nu$. For the normalized cross section we finally obtain

$$
\begin{equation*}
\rho\left(\mathrm{Q}^{2}, \nu, \nu^{\prime}, \kappa\right) \underset{\substack{\mathrm{Bj}}}{\simeq} \quad \mathrm{x}_{\mathrm{t}} \mathrm{f}_{\mathrm{t}}\left(\mathrm{x}_{\mathrm{t}}, \kappa\right) \tag{4.7}
\end{equation*}
$$

where $x_{t}=k \cdot q / p \cdot q$ is the new scaling variable. Using (3.17), we obtain

$$
\begin{aligned}
& \frac{1}{\mathrm{~m}} \nu \mathscr{V}_{1}=\frac{1}{\mathrm{Bj}} \mathrm{~F}_{1}(\omega) \rho^{\mathrm{T}}\left(\mathrm{Q}^{2}, \nu, \nu^{\prime}, \kappa\right)=\mathrm{F}_{1}(\omega) \mathrm{x}_{\mathrm{t}} \mathrm{f}_{\mathrm{t}}\left(\mathrm{x}_{\mathrm{t}}, \kappa\right) \\
& \frac{1}{\mathrm{~m} \mu} \nu^{2} \mathscr{T}_{2}=\frac{1}{\mathrm{Bj}^{2}} \mathrm{~F}_{2}(\omega) \rho^{\mathrm{T}+\mathrm{L}}\left(\mathrm{Q}^{2}, \nu, \nu^{\prime}, \kappa\right)=\mathrm{F}_{2}(\omega) \mathrm{x}_{\mathrm{t}} \mathrm{f}_{\mathrm{t}}\left(\mathrm{x}_{\mathrm{t}}, \kappa\right)
\end{aligned}
$$

Recalling that $\mathrm{F}_{1}$ is one power down in $\omega$, compared to $\mathrm{F}_{2}$, we see that the same is true for

$$
\begin{equation*}
\frac{1}{\mathrm{~m}} \nu \mathscr{N}_{1} \underset{\substack{\mathrm{Bj} \\ \mathrm{y}_{1}-\mathrm{y} \rightarrow \infty \\ \mathrm{y}-\mathrm{y}_{2} \text { finite }}}{\overrightarrow{\mathscr{F}_{1}}(\omega, \mathrm{x} ; \kappa)=\mathrm{F}_{1}(\omega) \mathrm{G}_{\mathrm{t}}\left(\mathrm{x}_{\mathrm{t}}, \kappa\right)} \tag{4.8}
\end{equation*}
$$

and

$$
\rightarrow \frac{1}{\mathrm{~m} \mu} \nu^{2} \mathscr{O \gamma _ { 2 }} \underset{\substack{\mathrm{Bj}} \mathscr{F}_{2}^{\mathrm{t}}(\omega, \mathrm{x} ; \kappa)=\mathrm{F}_{2}(\omega) \mathrm{G}_{\mathrm{t}}\left(\mathrm{x}_{\mathrm{t}}, \kappa\right)}{\mathrm{y}_{1}-\mathrm{y} \rightarrow \infty} \mathrm{y}-\mathrm{y}_{2} \text { finite } .
$$

Now the question is whether this particular single Regge limit corresponds to the limiting hadron fragmentation in the sense of Ref. [5]. In order to prove that this really is the case we have to show that the above limit includes the case where $k_{3}$ is small in the laboratory frame. Taking the extreme case $k_{3}=0$, we have

$$
\begin{aligned}
& \cosh \xi_{2}=\frac{\mu_{\perp}}{\mu}, \text { finite } \\
& \sinh \xi_{1}=\frac{\nu^{\prime}}{Q} \underset{B j}{\sim} \frac{\mu_{\perp}}{4 \mathrm{~m} \mu \omega} \mathrm{Q} \rightarrow \infty .
\end{aligned}
$$

So we see that the hadron produced in the region-a) has similar features to the target fragments of the purely hadronic processer and Eq. (4.7) shows that in the kinematical region which is a combination of ordinary hadron fragmentation region and the Bjorken scaling regions the quantity $\rho\left(\mathrm{Q}^{2}, \nu, \nu^{\prime}, \kappa\right)$ scales in a general way. The scaling variables are the old $\omega=Q^{2} / 2 p \cdot q$ and new $\mathrm{x}_{\mathrm{t}}=\frac{\mu \nu^{\prime}}{\mathrm{m} \nu}$. (Note that in the target fragmentation region we have enough freedom to choose another scaling variable, $2 \mu \nu^{\prime} / Q^{2}$, and these two scaling variables are related to each other as $\frac{2 \mu \nu^{\prime}}{\mathrm{Q}^{2}}=\frac{2 \mu \nu^{\prime}}{2 \mathrm{~m} \nu} \frac{2 \mathrm{~m} \nu}{\mathrm{Q}^{2}}=\frac{\mathrm{X}_{\mathrm{t}}}{\omega}$. But we shall see below that, our choice $\mathrm{x}_{\mathrm{t}}=\frac{\mu \nu^{\prime}}{\mathrm{m} \nu}$ has better features, that it is proportional to the celebrated Feynman scaling variable $\mathrm{x}_{\mathrm{F}}=2 \mathrm{k}_{3}^{*} / \sqrt{\mathrm{s}}$.) The meaning of the scaling variable $x_{t}$ is not immediately clear. In order to relate it to a more natural experimental variable let us consider the c.m. system of $p$ and $q$, in which (dropping the
asterisk)

$$
p=(p, 0,0, p), \quad q=\left[\left(p^{2}-Q^{2}\right)^{1 / 2}, 0,0,-p\right]
$$

Note that, using $\mathrm{s} \simeq 2 \mathrm{~m} \nu(1-\omega)$

$$
\mathrm{k} \cdot \dot{\mathrm{q}} \simeq \mathrm{k}_{3}\left(\mathrm{p}^{2}-\mathrm{Q}^{2}\right)^{1 / 2}+\mathrm{p} \simeq \mathrm{k}_{2} \sqrt{\mathrm{~s}}
$$

Therefore

$$
\begin{equation*}
x_{\mathrm{t}} \equiv \frac{\mathrm{k} \cdot \mathrm{q}}{\mathrm{p} \cdot \mathrm{q}} \simeq(1-\omega) \frac{2 \mathrm{k}_{3}}{(\mathrm{~S})^{1 / 2}} \equiv(1-\omega) \mathrm{x}_{\mathrm{F}} \tag{4.9}
\end{equation*}
$$

Since $0<1-\omega<1$ and $\left|x_{F}\right|<1$, we have $|x| \leq 1$. We also immediately obtain the following relation which is well known in purely hadronic inclusive reactions

$$
\frac{M^{2}}{S} \underset{B j}{\simeq} 1-\frac{1}{1-\omega} \frac{k \cdot q}{p \cdot q} \sim 1-x_{F}
$$

as $\mathrm{S} / \mathrm{M}^{2}$ becomes large $\mathrm{x}_{\mathrm{F}} \simeq 1$; this corresponds to the boundary of the phase space. When restated in terms of our new variable, $x_{t}$, this says that the phase space boundary in the hadronic side corresponds to $x_{t} \simeq 1-\omega$, i.e., it depends on $\omega$.

We also notice that, to leading order in terms of Regge singularities we considered, the scaling occurs in a factorized form: (Hadronic Feynman scaling) $\times$ (Leptonic Bjorken scaling). Adding secondary Regge trajectories spoils this form of factorization. Including only the next leading trajectories, $\mathscr{P}^{\prime}$ and $\mathrm{A}_{2}$, we have

$$
\mathrm{A} \underset{\xi_{1} \rightarrow \infty}{\simeq}\left(\cosh \xi_{1}\right) \beta_{\mathscr{P}}\left(\phi, \xi_{2}, \mathrm{Q}^{2}\right)+\left(\cosh \xi_{1}\right)^{1 / 2} \beta_{\mathscr{P}}\left(\phi, \xi_{2}, \mathrm{Q}^{2}\right)
$$

Using factorization of Regge residues and (2.12), and recalling $\cosh \xi_{1}=\frac{\nu^{\prime}}{Q}$, we get

$$
\begin{align*}
& \text { (Flux) } \gamma \frac{\mathrm{d} \sigma}{\mathrm{~d}^{3} \mathrm{k}_{\mathrm{k} / 2 \mathrm{k}_{0}}^{\underset{\mathrm{y}-\mathrm{y}_{2}}{ } \underset{\text { finite }}{ }} \underset{\omega}{ } \frac{1}{\omega} \mathrm{x}_{\mathrm{t}} \mathrm{f}_{\mathscr{P}}\left(\mathrm{x}_{\mathrm{t}}, \mathrm{t}\right)+\frac{1}{\sqrt{\omega}} \sqrt{\mathrm{x}_{\mathrm{t}} \mathrm{f} \mathscr{P}^{\prime}}\left(\mathrm{x}_{\mathrm{t}}, \mathrm{t}\right)}  \tag{4.10}\\
& \mathrm{y}_{1}-\mathrm{y} \rightarrow \infty
\end{align*}
$$

Arriving at this general scaling law we have used the factorization of the Pomeranchuk residue and the prescription (2.12) for the photon-photon-Reggeon vertex (the factorization of the Pomeranchuk residue also implies that the fragments of the hadron are essentially independent of the virtual photon beam).

## IV. 2 Central Region

We shall look at the central region of the single particle spectrum where both $\mathrm{y}_{1}-\mathrm{y}$ and $\mathrm{y}-\mathrm{y}_{2}$ are large, at high energies. From (4.4) we see that

$$
\begin{aligned}
& \sinh \xi_{1}=\frac{\mu_{1}}{\mu} \sinh \left(\mathrm{y}_{1}-\mathrm{y}\right)=\frac{\nu^{\dagger}}{\mathrm{Q}} \\
& \cosh \xi_{2}=\frac{\mu_{1}}{\mu} \cosh \left(\mathrm{y}-\mathrm{y}_{2}\right)=\frac{\kappa}{\mu}
\end{aligned}
$$

So both $\xi_{1}$ and $\xi_{2}$ are large. If the leading singularities in $\Lambda_{1}, \Lambda_{2}$ are simple poles, we get from (4.3) (Fig. 7)

$$
\begin{equation*}
\mathrm{A}\left(\mathrm{Q}^{2}, \xi_{2}, \phi, \xi_{1}\right) \underset{\xi_{1}, \xi_{2} \rightarrow \infty}{\sim}\left(\cosh \xi_{1}\right)^{\alpha}\left(\cosh \xi_{2}\right)^{\alpha} \beta_{\alpha_{1} \alpha}\left(\phi, Q^{2}\right) \tag{4.11}
\end{equation*}
$$

From (3.20) we have

$$
\frac{\mathrm{p} \cdot \mathrm{q}}{(\mathrm{k} \cdot \mathrm{q})(\mathrm{k} \cdot \mathrm{p})}=\frac{1}{\mu_{\perp}^{2}} \cdot \frac{\operatorname{Sh}\left(\mathrm{y}_{1}-\mathrm{y}_{2}\right)}{\operatorname{Sh}\left(\mathrm{y}_{1}-\mathrm{y}\right) \operatorname{Ch}\left(\mathrm{y}-\mathrm{y}_{2}\right)} \simeq \frac{1}{\mu_{\perp}^{2}}
$$

Also from (4.2) we obtain

$$
\frac{\mathrm{p} \cdot \mathrm{q}}{(\mathrm{k} \cdot \mathrm{q})(\mathrm{k} \cdot \mathrm{p})}=\frac{1+\cos \phi}{\mu^{2}}
$$

This gives us $1+\cos \phi \sim \mu^{2} / \mu_{\perp}^{2}$. This relation tells us that $\cos \phi$ is always finite.

The Pomeranchuk trajectory can always contribute, so that the largest $\alpha_{i}$ are 1. Assuming that the leading singularity is a simple pole probably means that it is factorizable in the usual sense in which case

$$
\beta_{\mathscr{P} \mathscr{P}}\left(\phi, \mathrm{Q}^{2}\right)=\beta_{\mathrm{pp}}^{\mathscr{P}}(0) \beta_{\gamma \gamma}^{\mathscr{P}}\left(\mathrm{Q}^{2}\right) \overline{\mathrm{g}}_{\mathscr{P} \mathscr{P}}(\phi)
$$

Again using our prescription (2.12) for the photon-photon-Reggeon vertex, we get

$$
\begin{equation*}
\text { (Flux) } \frac{\mathrm{d} \sigma}{\mathrm{~d}^{3} \mathrm{k} / 2 \mathrm{k}_{0}} \simeq\left(\frac{\nu^{\prime}}{\mathrm{Q}^{2}}\right)\left(\frac{\kappa}{\mu}\right) \beta_{\mathrm{pp}}^{\mathscr{P}}(0) \beta_{\gamma \gamma}^{\mathscr{P}}(0) \overline{\mathrm{g}}_{\mathscr{P} \mathscr{P}}(\phi) \tag{4.12}
\end{equation*}
$$

or, for the normalized distribution

$$
\rho\left(\mathrm{Q}^{2}, \nu, \nu^{\mathrm{t}}, \kappa\right) \underset{\substack{\mathrm{y}_{1}-\mathrm{y} \rightarrow \infty \\ \mathrm{y}-\mathrm{y}_{2} \rightarrow \infty}}{\sim} \underset{\mathrm{Bj}}{\sim} \mathrm{x}_{\mathrm{c}} \overline{\mathrm{~g}}_{\mathrm{c}}\left(\mathrm{x}_{\mathrm{c}}\right)
$$

where

$$
\mathrm{x}_{\mathrm{c}}=\frac{(\mathrm{k} \cdot \mathrm{q})(\mathrm{p} \cdot \mathrm{k})}{(\mathrm{p} \cdot \mathrm{q})} \simeq \mu_{\perp}^{2}
$$

We see that the general kinematical region which combines the double Regge and Bjorken scaling regions we get scaling and the new scaling variable is $\mathrm{x}_{\mathrm{c}}$. Recalling the definition $\cosh \xi_{2}=\frac{\mathrm{p} \cdot \mathrm{k}}{\mathrm{m} \mu}$ we notice that if we make $(\mathrm{p} \cdot \mathrm{k})$ also finite, which takes us from the double Regge region back to single Regge region, we recover the scaling variable $x_{t}$ again, which we found for the hadron fragmentation region. The function $\bar{g}_{c}(\phi)$ we found above is a universal function depending only on the type of particle observed, but not on the virtual photon beam or the target. The coordinate system described by (4.1) is not a very transparent one in terms of physical quantities. In particular the meaning of $\phi$ is not immediately
clear. In order to relate $\phi$ to a more natural experimental variable again let us consider the c.m. frame of p and q (again dropping the asterisk)

$$
\begin{equation*}
\mathrm{x}_{\mathrm{c}} \underset{\mathrm{Bj}}{\simeq} \frac{1}{2}(2 \omega+1)\left(\mathrm{k}_{0}-\mathrm{k}_{3}\right)\left[\left(\mathrm{k}_{0}-\mathrm{k}_{3}\right)-\frac{2 \omega}{1+2 \omega} \mathrm{k}_{0}\right] \tag{4.14}
\end{equation*}
$$

For small values of $\mathrm{k}_{3}$

$$
\mathrm{x}_{\mathrm{c}} \underset{\mathrm{k}_{3}^{*} \rightarrow 0}{\simeq} \mu_{\perp}^{2}
$$

This shows that for samll $k_{3}$, the $x_{c}$ dependence is equivalent to the $k_{\perp}^{2}$ dependence, which is transverse momentum of the produced particle in the c.m. frame of the virtual photon-hadron system.

In the purely hadronic production processes pionization products are those hadrons (pions) which maintain a finite momentum in the c.m. system of the initial particles, as the energy of these initial particles become very large. The most characteristic momentum value for the pionization products are those for which $\mathrm{k}_{3}=0$. Let us see whether the double Regge expansion (4.10) is valid for $\mathrm{k}_{3}=0$. We have

$$
\begin{equation*}
\sinh \xi_{1}=\frac{\nu^{\prime}}{Q} \underset{\mathrm{k}_{3}=0}{\simeq} \sqrt{\frac{1+2 \omega}{4 \omega}}\left(\frac{\mu_{\perp}}{\mu}\right) \tag{4.15}
\end{equation*}
$$

Bj
In order to have $\xi_{1}$ large we have to have $\omega \sim 0$, which is implicitly really the case in all our analysis, for finite $k_{\perp}$. So, central region contains particles which are slow in the barycentric frame. But of course there is nothing in the above argument which shows that these slow particles are the only ones produced in the central region.

If we include the next leading Regge trajectory also, we get for the normalized distribution

$$
\begin{aligned}
\rho\left(\mathrm{Q}^{2}, \nu, \nu^{\prime}, \kappa\right) \sim & \mathrm{x}_{\mathrm{c}}\left[\overline{\mathrm{~g}}_{\mathscr{P} \mathscr{P}^{\prime}}\left(\mathrm{x}_{\mathrm{c}}\right)+\sqrt{\frac{\mu}{\kappa}} \overline{\mathrm{g}}_{\mathscr{P} \mathscr{P}^{\prime}}\left(\mathrm{x}_{\mathrm{c}}\right) \frac{\beta_{\mathrm{pp}}^{\mathscr{P}}}{\beta_{\mathrm{pp}}^{\mathscr{P}}}\right. \\
& +\sqrt{\frac{\mathrm{x}_{\mathrm{c}} \omega}{\kappa}} \frac{\beta_{\gamma \gamma}^{\mathscr{P}^{\prime}}}{\beta_{\gamma \gamma}^{\mathscr{P}}} \overline{\mathrm{g}}_{\mathscr{P} \mathscr{P}^{\prime}}\left(\mathrm{x}_{\mathrm{c}}\right) \\
& \left.+\sqrt{\frac{\omega}{\mathrm{x}_{\mathrm{c}}}} \frac{\beta_{\mathrm{pp}}^{\mathscr{P}}}{\beta_{\mathrm{pp}}^{\mathscr{P}}} \frac{\beta_{\gamma \gamma}^{\mathscr{P}^{\prime}}}{\beta_{\gamma \gamma}^{\mathscr{P}}} \overline{\mathrm{g}}_{\mathscr{P}^{\prime} \mathscr{P}^{\prime}}\left(\mathrm{x}_{\mathrm{c}}\right)\right]
\end{aligned}
$$

or since the cross terms break the scaling, and we have two contributions

$$
\begin{align*}
\rho\left(\mathrm{Q}^{2}, \nu, \nu^{\prime}, \kappa\right) \sim & \left.\mathrm{x}_{\mathrm{c}}\left[\overline{\mathrm{~g}}_{\mathscr{P} \mathscr{P}^{( }} \mathrm{x}_{\mathrm{c}}\right)+\sqrt{\frac{\omega}{\mathrm{x}_{\mathrm{c}}}} \frac{\beta_{\mathrm{pp}}^{\mathscr{P}}}{\beta_{\mathrm{pp}}^{\mathscr{P}}} \frac{\beta_{\gamma \gamma}^{\mathscr{P}}}{\beta_{\gamma \gamma}^{\mathscr{P}}} \overline{\mathrm{g}}_{\mathscr{P} \mathscr{P}^{\prime}}\left(\mathrm{x}_{\mathrm{c}}\right)\right] \\
& +\frac{\mathrm{x}_{\mathrm{c}}}{\sqrt{\kappa}}\left(\frac{\beta_{\mathrm{pp}}^{\mathscr{P}^{\prime}}}{\beta_{\mathrm{pp}}^{\mathscr{P}}}+\sqrt{\omega \mathrm{x}_{\mathrm{c}}} \frac{\beta_{\gamma \gamma}^{\mathscr{P}}}{\beta_{\gamma \gamma}^{\mathscr{P}}}\right) \overline{\mathrm{g}}_{\mathscr{P} \mathscr{P}^{\prime}}\left(\mathrm{x}_{\mathrm{c}}\right) \tag{4.16}
\end{align*}
$$

one scaling, and one scale breaking. Equation (4.16) shows that to get the scaling, Pomeron dominance is crucial (Reggeon-Reggeon contribution is suppressed like $\sqrt{\omega}$, but increases with $Q^{2}$ causing charge asymmetries).

Note that when the contribution of the secondaries become comparable to the Pomeron contribution we are at the boundary of the central region, i.e., for

$$
\sqrt{\frac{\nu^{\prime}}{Q^{2}}} \sim 1, \quad \sqrt{\frac{\kappa}{\mu}} \sim 1, \quad \sqrt{\frac{\nu^{\prime} \kappa}{\mu Q^{2}}} \sim 1
$$

These mean

$$
\mathrm{m} \nu^{\prime}=\frac{\mathrm{m} \mu}{\mu} \mathrm{Q} \sinh \left(\mathrm{y}_{1}-\mathrm{y}\right) \sim \mathrm{Q}^{2}
$$

or

$$
\begin{align*}
& \operatorname{Sh}\left(\mathrm{y}_{1}-\mathrm{y}\right) \sim \frac{\mu}{\mu_{\perp}} \frac{\mathrm{Q}}{\mathrm{~m}}  \tag{4.17a}\\
& \kappa=\mu_{\perp} \cosh \left(\mathrm{y}-\mathrm{y}_{2}\right) \sim \mu
\end{align*}
$$

or

$$
\begin{equation*}
\cosh \left(\mathrm{y}-\mathrm{y}_{2}\right) \simeq \frac{\mu}{\mu_{\perp}} \tag{4.17b}
\end{equation*}
$$

Equations (4.17a) and (4.17b) show the passage to the current fragmentation and target fragmentation region respectively.

By using (3.17) we find the scaling behavior of the structure functions as follows

$$
\begin{align*}
& \frac{1}{\mathrm{~m}} \nu \mathscr{O}_{1} \underset{\begin{array}{c}
\text { central } \\
\text { region }
\end{array}}{\overrightarrow{\mathrm{Bj}} \quad \mathscr{\mathscr { F }}_{1}^{\mathrm{c}}\left(\omega, \mathrm{x}_{\mathrm{c}}\right)=\mathrm{F}_{1}(\omega) \mathrm{G}^{\mathrm{c}}\left(\mathrm{x}_{\mathrm{c}}\right)}  \tag{4.18}\\
& \frac{1}{\mathrm{~m} \mu} \nu^{2} \mathscr{O}_{2} \underset{\begin{array}{c}
\text { central } \\
\text { region }
\end{array}}{\overrightarrow{\mathrm{Bj}}} \mathscr{F}_{2}\left(\omega, \mathrm{x}_{\mathrm{c}}\right)=\mathrm{F}_{2}(\omega) \mathrm{G}^{\mathrm{c}}\left(\mathrm{x}_{\mathrm{c}}\right)
\end{align*}
$$

Here the function $G^{c}\left(\mathrm{x}_{\mathrm{c}}\right)$ is a universal function depending only on the type of particle observed, but not on the virtual photon beam or the target.

## IV. 3 Phase Space Boundary on the Target End (Triple Regge Limit)

If the produced particle is sufficiently near the hadron end of the spectrum, that is if, say $y-y_{2}$, is nearly as small as possible, it becomes possible to calculate $\mathrm{f}_{\mathrm{t}}\left(\phi, \mathrm{y}-\mathrm{y}_{2}\right)$ in (4.7) explicitly in the Bjorken limit. This is the route followed in the case of purely hadronic inclusive reactions. They show that phase space boundary corresponds to the Triple Regge Limit (TR). Here we shall follow the reverse route, by studying the mathematical TR limit, and then
looking what physical region does it correspond to. From Fig. 8 we see that in the $T R$ region $A^{T}, \mathrm{~L}\left(Q^{2}, \nu, \nu^{\prime}, \kappa\right)$ is given by

$$
\mathrm{A}^{\mathrm{T}, \mathrm{~L}}\left(\mathrm{Q}^{2}, \nu, \nu^{\prime}, \kappa\right) \underset{\mathrm{TR}}{\simeq}\left[\beta_{\mathrm{pp}}^{\alpha^{\prime}}(\mathrm{t})\right]^{2}\left(\cos \theta_{\mathrm{t}}\right)^{2 \alpha^{\prime}(0)}\left\{\begin{array}{l}
\left.\lim _{\mathrm{M}^{2} \rightarrow \infty} \mathrm{~A}^{\mathrm{T}, \mathrm{~L}_{\left(\mathrm{M}^{2}, \mathrm{Q}^{2}, \mathrm{t}\right)}}\right\} . \tag{4.19}
\end{array}\right\}
$$

where $\alpha^{\prime}(\mathrm{t})$ is the leading Regge trajectory in the t -channel, and $\overline{\mathrm{A}}^{\mathrm{T}, \mathrm{L}}$ stands for the absorptive part of the virtual photon-Reggeon forward elastic scattering amplitude, $\mathrm{M}^{2}$ is the missing mass, and $\theta_{\mathrm{t}}$ is the c . m . angle for the process $p+q \rightarrow k+$ anything in the $t$-channel, and for small $t$, and if the detected hadron is a nucleon given by

$$
\begin{equation*}
\cos \theta_{\mathrm{t}} \sim \sqrt{\frac{-\mathrm{t}}{4 \mathrm{~m}^{2}}} \frac{\omega+\mathrm{x}_{\mathrm{t}}}{\mathrm{x}_{\mathrm{t}}^{-1}} \tag{4.20}
\end{equation*}
$$

where $x_{t}=k \cdot q / p \cdot q$. When the detected hadron is a nucleon, then the dominant $\alpha^{\prime}$-trajectory is a Pomeron. So $\cos \theta_{\mathrm{t}}$ has a power of 2. A similar analysis gives

$$
\begin{equation*}
\mathrm{A}^{\mathrm{T}, \mathrm{~L}}\left(\mathrm{Q}^{2}, \mathrm{M}^{2}, \mathrm{t}\right) \underset{\mathrm{M}^{2}, \mathrm{Q}^{2} \rightarrow \infty}{\simeq} \mathrm{~g}_{\alpha^{\prime} \alpha^{\prime} \alpha^{\prime}}(\mathrm{t}, \mathrm{t}, 0) \beta_{\gamma \gamma}^{\mathrm{T}, \mathrm{~L}}\left(\mathrm{Q}^{2}\right)\left(\cos \theta_{\mathrm{M}}\right)^{\alpha(0)} \tag{4.21}
\end{equation*}
$$

where $g_{\alpha^{\prime} \alpha^{\prime} \alpha^{\prime}}(\mathrm{t}, \mathrm{t}, 0)$ is the triple Reggeon vertex, and $\theta_{M}$ is the scattering angle in the barycentric frame on the cross channel of the photon-photon channel for the forward process photon + Reggeon $\rightarrow$ photon + Reggeon, and is given by

$$
\begin{equation*}
\cos \theta_{M} \sim \frac{1}{(-t)^{1 / 2}} \frac{p \cdot q-k \cdot p}{Q} \tag{4.22}
\end{equation*}
$$

Pomeranchukon is the dominant $\alpha$-trajectory, and using (2.12) for the photonphoton Reggeon vertex, we get

$$
\begin{equation*}
\overline{\mathrm{A}}^{\mathrm{T}, \mathrm{~L}}\left(\mathrm{Q}^{2}, \mathrm{M}^{2}, \mathrm{t}\right) \underset{\substack{\mathrm{Q}^{2} \rightarrow \infty \\ \mathrm{M}^{2} \rightarrow \infty \\ \mathrm{t} \text { small }}}{\sim} \frac{\mathrm{g}_{\alpha^{\prime} \alpha^{\prime} \mathscr{P}} \beta_{\gamma \gamma}^{\mathrm{T}, \mathrm{~L}}}{(-\mathrm{t})^{1 / 2}}\left(\frac{\mathrm{p} \cdot \mathrm{q}-\mathrm{k} \cdot \mathrm{q}}{\mathrm{Q}^{2}}\right) \tag{4.23}
\end{equation*}
$$

Substituting this in (4.19) we get

$$
\begin{aligned}
& \text { Bj } \\
& \text { small t }
\end{aligned}
$$

where

$$
\Gamma^{\mathrm{T}, \mathrm{~L}}(\mathrm{t})=\frac{1}{2} \beta_{\gamma \gamma}^{\mathrm{T}, \mathrm{~L}}\left[\beta_{\mathrm{pp}}(\mathrm{t})\right]^{2} \mathrm{~g}_{\alpha^{\prime} \alpha^{\prime} \mathscr{P}}(\mathrm{t}) \cdot\left(\frac{\sqrt{-\mathrm{t}}}{4 \mathrm{~m}^{2}}\right)
$$

The region in which our expression is valid is given by the following kinematical constraints:
(a) $t$ small, fixed.
(b) $\frac{\omega+x_{t}}{1-x_{t}}$ large, which means $x_{t} \sim 1$. And this ratio is very sensitive to the variation of $x_{t}$ around $x_{t} \sim 1$. In terms of Feynman's variable this means $\mathrm{x}_{\mathrm{F}} \sim \frac{1}{1-\omega}$.
(c) $\frac{p \cdot q-k \cdot q}{Q}$ large. This condition is automatically satisfied in the Bjorken limit, as long as (b) is satisfied.

The constraint (b) implies $y_{1}-\mathrm{y} \simeq \mathrm{y}_{1}-\mathrm{y}_{2}$. This shows that indeed the the TR region corresponds to the boundary of the phase space available for the single particle spectrum.

So, we see that for a nucleon produced near the end of the phase space, the invariant distribution is

$$
\begin{equation*}
\text { (Flux) } \frac{\mathrm{d} \sigma}{\mathrm{~d}^{3} \mathrm{k} / 2 \mathrm{k}_{0}} \underset{\underset{\mathrm{TR}}{\mathrm{Bj}}}{\sim} \Gamma_{\mathrm{p}}^{\mathrm{T}, \mathrm{~L}} \mu_{\perp}^{2} \frac{\left(\omega+\mathrm{x}_{\mathrm{t}}\right)^{2}}{\omega}\left(\frac{1}{1-\mathrm{x}_{\mathrm{t}}}\right) \tag{4.25}
\end{equation*}
$$

i.e., there is a peak at the target and of the phase space in terms of the variable $x_{t}$. This agrees with the data beautifully (Fig.10) [9]. The existence of this peak (elastic peak) was also predicted, via quite formal arguments in Ref. [19].

If the detected hadron is a pion, then the dominant $\alpha^{\prime}$-trajectory is a baryon trajectory with the intercept, $\alpha_{\Delta}(0) \approx 0.3$. In this case,

$$
\underset{\mathrm{t}}{\cos \theta_{\mathrm{t}}^{(\pi)} \sim 1-\frac{\omega+\mathrm{x}_{\mathrm{t}}}{1-\mathrm{x}_{\mathrm{t}}} \underset{\underset{\mathrm{Bj}}{\sim}}{\underset{\mathrm{TR}}{ }} \frac{1}{1-\mathrm{x}_{\mathrm{t}}}}
$$

Substituting these in (4.19), we get

$$
\text { (Flux) } \frac{\mathrm{d} \sigma^{\mathrm{T}, \mathrm{~L}}}{\mathrm{~d}^{3} \mathrm{k} / 2 \mathrm{k}_{0}} \underset{\mathrm{TR}}{\underset{\mathrm{Bj}}{\approx}} \Gamma_{\pi}^{\mathrm{T}, \mathrm{~L}} \frac{1}{\omega}\left(1-\mathrm{x}_{\mathrm{t}}\right)^{0.4}
$$

This distribution vanishes at the boundary. Again, this result agrees with experiment (Fig. 11).

The scaling laws, for the structure function are, using (3.17)

$$
\begin{align*}
& \frac{1}{\mathrm{~m}} \nu \mathscr{W}_{1} \underset{\underset{\mathrm{Bj}}{\operatorname{TR}}}{\rightarrow} \mathscr{F}_{1}^{\mathrm{TR}}\left(\omega, \mathrm{x}_{\mathrm{t}}, \kappa\right)=\left\{\begin{array}{l}
\mathrm{B}_{\mathrm{N}}^{\mathrm{T}}\left(\mu_{\perp}^{2}\right) \frac{\mathrm{F}_{1}(\omega)}{\omega} \frac{1}{1-\mathrm{x}_{\mathrm{t}}} \\
\mathrm{~B}_{\pi}^{\mathrm{T}}\left(\mu_{\perp}^{2}\right) \frac{\mathrm{F}_{1}(\omega)}{\omega}\left(1-\mathrm{x}_{\mathrm{t}}\right)
\end{array} .\right. \\
& \frac{1}{m \mu} \nu \nu^{2} \mathscr{O}_{2} \underset{\substack{\operatorname{Bj} \\
\operatorname{TR}}}{\rightarrow \mathscr{F}_{2}^{\mathrm{TR}}\left(\omega, \mathrm{x}_{\mathrm{t}}, \kappa\right)}=\left\{\begin{array}{l}
\mathrm{B}_{\mathrm{N}}^{\mathrm{T}+\mathrm{L}}\left(\mu_{\perp}^{2}\right) \frac{\mathrm{F}_{2}(\omega)}{\omega} \frac{1}{1-\mathrm{x}_{\mathrm{t}}} \\
\mathrm{~B}_{\pi}^{\mathrm{T}+\mathrm{L}}\left(\mu_{\perp}^{2}\right) \frac{\mathrm{F}_{2}(\omega)}{\omega}\left(1-\mathrm{x}_{\mathrm{t}}\right)
\end{array}\right. \tag{4.27}
\end{align*}
$$

The first ones are for nucleon production, the second for pion production.

## IV. 4 Current Fragmentation

From (4.4) we see that when $y-y_{2}$ is large, while $y_{1}-\mathrm{y}$ is finite (Fig. 9) the $0(1,2)$ expansion yields

$$
\begin{equation*}
\mathrm{A}\left(\mathrm{Q}^{2} ; \xi_{2}, \phi, \xi_{1}\right) \simeq\left(\cosh \xi_{2}\right)^{\alpha} \beta\left(\mathrm{Q}^{2} ; \phi, \xi_{1}\right) \tag{4.28}
\end{equation*}
$$

where $\alpha$ is the leading singularity and assumed to be a simple pole. Again assuming that this leading singularity is the Pomeron trajectory, and it factorizes, we obtain

$$
\begin{equation*}
\text { (Flux) } \frac{\mathrm{d} \sigma}{\mathrm{~d}^{3} \mathrm{k} / 2 \mathrm{k}_{0}} \simeq\left(\frac{\kappa}{\mu}\right) \beta_{\gamma}^{\mathscr{P}}\left(\mathrm{Q}^{2} ; \mathrm{y}_{1}-\mathrm{y}\right) \beta_{\mathrm{pp}}^{\mathscr{P}} \tag{4.29a}
\end{equation*}
$$

where the function $\beta_{\gamma}^{\mathscr{P}}$ stands for the right-hand blob in Fig. 9.
Since in this particular region, we do not have the photon-photon-Reggeon vertex explicitly in a factorized form, this is as far as we can go in our approach, namely we cannot predict explicit scaling forms. For the normalized distribution we have

$$
\begin{equation*}
\rho\left(\mathrm{Q}^{2}, \nu, \nu^{\mathrm{y}}, \kappa\right) \underset{\substack{\mathrm{Bj}}}{\underset{\mathrm{y}_{1}-\mathrm{y} \text { finite }}{\sim}\left(\frac{\kappa}{\nu}\right)\left(\frac{\mathrm{Q}^{2}}{\mathrm{~m} \mu}\right) \frac{1}{\mathrm{y}^{2}-\mathrm{y}_{2} \rightarrow \infty}} \tag{4.29b}
\end{equation*}
$$

All we can do is really guess and find plausibility arguments for the scaling variables, and the distributions. Let us first study the possible candidates to be the scaling variables. Since $\mathrm{y}_{1}-\mathrm{y} \simeq \mathscr{O}(1)$, and $\mathrm{y}-\mathrm{y}_{2}$ large, we have in the Bjorken limit

$$
\begin{array}{cc}
\nu^{\prime} \sim \mathscr{O}(\mathrm{Q}) & \kappa \sim \mathscr{O}(\mathrm{Q} / \omega) \\
\frac{\kappa}{\nu} \sim \mathscr{O}\left(\mathrm{Q}^{-1}\right) & \frac{\nu^{\prime}}{\nu} \sim \mathscr{O}\left(\mathrm{Q}^{-1}\right)  \tag{4.30}\\
\frac{\kappa}{\mathrm{Q}^{2}} \sim \mathscr{O}\left(\mathrm{Q}^{-1}\right) & \frac{\nu^{\prime}}{\mathrm{Q}^{2}} \sim \mathscr{O}\left(\mathrm{Q}^{-1}\right) \\
\frac{\mathrm{m}^{\prime} \epsilon}{\mu \nu^{\prime}} \sim \frac{1}{\omega}\left(\frac{1+\operatorname{th}\left(\mathrm{y}-\mathrm{y}_{1}\right)}{2}\right)
\end{array}
$$

From this list we see that if we would like to have a scaling variable in the conventional form (conventional in the hadronic sense) the only likely
candidate is the last one. But unfortunately, that it is nothing but the Bjorken's scaling variable can be seen by explicitly calculating it out (as is already obvious from (4.30)) in any frame we like. Doing it in the c.m. frame we find

$$
\frac{\mathrm{m}^{\prime} \epsilon}{\mu \nu^{\prime}}=\frac{\mathrm{k}_{0} / \mathrm{k}_{3}}{1+\mathrm{k}_{0} / \mathrm{k}_{3}}\left[1-2 \omega-\frac{\mathrm{k}_{3}}{\mathrm{k}_{0}}\right] \underset{\mathrm{k}_{3}}{\underset{\text { large }}{\simeq} \omega}
$$

That eventually we took the limit $\mathrm{k}_{3} \rightarrow \infty$, was in accord with the definition that the detected hadron has a finite fraction of the photon mass. Since both photon and the detected hadron are well separated from the center of the rapidity space, this means that photon fragments have large longitudinal moments in the c.m. frame.

Since the photon fragmentation region is quite a peculiar region for high $Q^{2}$, we have to give up the conventions we made for the purely hadronic processes. To find the correct scaling variable let us review the scaling variables we have obtained for the neighboring regions:

$$
\begin{aligned}
& x_{\mathrm{t}}=\frac{\mathrm{k} \cdot \mathrm{q}}{\mathrm{p} \cdot \mathrm{q}}=\frac{\mu}{\mathrm{m}} \frac{\operatorname{Sh}\left(\mathrm{y}_{1}-\mathrm{y}\right)}{\operatorname{Sh}\left(\mathrm{y}_{1}-\mathrm{y}_{2}\right)} \approx \frac{\mu}{\mathrm{m}} \mathrm{e}^{-\left(\mathrm{y}-\mathrm{y}_{2}\right)} \\
& \mathrm{x}_{\mathrm{c}}=\frac{\mathrm{k} \cdot \mathrm{q})(\mathrm{p} \cdot \mathrm{k})}{(\mathrm{p} \cdot \mathrm{q})}=\mu_{1}^{2} \frac{\operatorname{Sh}\left(\mathrm{y}_{1}-\mathrm{y}\right) \operatorname{Sh}\left(\mathrm{y}-\mathrm{y}_{2}\right)}{\operatorname{Sh}\left(\mathrm{y}_{1}-\mathrm{y}_{2}\right)} \approx \mu_{1}^{2}
\end{aligned}
$$

Calling $\operatorname{Sh}\left(y_{1}-y\right)=z_{1}, \operatorname{Sh}\left(y-y_{2}\right)=z_{2}, \operatorname{Sh}\left(y_{1}-y_{2}\right)=z_{2}, \operatorname{Sh}\left(y_{1}-y_{2}\right)=z$ we see that

$$
\mathrm{x}_{\mathrm{t}}=\frac{\mu}{\mathrm{m}} \frac{\mathrm{z}_{1}}{\mathrm{z}}, \quad \mathrm{x}_{\mathrm{c}}=\mu_{1}^{2} \frac{\mathrm{z}_{1} \mathrm{z}_{2}}{\mathrm{z}}
$$

The scaling variable which would match with $x_{c}$ and $x_{t}$ and would carry the same meaning would be

$$
\begin{equation*}
\mathrm{x}_{\gamma}=\frac{\mathrm{z}_{2}}{\mathrm{z}}=\frac{\operatorname{Sh}\left(\mathrm{y}-\mathrm{y}_{2}\right)}{\operatorname{Sh}\left(\mathrm{y}_{1}-\mathrm{y}_{2}\right)}=\frac{\kappa / \mu}{\nu / \mathrm{Q}}=\left(\frac{\kappa}{\nu}\right)\left(\frac{\mathrm{Q}}{\mu}\right) \tag{4.31}
\end{equation*}
$$

In terms of this new scaling variable, our normalized distribution becomes

$$
\begin{align*}
& \rightarrow \rho_{\gamma}\left(\mathrm{Q}^{2}, \nu, \nu^{\prime}, \kappa\right) \underset{\mathrm{y}_{1}-\mathrm{y} \text { finite }}{\underset{\mathrm{Bj}}{\sim}}\left(\mathrm{x}_{\gamma} \frac{\mathrm{Q}}{\mathrm{~m}}\right) \frac{1}{\beta_{\gamma \gamma}} \cdot \beta_{\gamma \mathrm{P}}^{\mathscr{P}}\left(\mathrm{Q}^{2}, \mathrm{y}_{1}-\mathrm{y}\right)  \tag{4.29c}\\
& y-y_{2} \rightarrow \infty
\end{align*}
$$

As we see $\rho$ does not even have Bjorken scaling, as it is. To guess the scaling form, consider the inclusive sum rule [6]:

$$
\begin{equation*}
\sum_{\mathrm{i}} \int \frac{\mathrm{~d}^{3} \mathrm{k}_{\mathrm{i}}}{\mathrm{k}_{\mathrm{i} 0}} \rho_{\gamma}\left(\mathrm{Q}^{2} ; \nu, \nu^{\prime}, \kappa\right) \mathrm{k}_{\mathrm{i} 0}=\mathrm{E}_{\mathrm{tot}} \tag{4.32}
\end{equation*}
$$

We know that the total available energy is $\mathrm{E}_{\text {tot }} \simeq \nu$. We shall show in the next section quite generally that, the transverse momentum of outgoing particles is limited. From our detailed kinematical analysis above, we see that in the current fragmentation region $\mathrm{k}_{0} \sim \mathscr{O}(\nu)$, and it is at least one power of $\sqrt{\nu}$ suppressed in the neighboring region (because the boost parameter which relates the lab and the c.m. frame is $\operatorname{Ch} \beta=\sqrt{\nu / 2 \mathrm{~m}(1-\omega)})$. Assuming that cross sections are bounded as $Q^{2} \rightarrow \infty$ (i.e., multiplicities do not grow as powers of $Q^{2}$ ), then this sum rule is saturated by the contributions in the current fragmentation region. This means that the integral in rapidity extends only a finite region:

$$
\begin{equation*}
\sum_{\mathrm{i}} \int_{\left(\mathrm{y}_{1}-\mathrm{y} \simeq 2\right)} \mathrm{d}^{2} \mathrm{k}_{\perp \mathrm{i}} \int \mathrm{dy}_{\mathrm{i}} \rho_{\gamma}\left(\mathrm{y}_{1}-\mathrm{y}_{2} ; \mu_{\perp}^{2}\right) \frac{\mathrm{k}_{\mathrm{i} 0}}{\nu}=1 \tag{4.33}
\end{equation*}
$$

Since $\mathrm{k}_{\mathrm{i} 0} \sim \mathscr{O}(\nu)$, the simplest way of satisfying this sum rule would be to have $\rho\left(Q^{2}, \nu, \nu^{\prime}, \kappa\right)$ independent of $Q^{2}$, i.e., scale in $\omega$. (This would mean that $\nu \mathscr{T} V_{1}$ and $\nu^{2} \mathscr{Y}_{2}$ are the scaling structure functions, as in the neighboring region, and is in agreement with the parton model predictions [8] also, if this means anything). This is simply achieved if the explicit Q-dependence of the blob
$\beta_{\gamma \mathrm{P}}^{\mathscr{P}}\left(\mathrm{Q}^{2}, \mathrm{y}-\mathrm{y}_{1}\right)$ was

$$
\beta_{\gamma \mathrm{h}}^{\mathscr{P}}\left(\mathrm{Q}^{2}, \mathrm{y}-\mathrm{y}_{1}, \mu_{\perp}\right)=\frac{\mathrm{m}}{\mathrm{Q}} \beta_{\gamma \mathrm{h}}^{\mathscr{P}}\left(\mathrm{y}-\mathrm{y}_{1}, \mu_{\perp}\right)
$$

If this ansatz is correct, we finally obtain

$$
\begin{equation*}
\rho_{\gamma}^{\mathrm{T}, \mathrm{~L}}\left(\mathrm{Q}^{2}, \nu, \nu^{\prime}, \kappa\right) \underset{\substack{\mathrm{y}-\mathrm{y}_{2} \rightarrow \infty \\ \mathrm{y}_{1}-\mathrm{y} \text { finite }}}{\simeq} \mathrm{x}_{\gamma} \mathrm{f}_{\gamma \mathrm{h}}^{\mathscr{P}}\left(\mathrm{x}_{\gamma}, \mu_{\perp}\right) \tag{4.34}
\end{equation*}
$$

Where in the last step we also used the fact

$$
\mathrm{x}_{\gamma}=\frac{\operatorname{Sh}\left(\mathrm{y}-\mathrm{y}_{2}\right)}{\operatorname{Sh}\left(\mathrm{y}_{1}-\mathrm{y}_{2}\right)} \sim \mathrm{e}^{-\left(\mathrm{y}_{1}-\mathrm{y}\right)} \approx\left(\frac{\nu^{\prime}}{\mathrm{Q}}\right)^{-1}
$$

Adding the next leading trajectory breaks the scaling

As $\kappa$ gets large we approach scaling from above. Note that for $\mathrm{x}_{\gamma} \sim 1$, close to the end of the boundary, we have $\kappa / \mu \simeq \nu / Q$. Substituting this in (4.35) we get

$$
\begin{equation*}
\rho_{\gamma}\left(\mathrm{Q}^{2}, \nu, \nu^{\prime}, \kappa\right) \sim \mathrm{x}_{\gamma}\left[\mathrm{f}_{\gamma}^{\mathscr{P}}\left(\mathrm{x}_{\gamma^{\prime}}, \mu_{\perp}\right)+\sqrt{\frac{\mathrm{Q}}{\nu}} \mathrm{f}_{\gamma}^{\mathscr{P}^{\boldsymbol{\varphi}}}\left(\mathrm{x}_{\gamma^{\prime}}, \mu_{\perp}\right)\right] \tag{4.35b}
\end{equation*}
$$

For small Q (and fixed value of initial total energy) Pomeron contribution dominates. But if we increase $Q^{2}$, the contribution of the isospin carrying secondary trajectory $\left(\mathscr{P}^{1}+\mathrm{A}_{2}\right)$ becomes comparable to the $\mathscr{P}$-contribution. This implies that $\pi^{+} / \pi^{-}$asymmetry, which is due to isospin carrying secondary Regge trajectory increases as a function of $Q^{2}$, or as a function of $\omega$ for fixed value of $s$. This prediction although not as detailed as that of the parton model [20], is supported by the data (Fig. 12) [9].

Now, let us investigate the questions whether the hadron produced in the so defined current fragmentation region show the same features as in the purely
hadronic case. (In the purely hadronic case, particles which are slow in the beamrrest frame are the fragments of the beam.) Because it does not make sensc, to talk about the rest frame of a space-like photon, let us work in the barycentric frame:

$$
\begin{equation*}
\cosh \xi_{2}=\frac{\mathrm{p} \cdot \mathrm{k}}{\mathrm{~m} \mu} \underset{\mathrm{k}_{3} \rightarrow \infty}{\simeq} \frac{\nu}{\mathrm{~m}} \mathrm{x}_{\mathrm{F}} \rightarrow \infty \tag{4.36}
\end{equation*}
$$

So our expansion is valid for the fast particle production. Let us see now whether there are any slow particles produced in this region also:

$$
\begin{gather*}
\cosh \xi_{2} \underset{\mathrm{k}_{3} \rightarrow \infty}{\simeq} \frac{\mu_{\perp}}{\mu} \frac{(\nu / \mathrm{m})^{1 / 2}}{\sqrt{2(1-\omega)}} \rightarrow \infty  \tag{4.37a}\\
\sinh \xi_{1}=\frac{\mathrm{k} \cdot \mathrm{q}}{\mu \mathrm{Q}} \underset{\mathrm{k}_{3} \rightarrow 0}{\simeq} \frac{1-2 \omega}{2 \sqrt{\omega(1-\omega)}} \underset{\omega \rightarrow 0}{\rightarrow} \frac{1}{2 \sqrt{\omega}} \text { large } \tag{4.37b}
\end{gather*}
$$

This result shows that particles with $\mathrm{k}_{3}^{*} \simeq 0$ are produced in the central region, and the central region may be called the "pionization region" as in the hadronic case without any further reservation. The structure functions scale as follows:

$$
\begin{align*}
& \frac{1}{\mathrm{~m}} \nu \mathscr{W}_{1} \quad \overrightarrow{\mathrm{Bj}} \quad \mathscr{F}_{1}^{\gamma}\left(\omega, \mathrm{x}_{\gamma}, \mu_{\perp}\right)=\mathrm{F}_{1}(\omega) \mathrm{G}_{\gamma}^{\mathrm{T}}\left(\mathrm{x}_{\gamma}, \mu_{\perp}\right) \\
& y-y_{2} \rightarrow \infty \\
& y_{1}-\mathrm{y} \text { finite }  \tag{4.38}\\
& \begin{array}{c}
\frac{1}{\mathrm{~m} \mu} \nu^{2} \mathscr{W}_{2} \underset{\substack{ \\
\mathrm{Bj}-\mathrm{y}_{2} \rightarrow \infty}}{\overrightarrow{\mathrm{~F}_{2}}} \underset{2}{\gamma}\left(\omega, \mathrm{x}_{\gamma}, \mu_{\perp}\right)=\mathrm{F}_{2}(\omega) \mathrm{G}_{\gamma}^{\mathrm{T}+\mathrm{L}}\left(\mathrm{x}_{\gamma^{\prime}} \mu_{\perp}\right) \\
\mathrm{y}_{1}-\mathrm{y} \text { finite }
\end{array}
\end{align*}
$$

The hope that, we can find the form of $\mathrm{G}_{\gamma}\left(\mathrm{x}_{\gamma}, \mu_{\perp}\right)$ explicitly, by taking the single pion exchange diagram for pion production close to the boundary, fails when $Q^{2}$ large, because $(q-k)^{2} \simeq-Q^{2}$. This is a peculiarity of the highly space-like photon.

## V. AVERAGE MULTIPLICITIES

V. 1 Contributions of Each Kinematical Region to the Multiplicity

The average multiplicity of the produced particles is defined by

$$
\begin{equation*}
\sigma_{\text {tot }}\left(\mathrm{Q}^{2}, \nu\right) \overline{\mathrm{n}}\left(\mathrm{Q}^{2}, \nu\right)=\int \frac{\mathrm{d}^{3} \mathrm{k}}{2 \mathrm{k}_{0}}\left(\frac{\mathrm{~d} \sigma\left(\mathrm{Q}^{2}, \nu, \nu,\right)}{\mathrm{d}^{3} \mathrm{k} / 2 \mathrm{k}_{0}}\right) \tag{5.1}
\end{equation*}
$$

where $\sigma_{\text {tot }}$ is the total spin averaged virtual photon-proton cross section at a given $\nu$ and $\mathrm{Q}^{2}$ and the integral (5.1) extends over the allowed phase space. In our definition $\overline{\mathrm{n}}$ means the average multiplicity for a specific type of particle, say pion $\left(\pi^{+}, \pi^{-}\right.$, or $\left.\pi^{\circ}\right)$ to which $\mathrm{d} \sigma /\left(\mathrm{d}^{3} \mathrm{k} / 2 \mathrm{k}_{0}\right)$ refers. To get the average multiplicity for all kinds of pions produced in virtual inclusive photoproduction processes we have to add the contributions of all. Clearly if one does not have information about the transverse momentum distributions it is difficult to estimate the multiplicity of the pions. However, if $k_{\perp}$ dependence of the invariant distribution (explicitly $\mathrm{f}_{\mathrm{t}}, \mathrm{f}_{\gamma}$, and g ) falls faster than $\left(\mathrm{k}_{\perp}^{2}\right)^{-1}$, then the leading behavior of $\bar{n}\left(Q^{2}, \nu\right)$ can be calculated when the Pomeranchuk pole is the leading singularity.

At the end of this chapter we shall carefully study this point and will show that the suppression of transverse momentum comes out naturally as a result of other assumptions already made, and the data seems to be in perfect agreement with this (Fig. 12) [9, 25]. Making a variable change from $\left(\mathrm{k}_{3}, \mathrm{k}_{\perp}\right)$ to $\left(\mathrm{y}_{1} \mathrm{k}_{\perp}\right)$ we get

$$
\begin{equation*}
2 \overline{\mathrm{n}}\left(\mathrm{Q}^{2}, \nu\right)=\int \mathrm{d}^{2} \mathrm{k}_{\perp} \cdot \int \mathrm{dy} \rho_{\mathrm{t}}\left(\mathrm{x}_{\mathrm{t}}, \mu_{\perp}\right)+\int \mathrm{dy} \int \mathrm{~d}^{2} \mathrm{k}_{\perp} \rho_{\mathrm{c}}\left(\mu_{\perp}\right)+\int \mathrm{d}^{2} \mathrm{k}_{\perp} \int \mathrm{dy} \rho_{\gamma}\left(\mathrm{x}_{\gamma}, \mu_{\perp}\right) \tag{5.2}
\end{equation*}
$$

Because the $\mu_{\perp}$ behavior is damped, the total multiplicity is proportional to the length of the phase space, $\ln \left[(1-\omega) \frac{S}{\mathrm{~m}^{2}}\right]$.

Now let us calculate the shares of each region. The first integration is oyer a finite length of rapidity ( $\sim 2$ ), and therefore is a constant. The second integral is proportional to the length of the phase space of that region, because the integral of the universal function $\rho_{c}\left(\mu_{\perp}\right)$ over $k_{\perp}$ is constant:

$$
\int_{2}^{2+\mathrm{L}_{\mathrm{c}}} \mathrm{dy} \int \mathrm{~d}^{2} \mathrm{k}_{\perp} \rho_{\mathrm{c}}\left(\mu_{\perp}\right)=(\text { const }) \times \mathrm{L}_{\mathrm{c}}
$$

where $L_{c}$ is the length of the central region, in rapidity space.
Even though we assume that photon fragments very much like a hadron does, we know that certain things are different, because the photon is very highly space-like. To find the contribution of the current fragments, let us keep $Q^{2}$ fixed and large, and decrease s, until the projectile communicates with target, in the rapidity plane (this can be achieved, by keeping $Q^{2}$ fixed and letting $\omega$ to 1 also). The length of the projectile fragmentation region is

$$
\mathrm{L}_{\gamma}=\mathrm{Y}-\mathrm{L}_{\mathrm{t}} \simeq \ln \left[\frac{\mathrm{~S}}{\mu_{\perp}^{2}}(1-\omega)\right] .
$$

Substituting

$$
\mathrm{s} \simeq \mathrm{Q}^{2}\left(\frac{1}{\omega}-1\right) \underset{\omega \sim 1}{\simeq} \mathrm{Q}^{2}(1-\omega)
$$

we get

$$
\begin{equation*}
\mathrm{L}_{\gamma} \simeq \ln \left[(1-\omega)^{2} \frac{\mathrm{Q}^{2}}{\mu_{\perp}^{2}}\right] \tag{5.3}
\end{equation*}
$$

If we now let $s \rightarrow \infty$ (keeping $Q^{2}$ large, and fixed) central region reappears again. The size of the central region is, then

$$
\begin{align*}
\mathrm{L}_{\mathrm{C}} & =\mathrm{Y}-\mathrm{L}_{\gamma}-\mathscr{O}(1) \simeq \ln \left[\frac{\mathrm{S}}{\mathrm{Q}^{2}} \frac{1}{1-\omega}\right] \\
& \simeq \ln \left(\frac{1}{\omega}\right) \tag{5.4}
\end{align*}
$$

Finally, the contribution of each region to the total average charged multiplicity is (Fig. 14)

$$
\begin{align*}
& \overline{\mathrm{n}}_{\mathrm{t}}\left(\mathrm{Q}^{2}, \nu\right)=\text { constant } \\
& \overline{\mathrm{n}}_{\mathrm{c}}\left(\mathrm{Q}^{2}, \nu\right)=(\text { const }) \ln (1 / \omega)  \tag{5.5}\\
& \overline{\mathrm{n}}_{\gamma}\left(\mathrm{Q}^{2}, \nu\right)=(\text { const })+(\text { const }) \ln \left(\mathrm{Q}^{2} / \mu_{\perp}^{2}\right)
\end{align*}
$$

## V. 2 An Upper Bound on the Transverse Momentum.

Now it is well known that the logarithmic growth of the average multiplicity (at least in the purely hadronic case) arise from populating the longitudinal phase space $\mathrm{dk}_{3} / \mathrm{k}_{0}=\mathrm{dy}$ in a statistically independent manner [6]. We would like to point out here that for this type of longitudinal distribution the transverse momentum must be limited for the general case like the one in hand, where we have highly off shell virtual particles involved [21,22]. We shall make the following assumptions:
a) phase space is completely filled
b) transverse momentum is independent of rapidity .

Since the assumption (a) is the most crucial one for the processes involving highly virtual particles let us give a plausibility argument in support of it, recalling the kinematical study in Section III. 2. The quantity

$$
\begin{equation*}
\Delta \mathrm{y}=\mathrm{y}_{\max }-\mathrm{y}_{1}=\ln (1-\omega)+\ln \left(\frac{\mathrm{Q}}{\mu_{\perp}}\right) \tag{5.6}
\end{equation*}
$$

would measure the unfilled positions of the longitudinal phase space. Fortunately it is small $(\sim \mathscr{O}(1))$ for all values of $Q^{2}$. If $\Delta y$ was not small, it would mean that there are new regimes opened up in the phase space, other than the so defined target fragmentation, central and current fragmentation regimes.

That $\Delta \mathrm{y} \sim \mathscr{O}(1)$ means that phase space actually extends only between limits fixed by the rapidities of the leading particles, although it does not seem to be so
at first sight. Let us calculate the mass, $\mathscr{A}$, of the "Feynman gas" of length Y. Let us first translate the gas in $y$ by a Lorentz transformation so that it is centered at $y=0$. Let us call the average density of particles integrated over $k_{\perp}$, C, i.e., $d \bar{n} / d y=C$. Since $k_{3}=\mu_{\perp} \sinh y$, with the gas centered at $y=0$, its total momentum is zero; therefore its mass is its total energy. Assuming that $\bar{\mu}_{\perp}$ is independent of $y$, we obtain

$$
\begin{equation*}
\mathscr{M}=2 \mathrm{C} \bar{\mu}_{\perp} \sinh \left(\frac{\mathrm{Y}}{2}\right) \simeq \mathrm{C} \bar{\mu}_{\perp} \mathrm{e}^{\mathrm{Y} / 2} \tag{5.7}
\end{equation*}
$$

From (3.24) we immediately see that

$$
\begin{equation*}
\mathrm{Y}=\ln \left[(1-\omega) \frac{\mathrm{S}}{\mathrm{~m}^{2}}\right] \tag{5.8}
\end{equation*}
$$

Substituting this in (5.7), we get

$$
\begin{equation*}
\mathscr{M}=\mathrm{C} \bar{\mu}_{\perp} \sqrt{1-\omega}\left(\frac{\mathrm{S}}{\mathrm{~m}^{2}}\right)^{1 / 2} \tag{5.9}
\end{equation*}
$$

Since $\mathscr{M}$ cannot exceed the total available c.m. energy,

$$
\mathscr{M} \lesssim \mathrm{E}_{\mathrm{tot}}^{\mathrm{c} . \mathrm{m}} \simeq \sqrt{\mathrm{~s}}
$$

we get

$$
\begin{equation*}
\mathrm{C} \frac{\bar{\mu}_{\perp}}{\mathrm{m}} \underset{\mathrm{Bj}}{\lesssim} \frac{1}{1-\omega} \underset{\omega \rightarrow 0}{\simeq} 1 \tag{5.10}
\end{equation*}
$$

From (5.10) we see that if $C$ is to be of order -1 as is experimentally (Fig. 15) [9] then for $\omega \sim 0$ (deep Regge region) $\bar{\mu}_{\perp}$ is constrained to be of the order of a typical hadron mass or less, the same behavior we see in hadronic production processes [6]. As $Q^{2}$ gets very large, or as $\omega \rightarrow 1$, (5.10) predicts large deviation from this typical hadronic behavior. If C is again to be of order 1 , then $\bar{\mu}_{\perp} / \mathrm{m}$ may get very large with increasing $\mathrm{Q}^{2}$, for fixed s :

$$
\begin{equation*}
\frac{\bar{\mu}_{\perp}}{\mathrm{m}} \lesssim \frac{1}{\mathrm{C}} \frac{1}{1-\omega}=\frac{1}{\mathrm{C}}\left(1+\frac{\mathrm{Q}^{2}}{\mathrm{~S}}\right) \tag{5.11}
\end{equation*}
$$

This linear increase with $Q^{2}$ seems to be consistent with the data (Fig. 13). From the data (Fig. 15) we get $\mathrm{C} \simeq 1$. Se we see that this inequality is well satisfied by the data; for the left-hand side is 0.47 , but the right-hand side is 1.08. Notice also that (5.9) and (5.10) set a rough upper limit on the value of C, because $\mu \leq \bar{\mu}_{\perp}$. These upper limits are

$$
C_{\max }^{\pi} \simeq \frac{\mathrm{M}_{N}}{\mathrm{M}_{\pi}} \simeq 7, \quad C_{\max }^{N} \simeq \frac{\mathrm{M}_{N}}{\mathrm{M}_{\mathrm{N}}}=1
$$

and these give

$$
\mathrm{C}_{\max }^{\pi} / \mathrm{C}_{\max }^{\mathrm{N}} \simeq \mathrm{M}_{\mathrm{N}} / \mathrm{M}_{\pi} \approx 7
$$

Now it is the right place to note that if the central plateau is really two plateaus, hadronic plateau and the current plateau, lying between the current fragmentation and the hole fragmentation regions [23], our result (5.10) applies again. If they are of equal height $C$ is their common height; if they are of different height it is their average, $\mathrm{C}=\frac{1}{2}\left(\mathrm{C}_{\mathrm{h}}+\mathrm{C}_{\gamma}\right)$.

## VI. FINAL REMARKS

In the preceding chapters, following Mueller's analysis of purely hadronic processes closely, we calculated the momentum distribution of the final state hadrons in deep inelastic electroproduction processes, and obtained general scaling laws. Our approach to the problem is fairly model independent. We can list the ingredients that led up to the above result as follows:
(a) We assumed that the amplitude for the forward virtual Compton amplitude can be written as a sum over the leading Regge poles over a wide range of values of $Q^{2}$ and $\nu$ and not just the asymptotic limit (a plausibility argument for this is duality).
(b) The scaling law obtained by Bjorken for the ordinary deep inelastic electroproduction processes are consistent with all the experimental data available at present. Therefore, we takc it as an cxperimental fact, and combining this with (a) above we get a prescription for the $Q^{2}$ dependence of the virtual photon-, virtual photon-Reggeon vertex.
(c) We assume that Pomeranchukon singularity is a Regge pole, so that its residue factorizes. This assumption enables us to use the information we obtained in (b) for the ordinary deep inelastic electroproduction processes.
(d) We expand the absorptive part of the forward three-to-three amplitude, "virtual photon + nucleon $+\overline{\text { pion }} \longrightarrow$ virtual photon + nucleon $+\overline{\text { pion }}, "$ into harmonics of $0(1,2)$, as Mueller did in analyzing the purely hadronic inclusive reactions. Because of the existence of a highly spacelike virtual photon, this is not trivial exercise at all. Also, in this analysis, we did not assume the mass of the virtual photon, $\mathrm{Q}^{2}$, to be small compared to $\nu$.

Although we assumed that the virtual photon fragments very much like a hadron does, we get some interesting features in our case which are absent in purely
hadronic case. For instance, we observe that the size of the virtual photon fragmentation region in the rapidity plane changes with $Q^{2}$. Factorization of the Pomeronchukon residue insures that in our case also the target nucleon fragments exactly as it does in purely hadronic inclusive processes, i.e., its size in rapidity plane is a constant, so its contribution to the multiplicity is a constant. The average multiplicity again grows logarithmically with energy, as in the purely hadronic case. But this time the contribution from the photon fragmentation region is really large. If we fix $\omega$ at a not too small value, we see that the major contribution to the multiplicity comes from the photon fragmentation region, whereas in purely hadronic case it was coming from the central region.

We have, then, shown quite generally that the logarithmic increase of the multiplicity imposes a constraint on the average transverse momentums produced, and this upper limit increases with $Q^{2}$.

We have shown that, at not too high energies where the contribution of secondary Regge trajectories are comparable to the Pomeron, the charge asymmetries carried by these isospin carrying trajectories increase with increasing $Q^{2}$, both in the current fragmentation region and central region. This behavior for current fragmentation region is beatifully supported by the data. But since energies are not high enough to develop a plateau, the prediction for the central region is to be tested in the future.

We finally point out, as a final digression on the soft-pions (in Appendix 3), that the contributions of the soft-pions does not increase with energy, i.e., in the center of mass frame, it is the soft-pions' contribution which gives early onset of scaling.

Kinematics relevant to the problem is very thoroughly and carefully investigated throughout.

## APPENDIX 1

For $Q^{2} \rightarrow 0$ we should have $\frac{d \sigma^{L}}{d^{3} k / 2 k_{0}} \rightarrow 0$; this gives us the following relation

$$
\begin{equation*}
\lim _{Q^{2} \rightarrow 0} \mathscr{W}_{1}\left(Q^{2}, \nu, \nu^{\prime}, \kappa\right)=\lim _{Q^{2} \rightarrow 0}\left[\frac{\nu^{2}}{Q^{2}} \mathscr{V}_{2}\left(Q^{2}, \nu, \nu^{1}, \kappa\right)\right] . \tag{A1.1}
\end{equation*}
$$

Also $\frac{\mathrm{d} \sigma^{\mathrm{T}}}{\mathrm{d}^{3} \mathrm{k} / 2 \mathrm{k}_{0}}\left(\mathrm{Q}^{2}=0, \nu, \nu^{\prime}, \kappa\right)$ is the inclusive photoproduction.
We observe that as $Q^{2} \rightarrow 0, \sin ^{2} \frac{\theta}{2} \rightarrow 0$, and so $\cos ^{2} \frac{\theta}{2} \rightarrow 1$, therefore

$$
\begin{equation*}
\left\langle\frac{\mathrm{d} \sigma}{\mathrm{dQ}^{2} \mathrm{~d} \nu \mathrm{~d} \nu^{\prime} \mathrm{d} \kappa}\right\rangle_{\mathrm{Q}^{2} \rightarrow 0}^{\sim} 4 \pi^{2} \alpha\left(\frac{\mathrm{E}^{\prime}}{\mathrm{E}}\right) \lim _{\mathrm{Q}^{2} \rightarrow 0}\left[\frac{\mathscr{\mathscr { N }}_{2}\left(\mathrm{Q}^{2}, \nu, \nu^{\prime}, \kappa\right)}{\mathrm{Q}^{4}}\right] \tag{A1.2}
\end{equation*}
$$

Comparing this with the photo inclusive cross section

$$
\begin{equation*}
\left\langle\frac{\mathrm{d} \sigma}{\mathrm{~d}^{3} \mathrm{k} / 2 \mathrm{k}_{0},}\right\rangle \underset{\mathrm{Q}^{2} \rightarrow 0}{\sim} \frac{4 \pi^{2} \alpha}{\mathrm{~m} \nu} \lim _{\mathrm{Q}^{2} \rightarrow 0} \mathscr{N}_{1} \underset{\mathrm{Q}^{2} \rightarrow 0}{\sim} \frac{4 \pi^{2} \alpha \mu}{2 \mathrm{~m}} \frac{1}{\nu^{2}} \lim _{\mathrm{Q}^{2} \rightarrow 0}\left(\frac{\nu^{2}}{\mathrm{Q}^{2}} \stackrel{\omega}{2}^{2}\right) \tag{A1.3}
\end{equation*}
$$

we finally obtain

$$
\begin{array}{r}
\left\langle\frac{\mathrm{d} \sigma}{\mathrm{~d} Q^{2} \mathrm{~d} \nu \mathrm{~d} \nu^{\prime} \mathrm{d} \kappa}\right\rangle_{\mathrm{eh} \rightarrow \mathrm{e}^{\prime} \mathrm{h}^{\prime} \mathrm{H}} \mathrm{Q}^{2} \rightarrow 0 \\
 \tag{A1.4}\\
\times\left(\frac{\mathrm{d} \sigma}{\mathrm{~d} \nu^{\prime} \mathrm{d} \kappa}\right)_{\gamma \mathrm{h} \rightarrow \mathrm{~h}^{\prime} \mathrm{H}} .
\end{array}
$$

## APPENDIX 2

The relations between $W^{a b}$ and $W_{i}$ are as follows

$$
\begin{aligned}
& \mathrm{W}^{++}=\epsilon_{\mu}^{+} * \epsilon_{\nu}^{+} \mathrm{W}^{\mu \nu}=\mathrm{W}_{1}+\frac{1}{\mu^{2}}\left|\epsilon^{+} \cdot \mathrm{k}\right|^{2} \mathrm{~W}_{2}=\mathrm{W}^{--} \\
& \mathrm{W}^{\mathrm{oo}}=\epsilon_{\mu}^{\mathrm{o}} * \epsilon_{\nu}^{\mathrm{o}} \mathrm{~W}^{\mu \nu}=-\mathrm{W}_{1}+\left(1+\frac{\nu^{2}}{\mathrm{Q}^{2}}\right) \mathrm{W}_{2} \\
& +\frac{1}{\mu^{2}}\left|\epsilon^{\mathrm{o}} \cdot \mathrm{k}\right|^{2} \mathrm{~W}_{3}+\frac{1}{\mathrm{~m} \mu} \operatorname{Re}\left(\epsilon^{\mathrm{O}} \cdot \mathrm{p}\right)\left(\epsilon^{\mathrm{O}} \cdot \mathrm{k}\right) \mathrm{W}_{4} \\
& \mathrm{~W}^{+-}=\epsilon_{\mu}^{+} * \epsilon_{\nu}^{-} \mathrm{W}^{\mu \nu}=\frac{1}{\mu^{2}}\left(\epsilon^{+} * \cdot \mathrm{k}\right)\left(\epsilon^{-} \cdot \mathrm{k}\right) \mathrm{W}_{3} \\
& \operatorname{Re} \mathrm{~W}^{+o}=\frac{\mathrm{k}_{\mathrm{x}}}{\mu^{2}}\left(\epsilon^{\mathrm{o}} \cdot \mathrm{k}\right) \mathrm{W}_{3}+\frac{\mathrm{k}_{\mathrm{x}}}{2 \mathrm{~m} \mu}\left(\epsilon^{\mathrm{O}} \cdot \mathrm{p}\right) \mathrm{W}_{4}
\end{aligned}
$$

## APPENDIX 3: INCLUSIVE SOFT-PION ELECTROPRODUCTION

We shall now consider inclusive electroproduction process where the detected particle is a soft pion:

$$
\mathrm{e}(\ell)+\mathrm{N}(\mathrm{p}) \rightarrow \mathrm{e}^{\prime}\left(\ell^{\prime}\right)+\pi_{\mathrm{soft}}^{ \pm}(\mathrm{k})+\text { anything }\left(\mathrm{p}_{\mathrm{H}}\right)
$$

We shall first look at the problem as a special case of the general pion production process we investigated in the preceding chapters.

The notion of a soft meson depends on a particular Lorentz frame just as a soft (infrared) photon does. We are going to assume here that in some Lorentz frame, which we are going to specify explicitly below, all soft-pion momenta, say $k_{\mu}$, are so small, that they satisfy [24]

$$
\begin{equation*}
|\vec{k}| \ll \mu . \tag{A3.1}
\end{equation*}
$$

In order to get started we have to choose a special frame first. Special frames at our disposal are rest frames of the target and projectile, and the center-ofmass frame of the initial virtual photon-nucleon system. If we take the lab frame (target rest frame) as our special Lorentz frame, emitting soft pions correspond to target fragmentation, or, expressed in another way, we say in the lab frame target fragmentation limits and soft-pion limits are kinematically same (to be more precise it is not really the whole target fragmentation region but the end region of it, i.e., the triple Regge limit). We can rephrase this as follows: A soft meson is soft only in the rest frame of the primary particle that emits it (we must be careful at this point, because it may be difficult to identify the rest frame of the primary particle, since, in order to radiate, it must experience an acceleration, and therefore its rest frame changes). The invariant kinematical variables in this case are

$$
\kappa=\frac{\mathrm{p} \cdot \mathrm{k}}{\mathrm{~m}}-\mathrm{k}_{0}=\left(\mu^{2}+\overrightarrow{\mathrm{k}}^{2}\right)^{1 / 2} \simeq \mu+\frac{|\overrightarrow{\mathrm{k}}|^{2}}{2 \mu}
$$

and

$$
\nu^{\prime}=\frac{\mathrm{k} \cdot \mathrm{q}}{\mu} \underset{\overline{\mathrm{Bj}}}{\simeq}
$$

Therefore

$$
\begin{equation*}
\frac{\nu^{\prime}}{\nu} \simeq 1 \tag{A3.2}
\end{equation*}
$$

For a consistency check let us calculate the rapidity variable in this case:

$$
\cosh \left(\mathrm{y}-\mathrm{y}_{2}\right)=\frac{\kappa}{\mu_{\perp}} \simeq \frac{\mu}{\mu_{\perp}}
$$

This tells us that in the soft-pion limit indeed the rapidity difference cannot be large. So the Regge limits which involve large values of rapidity difference $\mathrm{y}-\mathrm{y}_{2}$ are inapplicable in this case, and the only allowed Regge limit is the one in which $y_{1}-y_{2} \rightarrow \infty, y_{1}-\mathrm{y} \rightarrow \infty$, and $\mathrm{y}-\mathrm{y}_{2}$ finite. So for the soft-pion production in the lab frame we have the scaling law (4.7) with the special values (A3.2) of the scaling variables. The particular value of $t \sim 1 \mathrm{GeV}^{2}$ shows that one nucleon exchange approximation is a fairly good approximation for the soft-pion production in the lab frame. We have shown in Chapter V that the contribution of the target fragmentation products to the multiplicity is a constant. Since the soft pions in the target frame are the fragments of the target their multiplicity is a constant.

Now we shall choose the c.m. frame of the initial system as our special Lorentz frame, as is usually done [5]. In this case the process of emitting soft real or virtual pions is known as pionization. The invariant variables in this case are

$$
\begin{aligned}
& \nu^{\prime}=\frac{\mathrm{k} \cdot \mathrm{q}}{\mu} \simeq \frac{\mu_{1}}{2 \mu}\left(\omega^{+} \frac{1}{2}\right)^{-1 / 2}(\mathrm{~m} \nu)^{1 / 2} \\
& \kappa=\frac{\mathrm{p} \cdot \mathrm{k}}{\mathrm{~m}} \simeq \sqrt{\omega+\frac{1}{2}} \mu_{\perp}\left(\frac{\nu}{\mathrm{m}}\right)^{1 / 2} .
\end{aligned}
$$

For the soft-pion production in the c.m. frame we have the scaling law (4.13) with the following special value for the scaling variable $\tau$ :

$$
\begin{equation*}
\tau=\frac{(\mathrm{k} \cdot \mathrm{q})(\mathrm{p} \cdot \mathrm{k})}{(\mathrm{p} \cdot \mathrm{q})} \underset{\substack{\text { soft } \\ \text { pion }}}{\simeq} \mu_{\perp}^{2} / 2 \tag{A3.3}
\end{equation*}
$$

For a consistency check let us calculate the rapidity variables in this case

$$
\begin{align*}
& \cosh \left(\mathrm{y}-\mathrm{y}_{2}\right)=\frac{\kappa}{\mu_{\perp}} \sim \sqrt{\omega+\frac{1}{2}}\left(\frac{\nu}{\mathrm{~m}}\right)^{1 / 2} \rightarrow \infty \\
& \cosh \left(\mathrm{y}_{1}-\mathrm{y}\right)=\frac{\mathrm{k} \cdot \mathrm{q}}{\mu_{\perp}} \sim \frac{1}{2 \mu}\left(\omega^{+} \frac{1}{2}\right)^{-1 / 2}(\mathrm{~m} \nu)^{1 / 2} \rightarrow \infty \tag{A3.4}
\end{align*}
$$

Again recalling the last chapter's prediction that the contribution of the pionization products to the multiplicity is $\ln (1 / \omega)$, i.e., scales, the multiplicity of the soft pions, in the c.m. frame, is $\ln (1 / \omega)$. So we see that which ever special Lorentz frame we choose for our definition the multiplicity for the soft pions does not grow with the energy, it is constant and it scales in the c.m. frame. T. D. Lee [8] showed that there is a general relation between the multiplicity and the energy scale (he defines the scale $S_{c}$, so that when $S$ is bigger than $S_{c}$ the structure functions scale). His prediction is that if the average multiplicity $\overline{\mathrm{n}}$ increases with energy, say, like

$$
\overline{\mathrm{n}} \underset{\mathrm{~S} \rightarrow \infty}{\sim} \mathrm{~K} \ln \left(\frac{\mathrm{~S}}{\mathrm{~m}_{\mathrm{N}}^{2}}\right)
$$

where $K$ is a number, then the scale $S_{c}$ for a channel with a large multiplicity should increase exponentially with $n$, i.e.,

$$
S_{c} \sim M_{N}^{2} e^{n / K}
$$

In the case of finite average multiplicity he finds that the scale for any channel of multiplicity $n$, is approximately

$$
S_{c} \sim\left(M_{N}+\lambda M_{\pi}\right)^{2}
$$

where $\lambda$ is some large factor, say 10. Applying his predictions to our problem, we claim that for the soft-pion production in the deep inelastic processes the scaling sets in early, i.e., the over all scale is low, for the multiplicity is finite irrespective of the Lorentz frame we choose. Then the possibility exists that if one excludes all the hard mesons in the deep inelastic electroproduction processes, the remaining hadron multiplicity at infinite energy may stay finite and not too high. Notice that our prediction is contrary to that of Lee's. He claims that the growing multiplicity is due to soft mesons, and our prediction is that it is rather due to hard mesons, and it is the soft mesons which set the scaling so early.

Now integrate the equation (3.10) over $\nu^{\prime}$ around the value $\nu^{\prime} \sim \nu$ in an interval of length $\frac{|\overrightarrow{\mathrm{k}}|}{\mathrm{m}}$ in the lab frame. Since the structure functions $\mathscr{W}_{1}$ and $\nu \mathscr{V _ { 2 }}$ scale with the special values, $\kappa=\mathrm{E}_{\pi} \sim \mu, \mathrm{x} \simeq \mu / \mathrm{m}$ of the variables, we get

$$
\begin{align*}
& \frac{\mathrm{d} \sigma}{\mathrm{~d} Q^{2} \mathrm{~d} \nu \mathrm{dE}} \underset{\pi}{\underset{\text { soft pion }}{\mathrm{Bj}}} \underset{\mathrm{Q}^{4}}{\sim} \frac{4 \pi \alpha^{2}}{\mathrm{E}}\left(\frac{\mathrm{E}^{\prime}}{\mathrm{E}}\right)\left[\mathscr{V}_{2}^{\mathrm{t}}\left(\omega, \mathrm{x}_{\mathrm{t}}, \kappa\right) \cos ^{2} \frac{\theta}{2}\right. \\
&+2 \mathscr{V _ { 1 } ^ { \mathrm { t } } ( \omega , \mathrm { x } _ { \mathrm { t } } , \kappa ) \operatorname { s i n } ^ { 2 } \frac { \theta } { 2 } ] | \vec { \mathrm { k } } |} \tag{A3.5}
\end{align*}
$$

and in the c.m. frame, again integrating over $\nu^{\prime}$ along an interval of length $\frac{|\vec{k}|}{m}, \rightarrow$ we get

$$
\begin{equation*}
\left.\frac{\mathrm{d} \sigma}{\mathrm{~d} \mathrm{Q}^{2} \mathrm{~d} \nu \mathrm{dE}} \pi \pi=\frac{4 \pi \alpha^{2}}{\mathrm{Q}^{4}}\left(\frac{\mathrm{E}^{\prime}}{\mathrm{E}}\right)\left[\mathscr{V}_{2}^{\mathrm{c}}\left(\mathrm{x}_{\mathrm{c}}\right) \cos ^{2} \frac{\theta}{2}+2 \mathscr{H}_{1}^{\mathrm{c}}\left(\mathrm{x}_{\mathrm{c}}\right) \sin ^{2} \frac{\theta}{2}\right] \right\rvert\, \overrightarrow{\mathrm{k} \mid} \tag{A3.6}
\end{equation*}
$$

where $\mathrm{x}_{\mathrm{c}} \sim \mu_{\perp}^{2}$ in the soft pion limit, and $\theta$ is the electron scattering angle.

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## FOOTNOTE

$\dagger \dagger$ The last reference in [11] was an attempt to modify the erroneous treatment of the current fragmentation (and some kinematical errors) in the second reference in [11]. Unfortunately they failed firstly to notice the $\omega$-dependence of the phase space, which is so crucial, especially in proving that no new regimes other than those studied in this work, open up at high $Q^{2}$; in other words the phase space is completely filled by the secondaries. Secondly, their scaling variable (and also scaling distribution) for the current fragmentation region is not a scaling variable for high $Q^{2}$, as is obvious from our list (4.30). So their only two new results being incorrect, they fail to outdate the second reference in [11]. In this work in addition to correctly and thoroughly treating the kinematics, we manage to get a unified scaling variable (which is related to Feynman's purely hadronic scaling variable) for both fragmentation regions, which is supported by the data beautifully.

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## FIGURE CAPTIONS

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Fig. 1


Fig. 2


Fig. 3


Fig. 4


Fig. 5


Fig. 6

Fig. 7

$\overline{\widetilde{T R}} \cdot \sum_{n}$


Fig. 8


Fig. 9


Fig. 10


Fig. 11


Fig. 12


Fig. 13


Fig. 14


Fig. 15


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