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PERSISTENT SELF INTERACTIONS IN SINE GORDON THEORY*

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ABSTRACT

The sine Gordon field in 3+1 dimensions is studied as an example of a quantum field theory with persistent self interactions. Exact, positive or negative frequency (solitary wave) solutions of the sine Gordon equation containing either annihilation or creation operators of the linear field theory are discussed. The sine Gordon solitary wave propagator closely resembles the solitary wave propagator of the $\lambda\phi^4$ theory.

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I. INTRODUCTION

The study of nonperturbative solutions of nonlinear field equations and their quantization is receiving considerable attention.[1] These theories are interesting as model field theories and in connection with various containment models.[2,3] In addition, several interesting results have been obtained in the study of polynomial field theories with persistent self interactions.[4,5,6]

The sine Gordon equation is a nonlinear field equation which is well known classically, with applications in solid state and optical phenomena.[7,8] Recently, the quantization of sine Gordon fields in 1+1 dimensions has been studied.[9,10,11] In this paper we consider the sine Gordon field in 3+1 dimensions as an additional example of a nonlinear theory describing a system with persistent self interactions. As in polynomial theories previously discussed,[4] exact, particular solutions of the sine Gordon field equation exist which contain the coupling constant for all times rather than reducing to in or out fields for asymptotic times.[12] These solutions contain either creation or annihilation operators of the linear theory and reduce to particular solutions of the linear field equations for vanishing coupling constant. Thus, they describe a persistent self interaction. The solutions are similar to solitary waves of classical theories[8] and will be referred to throughout this paper as solitary waves.

In succeeding sections of this paper the solitary wave solutions of the sine Gordon theory are discussed and a solitary wave propagator is constructed. The construction of the solitary wave propagator relies on a correspondence between sine Gordon solitary waves and those of the $\lambda\phi^4$ theory. The solitary wave propagator of both theories has poles independent of the coupling constant at $(2n+1)m$, where m is the mass of the associated linear theory and n is an integer.

II. SOLITARY WAVE SOLUTIONS TO THE SINE GORDON EQUATION

The sine Gordon equation may be written

$$\partial_\mu \partial^\mu \phi + (m^2/\lambda) \sin \lambda\phi = 0 \tag{1}$$

where λ is a self interaction coupling constant, m is the mass of the associated linear theory and $\hbar=c=1$. Examining the series expansion of $\sin \lambda\phi$ in Eq. (1), it is evident that the sine Gordon equation is a Klein Gordon equation modified by the addition of an infinite sequence of self interaction terms with a single coupling constant λ.

Exact, particular solutions of Eq. (1) have been obtained by direct integration.[12] These solutions, which are functions of $\check{k}\cdot\check{x}$ $(=k_0 x^o - \vec{k}\cdot\vec{x})$, are similar to solitary wave solutions of the nonlinear field equations of some polynomial field theories.[4, 13] A pair of positive and negative frequency solitary wave solutions used in this paper are

$$\phi_{SG}^{(\pm)} = (2/\lambda)\, \sin^{-1}\left[\frac{A_{\vec{k}}^{(\pm)}\, e^{\mp i\check{k}\cdot\check{x}}}{D(\omega V)^{1/2}} \left(1 + \frac{A_{\vec{k}}^{(\pm)2}\, e^{\mp 2i\check{k}\cdot\check{x}}}{4D^2\,\omega V}\right)^{-1}\right] \tag{2}$$

In Eq. (2) $A_{\vec{k}}^{(\pm)}$ are the annihilation or creation operators of the linear theory, $k_0 = \omega = (\vec{k}^2+m^2)^{1/2}$, D is an arbitrary constant and V is the volume of the system.

Using simple trigonometric identities the solutions in Eq. (2) may be written (see Appendix for details)

$$\phi_{SG}^{(\pm)} = \frac{4}{\lambda} \tan^{-1}\left[\frac{A_{\vec{k}}^{(\pm)}\, e^{\mp i\check{k}\cdot\check{x}}}{2D(\omega V)^{1/2}}\right] \tag{3}$$

These solutions may also be obtained by direct integration of Eq. (1) (Appendix). Since the solutions are to reduce to solutions of the Klein Gordon equation for

$\lambda=0$, we take

$$D = 2/\lambda \tag{4}$$

so

$$\phi^{(\pm)}_{SG} = \frac{4}{\lambda}\tan^{-1}\left[\frac{\lambda}{4}\,\frac{A^{(\pm)}_{\vec{k}}\,e^{\mp i\check{k}\cdot\check{x}}}{(\omega V)^{1/2}}\right] \tag{5}$$

These solutions are not hermitian, whereas for a theory describing neutral systems hermiticity is usually required. However, in the linear theory the particular solutions corresponding to those in Eq. (5) have the property

$$\phi^{(+)}_{\vec{k}}(\check{x}) = \frac{A^{(+)}_{\vec{k}}\,e^{-i\check{k}\cdot\check{x}}}{(\omega V)^{1/2}} = \phi^{(-)\dagger}_{\vec{k}}(x) = \frac{A^{(-)\dagger}_{\vec{k}}\,e^{-i\check{k}\cdot\check{x}}}{(\omega V)^{1/2}} \tag{6}$$

provided

$$A^{(-)\dagger}_{\vec{k}} = A^{(+)}_{\vec{k}} \tag{7}$$

This same requirement insures that for the nonlinear theory

$$\phi^{(+)}_{SG\,\vec{k}}(\check{x}) = \phi^{(-)\dagger}_{SG\,\vec{k}}(\check{x}) \tag{8}$$

Eq. (8) is adopted in place of hermiticity.

The solutions in Eq. (5) may be written in a formal series

$$\phi^{(\pm)}_{SG\,\vec{k}}(x) = \frac{A^{(\pm)}_{\vec{k}}\,e^{\mp i\check{k}\cdot\check{x}}}{(\omega V)^{1/2}} \sum_{n=0}^{\infty} \frac{(-)^n}{(2n+1)}\left(\frac{\lambda}{4(\omega V)^{1/2}}\right)^{2n}\left(A^{(\pm)}_{\vec{k}}\,e^{\mp i\check{k}\cdot\check{x}}\right)^{2n} \tag{9}$$

In this form it is evident that the solutions $\phi^{(\pm)}_{SG\,\vec{k}}(\check{x})$ have matrix elements which differ from those of an in or out field for asymptotic times. That is, by construction,[14, 15] an in or out field has matrix elements only with one particle states for $|t| \to \infty$, whereas the expression in Eq. (9) clearly has matrix elements with many particle states. As shown elsewhere,[16] operators of this form describe a system with a discrete mass spectrum independent of the coupling constant (see below, also). Furthermore, solitary wave fields such as those considered here are not unitary transformations of in or out fields[16]—hence the theory discussed here is not a canonical quantum field theory.

III. SINE GORDON SOLITARY WAVE PROPAGATOR

In the study of polynomial field theories with persistent self interactions a method of constructing solitary wave propagators has been developed.[4, 16] We will not repeat the arguments here. Instead, we observe that the field operators given in Eq. (9) closely resemble those of the polynomial field theories. Consequently, we interpret $\phi^{(\pm)}_{SG\,\vec{k}}(\check{x})$ as operators which annihilate or create sine Gordon solitary waves at position $\check{x}$ with quantum numbers $\vec{k}$. Choosing the commutator

$$\left[A^{(+)}_{\vec{k}},\ A^{(-)}_{\vec{q}}\right] = \delta_{\vec{n}_k, \vec{n}_q} \tag{10}$$

with

$$\vec{k} = 2\pi\, V^{-1/3}\left(n_1\hat{e}_1 + n_2\hat{e}_2 + n_3\hat{e}_3\right) = 2\pi\, V^{-1/3}\,\vec{n}_k \tag{11}$$

the solitary wave propagator in momentum space becomes

$$p^{sol}_{SG}(\check{k}) = \sum_{n=0}^{\infty}\left(\frac{\lambda^2}{16V}\right)^{2n} \frac{(2n+1)!\left[\check{k}^2 - (2n+1)^2\, m^2 + i\epsilon\right]^{-1}}{(2n+1)^{4-2n}\left[\vec{k}^{\,2} + (2n+1)^2\, m^2\right]^n} \tag{12}$$

Although we have omitted the detailed argument leading to this propagator it is worthwhile to emphasize that, as in the linear theory, the mathematical principle of superposition of solutions of a differential equation is unnecessary and not used. Instead, the construction depends upon the superposition principle of quantum theory—a principle which is independent of the form of field equations. As in previous theories, the solitary wave propagator is an asymptotic series.

Due to the presence of the term $(2n+1)!$ the nth term diverges for fixed $|\check{k}|$ and increasing n, while for n constant any term vanishes for large $|\check{k}|$. While the individual terms in the series contain different coefficients from the $\lambda\phi^4$ solitary wave propagator, the poles of the solitary wave propagators in both sine Gordon and $\lambda\phi^4$ theories occur for

$$|\check{k}| = (2n+1)m \tag{13}$$

Thus, as mentioned above, the sine Gordon solitary wave fields describe a many particle system with a mass spectrum independent of the coupling constant. Furthermore, the infinite sequence of interaction terms is, for persistent interactions, essentially equivalent to the $\lambda\phi^4$ persistent interactions. Thus, for example, potentials based on the solitary wave propagator will differ only in minor detail in the two theories. Consequently, the effects of sine Gordon solitary wave exchange in, for example, nucleon-nucleon scattering, should be similar to those in the $\lambda\phi^4$ theory. These applications will be considered in more detail elsewhere.

APPENDIX

In this appendix we give some of the details relating the forms of solutions of the sine Gordon equation. We first consider the connection between Eq. (2) and Eq. (3). Writing Eq. (2) as

$$\sin(\lambda\,\phi/2) = \psi\left(1+\frac{\psi^2}{4}\right)^{-1} \tag{A.1}$$

where

$$\psi = \frac{A^{(\pm)}_{\vec{k}}\, e^{\mp i\check{k}\cdot\check{x}}}{D(\omega V)^{1/2}} \tag{A.2}$$

and using the trigonometric identity

$$\sin(\lambda\,\phi/2) = 2\sin(\lambda\phi/4)\cos(\lambda\,\phi/4) \tag{A.3}$$

one has

$$2\sin(\lambda\,\phi/2)\cos(\lambda\,\phi/2) = \frac{2(\psi/2)}{\left(1+\frac{\psi^2}{4}\right)^{1/2}}\cdot\frac{1}{\left(1+\frac{\psi^2}{4}\right)^{1/2}} \tag{A.4}$$

for which

$$\sin(\lambda\,\phi/4) = (\psi/2)\left(1+\frac{\psi^2}{4}\right)^{-1/2} \tag{A.5}$$

and

$$\cos(\lambda\,\phi/4) = \left(1+\frac{\psi^2}{4}\right)^{-1/2} \tag{A.6}$$

or

$$\cos(\lambda\,\phi/4) = (\psi/2)\left(1+\frac{\psi^2}{4}\right)^{-1/2} \tag{A.7}$$

$$\sin(\lambda\,\phi/4) = \left(1+\frac{\psi^2}{4}\right)^{-1/2} \tag{A.8}$$

Solving for ϕ, Eq. (A.5) and Eq. (A.6) give

$$\phi = (4/\lambda)\ \tan^{-1} \frac{A^{(\pm)}_{\vec{k}}\, e^{\mp i\check{k}\cdot\check{x}}}{2D(\omega V)^{1/2}} \tag{A.9}$$

while Eq. (A.7) and Eq. (A.8) give

$$\phi = (4/\lambda)\ \mathrm{ctn}^{-1} \frac{A^{+}_{\vec{k}}\, e^{\mp i\check{k}\cdot\check{x}}}{2D(\omega V)^{1/2}} \tag{A.10}$$

Equation (A.9) has been used in the text in Eq. (3). The solution given in (A.10) is also a suitable form for a solitary wave field but will not be discussed in this paper.

The sine Gordon equation may be integrated directly with the assumption that $\phi = \phi(\chi)$, where

$$\chi = \check{k}\cdot\check{x} \qquad (\check{k}^2 \neq 0) \tag{A.11}$$

Writing $\eta = \lambda\,\phi/2$ and using Eq. (A.3), the sine Gordon equation becomes

$$\frac{d^2\eta}{d\chi^2} + \frac{m^2}{\check{k}^2}\ \sin\eta\,\cos\eta = 0 \tag{A.12}$$

Multiplying by $d\eta/d\chi$ and integrating one has

$$\left(\frac{d\eta}{d\chi}\right)^2 + \frac{m^2}{\check{k}^2}\ \sin^2\eta = E \tag{A.13}$$

where E is an arbitrary constant. For E=0, separation of Eq. (A.3) leads to the integral

$$\sqrt{\frac{-m^2}{\check{k}^2}}\ (\chi+\chi') = \int d\eta\ \csc\eta = \ln\,(\tan\ \eta/2) \tag{A.14}$$

where χ' is a constant. Thus, recalling the definition of η one has

$$\phi = \frac{4}{\lambda}\tan^{-1}\left(c\ e^{\left(-m^2/k^2\right)^{1/2}\chi}\right) \tag{A.15}$$

Finally, taking $k^2 = m^2$, for suitable choice of c Eq. (A.9) or Eq. (2) is recovered.

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