# SHOCK WAVE SOLUTIONS IN NONLINEAR FIELD THEORIES* 

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#### Abstract

Classical, particular solutions of some massless, nonlinear, relativistic field equations are constructed. These solutions have shocklike singularities.


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[^0]The field equations of several nonlinear relativistic field theories have simple classical solutions, both time-dependent and static. ${ }^{1,2}$ The timedependent solutions, characterized by a constant four-vector $\mathrm{k}_{\mu}$, can become singular for some values of the parameters in the theory. These singularities propagate with constant phase velocity $\mathrm{k}_{0} /|\overrightarrow{\mathrm{k}}|$ 。 The fact that the singularity propagates is suggestive of shock waves, but the constant speed rules out this interpretation. In this paper time-dependent, singular particular solutions of two massless self-interacting field theories are given. The singularities of these solutions propagate with time-dependent velocities, therefore resembling shock waves。 ${ }^{3}$ There are no arbitrary four vectors in the solutions.

Consider first the spin zero field theory with field equation ( $\hbar=\mathrm{c}=1$ )

$$
\begin{equation*}
\partial_{\mu} \partial^{\mu} \phi+4 \lambda \phi^{3}=0 \tag{1}
\end{equation*}
$$

Restricting the functions $\phi$ to depend only on $x_{\mu} x^{\mu}(=y)$, Eq. (1) simplifies to

$$
\begin{equation*}
\frac{d^{2} \psi}{\mathrm{dy}^{2}}+\lambda(\psi / \mathrm{y})^{3}=0 \tag{2}
\end{equation*}
$$

where

$$
\begin{equation*}
\psi=y \phi \tag{3}
\end{equation*}
$$

A particular solution of Eq. (2) for timelike and spacelike separations, obtained by the method of base equations, ${ }^{4}$ is

$$
\begin{equation*}
\psi=a y\left(1+\lambda a^{2} y / 2\right)^{-1} \tag{4}
\end{equation*}
$$

where a is an arbitrary constant. ${ }^{5}$ This function is singular for

$$
\begin{equation*}
r=\left(t^{2}+2 / \lambda a^{2}\right)^{\frac{1}{2}} \tag{5}
\end{equation*}
$$

provided

$$
\begin{equation*}
\mathrm{t}^{2} \geq-2 / \lambda \mathrm{a}^{2} \tag{6}
\end{equation*}
$$

The speed of propagation of the singularity, obtained from Eq. (5), is

$$
\begin{align*}
\mathrm{v}_{\mathrm{S}} & =\frac{\mathrm{dr}}{\mathrm{dt}}=\mathrm{t}\left(\mathrm{t}^{2}+2 / \lambda \mathrm{a}^{2}\right)^{-\frac{1}{2}}  \tag{7}\\
& \approx 1-1 / \lambda a^{2} \mathrm{t}^{2}
\end{align*}
$$

where the approximation is valid for large $t$. The cases $\lambda>0$ and $\lambda<0$ differ. For $\lambda>0$, Eq. (6) is valid for all times, consequently the singularity is present for all times. From Eq。(7) it is clear that the speed of propagation approaches the speed of light from below. For $\lambda<0$, however, the singularity does not develop until after a finite time $t^{2}=\left|2 / \lambda a^{2}\right|$. Furthermore, the propagation speed is initially infinite and subsequently slows down to light speed for large times. ${ }^{6}$ For either sign of $\lambda$ the propagation speed approaches the speed of light as $\lambda$ increases or as the constant a, related to the initial value of $\phi$, increases.

The second field theory considered is the self-interacting vector theory with field equation ${ }^{7}$

$$
\begin{equation*}
\partial_{\mu} \partial^{\mu} \mathrm{A}_{\nu}=\mathrm{g}^{2} \mathrm{~A}_{\mu} \mathrm{A}^{\mu} \mathrm{A}_{\nu} \tag{8}
\end{equation*}
$$

Letting $A_{\nu}=x_{\nu} \chi(y)$, one finds

$$
\begin{equation*}
\frac{\mathrm{d}^{2} \chi}{\mathrm{dy}^{2}}+\frac{3}{\mathrm{y}} \frac{\mathrm{~d} \chi}{\mathrm{dy}}=\mathrm{g}^{2} \chi^{3} \tag{9}
\end{equation*}
$$

for which a particular solution, using ref. 4, is ${ }^{8}$

$$
\begin{equation*}
x=a\left(1-g^{2} a^{2} y^{2} / 8\right)^{-1} \tag{10}
\end{equation*}
$$

with a an arbitrary constant. The singularity occurs for

$$
\begin{equation*}
r=\left[t^{2} \pm 2 \sqrt{2} /|\mathrm{ga}|\right]^{\frac{1}{2}} \tag{11}
\end{equation*}
$$

provided $0<8 / \mathrm{g}^{2} \mathrm{a}^{2}$, and propagates with speed

$$
\begin{equation*}
\mathrm{v}_{\mathrm{s}}=\mathrm{t}\left[\mathrm{t}^{2} \pm 2 \sqrt{2} /|\operatorname{ga}|\right]^{-\frac{1}{2}} \tag{12}
\end{equation*}
$$

As for the spin zero case, $\mathrm{v}_{\mathrm{s}}$ can be either greater or less than c . Again, the superluminal waves do not appear until after a finite time. The propagation speed approaches unity for strong coupling. For $\mathrm{g}^{2}<0$ the solution in Eq. (10) becomes singularity free.

Both solutions given here contain only one parameter and, consequently, are applicable only to a limited class of initial value problems. However, with the known particular solutions it is possible to develop a perturbation theory which gives a two-parameter solution valid approximately when one of the parameters is small. As an illustration consider the spinless case. A "natural" assumption is to write $\psi_{1} \approx \psi_{0}+\epsilon \mathrm{g}$, where $\psi_{0}$ is the solution given in Eq. (2) and $\epsilon$ is a small quantity. The resulting linear differential equation for $g$ has regular singular points at $y=0,-2 / \lambda_{a}{ }^{2}$ 。However, the series solution leads to a three-term recurrence relation. Consequently, we assume instead that

$$
\begin{equation*}
\psi=(2 / \lambda a) z(1+z+\epsilon g)^{-1} \approx(2 / \lambda a)^{-1}\left\{z /(1+z)-\epsilon g z /(1+z)^{2}\right\} \tag{13}
\end{equation*}
$$

where

$$
\begin{equation*}
z=\lambda a^{2} y / 2 \tag{14}
\end{equation*}
$$

The differential equation satisfied by $g$ is

$$
\begin{equation*}
\mathrm{z}(1+\mathrm{z}) \mathrm{d}^{2} \mathrm{~g} / \mathrm{dz}{ }^{2}+2(1-\mathrm{z}) \mathrm{dg} / \mathrm{dz}+2 \mathrm{~g}=0 \tag{15}
\end{equation*}
$$

This differential equation also has regular singular points at $z=0,-1$, so the analytic behavior of the correction function is the same as for the "natural" perturbation approach. The general solution of Eq. (15) is

$$
\begin{equation*}
g=d(1-z)+b\left[(1-z) \ln z-(6 z)^{-1}+3 z-z^{2} / 6\right], \tag{16}
\end{equation*}
$$

where d and b are arbitrary constants. Consequently, the corrected $\psi$ is $\psi_{1}=(2 / \lambda a) z\left\{1+\epsilon d+z(1-\epsilon d)+\epsilon b\left[(1-z) \ln z-(6 z)^{-1}+3 z-z^{2} / 6\right]\right\}^{-1}$

It now appears that there are too many arbitrary constants in the solution. However, if $\psi_{1}$ is "renormalized" so that $\psi_{1}(\mathrm{~b}=0)=\psi_{0}$, it is easy to see that this amounts to redefining the constant a. Since a is arbitrary, this does not change the solution. Thus, the corrected solution is

$$
\begin{equation*}
\psi_{1}=(2 / \lambda a) z\left\{1+z+\epsilon b\left[(1-z) \ln z-(6 z)^{-1}+3 z-z^{2} / 6\right]\right\}^{-1} \tag{18}
\end{equation*}
$$

For $\mathrm{z}<0$ the logarithm may be written $\frac{1}{2} \ln \mathrm{z}^{2}$ without changing the properties of $\psi_{1}$. In the vicinity of $z=-1$, ignoring the logarithm one finds the singularity at

$$
\begin{equation*}
z \approx-1+17 \epsilon b / 6 \tag{19}
\end{equation*}
$$

The terms in $\ell \mathrm{ln} \mathrm{z}$ and $\mathrm{z}^{-1}$ would appear to cause $\phi_{1}\left(=\psi_{1} / \mathrm{z}\right)$ to vanish at $\mathrm{z}=0$. However, this result is questionable since $\epsilon \mathrm{g}$ is no longer small compared with $1+z$ in the vicinity of $z=0$ 。 Thus, while the behavior of the corrected function can be estimated reliably in the vicinity of the original singularity, its behavior near the light cone must be determined in another manner.

The effect of a mass term in the field equation can also be calculated for small $\mathrm{m}^{2}$ using the perturbation theory described above. The details of this and applications of the solutions will be published elsewhere. ${ }^{9}$

REFERENCES

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5. There is also a solution containing no arbitrary parameters, $\psi= \pm(y / 4 \lambda)^{\frac{1}{2}}$. See G. Petiau, Suppl. Nuovo Cimento 9, 542 (1958). This is the spherically symmetric analogue of the Goldstone solution of the massive field theory.
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7. Gauge conditions are ignored. Consequently the field describes a mixture of spin one and spin zero systems.
8. As for the spin zero case, there is a solution with no free parameters, $\chi= \pm i / g y$. If the field equations are chosen to be $\partial_{\mu} \partial^{\mu} A_{\nu}-\partial_{\mu} \partial_{\nu} A^{\mu}+$ $+\mathrm{m}^{2} \mathrm{~A}_{\nu}=\mathrm{g}^{2} \mathrm{~A}_{\mu} \mathrm{A}^{\mu} \mathrm{A}_{\nu}$, with $\mathrm{A}_{\nu}=\mathrm{x}_{\nu} \chi(\mathrm{y})$, the only solution is $\chi= \pm\left[\mathrm{m}^{2} /\left(\mathrm{g}^{2} \mathrm{y}\right)\right]^{\frac{1}{2}}$ 。
9. P. B. Burt and G. Pocsik, to be published.

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