SLAC-PUB-1710
February 1976
(T)

EFIMOV EFFECT AND HIGHER BOUND STATES IN A THREE PARTICLE SYSTEM*

Antonio Carlos Antunes
Universidade Federal do Rio de Janeiro, Brazil

Vera Lucia Baltar†
Physics Department
University of California, Santa Barbara, California 93106

Erasmo M. Ferreira††
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

and

Pontifícia Universidade Católica do Rio de Janeiro, Brazil

ABSTRACT

We study the J=0 bound states for a system of three identical spinless particles interacting in pairs through delta-shell potentials. The Efimov states are identified, and their wavefunctions obtained. We have found a new family of bound states, which occur for higher values of the attractive coupling strength.

*Supported in part by the U. S. Energy Research and Development Administration.
†Supported by National Research Council, Brazil.
††Fellow of the John Simon Guggenheim Memorial Foundation.

(Submitted to Nucl. Phys. B.)

INTRODUCTION

Efimov [1] and Amado and Noble [2] have shown that in a system of three particles the number of bound states increases indefinitely as the two body binding strength approaches the value which is just necessary to keep two particles bound with zero binding energy. The condensation of energy levels can also be described as occurring as the binding energy approaches zero for a fixed two-body potential strength, which must have the right value just needed to bind two particles. The theorem was proved in the framework of nonrelativistic quantum mechanics, and is a consequence of the fact that the trace of the kernel of the integral equation which determines the bound states diverges under the conditions mentioned above.

The infinite number of three body bound states occurring for that specific value of the attractive coupling disappears rapidly as the coupling becomes more attractive. This remarkable result, which at first thought violates our physical intuition, can be understood if we note that the system does not become totally unbounded, but rather is brought to a continuum of states formed by two particles more strongly bound to each other, and one free particle. The increase in the attractive strength of the interaction in this region causes a faster increase in the two body than in the three body binding energy, and as a result the stable configuration of the system belongs to the continuum of one bound pair and a third particle.

Thus in a very restricted region of values of the pair interaction strength there is a whole family with an infinite number of three body bound states. The effect exists only for states with zero total orbital angular momentum, L=0. The quantum number of all states thus formed are the same, namely $J^P=0^+$ for spin zero particles and $J^P=1/2^+$ for fermions.

Let G be the two particle coupling strength for some finite range potential form, and let $G_F(s)$ be the value of G required to form a two body bound state with binding energy $-s$. For simplicity, suppose that the two particle system has only one bound state, so that there is just one value of $G_F(s)$ for each value of s. The evolution of the three body bound states as the coupling strength varies is shown in Fig. 1, which is completely out of scale.

Let $\boldsymbol{a}$ be a parameter indicating the typical range of the two body forces and let b be the S-wave scattering length measured in the same units. Call α the dimensionless quantity

$$\alpha = b/\boldsymbol{a} \tag{1}$$

Let α_N be the value of α corresponding to the value G_N of the coupling strength for which the N-th bound state is formed, with zero binding energy. Then, for values of G_N close to $G_0 \equiv G_F(0)$, two consecutive bound states are such that

$$\alpha_{N+1} = \alpha_N \exp(\pi) \tag{2}$$

On the other hand, if the coupling strength is kept at the value $G_0 \equiv G_F(0)$, the binding energies of consecutive bound states are related by

$$(-s)_{N+1} = (-s)_N \exp(-2\pi) \tag{3}$$

Relations (2) and (3) were derived by Efimov [1] and in a different way by Amado and Noble [2].

The Efimov states were observed in a practical computation by Stelbovics and Dodd [3] of the bound states of three spinless particles interacting in pairs through separable Yamaguchi interactions. In the present work we perform a more complete study of the same kind, describing the binding energies and the wavefunctions for the bound states occurring in a three body system.

To define a problem, we study a system of three identical spinless particles interacting in pairs through spherically symmetric delta shell potentials. The Faddeev equations are separated in angular momentum states, and we study the completely symmetric bound state with zero total angular momentum. The delta shell potential gives separable scattering amplitude in momentum space, and we are thus lead to a single integral equation in only one variable.

BINDING ENERGIES AND WAVEFUNCTIONS

We choose a spherically symmetric potential of the simple form

$$V(r) = \frac{\hbar^2}{m} \lambda \, \delta(r - a) \tag{4}$$

to describe the interaction of each pair of particles in the system. We consider the case of three identical spinless particles of mass m. The off-shell scattering amplitude in the angular momentum ℓ state can be factorized in the form

$$\tau_\ell(p, p'; E) = j_\ell(a p/\hbar) \, j_\ell(a p'/\hbar) \, t_\ell(E) \tag{5}$$

where

$$t_\ell(E) = (\lambda a^2/2\pi^2 \hbar m) / \left[1 + \lambda a \, I_{\ell+\frac{1}{2}}\left((-s)^{1/2}\right) K_{\ell+\frac{1}{2}}\left((-s)^{1/2}\right) \right] \tag{6}$$

with

$$s = m a^2 E/\hbar^2 \tag{7}$$

The symbols j_ℓ, $I_{\ell+\frac{1}{2}}$ and $K_{\ell+\frac{1}{2}}$ stand for the usual spherical Bessel and modified Bessel and Hankel functions respectively.

Let us define the dimensionless quantities

$$G = \lambda a \tag{8}$$

$$P = (a/\hbar) p \, , \qquad Q = (a/\hbar) q \tag{9}$$

where P is the relative momentum of two particles in a pair, and q is the momentum of the third particle with respect to the center-of-mass of the whole system. Then the integral equation for the J=0 bound states of the three-body system is

$$\psi_\ell(Q) = F_\ell(Q) \sum_{\substack{\ell'=0 \\ \text{even}}}^{\infty} \int_0^\infty M_{\ell\ell'}(Q,Q')\ \psi_{\ell'}(Q')\ dQ' \tag{10}$$

where

$$F_\ell(Q) = -(-1)^\ell (2\ell+1)(4G/\pi\sqrt{3}) / \left[1 + GI_{\ell+\frac{1}{2}}\left((-s)^{1/2}\right) K_{\ell+\frac{1}{2}}\left((-s)^{1/2}\right)\right] \tag{11}$$

and the index ℓ indicates the relative angular momentum for a pair of particles in the system. The functions $M_{\ell\ell'}$ in the integrand are given by

$$M_{\ell\ell'}(Q,Q') = \int_{(Q'-2Q)^2/3}^{(Q'+2Q)^2/3} A_{\ell\ell'}(Q,Q';P'^2)\ dP'^2$$

with

$$A_{\ell\ell'}(Q,Q';P'^2) = (-1)^{\ell'} \left[P'^2 + Q'^2 - s\right]^{-1} j_\ell(u)\ j_\ell(P')\ P_{\ell'}(\beta) \times$$

$$\times \left\{P_\ell(\gamma)\ P_\ell(\eta) + 2 \sum_{m=1}^{\ell} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\gamma)\ P_\ell^m(\eta)\right\} \tag{13}$$

and where

$$\beta = v/(2\sqrt{3}\ P'Q') \tag{14}$$

$$\gamma = (v-2P'^2)/(4uP') \tag{15}$$

$$\eta = \left[3P'^2 - Q'^2 + 4Q^2\right]/(4\sqrt{3}\ QP') \tag{16}$$

with

$$u = (P'^2 + Q'^2 - Q^2)^{1/2} \tag{17}$$

and

$$v = (Q'^2 + 3P'^2 - 4Q^2) \tag{18}$$

The symbols P_ℓ and P_ℓ^m stand for the Legendre polynomials and associated Legendre polynomials as usual.

According to Eq. (10), the J=0 wavefunction has contributions from states of all even values of the relative orbital angular momentum of a pair of particles. In our computations we have kept terms with ℓ=0 and ℓ=2 and have noticed that the ℓ=2 parts of the wavefunction are much smaller than the ℓ=0 components, and give very small correction to the calculated energy levels. For example, the ground state with energy $-s = 0.01$ occurs for $-G = 0.8234$ if the calculation is performed with the ℓ=0 contribution alone, and for $-G = 0.8226$ if the calculation keeps both ℓ=0 and ℓ=2 contributions. These results justify neglecting the terms with angular momenta equal to or higher than four. All the essential features of the Efimov states are determined by the ℓ=0 parts of the wavefunctions.

From Eq. (11) we obtain for ℓ=0,

$$F_0(Q) = -(4G/\pi\sqrt{3})\left[1 + G\,\frac{1-\exp\left[-2(Q^2-s)^{1/2}\right]}{2(Q^2-s)^{1/2}}\right]^{-1} \tag{19}$$

On the other hand, the coupling strength $G_F(s)$ which produces a two body bound state of binding energy $-E = -\hbar^2 s/m\,a^2$ is determined by

$$1 + G_F(s)\left[1-\exp(-2\sqrt{-s})\right]/2\sqrt{-s} = 0 \tag{20}$$

This is the equation for the frontier line, shown in Fig. 1, of the region where the three body bound states can exist. In this region $F_0(Q)$ is a regular function of Q for every real value of Q. At points of the boundary line, $F_0(Q)$ as a function of Q is singular at Q=0, behaving like $(8/\pi\sqrt{3})(\sqrt{-s}/Q^2)$. For values of G and s in the region of the three body bound states, the analytic structure of $F_0(Q)$ in the complex Q plane presents cuts along the positive and negative imaginary axis from $\pm i\sqrt{-s}$ to infinity, and two poles on the imaginary axis, symmetrically located with respect to the origin. As we approach the frontier line defined by $G = G_F(s)$, the two imaginary poles approach the origin of the Q plane, giving rise to the singularity in $1/Q^2$ mentioned above. On the other hand, as $s \to 0$ the two branch points meet at the origin of the Q-plane. To understand better what happens with $F_0(Q)$ at zero binding energy we can fix s=0 and leave G free, with $|G| < |G_0|$. (In Fig. 1 we are then at the boundary line between regions I and III.) As $Q \to 0$ we then have $F_0(Q) \to (-4/\pi\sqrt{3})G/(1+G)$. We here recognize the expression for the scattering length for the two-body potential, Eq. (4), which is $a G/(1+G)$ and goes to infinity as $G \to -1$.

As for the low momentum behavior of $M_{00}(Q,Q')$ we have $\lim_{Q\to 0} M_{00}(Q,Q')=0$ for arbitrary s and Q', except for s=0 and Q'=Q when we obtain $\lim_{Q\to 0} M_{00}(Q,Q')=$ ln 3.

A quantity useful to describe the location of Efimov bound states in the s-G plane is

$$\alpha(s) = -G/(|G_F(s)| + G) \tag{21}$$

For each given value of s this quantity $\alpha(s)$ tends to infinity as the value of G approaches the frontier line $G_F(s)$. For zero binding energy, $\alpha(0)$ coincides with the absolute value of the scattering length divided by the interaction radius a.

The binding energies of the bound states are obtained searching for the eigenvalues of Eq. (10) with the usual quadrature method, keeping only the ℓ=0 and ℓ=2 contributions to the kernel and to the wavefunctions. The results of our numerical computations of the binding energies are shown in Figs. 2 and 3. Figure 2 exhibits clearly the Efimov effect, showing the regularity, established by Eq. (2), in the values of the scattering length $\alpha(0)$ for the bound states in the limit of zero binding energy. The two lowest bound states are permanent, in the sense that they do not merge into the continuum as the coupling strength increases, and it is interesting to remark that the coupling strengths which form these states at zero binding energies also follow Eq. (2). The same Fig. 2 shows that the ratio of the binding energies with which bound states disappear in the continuum as the attractive interaction strength increases, obey Eq. (3). Only two of these Efimov states were followed quantitatively, due to the limitations in the accuracy of the numerical computations.

As we increase the strength of the attractive interaction beyond $G=G_0$, we are at first left with only two bound states. Our computations have shown for stronger couplings the existence of a set of three body bound states which begin their existence with binding energies different from zero, as they are not formed directly from three free particles. They arise from the two body continuum, through the capture of a third particle by the system of two bound particles. The first eight of these states are shown in Fig. 3, while Fig. 1, which is out of scale, shows only schematically their distribution in the s-G plane.

In our computations we have identified these eight higher states, but there may be many more in number, and even the set may become infinite as $|G|$ increases. Here we must remark that the two body problem for the same delta shell potential presents only one s-wave bound state, however strong the value of the attractive coupling may become. As $G=\lambda a$ is the product of the interaction strength by the shell radius, we see that the number of three body bound states increases with either the strength or the radius of the interaction.

To describe the wavefunctions, let us define

$$\chi_0(Q) = \psi_0(Q)/\left[QF_0(Q)\right] \tag{22}$$

and neglect the contributions of the ℓ=2, 4 ... components to the integral equation, Eq. (10). Then $\chi_0(Q)$ satisfies

$$\chi_0(Q) = \frac{1}{Q} \int_0^\infty M_{00}(Q,Q')\, Q' F_0(Q')\, \chi_0(Q')\, dQ' \tag{23}$$

Let us study the behavior of $\chi_0(Q)$ for small values of Q. Differentiating Eq. (23) twice with respect to Q we obtain

$$\frac{d^2\chi_0}{dQ^2} + \frac{2}{Q}\frac{d\chi_0}{dQ} = \frac{1}{Q}\int_0^\infty \frac{d^2 M_{00}(Q,Q')}{dQ^2}\, Q' F_0(Q')\, \chi_0(Q')\, dQ' \tag{24}$$

Expanding $M_{00}(Q,Q')$ in powers of Q, and noting that

$$M_{00}(0,Q') = 0$$

and

$$\frac{d}{dQ} M_{00}(Q,Q')\bigg|_{Q=0} \neq 0$$

we obtain for small values of Q

$$M_{00}(Q,Q') = Q\frac{d}{dQ} M_{00}(Q,Q') - \frac{1}{2} Q^2 \frac{d^2}{dQ^2} M_{00}(Q,Q') + O(Q^3) \tag{25}$$

Taking $d^2M_{00}(Q,Q')/dQ^2$ from Eq. (25) into Eq. (24) we obtain the simple result that

$$\chi_0''(Q) \xrightarrow[Q\to 0]{} \text{constant}$$

and therefore $\chi_0(Q)$ is well behaved at Q=0.

In Figs. 4 and 5 we plot $\chi_0(Q)$ for the four lowest bound states (the same shown in Fig. 2) in the limit of very low binding energy. The eigenfunctions are normalized so that $\chi_0(Q)$=1. Due to the very strong variations in the magnitudes and to the changes of sign of the wavefunctions as Q varies, peculiar logarithmic scales have been used in Figs. 4 and 5. The number of zeros of the wavefunctions increase by one from a bound state to the next. We may observe from the figures the peculiar fact that the zeros of all wavefunctions seem to occur at the same values of Q. Thus, all four wavefunctions have a zero at $Q \approx 3$. Then the functions labelled n=2, 3, and 4 have a zero for $Q \approx 0.2$, while both n=3 and n=4 wavefunctions change sign for $Q \approx 0.01$.

CONCLUDING REMARKS

In the present work we have identified the Efimov states for a three body problem defined in configuration space, and have verified the validity of the quantitative predictions regarding these states. To us this seems interesting in itself, as we have no acquaintance with a similar calculation. Of course the delta shell potential here used is singular, and thus may not be close to physical reality, but it allows a simple formulation of the three body bound state problem, and still is able to exhibit some geometric and physical features which may help the understanding of the Efimov effect and of other properties of the system.

We have shown that the energy spectrum consists of levels of three different kinds, according to their behavior as a function of the coupling strength. First we have the two lowest bound states, which are created at values of the coupling strengths which are not high enough to maintain only two particles bound, and they remain always as three body bound states as the attractive coupling is further increased.

The levels of second kind are the Efimov states. They exist only in a narrow range of values of the coupling strength, around the value where the two particle scattering length becomes infinite.

We then have a new family of states, whose existence has not been observed before. We can describe them as formed from a bound pair and a free particle, through the capture of the single particle by the previously bound pair as the coupling strength is increased. In our example this set of states does not coexist with states of the Efimov kind, as they start to be formed for values of the coupling constant higher than the one for which the last Efimov state merges into the continuum. This behavior is sketched (out of scale) in Fig. 1.

We have computed and drawn the momentum space wavefunctions at almost zero binding energies for the first four states which appear in the system as the attractive coupling is increased from zero. These states are the two permanent more fundamental states and the first two Efimov states. They are all formed at zero binding energies, by the sudden coalition of three particles. We have shown (Figs. 4, 5) that the momentum space structure of the wavefunctions for these four states tends to be concentrated at small values of the momentum. The highest energy levels present larger number of oscillations at smaller values of the momentum, which shows that these levels have a large structure in configuration space. This is in agreement with what is expected for Efimov bound states, as their existence is known to be the result of an effective long range two body interaction near the infinite value of the scattering length.

The third family of states arises as a remarkable consequence of the three body dynamics, as they do not have counterparts in the two body system, where only one bound state exists, whatever strong may be the attractive interaction. In contrast to the Efimov states they are not formed as a consequence of a resonant character of the two body force. The structure of the wavefunctions for these states deserves a special study, as they may be essentially different from the states of the first two kinds.

The very existence of this new family of states seems to us to be important for the description of nonrelativistic quantum mechanical systems of three and more particles, such as the systems of interest in nuclear physics.

These three particles states arising from the two body continuum with nonzero binding energies, have been identified in the particular example here treated, of the delta shell potential, but it is likely that they occur also for less singular potentials.

Finally, we wish to point out that our model allows a complete exact (although in part essentially numerical) treatment of a nonrelativistic three body system. The wavefunctions are obtained in both regions interior and exterior to the interaction radius. In this sense it is an essentially different model as compared to the usual models of hard core potentials and to the boundary condition model [4], as it allows the construction of the wavefunction for the whole range of values of the distance between the particles.

Acknowledgements

We are grateful to Professor H. P. Noyes for helpful discussions.

REFERENCES

[1] V. N. Efimov, Phys. Letters 33B (1970) 563;
V. N. Efimov, Sov. J. Nucl. Phys. 12 (1971) 589.

[2] R. N. Amado and J. V. Nobel, Phys. Letters 35B (1971) 25;
R. N. Amado and J. V. Nobel, Phys. Rev. D 5 (1972) 1992.

[3] A. T. Stelbovics and L. R. Dodd, Phys. Letters 39B (1972) 450.

[4] D. D. Brayshaw, Stanford Linear Accelerator Center report SLAC-PUB-1678 (1975)(to be appear in Phys. Rev. C).

FIGURE CAPTIONS

1. States of three particles, shown in the plane of the binding energy -s against the strength G of the two particle interaction. The plot is completely out of scale. The cross hatched border lines separate the regions of the three body bound states (I) and of the continuum. The frontier line $G_F(s)$ is defined by the coupling strength which forms a two body bound state with binding energy -s, and separates the region of the plane where the three particle bound states are found from the region (II) where the states of two bound and one free particle are found. Below the horizontal axis is the continuum (III) occupied by states of two bound and one free particle and by the states of three totally free particles. The lowest three body bound states are shown, out of scale. The first two of these states remain bound as the attractive strength is increased. An infinite number of states is formed as $-G$ approaches $-G_0 \equiv -G_F(0)$, according to Efimov theorem, and then disappears into the continuum, as they reach the frontier line. Our work has shown that as the coupling strength is further increased, other three body bound states are created, starting from the frontier line as shown in the figure. We cannot tell whether these states form a finite or an infinite set.

2. Dependence of the binding energies of the four lowest three body bound states on the two particle potential strength. The vertical axis gives in logarithmic scale a quantity s related to the binding energy by Eq. (7). The quantity α in the horizontal axis is defined by Eq. (21). At the threshold, $s \to 0$, the variable α coincides with the scattering length for the two particle system divided by the interaction radius. The horizontal distances between neighboring bound states tend to the same value $\pi \log e = 1.36$ as $s \to 0$, confirming

Efimov's theorem. The third and fourth states tend to the right in the above diagram, approaching the region of the continuum consisting of a bound pair and a free particle. The vertical distances between two neighboring states which move towards the continuum approach the same value 1.36, also in agreement with Efimov's results. Other bound states, which according to Efimov are infinite in number, would occupy the lower right part of the figure. They have not been quantitatively determined, due to limitations of accuracy in the numerical computations.

3. Dependence of the binding energies of the three particle bound states on the pair coupling strength. The horizontal axis uses in logarithmic scale the quantity α defined by Eq. (21). Only the two lowest bound states (curves labelled 1 and 2 above) remain always as three particle bound states as the coupling constant is increased. All other states which are formed for $|G| < |G_0|$ are broken up into states consisting of two bound plus one free particle, as the attractive strength is increased. This effect, which is shown in Fig. 1, is not seen above for scale reasons. As the coupling strength is further increased, new three particle bound states appear. The first eight of such states (curves labelled 3 to 10 above) which start with binding energies different from zero, are shown above, represented by the lines which come from $\alpha=\infty$. These states are the same shown qualitatively in Fig. 1 as curves starting at the border line between regions I and II.

4,5. Behavior of the function $\chi(Q)$ defined as an eigenfunction of Eq. (23), for the four lowest bound states. Q is a kinematical variable related by Eq. (9) to the magnitude of the momentum in the center-of-mass system. Due to the strong variations in the magnitudes and to the changes of sign of the wavefunctions as Q varies, and to the very different variations of amplitudes of the functions corresponding to the four bound states, we were forced to use rather peculiar scales for their plots. In Fig. 4 we use log $(\chi + 0.01)$ in the vertical axis and log $(Q + 0.01)$ in the abcissas. In Fig. 5 the ordinates follow log $(|\chi| + 0.001)$ and in the horizontal axis we use log $(Q + 0.001)$. The functions corresponding to the four states have a number of zeros which increases by one from the bound state to the next. The curves show a very peculiar behavior, namely, that all curves tend to pass through zero for the same values of Q. Note that this is true also if we join the information contained in Figs. 4 and 5.

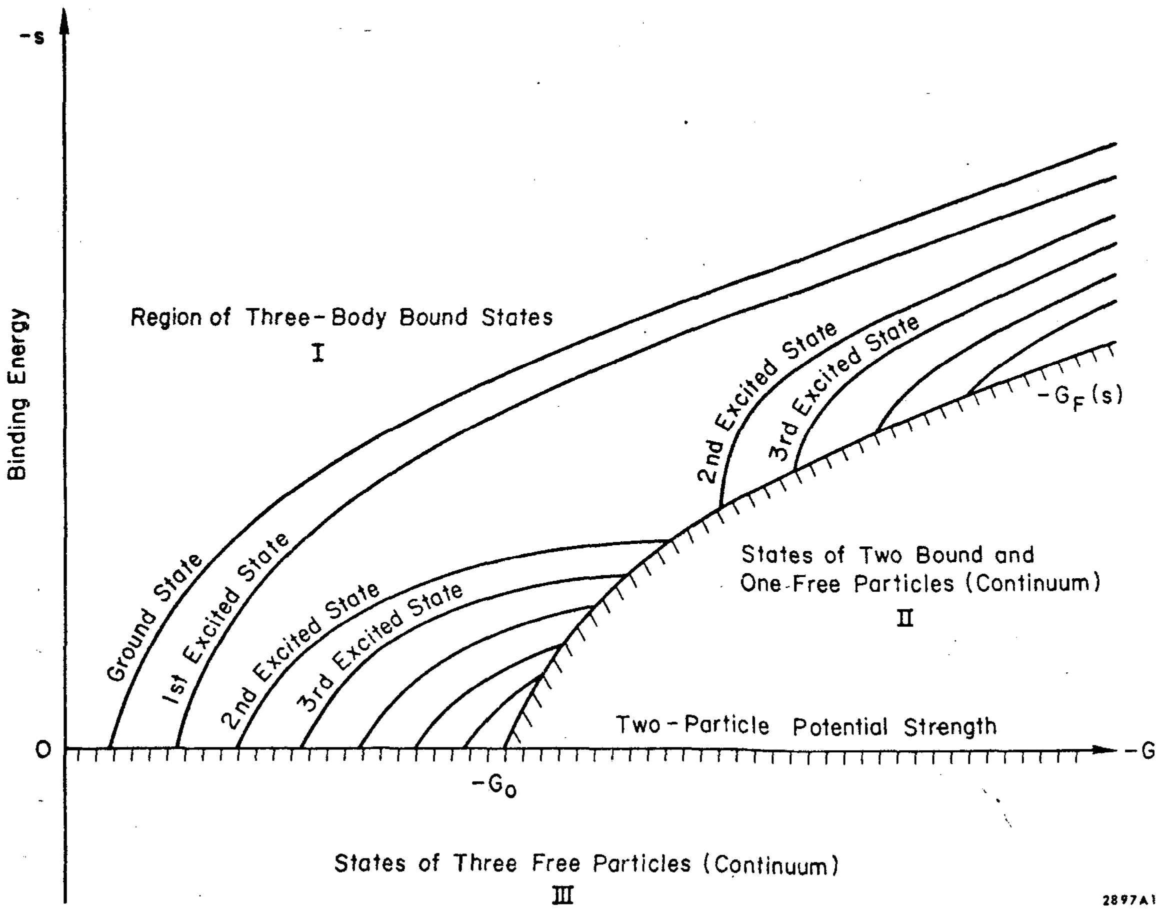
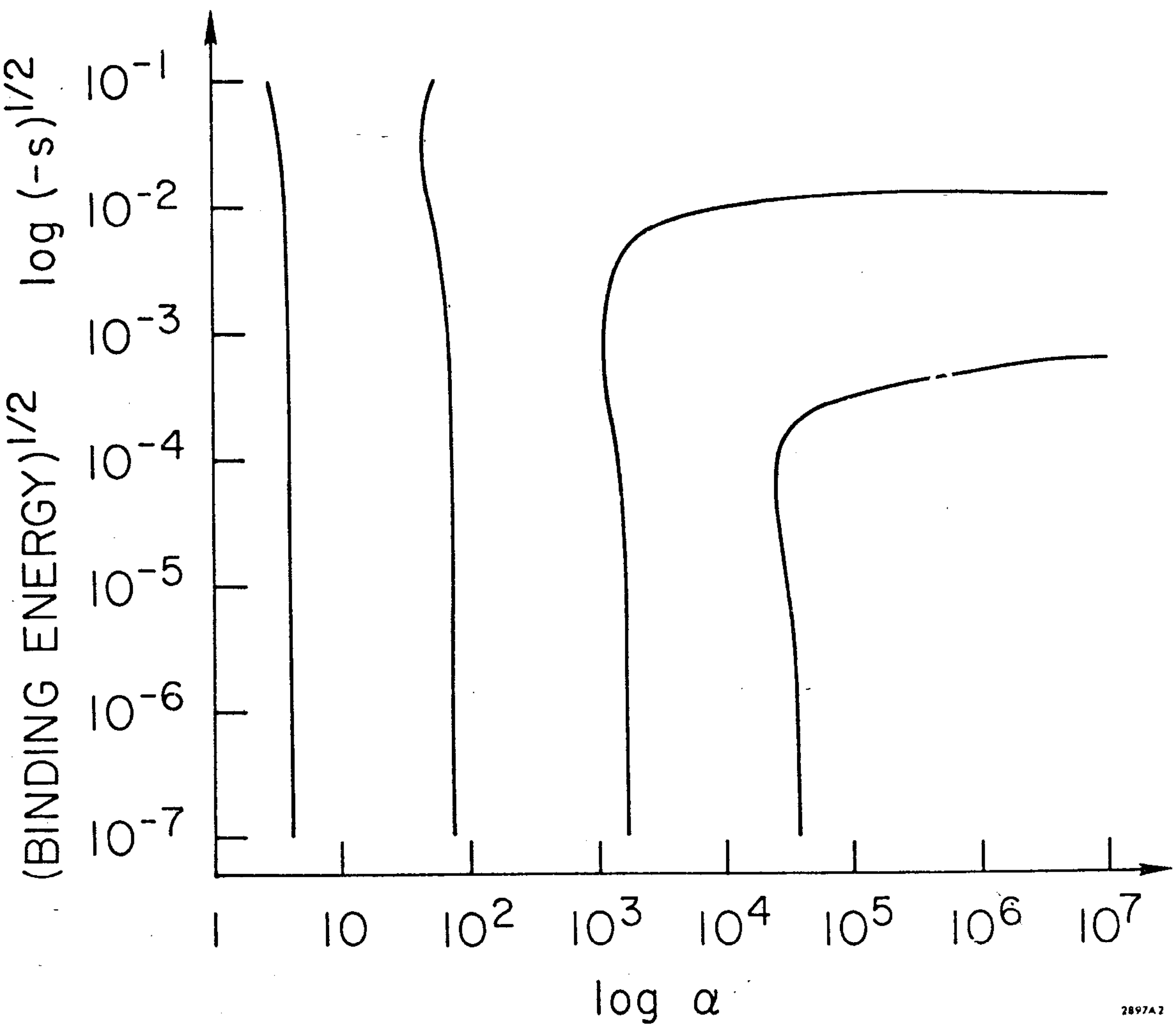

-s
Binding Energy
Region of Three-Body Bound States
I
Ground State
1st Excited State
2nd Excited State
3rd Excited State
2nd Excited State
3rd Excited State
$-G_F(s)$
States of Two Bound and One-Free Particles (Continuum)
II
Two-Particle Potential Strength
0
-G
$-G_0$
States of Three Free Particles (Continuum)
III
2897A1

Fig. 1



(BINDING ENERGY)^1/2
log (-s)^1/2
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
1
10
10^2
10^3
10^4
10^5
10^6
10^7
log α
2897A2

Fig. 2

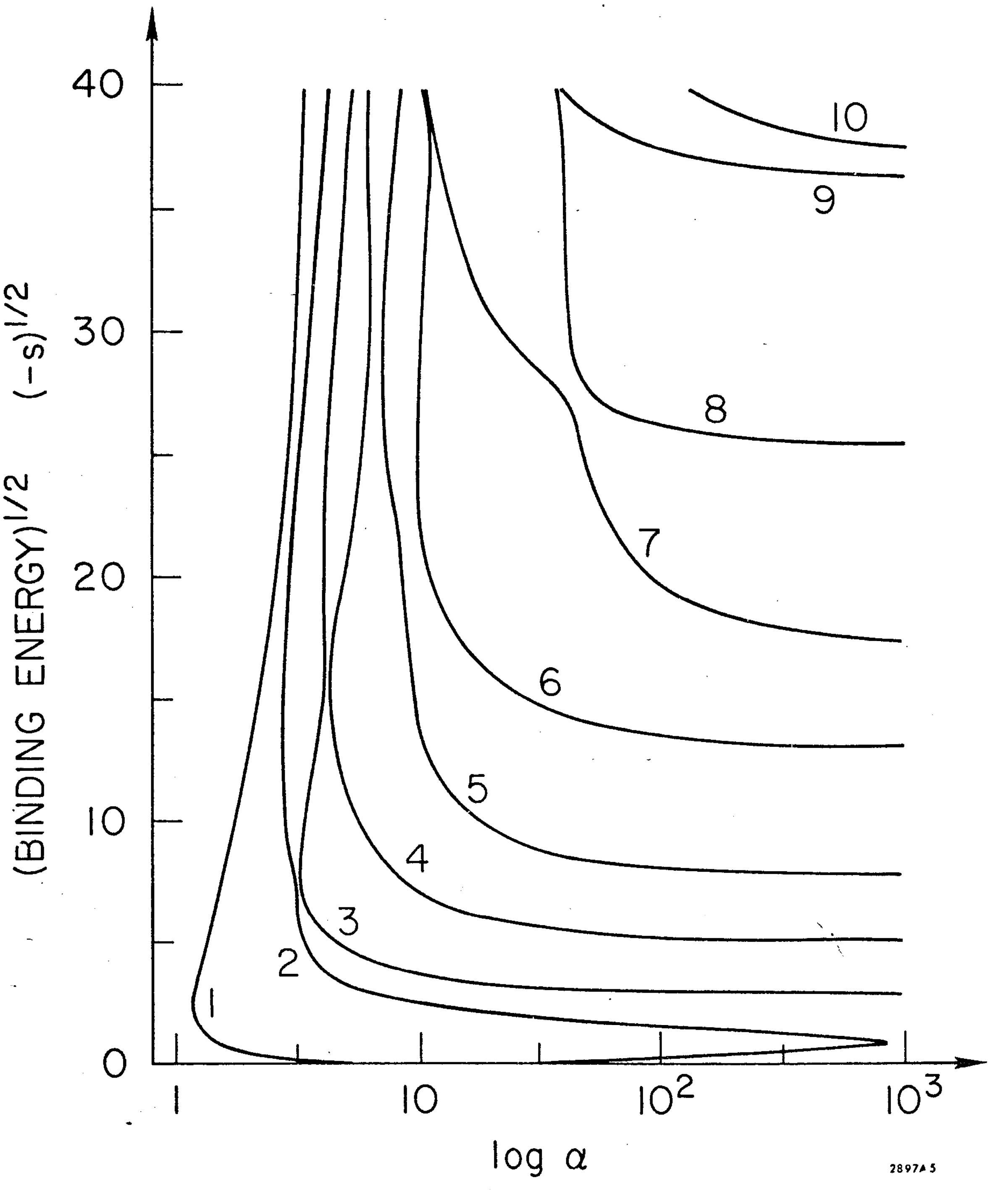

40
30
20
10
0
(BINDING ENERGY)$^{1/2}$ $(-s)^{1/2}$
1
10
10^2
10^3
log α
1
2
3
4
5
6
7
8
9
10
2897A5

Fig. 3

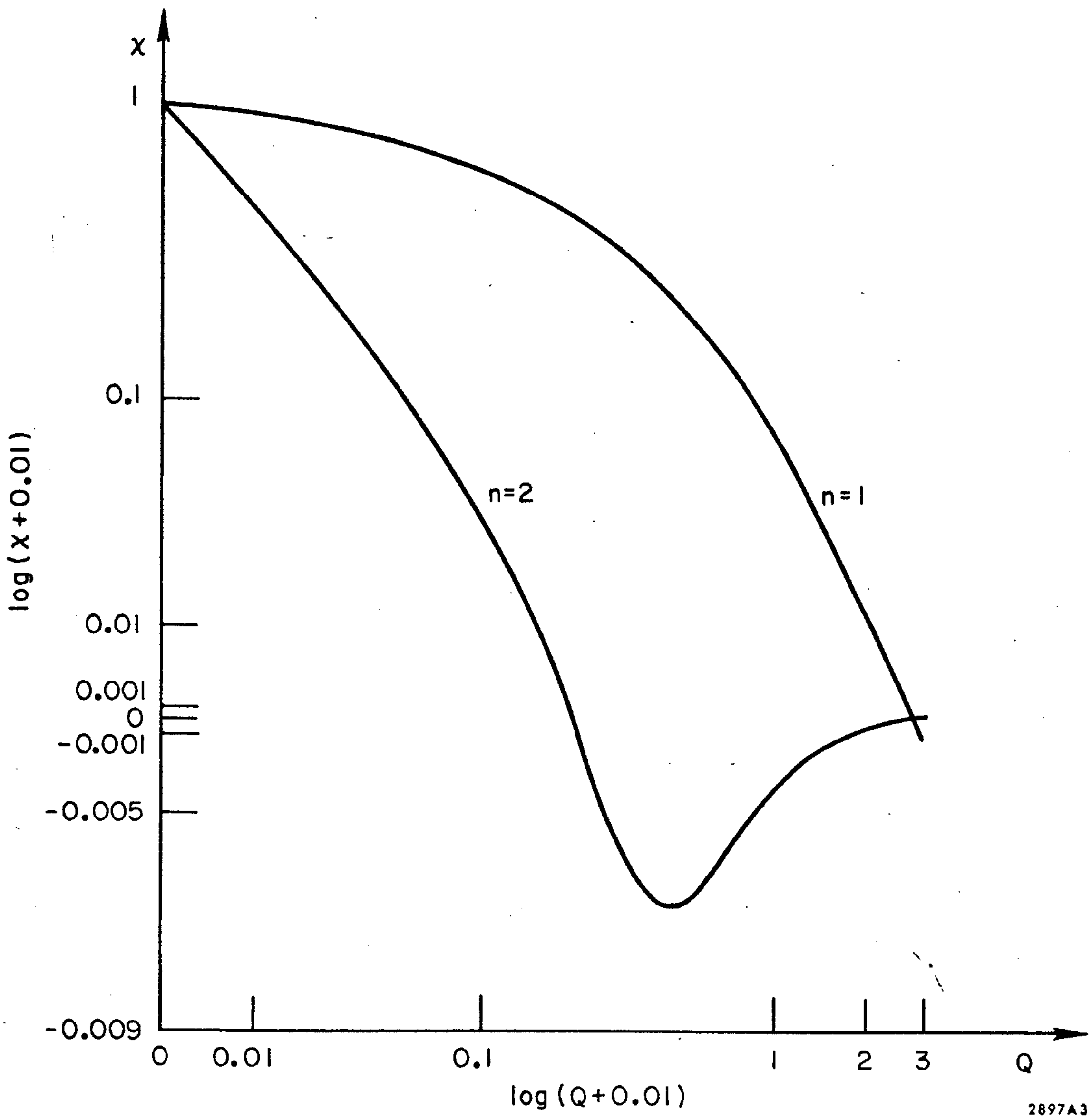

χ
1
0.1
0.01
0.001
0
-0.001
-0.005
-0.009
log (χ+0.01)
n=2
n=1
0
0.01
0.1
1
2
3
Q
log (Q+0.01)
2897A3

Fig. 4

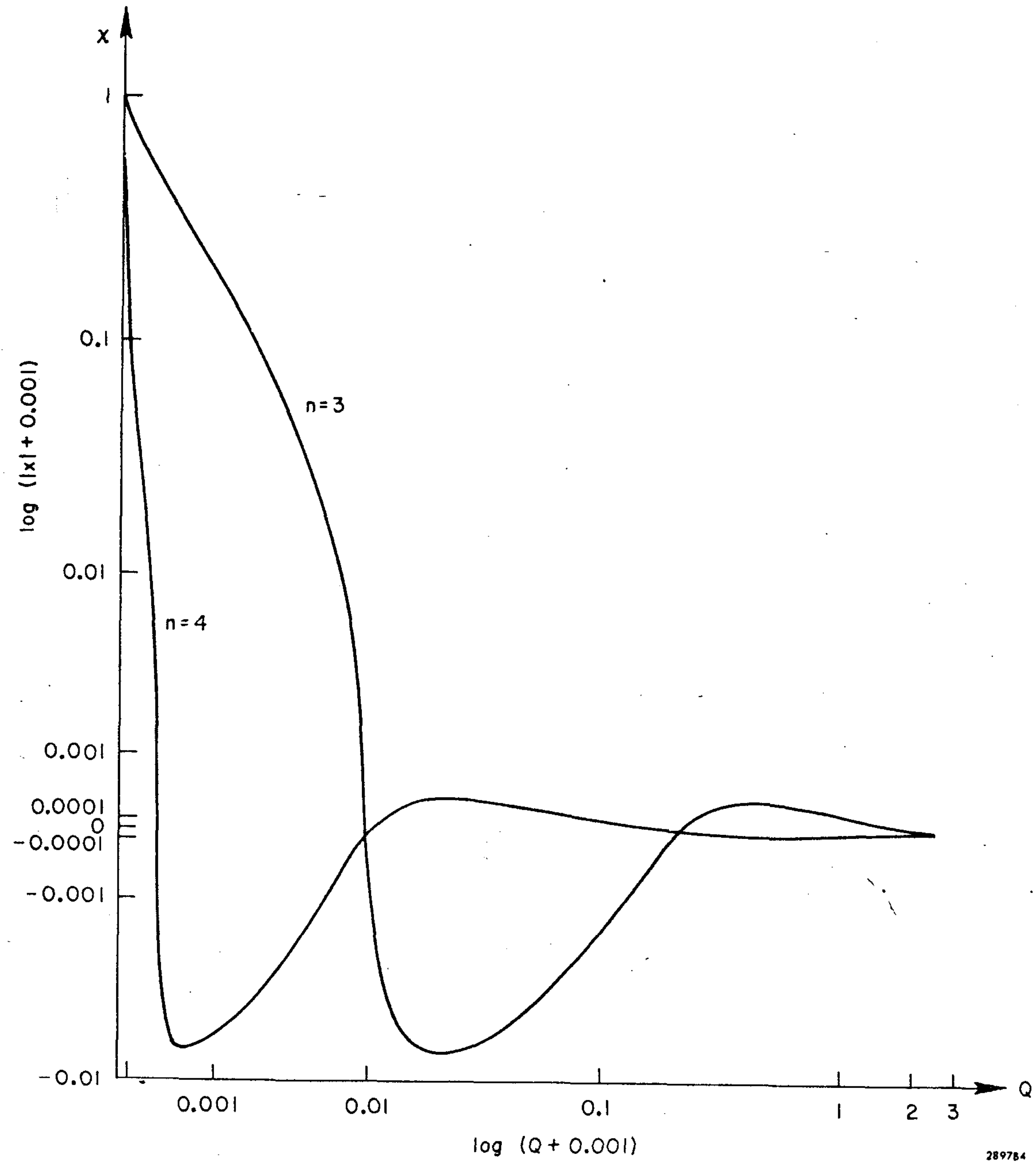

x
1
0.1
0.01
0.001
0.0001
0
-0.0001
-0.001
-0.01
log (|x| + 0.001)
n = 3
n = 4
0.001
0.01
0.1
1
2
3
Q
log (Q + 0.001)
289784

Fig. 5