# QUARK CONFINEMENT SCHEMES IN FIELD TIIEORY* 

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#### Abstract

These lectures describe some efforts to understand quark confinement in the context of quantum field theory. First we describe the formation of the SLAC "bag" ('bubble" or "shell") by the strong interaction of an elementary fcrmion quark field with a quartically self-coupled neutral scalar field. The SLAC bag is derived using a variational approach and a semi-classical ("tree approximation") treatment of the field theory. Physical properties of the SLAC bag are discussed. Next we show that the SLAC bag is "soft" to deformations and discuss the general formalism for studying bag deformations and the low lying bag excitations developed by R. Giles. Next we discuss the relation of the SLAC to the MIT bag. Finally we discuss the quantum corrections to the semi-classical treatment of field theory used in constructing the SLAC bag. Beyond the oneloop approximation we describe the initial progress in formulating the field theory on a discrete lattice and studying its quantum behavior and spontaneous symmetry breaking using variational methods.


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## QUARK CONFINEMENT SCHEMES IN FIELD THEORY*

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Evidence supporting the idea of quarks as the hadronic constituents has been strengthening since they were first proposed by Gell-Mann and Zweig in the 1960's. In particular there are the successes of the $\operatorname{SU}(3)$ symmetry scheme. Hadrons of the same spin and parity form multiplets each of which corresponds to an irreducible representation of $\mathrm{SU}(3)$. Experimentally the multiplets are observed to be octets, decuplets, and singlets, whereas the fundamental representation of $\operatorname{SU}(3)$ is a triplet. Therefore the simplest picture of hadrons is to build them as bound states of triplet quarks from which they derive their individual quantum numbers of electric charge, strangeness, etc.

Moreover, beyond their simple and attractive role in the group theoretic structure of hadrons, the quarks, though still unobserved, have acquired strong support from phenomenological analyses of hadronic properties and interactions. Among these successes of the quark model we include:

1) Static properties such as mass spectra of hadrons and low lying resonances
2) Transition matrix elements
3) Quark recombination rules (viz. Zweig's rule) for dual models and graphs
4) The quark-parton description of deep inelastic electron and neutrino scattering
5) Quark line counting rules for scaling laws in large transverse momentum exchange processes and the constituent interchange model
6) Constancy of the hadron to muon production ratio in electron-positron annihilation, in between thresholds for the onset of "new physics."
Despite these impressive successes several issues must be resolved before a real understanding of hadron dynamics based on the quark idea is possible:
7) Why do we not see isolated quarks?
8) In the observed hadronic spectroscopy, why do the quarks appear to obey symmetric statistics in spite of their half-integer spin?
The second of these issues is resolved by adopting the "color hypothesis." A hidden $\operatorname{SU}(3)$ of color is introduced together with the assertion that the dynamics permits only color singlet states to be bound to form hadrons. In effect the anomalous quark

[^1]statistics are avoided by tripling the number of quarks (analogously to the introduction of electron spin doubling the number of electron states).

However the nonobservation of isolated quarks is a new one to particle physics.* Are they permanently confined, never to be observed as isolated particles? Or are they so heavy when they are isolated from the extremely strong forces binding them as effectively light constituents within hadrons to form color singlet-or zero trialitystates that their production thresholds lie above presently observable energies $\left(M_{\text {quark }}>10 \mathrm{GeV}\right)$ ?

Theoretical efforts to understand quark confinement have developed along both lines-of permanent and of approximate quark confinement. I will be discussing in these lectures primarily an effort to understand approximate quark confinement that is based on the conservative approach of local canonical quantum field theory, with the quark fields included among the fundamental fields of the theory when it comes to writing interaction currents and forming asymptotic states. It is physically clear that weak coupling perturbative expansions are quite hopeless in such an approach. By no reasonable approximation can they span the gap between the starting point of a bare vacuum and a Fock space of free quark states created from this vacuum by interaction representation fields on one hand, and the observed hadronic spectrum of low lying quark bound states with a binding so strong that it essentially cancels their large bare masses.

An alternative class of models ${ }^{1)}$ that I will not discuss is that in which the fundamental fields do not create asymptotic states at all. In such "long range force models" one has contrasting behaviors at short distances and at long distances. At short distances the forces seem to weaken to the point that free field theory scaling laws are applicable and one speaks of asymptotic freedom at high energies. At large distances the opposite is assumed to occur: the forces grow so strong that the fundamental fields do not create asymptotic states. In particular the bonds between quarks cannot be broken and they are bound permanently to one another by flux lines due to their color charges or monopole moments. However, if a particle is in a color singlet state so that no flux lines are emerging from it it will not be bonded to an additional quark. This general idea was first explored by Schwinger ${ }^{1)}$ in two dimensional quantum electrodynamics. When summed to all orders in the coupling, the infrared singularities in Green's functions involving bare quark lines are so severe that they

[^2]prevent the existence of bare quark states. In a one space-one-time dimensional QED the flux lines between two charged quarks produce a constant "electric field,"i.e., a potential growing with separation. The analogous possibility exists for magnetic flux tubes in Lagrangians with Higgs fields. Or one can just speculate more generally that the infrared divergences in a massless non-Abelian gauge theory of color fields are very severe as well as complicated and generate long-range confining forces-i.e., infrared slavery.

Our concern here will be primarily with "bag models" derived from canonical field theory which reproduce a number of the successful naive quark model results and which provide a basis for additional speculations ${ }^{2)}$. The intuitive idea of the bag models is, for the present, clearer than their mathematical basis. The fundamental idea is that the vacuum is highly polarized in the presence of quarks. The "bags" themselves are extended, coherent vacuum excitations to which the quarks, which may have an extremely large bare mass, are bound. A non-Abelian colored gauge interaction is introduced following Nambu ${ }^{3)}$ so that this binding occurs only when the quarks form color singlet states. These color singlet bound states of low mass are the hadrons. Within the bag the interaction between the quarks is taken to be small in contrast to the strong interaction between the quarks and the bag. This gives a picture of hadrons as bound states of two or three almost free quasi-particles from which they derive their $\operatorname{SU}(3)$ properties. However there is an energy associated with the size of the bag-or vacuum excitation-containing the quarks which prevents their being separated.

The mechanism for polarizing the vacuum and forming "bags" is the same one giving rise to the abnormal states in the Lee-Wick theory of uniform nuclear matter ${ }^{4}$. In their example with many particles, a classical treatment with neglect of surface effects is valid. To illustrate this idea, consider the following classical energy exnression for a statistical ensemble of N nucleons, each of mass M, plus a scalar field with the self coupling illustrated in Fig. 1.

$$
\begin{equation*}
\mathrm{E}=\int \rho \mathrm{dV}[\mathrm{M}+\mathrm{g} \sigma(\mathrm{x})]+\int \mathrm{dV} \cup(\sigma) \tag{1}
\end{equation*}
$$

$\rho(\mathrm{x})$ represents the number density of nucleons,

$$
\mathrm{N} \equiv \int \rho \mathrm{dV}
$$

and $g$ is a measure of their interaction strength with the field strength $\sigma$. The "normal state" of matter is described


Fig. 1
by setting $\sigma=0$ so that the field state is at the minimum energy $U(0)=0$ and the assemblage of nucleons, neglecting surface and kinetic effects (i.e., the height of the fermi surface relative to their mass energy), is

$$
\mathrm{E}_{\text {normal }}=\mathrm{MN}=\rho_{0} \mathrm{~V}_{0} \mathrm{M}
$$

in terms of the uniform nucleon density $\rho_{0}$ within the volume $\mathrm{V}_{0}$.
The "abnormal state" is described by "polarizing the field vacuum" and exciting the field strength $\sigma=-\sigma_{c}$, the value at the local minimum in Fig. 1, within the volume $\mathrm{V}_{0}$ of nucleons, i.e.,

$$
\begin{array}{ll}
\sigma=-\sigma \mathrm{c} & \\
\text { inside } \mathrm{V}_{0} \\
\sigma=0 & \\
\text { outside } \mathrm{V}_{0}
\end{array}
$$

The total energy is now given by

$$
\begin{align*}
E_{\text {abnormal }} & =\rho_{0} V_{0}\left(M-g \sigma_{c}\right)+U\left(-\sigma_{c}\right) V_{0} \\
& =E_{\text {normal }}+V_{0}\left[-g \sigma_{c} \rho_{0}+U\left(-\sigma_{c}\right)\right] \tag{2}
\end{align*}
$$

and evidently at large enough densities $\rho_{0}$ the abnormal state (2) will be at a lower energy than the normal one.

The Nambu mechanism is a generalization of the observation that the classical electrostatic energy of a charged system is positive, corresponding to the repulsion between charges of the same sign. As an example, consider first the effect of coupling nucleons via the vector isospin current. This interaction leads to a nonrelativistic description of the isospin coupling in terms of two-body potentials of the form

$$
v_{i j}=v \vec{t}_{i} \cdot \vec{t}_{j}, \quad v>0
$$

where $\vec{t}_{i}$ is the isospin of the ith particle and $V$ contains the dependence on the other degrees of freedom. The potential energy of an n-nucleon system may be estimated as

$$
\begin{aligned}
V(n) & =\frac{1}{2} \sum_{i \neq j} V_{i j} \\
& =\frac{1}{2} V \sum_{i \neq j} \overrightarrow{\mathrm{t}}_{i} \cdot \vec{t}_{j} \\
& =\frac{1}{2} V[I(I+1)-n t(t+1)],
\end{aligned}
$$

where I is the total isospin of the system and $t$ is the nucleon isospin. This force is seen to be attractive for the deuteron ( $\mathrm{I}=0$ ) and repulsive for the dineutron system ( $\mathrm{I}=1$ ).

To extend these ideas to quark bound states, the quarks are endowed with the additional internal quantum numbers of color so that there are three triplets of quarks: red, blue, and yellow. The color interaction is mediated by an octet of non-Abelian gauge bosons coupled to the $\mathrm{SU}(3)$ vector currents of the color symmetry.

In analogy with the isospin interaction, the effective potential energy for an n quark system is

$$
V(n)=\frac{1}{2} v \sum_{i \neq j} \sum_{a=1}^{8} \lambda_{i}^{a} \lambda_{j}^{a}
$$

where $\left\{\lambda_{i}^{a}\right\}$ are octet coupling matrices to the quarks. The potential energy may be reduced to the form

$$
\begin{equation*}
\mathrm{V}(\mathrm{n})=\frac{1}{2} \mathrm{~V}(\mathrm{C}-\mathrm{nc}), \quad \mathrm{V}>0 \tag{3}
\end{equation*}
$$

where $C$ is the eigenvalue of the Casimir operator for $\operatorname{SU}(3)$ of color for the n-particle system

$$
C=\sum_{a}\left(\sum_{i} \lambda_{i}^{a}\right)^{2}
$$

and $c=4 / 3$ is the equivalent eigenvalue for the quark. Since $C$ is positive-definite and has zero eigenvalue only for color-singlet states, the strongest attractive interaction occurs for those states which are color singlets.

Returning to a field theoretic description of the color interaction in terms of the color current operators and their contributions to the self energies as well as interaction strengths, we recall that for an electrically neutral system in a state of definite charge conjugation the expectation value of the charge density operator vanishes. Hence, so does the electrostatic energy in a semi-classical approximation which neglects higher order fluctuation energies. By the Wigner-Eckart theorem this is also true for color singlet states in the same approximation.

Hence for hadronic states that are color singlets, the color interaction plays no direct role and we can concentrate on the binding mechanism responsible for bag formation. In a literal sense, quark confinement to form hadrons is the result of the color interactions with their local gauge invariance property. Since the color interaction is presumably a strong one, if the thresholds for forming states of colored matter are to be large on a scale of $\sim \mathrm{GeV}$, this semi-classical approximation is of course dangerous and not defensible. In general, interaction energies are given by the expectation value of squares of current operators and important contributions to the energy due to the quantum fluctuations are neglected when we replace such an expression by the square of their expectation value. Efforts and problems in constructing a fully quantized "bag theory" will be described in my third lecture. For the essentially classical
picture being developed here, we can neglect the color interaction for hadronic states that are color singlets and concentrate on the mechanism forming the bags, i.e., the extended low mass bound states that are color singlets.
THE SLAC BAG ${ }^{5}$ )
Our first task is to show how the bag is created by the strong interaction of an elementary fermion quark field with a self-coupled scalar field. Our method is to do a variational calculation over a class of trial states and to show that the Hamiltonian of this model possesses low energy bound states of quarks and of coherent excitations of the scalar field.

The basic idea of this approach is illustrated by the following simple semiclassical model. Consider a quark described by wave function $\psi$ interacting with a neutral scalar field $\sigma$ with the Hamiltonian

$$
\begin{align*}
\mathrm{H} & =\int \mathrm{d}^{3} \mathrm{x} \mathscr{H}(\mathrm{x}) \\
\mathscr{H}(\mathrm{x}) & =\psi^{\dagger}\left(\frac{\vec{\alpha} \cdot \vec{\nabla}}{\mathrm{i}}+\mathrm{G} \beta \sigma\right) \psi+\frac{1}{2} \dot{\sigma}^{2}+\frac{1}{2}|\vec{\nabla} \sigma|^{2}+\lambda\left(\sigma^{2}-\mathrm{f}^{2}\right)^{2}, \tag{4}
\end{align*}
$$

where $G, \lambda \gg 1$ are large dimensionless coupling constants, and $f$ has the dimension of a mass. The form of the quartic self-interaction term exhibits the invariance of the theory under the discrete transformation $\sigma \rightarrow-\sigma$. In a quantum field-theory description, Eq. (4) describes a spontaneously broken theory and $\sigma$ has a nonvanishing vacuum expectation value. In the vacuum state the field, $\sigma$, takes one of two values, $\pm f$. Small vibrations about one of these ground states are usually studied by making the translation $\sigma \rightarrow \sigma^{r}=\sigma+\mathrm{f}$. One readily finds that the small $\sigma$ vibrations have the mass $\mathrm{m}_{\sigma}^{2}=8 \lambda \mathrm{f}^{2}$ and the small $\psi$ vibrations have mass $\mathrm{M}_{\mathrm{Q}}=\mathrm{Gf}$. By assumption $\mathrm{m}_{\sigma}$ and $\mathbb{M}_{Q}$ are $\gg 1 \mathrm{GeV}$. Our choice of the specific Hamiltonian (4) is arbitrary. We consider it as typical of a class of renormalizable field theories exhibiting spontaneous breakdown.

Our key question is, "Do these theories also have quark states with much lower energy than indicated by the bare quark mass?"

For the purpose of developing an intuitive picture of nonperturbative solutions to the field equations, we approach this problem classically, although this is no longer a purely classical question when fermions are present. The point is that in the onefermion sector when the charge

$$
\mathrm{Q}=\int \psi^{\dagger} \psi \mathrm{d}^{3} \mathrm{x}
$$

has unit eigenvalue we are solving a Dirac equation for the quark in the presence of a scalar potential $\sigma$. We are faced with the usual question of negative-energy states
and must specify that all the negative-energy states in the presence of this potential are filled, and then focus our attention on the lowest positive-energy eigenvalue. Since we are solving for the quark energy in a scalar potential, there is no Klein paradox of the familiar type encountered in the presence of strong, sharp vector potentials and therefore no ambiguity in identifying and interpreting the desired positiveenergy "one-particle" solutions.

We proceed classically therefore with $\mathrm{Q}=1$. Classically, we expect that the quarkwave function and the ficld amplitude $v$ will avoid one another as indicated in Fig. 2, so as to escape the high-mass energy $M_{Q}$.

The importance of this effect increases with the magnitude of $M_{Q}=G f$. At the same time, working against the formation of such a hole into which the quark will trap itself


Fig. 2 are the energies associated with the curvature of the localized quark-wave function, with the curvature of the $\sigma$ field as it changes its value, and the energy associated with the potential term $\lambda\left(\sigma^{2}-\mathrm{f}^{2}\right)^{2}$ extending over the volume where $\sigma \neq \pm$. As a simple illustrative example of how these contributions balance, consider a solution as in Fig. 2 with $\sigma \rightarrow 0$ within a volume of radius R. Denoting by $D$ the thickness of the shell in which the $\sigma$-field amplitude falls from +f to 0 , we estimate by the uncertainty principle

$$
\begin{align*}
& \int \psi^{\dagger} \frac{\vec{\alpha} \cdot \vec{\nabla}}{i} \psi d^{3} x \sim 1 / R  \tag{5}\\
& \int \frac{1}{2}|\vec{\nabla} \sigma|^{2} d^{3} x \sim \frac{1}{2}(f / D)^{2} 4 \pi R^{2} D \\
& \int \lambda\left(\sigma^{2}-f^{2}\right)^{2} d^{3} x \sim \lambda f^{4}\left(\frac{4}{3} \pi R^{3}+4 \pi k R^{2} D\right), \tag{6}
\end{align*}
$$

where $\mathrm{k} \sim 1$ is a shape-dependent number. The energy of this configuration is given by the sum of these contributions

$$
\begin{equation*}
E(R, D) \sim \frac{1}{R}+2 \pi R^{2} f^{2} / D+\lambda r^{4}\left(\frac{4}{3} \pi R^{3}+4 \pi k R^{2} D\right) \tag{7}
\end{equation*}
$$

Minimizing with respect to D and R , we find a surface thickness given dimensionally by

$$
\frac{\partial \mathrm{E}}{\partial \mathrm{D}}=0 \Rightarrow \mathrm{D} \sim 1 / \lambda^{1 / 2} \mathrm{f}
$$

and if $\lambda^{1 / 2} \mathrm{f} \gg 1 / \mathrm{R}$, i.e., if the volume energy dominates the surface energy, then

$$
\frac{\partial \mathrm{E}}{\partial \mathrm{R}}=0 \Rightarrow \mathrm{R} \sim 1 / \lambda^{1 / 4} \mathrm{f} .
$$

Hence the lowest possible energy is given by

$$
\begin{equation*}
E \equiv \min E(R, D)=\frac{4}{3 R} \sim f \lambda^{1 / 4} \tag{8}
\end{equation*}
$$

In this case

$$
\mathrm{D} / \mathrm{R} \sim \lambda^{-1 / 4} \ll 1,
$$

which is consistent with a thin transition-shell region in the strong-coupling limit. Comparing with $M_{Q}=G f$, we see that a localized bound state is formed if $G \gg \lambda^{1 / 4}$.

According to Fig. 2, the quark moves as a free massless quantum within the sharp well boundaries, suggesting some of the popular quark-parton model ideas. However, it turns out that this treatment is much too naive and crude, although it illustrates the basic idea. We learn from a more systematic and careful treatment of Hamiltonian (4) that what actually emerges for the classical theory is a thin-shell model of the hadron, with the field rapidly changing from $\sigma=+\mathrm{f}$ outside to $\sigma=-\mathrm{f}$ in a region of thickness $\mathrm{D} \sim 1 / \lambda^{1 / 2} \mathrm{f} \ll \mathrm{R}$, and with the quark confined to a thin shell within a distance $1 / \mathrm{Gf}$ of R as illustrated in Fig. 3.

This comes about as follows. With a solution as drawn in Fig. 3, the first term in (6) representing the large volume energy of the field (the "polarized vacuum state") is not present. Furthermore, although the fermion is now confined to a thin shell of thickness $\mathrm{D} \ll \mathrm{R}$, its energy remains of the same order as in (5). This result which is surprising at first can be confirmed by direct calculation which can be done exactly and simply in one space dimension (in which case the contribution of (5) vanishes iden-


Fig. 3 tically). Formally, it follows from the observation that in a representation for an S-wave $j=1 / 2$ quark solution

$$
\psi=\binom{\mathrm{iG}(\imath) / z}{(\mathrm{~F}(\imath) / \imath) \frac{\sigma \cdot \imath}{v}}
$$

the energy (5) is expressed by

$$
\begin{equation*}
\int \psi \frac{\alpha \cdot \nabla}{\mathrm{i}} \psi \mathrm{~d}^{3} \mathrm{x}=4 \pi \int_{0}^{\infty} \mathrm{d} \imath\left\{\mathrm{~F}^{2} \frac{\mathrm{~d}}{\mathrm{~d} \imath}(\mathrm{G} / \mathrm{F})-\frac{2 \mathrm{FG}}{\imath}\right\} \tag{9}
\end{equation*}
$$

i.e., the rate of change of the ratio of components is what matters, and not their individual curvatures. Physically what is happening is that the fermion energy is given relativistically by $\sim \vec{p} \cdot \vec{\alpha}$ where $\overrightarrow{\mathrm{p}}$ and $\vec{\alpha}$ are the momenta and velocities in the localized quark packet and their relative sign changes as we go from the positive to the negative mass region in the thin shell of Fig. 3. This leads to the calculated cancellation. Hence with the thin shell solution, the volume energy term is omitted from (7) and the energy found by repeating the steps subsequent to (7) is

$$
\mathrm{E}=\frac{2}{2 \mathrm{R}} \sim \mathrm{f} \lambda^{1 / 6} \ll \mathrm{f} \lambda^{1 / 4} \quad \text { if } \lambda \gg 1
$$

for strong coupling. Further details of this classical solution can be found in Ref. 2.
We turn next to a brief description of the systematic steps leading to the above heuristic picture from a canonical quantum field theory. This analysis makes use of the variational principle for the expectation value of the Hamiltonian (4) in a trial state. In addition to making a suitable guess for the trial state, we must make one crucial approximation involving normal ordering. This is the "tree" approximation. Corrections to this will be described in the third lecture.

To start, we must construct a Fock space trial state, |s>, determine an upperbound on the ground state energy by varying the trial parameters

$$
\begin{equation*}
\delta\langle\mathrm{s}| \mathrm{H}|\mathrm{~s}\rangle=0 \tag{10}
\end{equation*}
$$

and show that the energy found this way lies much lower than Gf. We demonstrate this first for a bound state of a single quark using (4) and ignoring color $\operatorname{SU}(3)$.

Fock space
To perform the variation, (10) we require only the canonical equal sime commutation relations and a Fock space basis in terms of which to expand the field operators.

For the scalar field, we choose a plane-wave expansion

$$
\begin{equation*}
\sigma(x)=\int \frac{d^{3} k}{\left[(2 \pi)^{3} 2 \omega_{k}\right]^{1 / 2}}\left(a_{k} e^{i \vec{k} \cdot \vec{x}}+a_{k}^{\dagger} e^{-i \vec{k} \cdot \vec{x}}\right) \tag{11}
\end{equation*}
$$

where

$$
\omega_{\mathrm{k}}=\left(\overrightarrow{\mathrm{k}}^{2}+\mathrm{m}_{\sigma}^{2}\right)^{1 / 2}, \quad \mathrm{~m}_{\sigma}^{2}=8 \lambda \mathrm{f}^{2}
$$

and the operators are quantized by imposing the usual canonical commutation relations. In (11) we have used the mass $\mathrm{m}_{\sigma}=2 \sqrt{2 \lambda f}$ for the small $\sigma$ oscillations after making the translation $\sigma \rightarrow \sigma+\mathrm{f}$. For the fermion field, we make an expansion in terms of the eigenfunctions of the Dirac equation in an external potential to be determined
self-consistently by the variational procedure

$$
\begin{equation*}
\psi(x)=\sum_{n}\left[B_{n} U_{n}(x)+D_{n}^{\dagger} V_{n}(x)\right] \tag{12}
\end{equation*}
$$

The positive-and negative-energy eigenfunctions $U_{n}$ and $V_{n}$ satisfy the usual orthonormality relations and the expansion coefficients satisfy the anticommutation relations

$$
\left\{B_{n}, B_{m}^{\dagger}\right\}=\left\{D_{n}, D_{m}^{\dagger}\right\}=\delta_{n m}
$$

The Hilbert space at $\mathrm{t}=0$ is constructed by applying the creation operators $\mathrm{a}_{\mathrm{k}}^{\dagger}$ and $B_{n}^{\dagger}, D_{m}^{\dagger}$ to the translationally noninvariant no-particle state $1 O_{L}>$ characterized by

$$
\mathrm{a}_{\mathrm{k}}\left|\mathrm{O}_{\mathrm{L}}\right\rangle=\mathrm{B}_{\mathrm{n}}\left|\mathrm{O}_{\mathrm{L}}\right\rangle=\mathrm{D}_{\mathrm{m}}\left|\mathrm{O}_{\mathrm{L}}\right\rangle=0
$$

The relation of this expansion to the usual one in terms of plane waves and a translationally invariant trial vacuum can be clarified in terms of the Bogoliubov transformation.

Normal-ordering and definition of the Hamiltonian
The field-theory model with H given by (4) is a renormalizable theory. Because of the divergences inherent in any renormalizable quantum field theory, the meaning of a product of field operators at the same space-time point is ambiguous and has to be properly defined via the renormalization program. Within our semi-classical framework, we define the Hamiltonian by a naive normal-ordering prescription. The prescription depends on the particular expansion chosen for the field operators. Hamiltonians normal-ordered with respect to two different expansions, such as (12) vs. a plane-wave expansion for the quark fieid, differ by a c-number contribution which is usually a difference of two infinite constants. In order to give such a difference a precise meaning, it would be necessary to regulate and properly renormalize the quantum field theory.

The very fundamental approximation we are here making is to ignore these differences in normal-ordering prescriptions. In other words, the Hamiltonian we are working with its correct only in the so-called "tree" approximation. In the same approximation, the true vacuum state also coincides with the free-field vacuum as defined for small oscillations about $\sigma=\mathrm{f}$. Our hope is that when renormalization effects are included, the conclusions will be qualitatively similar although they may be quantitatively different. We return to this approximation in the third lecture.

## Boson coherent states

The construction of the trial state is guided by our intuitive idea that the boson field develops a localized expectation value in the neighborhood of the fermion source.

To describe such a situation, we employ the so-called boson coherent states

$$
|g\rangle=U(g)\left|O_{L}\right\rangle,
$$

where $U(g)$ is a unitary transformation

$$
\begin{equation*}
\mathrm{U}(\mathrm{~g})=\exp \left(-\mathrm{i} \int \mathrm{~d}^{3} \mathrm{x} \mathrm{~g}(\mathrm{x}) \dot{\sigma}(\mathrm{x})\right) \tag{13}
\end{equation*}
$$

which displaces the field operator $\sigma$

$$
\begin{gathered}
\mathrm{U}^{-1}(\mathrm{~g}) \mathrm{f}(\sigma(\mathrm{x})) \mathrm{U}(\mathrm{~g})=\mathrm{f}(\sigma(\mathrm{x})+\mathrm{g}(\mathrm{x})), \\
\mathrm{U}^{-1}(\mathrm{~g}) \dot{\sigma} \mathrm{U}(\mathrm{~g})=\dot{\sigma}
\end{gathered}
$$

Thus, if $\mathrm{f}(\sigma)$ is any polynomial function of $\sigma$ which is normal-ordered term by term, then

$$
\begin{aligned}
\langle\mathrm{g}| \mathrm{f}(\sigma)|\mathrm{g}\rangle & =\left\langle\mathrm{O}_{\mathrm{L}}\right| \mathrm{f}(\sigma+\mathrm{g})\left|\mathrm{O}_{\mathrm{L}}\right\rangle \\
& =\mathrm{f}(\mathrm{~g})
\end{aligned}
$$

The tree approximation rule for taking the expectation value of a function of $\sigma$ in a coherent state is to replace $\sigma$ by the c-number amplitude $\mathrm{g}(\mathrm{x})$. This procedure gives a concrete realization of the intuitive picture presented earlier.

Fermion states and the Bogoliubov transformation
We shall also want to replace the fermion field operator by an arbitrary c-number Dirac spinor wave function when we take the expectation value of H in our trial state. For a trial state of fermion number one, we do this by constructing

$$
\begin{equation*}
|\mathrm{s}\rangle=\mathrm{B}_{\mathrm{n}}^{\dagger}\left|\mathrm{O}_{\mathrm{L}}\right\rangle \tag{14}
\end{equation*}
$$

where $B_{n}^{\dagger}$ is the creation operator for a fermion in an arbitrary state $n$ and $1 O_{L}>$ is the no-particle state in the basis formed as shown in (12). With this procedure, the expectation value of an operator biljnear in the fermion field and normal-ordered in this basis is

$$
\langle\mathrm{s}|: \psi^{\dagger}(\mathrm{x}) \Gamma \psi(\mathrm{x}):|\mathrm{s}\rangle=\mathrm{U}_{\mathrm{n}}^{\dagger}(\mathrm{x}) \Gamma \mathrm{U}_{\mathrm{n}}(\mathrm{x})
$$

where the arbitrary wave function is to be determined self-consistently by the variational calculation.

## Derivation of classical field equations from the variational principle

We now apply the variational principle guessing as the trial state

$$
\begin{aligned}
|s\rangle & =\mathrm{U}(\mathrm{~g}) \mathrm{B}_{0}^{\dagger}\left|\mathrm{O}_{\mathrm{L}}\right\rangle \\
& =\exp \left(-\mathrm{i} \int \mathrm{~d}^{3} \mathrm{x} \mathrm{~g}(\mathrm{x}) \dot{\sigma}(\mathrm{x})\right) \mathrm{B}_{0}^{\dagger}\left|\mathrm{O}_{\mathrm{L}}\right\rangle,
\end{aligned}
$$

where $\mathrm{B}_{0}^{\dagger}$ is the creation operator associated with the ground-state wave function in (12), and $U(g)$ creates the coherent boson state (13). This procedure reduces the quantum field-theory problem to a classical form to which we can apply the earlier heuristic discussion. It can also be solved by mathematical analysis as described in detail in Ref. 2 in the tree approximation. In particular, the amplitude $g(x)$ for the boson field and the ground state quark wave function $\chi(x)$ created by $B_{0}^{\dagger}$ in (12) satisfy the coupled field equations

$$
\begin{align*}
& \nabla^{2} \mathrm{~g}-4 \lambda \mathrm{~g}\left(\mathrm{~g}^{2}-\mathrm{f}^{2}\right)=\mathrm{G} \bar{\chi} \chi  \tag{15}\\
& \left(\frac{\alpha \cdot \nabla}{\mathrm{i}}+\mathrm{G} \beta \mathrm{~g}\right) \chi=\epsilon \chi \text { with } \int \chi^{\dagger} \chi \mathrm{d}^{3} \mathrm{x}=1
\end{align*}
$$

and the energy of the ground state is expressed in terms of the lowest positive eigenvalue $\epsilon_{0}$ by

$$
\begin{equation*}
E=\int d^{3} x\left[\frac{1}{2}(\nabla g)^{2}+\lambda\left(g^{2}-f^{2}\right)^{2}\right]+\epsilon_{0} \tag{16}
\end{equation*}
$$

In agreement with the form in (5) and (6) this gives for the energy

$$
\mathrm{E}-\frac{1}{\mathrm{R}}+\frac{16 \pi}{3} \sqrt{2 \lambda} \mathrm{R}^{2} \mathrm{f}^{3}
$$

which has the minimum at the value

$$
\begin{equation*}
\frac{1}{\mathrm{R}_{0}}=\lambda^{1 / 6} \mathrm{f}\left(\frac{32}{3} \pi \sqrt{2}\right)^{1 / 3} \tag{17}
\end{equation*}
$$

The total energy is then

$$
\begin{equation*}
\mathrm{E}_{0}=3 / 2 \mathrm{R}_{0} \tag{18}
\end{equation*}
$$

To confront the ideas developed so far for single quark states with physical parameters, we must extend this scheme by constructing multiquark states and compare with observed hadronic properties. We must answer the question: "If a single quark prefers to dig a hole in the vacuum and trap itself, what happens if a three quark state or a quark-antiquark pair state is formed, as required for color singlets when the usual ideas of the $\operatorname{SU}(3)$ of color are introduced?"

Multiquark bound states may be constructed using the same variational method discussed earlier. The variational state consists of a coherent scalar field plus quarks and antiquarks. As in the Hartree-Fock approximation, the quarks and antiquarks are assumed to move in the self-consistent scalar field, the source of which contains contributions from all of the quarks and antiquarks in the state. As for the single quark, the multiquark states are those which minimize the expectation value of the energy. The potential $g(x)$ is similar to the Hartree-Fock field in atomic physics and the (anti) quarks move in the ground states of this self-consistent potential.

To be more explicit, we consider multiquark states of the type

$$
\left|\mathrm{S}_{\mathrm{N}}\right\rangle=\mathrm{U}(\mathrm{~g}) \mathrm{C}_{1}^{\dagger} \ldots \mathrm{C}_{\mathrm{N}}^{\dagger}\left|\mathrm{O}_{\mathrm{L}}\right\rangle
$$

where $\mathrm{C}^{\dagger}$ creates quarks $\left(\mathrm{B}^{\dagger}\right)$ or antiquarks $\left(\mathrm{D}^{\dagger}\right)$ in states corresponding to the potential $g(x)$, which defines the coherent state for the scalar field. The energy functional becomes

$$
\begin{align*}
\mathrm{E} & \equiv\left\langle\mathrm{~S}_{\mathrm{N}}\right| \mathrm{H}\left|\mathrm{~S}_{\mathrm{N}}\right\rangle \\
& =\sum_{\mathrm{i}=1}^{\mathrm{N}} \mathscr{E}_{\mathrm{i}}+\int \mathrm{d}^{3} \mathrm{x}\left[\frac{1}{2}(\vec{\nabla} \mathrm{~g})^{2}+\lambda\left(\mathrm{g}^{2}-\mathrm{f}^{2}\right)^{2}\right] \tag{19}
\end{align*}
$$

where the quark energies are given by the solution to the Dirac equation

$$
\left(\frac{1}{i} \vec{\alpha} \cdot \vec{\nabla}+G \beta \mathrm{~g}\right) \chi_{\chi_{i}}=\mathscr{E}_{\mathrm{i}} \chi,
$$

and $\mathrm{g}(\mathrm{x})$ is determined by

$$
\nabla^{2} g-4 \lambda g\left(g^{2}-f^{2}\right)=G \sum_{i=1}^{N} \bar{\chi}_{i} x_{i}
$$

The solution we obtain from this system is identical in structure to the solution found for the single-quark system.

Following our discussion of the single-quark system, we find the energy of a state with N quarks or antiquarks in the ground state to be

$$
\begin{align*}
E_{N} & =\frac{3}{2} \frac{N}{R_{N}} \\
& =\frac{3}{2} \frac{N^{2 / 3}}{R_{0}}, \tag{20}
\end{align*}
$$

where the radius of the system $R_{N}$ is given in terms of the radius for a single quark, $\mathrm{R}_{0}$, by

$$
\begin{equation*}
\mathrm{R}_{\mathrm{N}}=\mathrm{N}^{1 / 3} \mathrm{R}_{0}, \quad \mathrm{R}_{0}=\left(\frac{32}{3} \pi \sqrt{2 \lambda}\right)^{-1 / 3} \mathrm{f}^{-1} \tag{21}
\end{equation*}
$$

We will now discuss a few of the consequences of applying these results to hadron properties.
A. $\mathrm{q} \bar{q}$ system

Both $q$ and $\bar{q}$ are in the $\ell=0$ states. These states have odd parity since $q \bar{q}$ has an odd intrinsic parity. They consist of the $0^{-}$pseudoscalar and $1^{-}$vector mesons.

These are the $\underline{35}$ in $\operatorname{SU}(6)$ classification, and are degenerate with the energy

$$
\begin{equation*}
\mathrm{E}_{\mathrm{M}}=\frac{3}{2} \frac{1}{\mathrm{R}_{0}}(2)^{2 / 3} \tag{22}
\end{equation*}
$$

B. qqq system

All the three quarks are in $\ell=0$ states. These are the positive-parity (by definition) states with $J=3 / 2$ and $1 / 2$, namely the 56 in $\operatorname{SU}(6)$ classification. Their common energy is

$$
\begin{equation*}
\mathrm{E}_{\mathrm{B}}=\frac{3}{2} \frac{1}{\mathrm{R}_{0}}(3)^{2 / 3} . \tag{23}
\end{equation*}
$$

Thus $E_{B} / E_{M}$ is fixed at $(3 / 2)^{2 / 3}$.
C. Exotic states

One can form color-singlet states with more than one quark-antiquark pair or three quarks. These are the exotic states. So far, there is no experimental evidence for the existence of the exotic sta\%s. According to Eqs. (22) and (23), exotic states appear in our spectrum. For example, a noninteracting two-nucleon system has a mass given by

$$
\begin{aligned}
\mathrm{E}_{2 \mathrm{~B}} & =2 \mathrm{E}_{\mathrm{B}} \\
& =\frac{3}{2} \frac{1}{\mathrm{R}_{0}} 2(3)^{2 / 3},
\end{aligned}
$$

while a color-singlet 6-quark state has a mass given by

$$
\mathrm{E}_{6 \mathrm{q}}=\frac{3}{2} \frac{1}{\mathrm{R}_{0}} 6^{2 / 3}
$$

In general, for an $N$ quark system

$$
\mathbf{E}_{\mathrm{N}} \propto \mathrm{~N}^{2 / 3}
$$

## D. Magnetic moments

We are also led to a prediction for the proton magnetic moment that is in close accord with experiment. Using the thin shell wave function for the quarks illustrated in Fig. 3, and as described earlier as the solution of (15), we find, using the moment operator

$$
\begin{aligned}
\overrightarrow{\mathrm{M}} & =\frac{1}{2} \int \mathrm{~d}^{3} \mathrm{x} \vec{r} \mathrm{x} \overrightarrow{\mathrm{j}}(\mathrm{x}) \\
\mathrm{j} & =\frac{2}{3} \mathrm{e} \psi_{\rho}^{\dagger} \vec{\alpha} \psi_{\rho}-\frac{1}{3} \mathrm{e} \psi_{\mathrm{n}}^{\dagger} \vec{\alpha} \psi_{\mathrm{n}}-\frac{1}{3} \mathrm{e} \psi_{\lambda}^{\dagger} \vec{\alpha} \psi_{\lambda}
\end{aligned}
$$

$$
\begin{align*}
M_{\rho} & =\left\langle P, j_{z}=\frac{1}{2}\right| M_{z}\left|P, j_{z}=\frac{1}{2}\right\rangle \\
& =\frac{1}{3} e\left(3^{1 / 3} R_{0}\right)=3\left(\frac{e}{2 M}\right) \tag{23}
\end{align*}
$$

where $M$ is the ground state mass of the baryon 56 . The experimental proton moment is

$$
\left(M_{p}\right)_{\operatorname{expt}}=2.79\left(\frac{e}{2 M_{p}}\right)
$$

Additional results on M1 transition moments for baryon radiative decay (viz. $\Delta^{+} \rightarrow \mathrm{p}+\gamma$; $\omega \rightarrow \pi^{0}+\gamma$ ), for baryon radii, and for the $F / D$ ratios are in reasonable accord with data (see Ref. 2).

Although this theory as written does not have a conserved (or almost conserved) axial-vector current, we attempt to identify the axial coupling for neutron $\beta$ decay, $\mathrm{g}_{\mathrm{A}}$, through the matrix elements of the quark current

$$
\begin{equation*}
\mathrm{j}_{\mathrm{A} \mu}=\bar{\psi} \gamma_{\mu} \gamma_{5} \frac{1}{2}\left(\lambda_{1}+\mathrm{i} \lambda_{2}\right) \dot{\psi} \tag{24}
\end{equation*}
$$

in order to calculate the value of the axial charge. This is a natural choice for the axial-vector current of the weak interactions since it satisfies the usual commutation rules of current algebra. The result we obtain is

$$
\mathrm{g}_{\mathrm{A}}=\langle\mathrm{p}| \int \mathrm{d}^{3} \mathrm{x}\left[j_{A}^{3}(\mathrm{x})\right]|\mathrm{n}\rangle=5 / 9
$$

where both the proton and the neutron are in the $j_{z}=+1 / 2$ state. This value of $g_{A}$ is less than one-half the observed value, 1.25.

We can't say whether this unsatisfactory result is an argument against models of this type because the models being studied do not incorporate partially conserved axial-vector current (PCAC). This is evident from the fact that the $\pi$ and $\rho$ mesons are degenerate, although the $\pi$ should be a Goldstone boson associated with chiral symmetry. Whether or not proper inclusion of PCAC will sufficiently modify the axial-vector current in this model is an open question.

PCAC and the role of the pion present a fundamental challenge to all quark models of hadrons. It is very attractive to suppose that the successes of $\operatorname{SU}(2) \times \operatorname{SU}(2)$ are explained by viewing the pion as a Goldstone boson. On the other hand, in a quark model with $\operatorname{SU}(6)$ mass spectra, the pion is simply a $q \bar{q}$ bound-state partner of the $\rho$ meson in the 35 and is accorded no special role. How to make these two different viewpoints mutually compatible is at present an unsolved problem.

In our theory we do not have PCAC because the divergence of the axial-vector current $\partial_{\nu} A_{\alpha}^{\nu}=G \sigma \bar{\psi} \gamma_{5} \lambda_{\alpha} \psi$ is nonvanishing, and, in the strong-coupling limit with $G \gg 1$, is in no sense a "small operator". These difficulties with PCAC may be related to the unsatisfactory result for $g_{A}$ which we have obtained since (24) may very well define the wrong operator in contrast with the magnetic moment operator which is constructed from the known and conserved electromagnetic current.

We turn next to the question of excitations of the hadron. So far we have considered only the spherical ground state of the hadronic shell, or bubble, and now we ask: "How soft is this bubble when one of the quarks is excited?"

The key question in the treatment of excited states is how rigidly the "classical" potential $\mathrm{g}(\mathrm{x})$ (the $\sigma$ expectation value) resists changing when a quark is excited. If $\mathrm{g}(\mathrm{x})$ remains very nearly spherically symmetric, then a quark with nonzero orbital angular momentum $\ell$ will have an cnergy $\mathrm{M}_{\ell}=(\ell+1)^{2 / 3} \mathrm{M}_{0}$ which is the spectrum for the Dirac equation in the potential $G g(x)=G f \tanh (\sqrt{\lambda f}(r-R))$, the approximate spherical solution to (15). However it is evident from (15) that $\mathrm{g}(\mathrm{x})$ will not remain exactly spherical when the quark is in an $\ell \neq 0$ state.

In fact, if angular momentum is imparted to a quark along, say, the $z$ direction, its wave function will develop nodes along this direction and extend primarily in orthogonal directions as illustrated in Fig. 4. We, therefore, expect the scalar potential to collapse in shape around the quark while extending along the direction of momentum $\vec{p}$ since it can reduce the surfacc area of the confining bubble and, thereby, also the field energy carried by the scalar field $g(x)$. At the same time, this deformation will not further squeeze the quark-wave function which, when $\ell>0$, is not using all the space available to it and so it will not increase its energy. Thus, we intuitively expect that the shape of the sclf-consistent scalar field will be distorted when the confined quarks carry angular momentum reflecting the softness of the bag. This speculation can be extended further to a description of final hadronic states and of the origins for Zweig's Rule. Basically, the idea is that the quark in the hadronic shell that is struck by an incident (virtual $\gamma$ or $\mathrm{W}^{+}$) current recoils stretching the surface (and also destroying local color neutrality and unshielding large color current densities when the color gauge interactions are included).

As the surface of the shell increases in area, the threshold for producing quarkantiquark pairs decreases since they have more space in which to move, and so there will be a critical distance at which the


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Fig. 4
energy stored in the confined color fi.eld will exceed the $q \bar{q}$ production threshold. At this moment, a $q \bar{q}$ pair will be produced and the color field will break and join separating sets of quarks. The shell will then break in two, corresponding to two states having the quantum numbers of a baryon and a meson for the example illustrated in Fig. 4. The process will repeat itself until the resulting fragments no longer have enough energy to separate any further as in


Fig. 5 Fig. 5. They will then oscillate and decay into hadrons via a different mechanism. As a consequence of the existence of these two different mechanisms, one would expect to have a set of excited clusters formed, possibly spaced by a fixed distance in rapidity, which would decay into ordinary hadrons. Hence the general picture of an inside-outside cascade producing a plateau with short-range correlations in rapidity would seem natural from this point of view.

This mechanism for the breakup of a hadronic shell illustrates Zweig's Rule because the basic process corresponds to a local $q \bar{q}$ pair production as illustrated. This is precisely the mechanism as illustrated in the usual quark line graph also shown in Fig. 5 corresponding to the shell breakup.

A formal development of the theory of deformed shells has been given by Roscoe Giles (to be published; Stanford Ph. D. Dissertation 1975), the main points of which I will now discuss.

## DEFORMED BAGS ${ }^{6)}$

The basic idea of this approach is to take advantage of the thin shell character of the semi-classical solution described so far; i.e., of the smallness of the ratio of the surface thickness $D \sim 1 / \lambda^{1 / 2}$ f to the bubble radius $R \sim 1 / \lambda^{1 / 6_{f}}$. Generalizing to arbitrary bubble shape we shall keep $\left(\lambda^{1 / 6} \mathrm{f}\right.$ ) fixed with $\lambda, \mathrm{G} \rightarrow \infty$ and $\mathrm{f} \rightarrow 0$ so that the thickness of the transition region $1 / \lambda^{1 / 2} \mathrm{f} \rightarrow 0$; we call this the strong coupling limit.
Retaining the inequality $\lambda^{1 / 6} \ll G$ as required for strong binding it can be shown that for an arbitrary shape the bubble can be pictured as an infinitely thin shell (i.e., a spacially closed hypertube in space-time for the general nonstatic case) upon which free quark fields are defined.

The procedure for solving the coupled field equations (15) is the following generalization of that used for the static spherical solution:
(i) Assume the solution will be a bubble of some as yet undetermined shape and solve the field equation for $g(x)$ approximately for such a configuration.
(ii) Construct the approximate solution to the Dirac equation in the presence of this $g$ field to leading order for large $\lambda$ and $G$. This gives the Dirac energy up to corrections which vanish in the strong coupling limit.
(iii) Show that if the shape of the bubble surface is chosen to minimize the total energy, all further corrections to the fields give vanishingly small corrections to the total energy in the strong coupling limit.

To begin with we define the bubble surface-i.e., the narrow transition region of space inside of which $g=-f$, and outside of which $\mathrm{g}=+\mathrm{f}$ as in Fig. 6. The boundary surface of the bubble is defined by giving its points as functions of three "internal" coordinates. $u^{0}, u^{1}, u^{2}$; i.e., by

$$
\begin{array}{ll}
\mathrm{R}^{\mu}\left(\mathrm{u}^{\alpha}\right) & \alpha=0,1,2 \\
& \mu=0,1,2,3 \tag{25}
\end{array}
$$

In the static case there are two coordinates $u^{1}, u^{2}$ and $\vec{R}_{\text {is }}$ defined as that closed surface in space at which the $g$ field goes through zero. Let us first work out explicitly the static solution.


Fig. 6

Because all fields will have a nontrivial spacial dependence only in a very thin shell about this surface, it is convenient to use a set of (non-cartesian) spacial coordinates centered about it:

$$
\overrightarrow{\mathrm{x}}\left(\mathrm{u}^{\alpha}, \xi\right)=\overrightarrow{\mathrm{R}}\left(\mathrm{u}^{\alpha}\right)+\xi \hat{\mathrm{n}}\left(\mathrm{u}^{\alpha}\right)
$$

where $\hat{n}\left(u^{\alpha}\right)=$ unit normal to surface at point $u^{\alpha}$.
rne coordinates ( $u^{1}, u^{2}, \xi$ ) are well defined only within a distance on the order of one radius of curvature away from the surface. Henceforth it is assumed that the radii of curvature of the bubble surface, $\overrightarrow{\mathrm{R}}$, are always large compared to D; i.e., the surface has no sharp corners. This assumption has no effect on the spcctrum of low-lying excitations of the theory in the strong coupling limit.

In the new coordinate system, we can write the gradient:

$$
\vec{\nabla}=\vec{\nabla}_{\|}+\hat{n} \frac{\partial}{\partial \xi}
$$

where $\vec{\nabla}_{\|}$is the "tangential" gradient which, though it depends on $\xi$, involves only differentiations with respect to the $u^{\alpha}$, and is tangent, as a vector, to the surface.

Consider the field equation for $g$. We will choose, as our first approximation to $g$, a function that satisfies the "largest" part of Eq. (15) near the surface. Because $g$
makes its transition from $-f$ to $+f$ in a distance $D$, we expect:

$$
\frac{\partial g}{\partial \xi} \sim \frac{1}{D} f
$$

while

$$
\nabla_{\|} \mathrm{g} \sim \frac{1}{\mathrm{R}} \mathrm{f} \ll \frac{1}{\mathrm{D}} \mathrm{f}
$$

In the limit of infinite radius of curvature, $\mathrm{R} \rightarrow \infty$ the bubble surface becomes a kink in one space dimension and the exact solution to Eqs. (15) can be constructed. We reproduce this solution here since it contains all essential features of the general problem. We have to solve the coupled equations

$$
\begin{equation*}
\frac{d^{2}}{d x^{2}} g(x)-4 \lambda g\left(g^{2}-f^{2}\right)=G \bar{\chi} \chi(x) \tag{26}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\frac{1}{i} \alpha \frac{d}{d x}+G \beta g(x)\right) \chi(x)=\mathscr{E} \chi \tag{27}
\end{equation*}
$$

Since there is no spin in 1 space dimension, we have the two-component form for $\chi(x)$

$$
\chi(\mathrm{x})=\binom{\chi_{\mathrm{u}}(\mathrm{x})}{\chi_{\ell}(\mathrm{x})}
$$

and choose a convenient representation in terms of Pauli matrices

$$
\beta=\sigma_{3} ; \quad \alpha=\sigma_{1}
$$

First observe that if we set $G \bar{\chi} \bar{\chi}=0$ in (26), then the resulting equation admits the exact solution

$$
\begin{equation*}
\mathrm{g}(\mathrm{x})=\mathrm{f} \tanh \left(\sqrt{2 \lambda} \mathrm{f}\left(\mathrm{x}-\mathrm{x}_{0}\right)\right) \tag{28}
\end{equation*}
$$

If we now consider this as the input potential, we find that (27) then admits the exact solution

$$
\begin{equation*}
\chi(\mathrm{x})=\mathrm{N}\left[\cosh \left(\sqrt{2 \lambda} \mathrm{f}\left(\mathrm{x}-\mathrm{x}_{0}\right)\right)\right]^{-\mathrm{G} / \sqrt{2 \lambda}}\binom{1}{\mathrm{i}} \tag{29}
\end{equation*}
$$

with $\mathscr{E}=0$. Now, using (29) to compute $\bar{\chi} \chi=\chi^{\dagger} \beta \chi$, we find

$$
\begin{equation*}
\bar{\chi} \chi=0 \tag{30}
\end{equation*}
$$

and so (28) and (29) provide exact solutions to the coupled equations. The general form of this solution is as shown in Fig. 7. Although the quark


Fig. 7
is confined to a narrow region, the total energy corresponding to the quark part of the Hamiltonian manages to be $\mathscr{E}=0$. The reason for this is basically the same as discussed in the three dimensional solution and shown in Eq. (9).

The fermion source term on the right hand side of (26) vanishes in the case of one space dimension according to (30). In the case of the spherical solution it is relatively unimportant in (15) and we anticipate that it remains small for arbitrary shapes. This assertion, which must be verified to insure self-consistency, lcads our first approximation to (15) in the neighborhood of the surface:

$$
\begin{equation*}
\frac{\partial^{2} g}{\partial \xi^{2}}+4 \lambda g\left(f^{2}-g^{2}\right)=0 \tag{31}
\end{equation*}
$$

This is the same as (26) and (30) for the kink of the one dimensional theory. The solution of (31) which satisfies the boundary conditions and vanishes on the surface is unique:

$$
\mathrm{g}(\mathrm{x})=\mathrm{g}(\xi)=\mathrm{f} \tanh \sqrt{2 \lambda} \mathrm{f} \xi
$$

Next, we solve the Dirac equation in the presence of this g field.

$$
\begin{equation*}
\left[-\mathrm{i} \vec{\alpha} \cdot \vec{\nabla}_{\|}-\mathrm{in} \cdot \vec{\alpha} \frac{\partial}{\partial \xi}+\gamma^{o} \mathrm{Gf} \tanh \sqrt{2 \lambda} \mathrm{f} \xi\right] \psi=\mathrm{E} \psi \tag{32}
\end{equation*}
$$

We expect that $\psi$ will fall off exponentially as $\sim \mathrm{e}^{-\mathrm{Gf}|\xi|}$ away from the surface as in the spherical and the one dimensional example (29). This suggests that we attempt to factor out the leading exponential behavior and then expand its coefficient in $1 / \mathrm{G}$.

We write:

$$
\begin{gathered}
\psi\left(\mathrm{u}^{\alpha}, \xi\right)=\mathrm{Ne} \\
\mathrm{EGF}(\lambda, \xi)\left[\psi_{0}\left(\mathrm{u}^{\alpha}, \xi\right)+\frac{1}{\mathrm{G}} \psi_{1}\left(\mathrm{u}^{\alpha}, \xi\right) \ldots\right] \\
\mathrm{E}=\mathrm{E}_{0}+\frac{1}{\mathrm{G}} \mathrm{E}_{1}
\end{gathered}
$$

where $\mathrm{F}, \psi_{0}, \mathrm{E}_{0}$ are independent of $\mathrm{G}, \mathrm{N}$ is a normalization constant, $\psi_{0}, \psi_{1}$ are finite near $\xi=0$ as $G \rightarrow \infty, E_{0}, E_{1}$ are finite as $G \rightarrow \infty, \psi_{0}+\frac{1}{G} \psi_{1}$ is the beginning of an expansion of the field in powers of $1 / \mathrm{G}$. As will become evident, only the properties of the first term will be important.

Substituting this form in the Dirac equation (15) give

$$
\begin{align*}
\mathrm{G}\left[-\hat{\mathrm{n}} \cdot \vec{\alpha} \frac{\mathrm{dF}}{\mathrm{~d} \xi}\right. & \left.+\gamma^{\mathrm{o}} \mathrm{f} \cdot \tanh \sqrt{2 \lambda} \mathrm{f} \xi\right] \psi_{0}\left(\mathrm{u}^{\alpha}, \xi\right)+\left[-\mathrm{in} \cdot \vec{\alpha} \frac{\partial}{\partial \xi}-\vec{i} \vec{\alpha} \cdot \vec{\nabla}_{\|}\right] \psi_{0}\left(u^{\alpha}, \xi\right) \\
& +\left[\gamma^{\left.o_{\mathrm{f}} \tanh \sqrt{2 \lambda} \mathrm{f} \xi-i \hat{n} \cdot \vec{\alpha} \frac{\mathrm{dF}}{\mathrm{~d} \xi}\right] \psi_{1}\left(\mathrm{u}^{\alpha}, \xi\right)=\mathrm{E}_{0} \psi_{0}\left(\mathrm{u}^{\alpha}, \xi\right)+\mathrm{O}\left(\frac{1}{\mathrm{G}}\right)}\right. \tag{33}
\end{align*}
$$

To solve we first equate the coefficient of order $G$, and then of order unity. The cocfficient of $G$ is

$$
\left[-\hat{\mathrm{in}} \cdot \vec{\alpha} \frac{\mathrm{dF}}{\mathrm{~d} \xi}+\gamma^{\mathrm{o}} \mathrm{f} \tanh \sqrt{2 \lambda} \mathrm{f} \xi\right] \psi_{0}\left(\mathrm{u}^{\alpha}, \xi\right)=0
$$

In order for there to be any solution of this matrix equation such that $\psi_{0} \neq 0$, we must have

$$
\frac{\mathrm{dF}}{\mathrm{~d} \xi}=-\mathrm{f} \tanh \sqrt{2 \lambda} \mathrm{f} \xi
$$

where the minus sign is fixed by the requirement that $F$ decrease with $|\xi|$. Hence

$$
\begin{equation*}
\left(\gamma^{\circ}+\hat{\mathrm{n}} \cdot \vec{\alpha}\right) \psi_{0}=0 \quad \text { and } \quad \mathrm{e}^{+\mathrm{GF}(\xi)}=[\cosh \sqrt{2 \lambda} \mathrm{f} \xi]^{-\mathrm{G}} \sqrt{2 \lambda} \tag{34}
\end{equation*}
$$

The equation between the terms of order unity in (33) becomes

$$
\begin{align*}
& {[-\mathrm{i} \alpha} \\
& \nabla_{\|}\left.-\hat{\mathrm{in}} \cdot \vec{\alpha} \frac{\partial}{\partial \xi}\right] \psi_{0}\left(\mathrm{u}^{\alpha}, \xi\right)  \tag{35}\\
&+\mathrm{f} \tanh \sqrt{2 \lambda} \mathrm{f} \xi\left(\gamma^{o}+\hat{\mathrm{in}} \cdot \vec{\alpha}\right) \psi_{1}\left(\mathrm{u}^{\alpha}, \xi\right) \\
&= \mathrm{E}_{0} \psi_{0}\left(\mathrm{u}^{\alpha}, \xi\right)
\end{align*}
$$

Multiplying by $\left(\gamma^{\mathrm{o}}+\mathrm{in} \cdot \vec{\alpha}\right)$ and using (34) and the fact that $\left(\gamma^{\mathrm{o}}+\hat{\mathrm{in}} \cdot \vec{\alpha}\right)^{2}=0$ gives

$$
\frac{\partial \psi_{0}}{\partial \xi}=-\mathrm{k} \psi_{0}
$$

where

$$
\mathrm{k} \equiv \frac{1}{2}\left(\vec{\nabla}_{\mathrm{ll}} \cdot \hat{\mathrm{n}}\right)
$$

The quantity k depends on the geometry alone being proportional to the mean curvature of the surface at each point. At $\xi=0$, where the term in (35) involving $\psi_{1}$ vanishes, we have

$$
\begin{equation*}
\left[-\mathrm{i} \vec{\alpha} \cdot \vec{\nabla}_{\|}+\mathrm{ikn} \cdot \vec{\alpha}\right] \psi_{0}\left(\mathrm{u}^{\alpha}, 0\right)=\mathrm{E}_{0} \psi_{0}\left(\mathrm{u}^{\alpha}, 0\right) \tag{36}
\end{equation*}
$$

This is an eigenvalue equation for $\mathrm{E}_{0}$ involving only the Dirac field on the surface. Thus, given only the geometry of the bubble surface, the Dirac energy can be computed, up to terms that vanish in the strong coupling limit, by solving (34) and (36).

The final step in this program is to show that when the bubble shape is determined by requiring the total energy in the quark plus the $g$ field to be stationary under variations of the bubble configurations, further corrections to the above solutions vanish in the strong coupling limit. This condition is the generalization of the energy minimization principle used to determine the radius of the spherically symmetric solution earlier in (16) and (17).

We only sketch this result here. Formally one finds ${ }^{6)}$ that the only corrections to the form of the scalar field, $g$, that lead to corrections to the energy which are finite in the strong coupling limit correspond to motions of the surface itself, rather than to changes in the shape of the $g$ field near the surface. The calculation of the total energy is accurate in the strong coupling limit, then, if and only if the total energy that is computed is stationary under all local variations of the bubble surface.

The total field energy is the sum of the Dirac energy, E, and the energy associated with the g field configuration. To lowest order in $\mathrm{D} / \mathrm{R}$, the g field energy is given by

$$
\begin{align*}
\mathrm{E}_{\sigma}= & \int \mathrm{d}^{3} \mathrm{x}\left[\frac{1}{2}(\nabla \mathrm{~g})^{2}+\lambda\left(\mathrm{g}^{2}-\mathrm{f}^{2}\right)^{2}\right] \\
& \int \mathrm{d} \mathscr{A}\left[\int \mathrm{~d} \xi\left\{\frac{1}{2}\left(\frac{\partial \mathrm{~g}}{\partial \xi}\right)^{2}+\lambda\left(\mathrm{g}^{2}-\mathrm{f}^{2}\right)^{2}\right\}\right]  \tag{37}\\
= & C \mathscr{A}
\end{align*}
$$

where g is the solution to (31), $\mathrm{C} \equiv \frac{4}{3} \sqrt{2 \lambda} f^{3}$, and $\mathscr{A}=$ area of bubble. Thus, the g field energy is simply proportional to the area of the bubble surface, with the combination of the couplings

$$
\mathrm{C}=\frac{4}{3} \sqrt{2 \lambda} \mathrm{f}^{3}
$$

playing the role of a constant energy density per unit area.
Physically what has been shown is that, in the strong coupling limit, only a very special class of solutions exist which retain low energies. These low energy solutions are, locally, just like the one dimensional kink. The only degrees of freedom that remain are those that describe how these local one dimensional kinks are patched together continuously in space time-i.e., $\overline{\mathrm{R}}\left(\mathrm{u}^{\alpha}\right)$, and the surface Dirac field, $\psi_{0}\left(\mathrm{u}^{\alpha}, 0\right)$, which defines how the quark is apportioned among kinks.

To summarize the static field equations can be written in terms of this reduced set of variables as follows:

$$
\begin{align*}
& \left(\gamma^{o}+\hat{\mathrm{in}} \cdot \vec{\alpha}\right) \psi_{0}\left(\mathrm{u}^{\alpha}\right)=0  \tag{38}\\
& \left(-\mathrm{i} \vec{\alpha} \cdot \vec{\nabla}_{\|}+\mathrm{ikn} \hat{\mathrm{n}} \cdot \vec{\alpha}\right) \psi_{0}\left(\mathrm{u}^{\alpha}\right)=\mathrm{E}_{0} \psi_{0}\left(\mathrm{u}^{\alpha}\right)  \tag{39}\\
& \delta_{\text {geometry }}\left(\mathrm{E}_{0}+\mathrm{C} \mathscr{A}\right)=0 \tag{40}
\end{align*}
$$

As a consequence of (38) we can write the Dirac equation in a two-component Hamiltonian form. The spinor structure dictated by (38) is

$$
\begin{equation*}
\psi_{0}=\binom{1}{\hat{\mathrm{n}} \cdot \sigma} \chi\left(\mathrm{u}^{\alpha}\right) \tag{41}
\end{equation*}
$$

in terms of two-component Pauli $\sigma$ matrices, in the representation

$$
\beta=\left(\begin{array}{c|c}
1 & 0 \\
\hline 0 & -1
\end{array}\right) ; \quad \vec{\alpha}=\left(\begin{array}{l|l}
0 & \vec{\sigma} \\
\hline \vec{\sigma} & 0
\end{array}\right)
$$

equation (39) can now be written $\mathrm{H}_{\chi}=\mathrm{E}_{0} \chi$ with

$$
\begin{equation*}
\mathrm{H} \equiv \frac{1}{\mathrm{i}} \sigma \cdot\left(\hat{\mathrm{n}} \times \vec{\nabla}_{\|}\right)+\mathrm{k} \tag{42}
\end{equation*}
$$

The first term of $H$ is recognized as the angular momentum and the second term shows the local mean curvature of the surface playing the role of a mass.

An exact static solution to (10) and (42) can be constructed in two space dimensions, in which case the bubble is a simple closed curve. Giles has shown in this case that the total energy is independent of the shape of the curve and depends only on its perimeter. There is thus a degeneracy in the energy for all curves of a given perimeter showing that this two dimensional bubble is extremely soft to shape distortions. This result is simply exhibited. We choose the length $\ell$ as the single parameter describing this curve

$$
\overrightarrow{\mathrm{R}}=\overrightarrow{\mathrm{R}}(\ell)
$$

and introduce the unit vectors $\hat{e}=d \vec{R} / d l$ and $\hat{n}$ defined by $\hat{n} \hat{e}=\hat{Z}$ in the plane of the curve as illustrated in Fig. 8. The curvature is

$$
\mathrm{k}=\frac{1}{2} \hat{\mathrm{e}} \cdot \frac{\mathrm{~d} \hat{\mathrm{n}}}{\mathrm{~d} \ell}=\frac{1}{2} \frac{\mathrm{~d} \phi}{\mathrm{~d} \ell}
$$

where $\phi$ is the angle of the normal with respect to some fixed direction in the plane.

The Dirac equation is

$$
\left[\frac{1}{2} \frac{\mathrm{~d} \phi}{\mathrm{~d} \ell}-\mathrm{i} \sigma_{3} \frac{\mathrm{~d}}{\mathrm{~d} \ell}\right] \chi=\mathrm{E} \chi
$$

which may be integrated immediately to yield


Fig. 8

$$
\begin{equation*}
\chi(\ell)=\mathrm{e}^{\mathrm{i} \sigma_{3}\left[\mathrm{E} \ell-\frac{1}{2}(\psi(\ell)-\phi(0))\right]_{\chi(0)}} \tag{43}
\end{equation*}
$$

$\chi$ must be single valued, so we have $\chi(\mathrm{L})=\chi(0)$ where $\mathrm{L}=$ total length or

$$
2 \pi \mathrm{n}=\mathrm{EL}-\frac{1}{2}[\phi(\mathrm{~L})-\phi(0)]=\mathrm{EL}-\pi
$$

where n is an integer. The Dirac energy is

$$
\begin{equation*}
\mathrm{E}=\frac{2 \pi \mathrm{~m}}{\mathrm{~L}}, \quad \mathrm{~m} \equiv \mathrm{n}+\frac{1}{2} \tag{44}
\end{equation*}
$$

and the normalized Dirac wave function can be written

$$
\chi=\frac{1}{\sqrt{L}} e^{\mathrm{i} \sigma_{3}\left(\mathrm{El}-\frac{1}{2} \phi(\ell)\right)} \mathrm{u}
$$

where $u$ is a fixed unit spinor.
The Dirac energy is seen to depend only on the perimeter of the bubble, $L$, not on its shape. There are paired positive and negative energy levels of the same magnitude. There is no zero energy mode.

We interpret negative energy quark states as positive energy antiquarks. The total bubble energy is, then,

$$
\mathrm{U}=\frac{2 \pi|\mathrm{~m}|}{\mathrm{L}}+\mathrm{CL}
$$

Minimizing over L, we have

$$
\begin{aligned}
& L=\frac{2 \pi|M|^{1 / 2}}{C} \\
& U=(8 \pi C)^{1 / 2}|M|^{1 / 2}
\end{aligned}
$$

It is straightforward to check that, if $L$ is chosen to minimize $U$ as above, Eq. (42) is satisfied at each point on the bubble surface.

The two dimensional bubble is, then, extremely soft. Static bubble states occur only with perimeters fixed by the Dirac quantum number $m$; but bubbles of all shapes with this perimeter are degenerate classically. This shape independence is an expression of the fact that there is no intrinsic curvature on the one dimensional manifold around which the spinor is being transported by the Dirac equation (42). In the language of quantum theory the bubble's softness is reflected in the large quantum fluctuations of the surface. The three dimensional bubble is also soft, but not so soft that all shapes are degenerate.

The angular momentum, $\mathrm{J}_{3}$, can be shown to depend on the bubble shape as

$$
\begin{align*}
\mathrm{J}_{3} & \equiv \mathrm{M}^{12}=\int \mathrm{d} l\left[\mathrm{R}^{1} \mathrm{~T}^{0(2)}-\mathrm{R}^{2} \mathrm{~T}^{0(1)}\right]  \tag{45}\\
& =\frac{\mathrm{E}}{\mathrm{~L}}<\sigma_{3}>\int \mathrm{dl}[\overrightarrow{\mathrm{R}} \times \hat{\mathrm{e}}]_{3}=\frac{\mathrm{E}}{\mathrm{~L}}<\sigma_{3}>\mathrm{A}
\end{align*}
$$

where $\mathrm{T}^{0(\mathrm{i})}$ are the components of the stress tensor and A is the total area of the bubble, which of course depends on its shape. Using the expression (44) for E, we can rewrite this result:

$$
\begin{equation*}
\left.\mathrm{J}_{3}=|\mathrm{m}|<\sigma_{3}\right\rangle\left[\frac{\mathrm{A}}{\pi\left(\frac{\mathrm{~L}}{2 \pi}\right)^{2}}\right] \tag{46}
\end{equation*}
$$

or

$$
J_{3}=(8 \pi \mathrm{C})^{-1} \mathrm{U}^{2}<\sigma_{3}>\left[\frac{\mathrm{A}}{\pi\left(\frac{\mathrm{~L}}{2 \pi}\right)^{2}}\right]
$$

The ratio $A /\left[\pi(\mathrm{L} / 2 \pi)^{2}\right]$ is the ratio of the area of the bubble to the maximum area it could have, given perimeter $L$. The state of maximum area is a circle, which is unique. Thus, the maximum possible angular momentum of a state of energy $U$ is

$$
J_{3 \text { max }}\left(U^{2}\right)=(8 \pi C)^{-1} U^{2}
$$

In a Regge picture, this is the statement that the leading Regge trajectory is nondegenerate, and linear in (mass) ${ }^{2}$ with slope $(8 \pi \mathrm{C})^{-1}$.

Unfortunately, the static bubble equations in three dimensions are not so easily solved. In practice one can however carry out variational calculations by choosing a form for the bubble surface that depends on several real parameters, solve for the Dirac energy as a function of these parameters, then minimize the total energy over the parameters that define the surface. Because the total energy functional is positive definite, this variational estimate of the energy is always an upper bound on the energy of the lowest bubble state. The accuracy of such a variational estimate depends entirely on how good a guess is made for the trial surfaces.

I will cite briefly several of the results obtained by Giles using this procedure. A simple two-parameter trial surface that is smooth and flattened at the poles is the oblate spheroid, Fig. 9:

$$
\overrightarrow{\mathrm{R}}(\theta, \psi)=\mathrm{R}\left[\sin \theta \cos \psi, \sin \theta \sin \psi, \sqrt{1-\mathrm{d}^{2}} \cos \theta\right]
$$

$R$ determines the overall size, and $d$ the flatness ( $d=0$ for a sphere and $d=1$ for an infinitely thin pancake): It is found that the spheroid is not an adequate trial sur-


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Fig. 9 face since the resulting energy upper bound decreases monotonically with increasing $d$. Although the spheroid allows the surface to spread at the equator it does not allow for a depression at the poles, and hence for an energy minimum. In any event however the energy change is very small showing a decrease of less than $10 \%$ from the estimate for a sphere as illustrated in Fig. 10 for z-axis angular momentum projections of $\mathrm{m}=3 / 2$ and $\mathrm{m}=5 / 2$.

A two parameter surface which does allow for a polar depression is a torus as illustrated in Fig. 11. The corresponding results for this choice shown in Fig. 12 reveal minima for spin $>1 / 2$ states. Furthermore the energy estimate in the $m=5 / 2$


Fig. 10


Fig. 11


Fig. 12
state is lower for the torus than for the spheroid flattened to a dish in Fig. 9 suggesting a toroidal shape for single quark bubbles with spin $\geq 5 / 2$.

Despite the radical differences in their shapes and topologies, we see that the energics of low-lying single quark states on spheres and on toruses are not very different. We interpret this as a reflection of the "softness" of the three dimensional bubble. This three dimensional result is analogous to the complete shape degeneracy of the two dimensional bubble.

We turn now briefly to timedependent bubbles and generalize the static discussion to include the internal time coordinate $u^{\circ}$ and a boundary surface $R^{\mu}\left(u^{\alpha}\right)$ as in (25) that is a hypertube forming a closed surface in space and infinitely extended in time. The covariant generalization of (38), (39), and (40) has been given by Giles and requires introduction oi a unit spacelike outward normal $\mathrm{n}^{\mu}$ to the bubble surface. In the static case $\mathrm{n}^{\mu} \rightarrow(0, \overrightarrow{\mathrm{n}})$, and generally $\mathrm{n}^{\mu}$ has the following properties:

$$
\begin{gathered}
\mathrm{n}_{\mu} \mathrm{n}^{\mu}=-1 \\
\mathrm{n} \cdot \tau_{\alpha}=\mathrm{n}_{\mu} \tau_{\alpha}^{\mu}=0
\end{gathered}
$$

where

$$
\tau_{\alpha}^{\mu} \equiv \frac{\partial \mathrm{R}^{\mu}}{\partial \mathrm{u}^{\alpha}}
$$

are the tangent vectors to the surface. (The remaining fundamental tensors
describing the surface are the induced metric $\mathrm{g}_{\alpha_{\beta}} \equiv\left(\tau_{\alpha \alpha}\right)^{\mu}\left(\tau_{\beta}\right)_{\mu} \equiv \tau_{\alpha} \cdot \tau_{\beta}$ and the coefficient of curvature is $\mathrm{h}_{\alpha \beta} \equiv-\mathrm{n} \cdot \partial \tau_{\alpha} / \partial \mathrm{u}^{\beta}=\mathrm{h}_{\beta \alpha}$ in terms of which the local mean curvature is expressed by

$$
\mathrm{k}=\frac{1}{2} \mathrm{~g}^{\alpha \lambda} \mathrm{h}_{\lambda \alpha}=\frac{1}{2} \mathrm{~h}_{\alpha}^{\alpha}=\frac{1}{2}\left(\partial_{\|}\right)_{\mu} \mathrm{n}^{\mu}
$$

where $\left(\partial_{\|}\right)_{\mu}=\left(\frac{\partial}{\partial t}, \vec{\nabla}_{\|}\right)$is the generalization of the tangential gradient.) Multiplying (38) and (39) by $\gamma_{0}$ and introducing the time dependent Dirac wave function

$$
\begin{equation*}
\psi\left(u_{\alpha}\right)=\psi_{0}\left(u_{\alpha}\right) e^{-i E t} \tag{47}
\end{equation*}
$$

gives the covariant equations

$$
\begin{gather*}
i \not n \psi=\psi  \tag{48}\\
\left(i \not \varnothing_{\|}+i k \not n^{\prime}\right) \psi=0 \tag{49}
\end{gather*}
$$

Equation (40) which expresses that the total energy is stationary under arbitrary variations of the static surface geometry is naturally generalized to an action principle with the general bubble having a constant action density, -C. Thus we have

$$
\begin{equation*}
\delta \int \mathrm{d} \mathscr{A}\left\{\bar{\psi}\left(\not x_{\|}+\mathrm{kxx}\right) \psi-\mathrm{C}\right\}=0 \tag{50}
\end{equation*}
$$

where $\mathrm{d} \mathscr{A}$ is the invariant area element on the hypertube and the variation can include the geometric variables defining the bubble surface as before; a variation $\delta$ in (50) with respect to $\psi$ just reproduces (49).

Application of this formalism to the special excitation of the three dimensional bubble in a spherically symmetric "breathing" mode has been made by Giles ${ }^{6}$ ) and shows that such surface oscillations have large amplitudes indicating once again in a qualitative way that the bubble surface is "soft" to distortions.

## RELATION TO THE MTT BAG ${ }^{7}$ )

In the MIT model a hadron is a finite region of space to which almost free quanta of the hadronic fields (quarks or partons) are confined. It is obtained from free-field theory with two modifications:
(1) adding to the stress tensor $\mathrm{T}^{\mu \nu}$ a term $\mathrm{g}^{\mu \nu} \mathrm{B}$, called the volume tension, which acts to compress the bag against the outward pressure of the quark gas;
(2) imposing boundary conditions such that the hadronic fields be confined in a finite region of space: the interior of a hadron or the bag.
Because of the boundary conditions, the MIT bag model is not a local field theory as is the SLAC bag. In contrast, however, it builds in permanent quark confinement as a result of the boundary conditions.

Formally, for massless fermion quarks moving within the bag, which is the hadron, one writes the stress tensor

$$
\begin{equation*}
\mathrm{T}^{\mu \nu}=\mathrm{T}^{\mu \nu}+\mathrm{Bg}^{\mu \nu} \text { within the bag } \tag{51}
\end{equation*}
$$

where $\mathrm{T}^{\mu \nu}$ is the usual stress tensor for a spin $1 / 2$ massless Dirac spinor field and the positive constant $B$ represents the volume tension. The boundary conditions are constructed by the requirement that the energy and momentum be conserved within the surface of the hypertube representing the trajectory of the bag in space-time. These are given by

$$
\begin{equation*}
\mathrm{n}_{\mu} \mathrm{T}^{\mu \nu}=0 \text { on the surface } \tag{52}
\end{equation*}
$$

as illustrated in Fig. 13. In terms of the spinor fields (52) becomes

$$
\begin{gather*}
\operatorname{in} \psi=\psi \quad \text { on the surface } \\
2 \mathrm{~B}=\mathrm{n}^{\mu} \partial_{\mu}(\bar{\psi} \psi) \tag{53}
\end{gather*}
$$

The first of (53) coincides with (48). Within the bag the Hamiltonian derived from (51) for a single fermion quark is just that of a massless spin $1 / 2$ particle plus a constant


Fig. 13 term given by the positive constant $B$ multiplied by volume. Thus in a one-particle semi-classical treatment the energy for a static spherical bag of radius $R$ is

$$
\begin{equation*}
E=\frac{c}{\mathrm{R}}+\mathrm{B} \frac{4 \pi}{3} \mathrm{R}^{3} \tag{54}
\end{equation*}
$$

where $c \approx 2$ is a constant characterizing the energy of a massless fermion in a spherical well of radius $R$ with infinitely high walls. Minimizing (54) with respect to $R$ gives a bag as described earlier in (8) with

$$
\mathrm{E}=\frac{4}{3} \mathrm{C} / \mathrm{R}_{0}
$$

The volume tension $B$ corresponds to $f^{4} \lambda$ in (8). The numerical predictions of the MIT bag have generally been very successful ${ }^{8)}$ in reproducing static properties such as described in my first lecture. In particular a value has been found for the axial charge ( $\sim 1.1$ ) which is much closer to experiment than found by the SLAC model although here too one does not have an almost conserved axial current. Additional calculations including $\mathrm{SU}(3)$ breaking effects have had considerable success (see lecture in this volume by V. F. Weisskopf).

There is an obvious difference between the MIT and SLAC bags when we compare their semi-classical solutions. Whereas the SLAC bag is a thin shell with a surface tension created by the quartically self-coupled scalar field to contain the quarks, the MIT bag forms a volume within which the free massless quarks are contained by a volume tension $B$. At a more fundamental level it is important to understand the relation between the two approaches since one (the SLAC model) is based on conventional local field theory treated nonperturbatively, whereas the other (the MIT modcl) modifies the familiar stress tensor within a finite region of space and imposes boundary conditions turning the model into an effectively nonlocal theory. Is conventional field theory adequate or not to describe hadrons and their confined constituents? Or must we make fundamental new dynamical models a'la the MIT bag? Or is the MIT model a valuable phenomenology in the same spirit as the Feshbach-Lomon boundary condition model for nuclear reactions?

A possible connection between the two approaches has been discussed by M. Creutz and K. Soh ${ }^{9}$ ) who have shown at the semi-classical level (i.e., a tree-approximation calculation as described in the first lecture) that the MIT bag can be derived as a limiting case from local field theory if we assume a quartic potential that does not exhibit the symmetry of the self-interaction term in (4) but has the similar form illustrated in Fig. 1 in dcscribing the Lee-Wick model. Essentially a large bare quark mass $M_{Q}$ is added to (4) with a value to cancel precisely the field strength- $G \sigma_{c}$ corresponding to the abnormal vacuum state inside the bag-i.e.,

$$
\mathrm{M}_{\mathrm{Q}}-\mathrm{G} \sigma_{\mathrm{c}}=0
$$

so that inside the quark is massless. However at large distances we must revert to the normal vacuum at $\sigma=0$ in order to avoid an infinite field energy proportional to $U\left(-\sigma_{c}\right)$ times the volume. The limiting parameters to make this transition very sharp with strongly rising potential walls in Fig. 1 and the character of free massless quark motion within the bag can then be readily derived. ${ }^{9)}$ Note only that the value of the quartic self-coupling potential within the bag must be nonzero or else the ground state will correspond to $\sigma=-\sigma_{\mathrm{c}}$ everywhere and there will be no bag. It is easy to see that the MIT volume tension is just the energy density in the abnormal vacuum state, i.e.,

$$
\mathrm{B}=\mathrm{U}\left(-\sigma_{\mathrm{c}}\right)
$$

In contrast to the SLAC bag according to which quark confinement is approximatei.e., isolated quarks as well as color nonsinglet states exist in principle but with (arbitrarily) high production thresholds-the MIT bag model admits only color singlet states when the quarks (i.e., the spinor fields) are coupled to non-Abelian gauge fields associated with color. This is an exact selection rule which follows from the boundary conditions for the color gauge fields in the MIT bag model and from Gauss's law. We
can illustrate it most simply for an Abelian gauge field ${ }^{7}$ ) by writing the extension of (52) and (53) to a vector gauge field

$$
\begin{equation*}
\mathrm{n}_{\mu} \mathscr{\mathscr { F }}_{\text {surface }}^{\mu \nu}=0 \tag{55}
\end{equation*}
$$

The total (color) charge is then

$$
\begin{equation*}
\mathrm{Q}_{\text {color }}=\int_{\substack{\text { Bag } \\ \text { volume }}} \mathrm{d}^{3} \mathrm{x} \partial_{\mathrm{k}} \mathscr{\mathscr { F }}^{\mathrm{ko}}=\int_{\substack{\mathrm{Bag} \\ \text { surface }}} \mathrm{dS} \mathrm{n}_{\mu} \mathscr{\mathscr { Y }}^{\mu \mathrm{o}}=0 . \tag{56}
\end{equation*}
$$

Hence only color singlet bags can be formed. Note however that (55) corresponds to $B_{\tan }=E_{\text {normal }}=0$ on the surface in contrast to the usual condition of $B_{\text {normal }}=0$ by Gauss's law for no magnetic monopoles. The problems that this switch of the roles of $E$ and $B$ creates in the non-Abelian case for local field theory are discussed in Ref. 9.

## QUANTUM CORRECTIONS

Finally we come to a discussion of the quantum corrections to our semi-classical discussion of field theory. We can illustrate the importance of this question by considering the Hamiltonian (4) in one space-one time dimension. In the classical or tree approximation this is exactly soluble as shown in (26)-(30) and the ground state energy is given by neglecting the vacuum fluctuations; according to (37) it is just

$$
\begin{equation*}
\mathrm{E}_{0}=\mathrm{C}=\frac{4}{3} \sqrt{2 \lambda} \mathrm{f}^{3} \tag{57}
\end{equation*}
$$

in $1 \mathrm{x}-1 \mathrm{t}$ dimensions. The lowest order corrections to this result in an expansion in powers of the coupling parameters $\lambda$ and $G$ is given by the one loop corrections to the tree approximation. Physically these corrections correspond to the shift in the zero point fluctuations of the quantized scalar field for small oscillations about the kink solution (28) relative to the constant $g(x)=f$, plus the shift in the energy of the filled negative energy quark sea when the potential (28) is introduced into (27) relative to the energy of sea with $g=f$. These energy shifts are dropped in a tree approximation in which, as we saw, one assumes H to always be normal ordered in whatever basis. However they must be included in comparing the energy of excitation of a kink relative to the constant solution $g=\int$ which leads to the bare quark mass $M_{Q}=G f$.

Formally we arrive at this result by a small fluctuation expansion in (4) in $1 \mathrm{x}-1 \mathrm{t}$ dimension

$$
\sigma(\mathrm{x}, \mathrm{t})=\mathrm{g}(\mathrm{x})+\delta \sigma(\mathrm{x}, \mathrm{t})
$$

with $\mathrm{g}(\mathrm{x})$ defined as a solution of

$$
\frac{d^{2} g}{d x^{2}}-4 \lambda g\left(g^{2}-f^{2}\right)=0
$$

and quantum field amplitudes $\delta 0^{\circ}$ satisfying the usual boson commutation relations. This leads to a Hamiltonian density

$$
\begin{align*}
\mathscr{H}=\left\{\frac{1}{2}\right. & \left.\frac{\mathrm{dg}}{\mathrm{~g}}^{2}+\lambda\left(\mathrm{g}^{2}-\mathrm{f}^{2}\right)^{2}\right\} \\
& +\left\{\frac{1}{2}(\delta \dot{\sigma})^{2}+\frac{1}{2} \frac{\mathrm{~d}(\delta \sigma)}{\mathrm{dx}}{ }^{2}+\frac{1}{2}(\delta \sigma)^{2}\left[4 \lambda\left(3 \mathrm{~g}^{2}-\mathrm{f}^{2}\right)\right]\right\} \\
& +4 \lambda \mathrm{~g}(\delta \sigma)^{3}+\lambda(\delta \sigma)^{4}  \tag{58}\\
& +\psi^{\dagger}\left(\frac{\alpha}{\mathrm{i}} \frac{\mathrm{~d}}{\mathrm{dx}}+\mathrm{G} \beta \mathrm{~g}\right) \psi \\
& +\psi^{\dagger} \mathrm{G} \beta \delta \sigma \psi
\end{align*}
$$

The first line of (58) gives the classical kink energy. The second line of (58) can be expanded in normal modes

$$
\begin{aligned}
& \delta \sigma=\sum_{\mathrm{n}} \frac{1}{\sqrt{2 \mathrm{E}_{\mathrm{n}}}}\left(\phi_{\mathrm{n}} \mathrm{a}_{\mathrm{n}}+\phi_{\mathrm{n}}^{*} \mathrm{a}_{\mathrm{n}}^{\dagger}\right) \\
& \delta \dot{\sigma}=\sum_{\mathrm{n}}-\mathrm{i} \sqrt{\frac{E_{\mathrm{n}}}{2}}\left(\phi_{\mathrm{n}} \mathrm{a}_{\mathrm{n}}-\phi_{\mathrm{n}}^{*} \mathrm{a}_{\mathrm{n}}^{\dagger}\right)
\end{aligned}
$$

with $\phi_{n}$ defined by the wave equation

$$
\left[-\frac{\mathrm{d}^{2}}{\mathrm{dx}^{2}}+4 \lambda\left(3 \mathrm{~g}^{2}-\mathrm{f}^{2}\right)\right] \phi_{\mathrm{n}}=\mathrm{E}_{\mathrm{n}} \phi_{\mathrm{n}}
$$

The zero point energies of these small oscillations sum to

$$
\frac{1}{2} \sum_{\mathrm{n}} \mathrm{E}_{\mathrm{n}}
$$

and the difference

$$
\begin{equation*}
\Delta E^{\text {boson }}=\frac{1}{2} \sum_{\mathrm{n}} \mathrm{E}_{\mathrm{n}}^{(\mathrm{kink})}-\frac{1}{2} \sum_{\mathrm{n}} \mathrm{E}_{\mathrm{n}}^{(\mathrm{no} \mathrm{kink})} \tag{59}
\end{equation*}
$$

gives the shift in energy due to small boson fluctuations about the kink solution with $g$ given by (28) relative to the free solution $\mathrm{g}=\mathrm{f}$.

The third line of (58) gives higher order boson corrections. The fourth line of (58) gives the fermion or quark energy as the one particle ground state plus the sum over the filled negative energy Dirac sea and again the difference between the kink solution (28) and the constant $\mathrm{g}=\mathrm{f}$ gives the fermion contribution to the kink energy-i.e.,

$$
\begin{equation*}
\Delta E^{\text {fermion }}=-\sum_{\mathrm{n}}\left|\epsilon_{\mathrm{n}}^{(\text {kink })}\right|+\sum_{\mathrm{n}}\left|\epsilon_{\mathrm{n}}^{(\text {no kink })}\right| \tag{60}
\end{equation*}
$$

Finally the last line of (58) gives higher order corrections.

The sums (59) and (60) can be performed exactly ${ }^{10}$ ) and display two important properties for us to consider;
(1) The boson fluctuations (59) are lowered in energy by an amount proportional to $\sqrt{\lambda} f$ whereas the negative energy sea (60) is pulled up in energy by the binding of the low lying states to the kink by an amount proportional to Gf. Therefore since the zero point fluctuations are shifted in opposite directions it is possible by a proper choice of the ratio $G / \sqrt{\lambda}$ to arrange for their precise cancellation.
(2) However the individual shifts, being of order of $\sqrt{\lambda} f$ and Gf, which are the large bare masses in the theory, show that in the region of interest a weak coupling expansion about these kink (or "solition" like) solutions does not converge and therefore we cannot proceed in this manner.
We are now faced inescapably with the challenge of strong coupling field theory if wo are to hope to push on beyond our semi-classical solutions and verify the results obtained thus far by our semi-classical approximation. We adopt the following approach. We shall again make variational guesses for the state functions but will keep quantum fluctuations and not resort to a tree approximation as in the first lecture. Since we avoid an iterative weak coupling expansion and the actual implementation of a renormalization procedure which is usually performed in the Feynman graph expansion, we work with a cutoff field theory that is finite at each stage. This cutoff in practice can be expressed either by introducing a finite maximum momentum in the Fourier expansion of the field amplitudes or by formulating the field theory on a lattice. To be specific I will work on a Iattice of finite length $L$ (i.e., volume $L^{3}$ in $3 \overrightarrow{\mathrm{x}}$ dimensions), with $\mathrm{L}_{+} \rightarrow \infty$ eventually, and with a finite Iattice spacing, $1 / \mathrm{m}$, as illustrated in Fig. 14 such that

$$
\begin{equation*}
\mathrm{L}=(2 \mathrm{~N}+1) / \mathrm{m} \tag{61}
\end{equation*}
$$

In the remaining portion of this lecture I will describe work in progress with M. Weinstein and S. Yankielowicz at SLAC in which we have formulated the self-coupled neutral sca-
 lar boson field on a lattice and studied the quantum behavior and spontaneous symmetry breaking using a variational method for bounding the energy from above. To illustrate our method and the mathematical diseases that are pitfalls on the way, we concentrate on the strong coupling behavior
of the boson Hamiltonian

$$
\begin{equation*}
\mathrm{H}=\int \mathrm{dx}\left[\frac{1}{2} \pi^{2}+\frac{1}{2}\left(\frac{\partial \phi}{\partial \mathrm{x}}\right)^{2}+\lambda_{0}\left(\phi^{2}-\mathrm{f}^{2}\right)^{2}\right], \tag{62}
\end{equation*}
$$

ignoring couplings to the spinor quark field. Equation (62) is re-expressed on a lattice by

$$
\begin{equation*}
\frac{1}{m} H=\sum_{j=-N}^{N}\left\{\frac{1}{2} p_{j}^{2}+\lambda\left(x_{j}^{2}-f^{2}\right)^{2}\right\}+\sum_{j, j^{\prime}} \Delta\left(j-j^{\prime}\right) x_{j^{\prime}} x_{j} \tag{63}
\end{equation*}
$$

where for simplicity we work in one space dimension, defining dimensionless variables

$$
\begin{align*}
& \mathrm{x}_{\mathrm{j}} \equiv \phi(\mathrm{x}) \text { at } \mathrm{x}=\mathrm{x}_{\mathrm{j}}, \text { the } \mathrm{j} \text { th lattice site } \\
& \mathrm{p}_{\mathrm{j}} \equiv \frac{1}{\mathrm{~m}} \pi_{\mathrm{j}} \equiv \frac{1}{\mathrm{~m}} \pi(\mathrm{x}) \text { at } \mathrm{x}=\mathrm{x}_{\mathrm{j}}, \text { the } \mathrm{jth} \text { lattice site } \\
& {\left[\mathrm{p}_{\mathrm{j}}, \mathrm{x}_{\mathrm{j}^{\prime}}\right]=-\mathrm{i} \delta_{\mathrm{j}} \mathrm{j}^{\prime}} \tag{64}
\end{align*}
$$

and

$$
\lambda \equiv \frac{1}{\mathrm{~m}^{2}} \lambda_{0} \text { is dimensionless. }
$$

According to the usual approximation of the gradient as a finite difference operator

$$
\Delta\left(\mathrm{j}-\mathrm{j}^{\prime}\right)=\left\{\begin{aligned}
1 & \text { if } \mathrm{j}=\mathrm{j}^{\prime} \\
-1 & \text { if } \mathrm{j}=\mathrm{j}^{\prime}-1 \\
0 & \text { otherwise }
\end{aligned}\right.
$$

An alternative definition of the gradient in terms of the Fourier expansion

$$
\frac{d f(x)}{d x} \equiv \sum_{k=-\frac{2 \pi N}{L}}^{+\frac{2 \pi N}{L}} i k f_{k} e^{i k j / m}
$$

leads to long range correlations via

$$
\frac{1}{2 m} \int d x\left(\frac{\partial \phi}{\partial x}\right)^{2} \rightarrow \frac{2 N+1}{2} \sum_{k=-\frac{2 \pi N}{L}}^{+\frac{2 \pi N}{L}}\left(\frac{k}{m}\right)^{2} \phi_{k} \phi_{-k}=\sum_{j, j^{\prime}-N}^{N} \Delta\left(j-j^{\prime}\right) x_{j} x_{j}
$$

where

$$
\Delta\left(j-j^{\prime}\right) \equiv \frac{1}{2(2 N+1)} \sum_{k}\left(\frac{k}{m}\right)^{2} e^{i k\left(j-j^{\prime}\right) / m}=\left\{\begin{array}{l}
\pi^{2} / 6 \text { if } j=j^{\prime}  \tag{65}\\
\frac{(-1)^{j}-j^{\prime}}{\left(j-j^{\prime}\right)^{2}} \text { if } j \neq j^{\prime}
\end{array} \quad \text { for } N \rightarrow \infty\right.
$$

Either form for $\Delta$ leads to the same conclusions in the following. We can illustrate the essential points most simply by further simplifying (63) to a Schrödinger problem for the anharmonic oscillator in order to recall general properties of the solution, i.e., consider

$$
\begin{align*}
H & =\frac{\mathrm{p}^{2}}{2 \mathrm{M}}+\lambda_{\mathrm{S}}\left(\mathrm{X}^{2}-\mathrm{f}^{2}\right)^{2} \mathrm{M} \\
& =\frac{1}{2 \mathrm{Mf}^{2}}\left\{\mathrm{p}^{2}+\mathrm{g}^{2}\left(\mathrm{x}^{2}-1\right)^{2}\right\} \tag{66}
\end{align*}
$$

where $p \equiv f P$, and $x \equiv \frac{1}{f} X$ are canonical variables and $g^{2}=2 M^{2} f^{6} \lambda_{s}$ is the effective coupling strength. Relative to the field theory problem we have dropped the gradients in (63). Recall that for the anharmonic oscillator of positive mass-i.e.,

$$
\begin{equation*}
\widetilde{\mathrm{H}}=\frac{1}{2 \mathrm{Mf}^{2}}\left\{\mathrm{p}^{2}+\mathrm{g}^{2}\left(\mathrm{x}^{2}+1\right)^{2}\right\} \tag{67}
\end{equation*}
$$

a perturbation expansion in $g^{2}$ does not converge but a variational solution can be made very accurate with very little work. In particular a gaussian packet for a ground state trial solution

$$
\begin{equation*}
\psi \sim \mathrm{e}^{-\alpha \mathrm{x}^{2} / 2} \tag{68}
\end{equation*}
$$

with width $1 / \sqrt{\alpha}$ varied to minimize the energy, gives a ground state energy in error by $<2 \%$ over the entire coupling constant range.

Let us derive this in field theory language and illustrate the weak and strong coupling limiting behavior as a guide to the field theory (63). The trial state for the vacuum or ground state energy is constructed by expressing the coordinate in terms of creation and annihilation operators

$$
\begin{aligned}
& \mathrm{x} \equiv-\frac{1}{\sqrt{2 \alpha}}\left(\mathrm{a}_{\alpha}+\mathrm{a}_{\alpha}^{\dagger}\right) \\
& \mathrm{p} \equiv-\mathrm{i} \sqrt{\alpha / 2}\left(\mathrm{a}_{\alpha}-\mathrm{a}_{\alpha}^{\dagger}\right)
\end{aligned}
$$

and defining the ground state $1 \mathrm{O}_{\alpha}>$ by

$$
\begin{equation*}
\mathrm{a}_{\alpha}\left|\mathrm{O}_{\alpha}\right\rangle=0 \tag{69}
\end{equation*}
$$

The "mass" $\alpha$ is the width parameter of the gaussian (68) and by varying $\alpha$ we find the best energy:

$$
\begin{equation*}
\left.\delta_{\alpha}<\mathrm{O}_{\alpha}|\widetilde{\mathrm{H}}| \mathrm{O}_{\alpha}\right\rangle=0 \tag{70}
\end{equation*}
$$

The weak coupling limit corresponds to $\mathrm{g}^{2} \rightarrow \infty$; i.e., by (67) the oscillator stays very near $x \sim 0$ and the motion and energy are controlled by the curvature near the minimum. In terms of the constants in (66) weak coupling means $\lambda_{S} \rightarrow 0$ for fixed $\lambda_{s} f^{2}$ so that $\mathrm{g}^{2}=2 \mathrm{M}^{2} \mathrm{f}^{6} \lambda_{\mathrm{s}} \rightarrow \infty$. In this case (70) gives $\alpha^{2}=2 \mathrm{~g}^{2}$ for the minimum, corresponding to a narrow oscillator packet at the bottom of the potential well, and the zero point energy is to a leading approximation just that for the simple harmonic motion of small amplitude. In the strong coupling limit $\mathrm{g}^{2} \rightarrow 0$ corresponding to $\lambda_{\mathrm{s}} \gg 1$ for fixed $\lambda_{\mathrm{s}} \mathrm{f}^{2}$. In this case the simple solution of (70) gives $\alpha^{3}=3 g^{2}$ and a ground state energy of

$$
\begin{align*}
\mathrm{E}_{0} & =\frac{3}{8}\left(6 \lambda_{\mathrm{S}} / \mathrm{M}\right)^{1 / 3} \\
& =0.68142\left(\lambda_{\mathrm{s}} / \mathrm{M}\right)^{1 / 3} \tag{71}
\end{align*}
$$

which is less than $2 \%$ higher than the exact result

$$
\mathrm{E}_{0}^{\text {exact }}=0.66799 \ldots\left(\lambda_{\mathrm{s}} / \mathrm{M}\right)^{1 / 3} .
$$

The solution also has the physical property of a broad packet of width $1 / \sqrt{\alpha} \sim \mathrm{g}^{-1 / 3}$.
So much for the obvious. Let us return now to (66) with the double minimum in the potential as illustrated in Fig. 15. In this case we expect physically that the strong coupling behavior of a broad packet centered at the origin $\mathrm{x}=0$ for $\mathrm{g}^{2} \rightarrow 0$ will change to a narrow packet centered at the two minima at $x= \pm 1$ for weak coupling. How does this transition actually occur and how can we iliustrate it by the variational calculation?

Evidently the above procedure used in the case of positive oscillator mass van be


Fig. 15 repeated and the good result of (71) will be obtained again in the strong coupling regime of $\mathrm{g}^{2} \ll 1$ since the small bump at the center of the well has negligible impact on the eigenstate and on the energy eigenvalue in this limit. It is also clear that the trial form (69) is inadequate to give a low lying energy state near $\mathrm{x}= \pm 1$ for $\mathrm{g}^{2} \rightarrow \infty$ and therefore it is a very poor guess. In fact a simple cal. culation gives $\alpha=3 / 2$ in the $\mathrm{g}^{2} \rightarrow \infty$ limit corresponding to a ground state energy growing as $\mathrm{g}^{2}$ since there is a finite probability for the oscillator to be found at $\mathrm{x}=0$ where the potential energy is increasing with $g^{2}$ according to Fig. 15. This then gives us the clue that a much better trial state is one that is displaced away from the origin at $x=0$
and suggests the form

$$
\begin{equation*}
\mathrm{S}_{\alpha}>=\mathrm{e}^{-\mathrm{ipc}} 1 \mathrm{O}_{\alpha}> \tag{72}
\end{equation*}
$$

so that

$$
\left\langle\mathrm{S}_{\alpha}\right| \mathrm{x}\left|\mathrm{~S}_{\alpha}\right\rangle=\mathrm{c} .
$$

Equation (72) has two parameters: mass, $\alpha$, and displacement, c, which are varied to determine the ground state energy via (70). In particular for $g \rightarrow \infty$ in the weak coupling limit the minimum will occur for $\mathrm{c} \rightarrow \pm 1$ at one of the potential minima.

To illustrate the behavior for weak and strong coupling regimes it is convenient to plot the ground state energy $E_{0}$ as a function of the displacement $c$, as derived by minimising $\left\langle\mathrm{S}_{\alpha}\right| \mathrm{H} \mid \mathrm{S}_{\alpha}>$ with respect to the mass parameter $\alpha$. A series of graphs for the weak, intermediate, and strong coupling regions is shown in Fig. 16 and summarizes the results of straightforward calculations using (66), (69), and (72). The important feature of this result is that the energy always shows a local minimum for a centered oscillator state at $c=0$. It also exhibits a point of inflection at finite c when $\mathrm{g}^{2}>1$ which develops into the true minimum in the weak coupling limit as c approaches $\pm 1$, and the oscillator falls into the bottom of the well. The existence of the local minimum at $c=0$ suggests at first that the oscillator undergoes a phase transition from the solution with $\mathrm{c}=0$ which describes the ground state when the coupling is sufficiently strong to a displaced solution with $c \neq 0$ when $\mathrm{g}^{2}$ exceeds a critical value $\mathrm{g}^{2}=\mathrm{g}_{\mathrm{cr}}^{2}$ as illus-


Fig. 16 trated in Fig. 16. Intriguing as this may be such a first order phase transition violates a general property of solutions of the Schrödinger equation.

It can be proved rigorously for this Hamiltonian that a behavior as in Fig. 16 for the ground state which exhibits a sudden shift in the mean value of $\langle\mathrm{x}\rangle$ from $\langle\mathrm{x}\rangle=0$ for $\mathrm{g}^{2}<\mathrm{g}_{\mathrm{cr}}^{2}$ to $<\mathrm{x}>\neq 0$ for $\mathrm{g}^{2}>\mathrm{g}_{\mathrm{cr}}^{2}$ is not compatible with the mathematical properties of the Hamiltonian (66) and is hence simply a disease of our particular trial wave
function. ${ }^{11)}$ away from $\langle x\rangle=0$ by the displacement in (72) with $c \neq 0$ we initially increase the energy because of greater overlap of the trial function with the quartically rising cutside potential wall toward which it has been displaced. A packet with $\langle x\rangle \neq 0$ seems to illustrate spontaneous symmetry breaking since it can move in either of the two degenerate directions toward $\langle\mathrm{x}\rangle= \pm 1$. This however is a false evidence of spontaneous breakdown of the underlying symmetry of the Hamiltonian since as we well know from the exact eigensolutions of the Schrödinger problem with a symmetric well as in Fig. 15 the ground state eigenfunction is symmetrical about $x=0$, and for a high barrier at $x=0$ (i.e., large $\mathrm{g}^{2}$ or weak coupling) the probability density is peaked at both $\mathrm{x}=+1$ and $x=-1$. The first excited eigensolution is antisymmetric with a node at $x=0$ and is peaked again at the bottoms of the well. However these two solutions become degenerate only in the limit of infinite height of the central bump or of infinite separation between the two wells so that there is no tunnelling between the two solutions with $\langle x\rangle \rightarrow+1$ and $\langle\mathrm{x}\rangle \rightarrow-1$. Thus the spontaneous symmetry breakdown occurs only in the $\mathrm{g}^{2} \rightarrow \infty$ weak coupling limit. For finite $\mathrm{g}^{2}$ there is always tunnelling, the symmetric solution lies lower than the antisymmetric one, $\langle x\rangle=0$, and there is no symmetry breakdown.

One can of course make more elaborate variational guesses than (72) (such as, for example, by making a symmetrical displacement

$$
\begin{equation*}
\left(e^{i p c} \pm e^{-i p c}\right) \mid O_{\alpha}> \tag{73}
\end{equation*}
$$

this removes the minimum at $\mathrm{c}=0$ and with it the disease), but the important point of this example is to display and recognize an incorrect result so we can avoid it in the more difficult field theory problem (63).

Returning to the field theory example of interest and setting up a variational calculation for the ground state energy for Hamiltonian (63), a natural first guess is the mean field approximation. To do this we expand the canonical fields and momenta in a Fourier series with arbitrary "energies" $\alpha$

$$
\begin{align*}
& x_{j}=\sum_{k} \sqrt{\frac{1}{2 \alpha_{k} L}}\left(a_{k}+a_{-k}^{\dagger}\right) e^{i k j / m} \\
& p_{j}=\sum_{k}(-i) \sqrt{\frac{\alpha_{k}}{2 L}}\left(a_{k}-a_{-k}^{\dagger}\right) e^{i k j / m} \tag{74}
\end{align*}
$$

where the discrete momenta are given by $k=-\frac{2 \pi N}{L}, \ldots, \frac{2 \pi n}{L}, \ldots,+\frac{2 \pi N}{L}$ and

$$
\left[a_{k}, a_{k^{\prime}}^{\dagger}\right]=\delta_{k_{k}}
$$

In terms of this basis the trial ground state analugous to the use of (69) in the Schrödinger problem is defined by

$$
\begin{equation*}
a_{k} \mid 0_{\alpha}>=0 \tag{75}
\end{equation*}
$$

for all k . The ground state energy is then determined by taking the expectation value of (63) in the state $10>$ and varying with respect the $\alpha_{k}$. One immediately finds upon carrying out this procedure that the parameters $\alpha_{k}$ for different $k$ are related by

$$
\begin{equation*}
\alpha_{\mathrm{k}}^{2}=\mathrm{k}^{2}+\alpha_{0}^{2} \tag{76}
\end{equation*}
$$

to a single variational mass parameter $\alpha_{0}$. Equivalently we can ab initio introduce the $\alpha_{k}$ in (74) in terms of a single variational mass parameter $\alpha_{0}$ via the energy-mass relation (76). The discussion of the ground state energy for this field theory example is entirely parallel to that already given for the Schrödinger problem. The next natural step is to introduce a two parameter trial state with the field displaced from zero expectation value in order to study the spontaneous symmetry breaking. Corresponding to (72) we introduce a constant displacement

$$
\begin{equation*}
s>=e^{-i c \Sigma_{j} p_{j}} \mid 0> \tag{77}
\end{equation*}
$$

Both trial forms (75) and (77) are products over the individual momentum cells for each k and are equivalent to the mean field replacement

$$
\begin{aligned}
& x^{3} \rightarrow 3<x^{2}>x \\
& \left.x^{4} \rightarrow 6<x^{2}>x^{2}+3<x^{4}\right\rangle
\end{aligned}
$$

in (62). The results of a variation calculation based on these trial forms are the same in all essentials, including the disease, as found in the Schrodinger problem. Furthermore the symmetrical analogue of (73) fails to remove the illness in this case. Note, hovever, that in the limit $L \rightarrow \infty$ and $\langle x\rangle=c \neq 0$ this wave function describes spontaneous symmetry breaking (unlike the corre-
sponding wave function in the Schrodinger problem) since $\langle\mathrm{s}(+\mathrm{c})| \mathrm{s}(-\mathrm{c})>=$ $\mathrm{e}^{-\mathrm{L} \mathrm{c}^{2} \alpha_{0}} \underset{(\mathrm{~L} \rightarrow \infty}{\longrightarrow} 0$ and there is no tunneling between the two minima at $+c$ and at $-c$.

Let us try now to understand physically what is going on here; why we are led to a "disease" by our momentum base procedure via the trial state (77) and what can we do to avoid the pitfall and do better. One way to illustrate the problem is to


Fig. 17
observe from (63) that the potential has a bump of height $\lambda f^{4}$ near its center at $x_{j} \rightarrow 0$ whereas for small oscillations about the minimum near $x_{j} \rightarrow f$, the zero point energy is $\sim \sqrt{2 \lambda} \mathrm{f}$, as illustrated in Fig. 17. For weak coupling, i.e., for small field amplitudes and harmonic motion near the bottom of the potential well we have the inequality

$$
\begin{equation*}
\lambda f^{4} \gg \lambda^{1 / 2} f \quad \text { or } \quad \lambda^{1 / 2} f^{3} \gg 1 . \tag{78}
\end{equation*}
$$

In this limit the two lowest energy levels are the symmetric and antisymmetric solutions, as described above Eq. (73) for the Schrödinger equation. Their energy splitting is very small due to the suppressed tunnelling of the oscillator amplitudes through the center bump, and hence mixing between these two even and odd oscillator solutions due to the gradient term in (63) will be large. Therefore, it becomes important to treat the last term of (63) accurately, which of course is precisely what we do by working in $\overrightarrow{\mathrm{k}}$-space by the expansion (74) and the choice of ground state (77). This basis diagonalizes the kinetic energy or gradient term. It is also in this weak coupling region described by parameters (78) that the system is safely in the c$\neq 0$ phase for its ground state; this is the $\mathrm{g}^{2}>1$ region of Fig. 16. The trouble arises as we approach the strong coupling region. Formally the energy difference between even and odd solutions increases as $\lambda^{1 / 2_{f}}{ }^{3}$ decreases into the intermediate coupling region, the gradient term of (73) also decreases, and a site basis-i.e., a trial state diagonalizing the terms at individual lattice sites $j$-is the more natural approach for guessing a solution of (63). Indeed the region of $\lambda^{1 / 2_{f}}<1$ in the strong coupling domain is precisely the one that our earlier classical discussions lead us to look to as the one appropriate to our kink solutions. In fact for the kink energy in $1 \mathrm{x}-1 \mathrm{t}$ dimensional case we deduce from (37) that

$$
\begin{equation*}
\lambda^{1 / 2} \mathrm{f}^{3} \ll 1 \tag{79}
\end{equation*}
$$

or in terms of the original dimensional coupling in (73) and (64)

$$
\lambda_{0}^{1 / 2} \mathrm{f}^{3} \ll \mathrm{~m}
$$

if the kink energy is to be finite ( $\approx 1 \mathrm{GeV}$ ) and therefore very small on the scale of the arbitrarily large cutoff mass $m$ introduced via the lattice.

We are led therefore to introduce a trial form for the variational calculation that is diagonal in a single-site basis on the lattice. To do this we introduce creation and annihilation operators at each site $j$

$$
\begin{aligned}
& x_{j}=\frac{1}{\sqrt{2 \alpha_{j}}}\left(a_{j}+a_{j}^{\dagger}\right) \\
& p_{j}=-i \sqrt{\frac{\alpha_{j}}{2}}\left(a_{j}-a_{j}^{\dagger}\right)
\end{aligned}
$$

$$
\begin{equation*}
\left[a_{j}, a_{j^{\prime}}^{\dagger}\right]=\delta_{j j^{\prime}} \tag{80}
\end{equation*}
$$

where $\alpha_{j}$ is a parameter to be varied for the energy extremum, and write for the trial state

$$
\begin{equation*}
|S\rangle=N \prod_{j}\left\{\cos \theta_{j} e^{-i p_{j} C_{j}}+\sin \theta_{j} e^{+i p_{j} C_{j}}\right\}\left|0_{j}\right\rangle \tag{81}
\end{equation*}
$$

with

$$
a_{j}\left|0_{j}\right\rangle=0
$$

and $N$ a normalization constant. $C_{j}$ and $\theta_{j}$ are also variation parameters in terms of which the mean field amplitude is measured; viz.

$$
\left\langle x_{j}\right\rangle=\frac{C_{j} \cos 2 \theta_{j}}{1+e^{-\alpha_{j} C_{j}^{2}} \sin 2 \theta_{j}}
$$

Calculations based on (81) are in progress at the time of these lectures (July 1975) with M. Weinstein and S. Yankiclowicz and we are learning simpler methods of analyzing the ground state properties using (81). However several important properties have been found already that I want to report in concluding these lectures:
(1) The disease of an illegal first order phase transition encountered earlier has now been avoided and a transition from $\left\langle x_{j}\right\rangle=0$ for strong coupling to an arbitrarily small $\left\langle\mathrm{x}_{\mathrm{j}}\right\rangle \neq 0$ as $\lambda$ or $\mathrm{f}^{2}$ increases has been exhibited. Therefore we have displayed that spontaneous symmetry breaking actually occurs in this model. We have also constructed (since these lectures) a low lying energy state with a kink structure analogous to the classical result as in (78).
(2) A "good" energy for the ground state can be achieved. In particular Weinstein and Yankielowicz have explicitly established that this approach in terms of a single site basis gives lower (i.e., better) ground state energies for the parameters $\lambda$ and $f^{2}$ of interest than found in the momentum basis (77) if several higher configurations are mixed in with the ground state oscillator (81).
We believe therefore we have now a valid basis for studying low lying coherent excitations in strong coupling field theory and are presently engaged in studying the coupled, fermion (quark)-scalar field Hamiltonian. Eventually it is our goal to work with gauge theories and to view the self coupled scalar field that gives rise to the spontaneous symmetry breaking in the present treatment as no more than a phenomenological crutch.

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[^0]:    *Work supported by the U. S. Energy Research and Development Administration.

[^1]:    *Work supported by the U. S. Energy Research and Development Administration.

[^2]:    *Although not a new one to the Bible, I thank V. F. Weisskopf for pointing out to me the following quote from Chapter 11, Verse 3, of the "Book of the Hebrews": "By faith we understand that the world was created by the word of God, so that what is seen was made out of things which do not appear."

