

QUARKS AT THE ENDS OF THE STRING

Itzhak Bars*

Department of Physics, Yale University
New Haven, Connecticut 06520

and

Andrew J. Hanson**

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

We develop a technique for attaching quark quantum numbers to worldlines joined by relativistic strings.

We are able to describe spin 0 and spin $1/2$ $U(n)$ quarks attached to worldlines. One spin $1/2$ theory based on the Dirac equation yields a classical particle with helical motion, interpretable as Zitterbewegung. Another spin $1/2$ model has no helical motion, but yields an algebra resembling that of supersymmetry.

Motivated by duality diagrams and some general properties of quark-gluon models, we then construct quark-string models of mesons and baryons. The analysis of the meson model with unequal quark masses implies a stringlike spectrum with broken trajectory intercepts. A simple baryon model suggests a dynamical reason for diquark configurations in the lowest states. Physical weak and electromagnetic currents for the quark-string system follow from a minimal-coupling scheme as in gauge field theories.

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1. Introduction

The quark model has provided a successful framework for understanding many properties of elementary particles and their interactions, such as classification and spectroscopy, deep inelastic scattering from nucleons, and possibly e^+e^- annihilation into hadrons. The details of the strong interactions between quarks, and consequently between elementary particles, are presently unknown. However, some general features of the interaction such as Regge behaviour and approximately linear trajectories have emerged. There are at present two philosophical approaches to the strong interactions, the dual resonance models⁽¹⁾ and the quark-gluon color gauge theories⁽²⁾.

The dual resonance model provides a reasonably successful approximation to the required properties of the strong interaction S-matrix even in the "Born-approximation". Mandelstam⁽³⁾ has shown a direct connection between dual models and interacting strings⁽⁴⁾, thus providing an appealing physical picture for the dynamics underlying the dual models.

In the interacting string picture a meson is represented by a string with two free ends. During interaction, two such strings (mesons) join ends to form a single continuous string with two free ends; this object is interpretable as a resonance which can in turn split (decay) into two or more strings.

As shown in Fig. 1, the spacetime paths of the string ends thus form a realization of the duality diagrams which were used originally⁽⁵⁾ to keep track of the quark model quantum numbers of the intermediate states in dual scattering amplitudes.

This picture strongly suggests that the quark and anti-quark inside the meson are bound together by a string. Then we are led to the interpretation that in the interacting string model the propagation of the string is dictated by the string action of Nambu, while the interaction between strings is a local interaction between quarks on different strings. Two end-quarks annihilate during the formation of a larger string, and a pair of quarks is created when the string splits. Thus the meson interacts only through its "valence" quarks. We note that in Mandelstam's model, the interior of the string does not interact. Although

the local quark interaction is sufficient to reproduce the dual model amplitudes, it is not clear a priori that the string does not interact also in other ways.

Another popular approach to the strong interactions of elementary particles is the non-Abelian gauge theory of $SU(3)$ -colored gluons and quarks. Calculations in this framework are still at an elementary stage, but there are several indications in the literature that some form of the string model may emerge from such theories: Nielsen and Olesen⁽⁶⁾ have argued that vortex-like classical solutions of field theories may be identified with dual strings, and Wilson⁽⁷⁾ and Kogut and Susskind⁽⁸⁾ have suggested a form of string in their lattice formalism for gauge theories; both proposals imply that strings are made of color glue. If this picture is correct, then color singlet mesons can be made only if the string terminates on color "monopoles" - that is, the quarks must be attached to the ends of the string. The endpoints would therefore carry all observable quantum numbers such as spin, charge, isospin, etc., while the body of the string carries none. In addition, 't Hooft⁽⁹⁾ has proposed a two-dimensional color gauge theory model for mesons. The model which will be presented here yields results very close to those of 't Hooft, thus establishing a further connection between the string picture and color gauge theory. Other pictures may also emerge from the color quark-gluon model, but the color model does not seem to be inconsistent with the idea of placing quarks on the ends of strings.

With these motivations in mind, we will propose here models for mesons as strings with quarks at the ends and models for baryons as strings with 3-quarks. The new feature in our approach is the introduction of quark spin and internal symmetry degrees of freedom in the string formalism. Our new variables are not related to the Neveu-Schwarz or Bardakci-Halpern variables⁽¹⁰⁾ previously introduced in the string formulation, but rather they directly correspond to the usual spin and internal symmetry of the quark fields. Our formulation makes a close connection between the standard phenomenological quark model and the string model.

The physical picture that emerges is appealing: the string action produces a relativistic potential which binds the quarks together. Furthermore,

a two-dimensional analysis indicates that the potential energy of the system depends linearly on the separation between the quarks; the quarks are thus trapped in a manner reminiscent of the proposals of 't Hooft⁽⁹⁾, Wilson⁽⁶⁾, Kogut and Susskind⁽⁷⁾.

Our formulation has non-trivial implications that follow from the introduction of quark internal symmetry variables. First, the internal symmetry is broken by the unequal masses of the quarks. This then leads to a spectrum of Regge trajectories with nondegenerate intercepts. These trajectories curve at low energies but are asymptotically essentially linear. The amount of curvature increases with the masses of the quarks determining the quantum numbers of the trajectories. Second, weak and electromagnetic interactions can be coupled directly to the quarks following the same prescription as unified gauge field theories. This then leads to the definition of the physical currents in the string formalism. Weak and electromagnetic interactions couple only to the quarks, not to the string, just as in the quark-gluon model where the colored gluons do not possess weak and electromagnetic interactions. The string, just like the gluons, is the medium of strong interactions between the quarks. Weak and electromagnetic interactions can be treated perturbatively as in the standard field theory approach.

We remark that there are two very different ways of regarding our model. On the one hand, the quarks and the color gluons interacting with them could be considered as the fundamental basis for strong interactions. Then our picture would be a phenomenological approximation to the string-like vortices of, say, Nielsen-Clesen⁽⁶⁾. On the other hand, one might believe that some more sophisticated version of the interacting string model will give the exact solution to the strong interaction problem. If this were the case, then the known connection between the zero-slope limit of dual models and non-Abelian gauge theories would suggest a different viewpoint: vector gluon field theories of strong binding would be phenomenological approximations to the richer structure of a string-like theory. From this second point of view our model is an attempt to incorporate the quark quantum numbers into the picture.

We should point out that the observable quantum numbers of the quarks

could conceivably arise from topological properties of more complex geometrical models, e.g., "membrane" or "jelly" models. For example, the left-twisted and right-twisted ["]Möbius strips could be associated with two different values of a quantum number. Such connections between topology and internal quantum numbers also occur in field theory⁽¹¹⁾. Thus our model with quarks on the ends of the string could be an approximation to a theory based on a geometrical structure more complicated than the string. In such a theory purely topological quark quantum numbers might appear in some limit to be joined by an extended stringlike structure.

In the end, it might even happen that a perfect quark-gluon model and a perfect geometrical model were "dual" to one another in the sense that both would give equivalent descriptions of physical processes.

The present paper deals mainly with the basic principles of our general formalism. We will discuss various simple examples to develop intuition, but will leave for later work a number of difficult problems presented by the most realistic models. We begin in section II by discussing a new approach⁽¹²⁾ to the incorporation of field-theoretic degrees of freedom into point particles lying on a world line. We develop models for spin-0 and spin-1/2 particles carrying internal symmetry. These then form the basis of our technique for attaching point quarks to the string. In section III we summarize what is known about the relativistic string with massive ends⁽¹³⁾, since some cases of our model reduce effectively to this one. Much of our intuition is based upon our knowledge of the string with massive ends. Section IV deals with our essential problem - that of building mesons by replacing the ends of the string with massive quark-like point particles of the type discussed in section II. We also suggest a model of baryons with 3 quarks. In section V, we generalize the field-theoretical minimal-coupling principle to couple external electromagnetic and non-Abelian gauge fields to our point quarks. We are then able to define the physical currents of our model in a natural way. Suggestions for future investigations and a summary of the current work are contained in the final section. An Appendix is devoted to a general technique for restricting fields to a subspace of the physical spacetime.

II. Point Particles with Internal Symmetry and Spin

In order to describe quarks as point particles following world lines attached to the ends of the string, we must find a way of attaching spin and internal symmetry indices to a world line. Since it is clear how to describe conventional fields possessing these extra indices, we will accomplish our goal by starting with conventional fields and restricting them to a world line. We begin for simplicity with a free spinless $U(n)$ quark. Next, we treat the more realistic case of a spin $1/2$ $U(n)$ quark. Here two models are considered: the first, following directly from the Dirac equation, possesses a classical Zitterbewegung, while the other does not. The quantum theory of the second model leads to canonical quantization rules reminiscent of supersymmetry.

A. Spinless particle with $U(n)$

The standard τ -reparametrization invariant action for a free, spinless, relativistic point particle is

$$S = - \int_{\tau_1}^{\tau_2} d\tau \mu \sqrt{-x_\tau^2} \quad (2.1)$$

where μ is the mass of the particle and $x_\tau^\mu = \frac{\partial x^\mu}{\partial \tau}$. Our metric is such that $x^2 = -x_0^2 + x_\sim^2$. The canonical momentum is

$$p^\mu = \mu x_\tau^\mu / \sqrt{-x^2}$$

and obeys the constraint $p^2 + \mu^2 = 0$. Minimizing the action, one finds that p^μ is a constant of motion. We may thus solve the equations of motion for x^μ in the form

$$x^\mu = q^\mu + p^\mu s(\tau) / \sqrt{-p^2}. \quad (2.2)$$

Here the τ -reparametrization invariant function $s(\tau)$ may depend also on the

integration constants q^μ and p^μ . Choosing a gauge, for example the proper time gauge $x^0 = p^0 \tau / \sqrt{-p^2}$, fixes the form of $s(\tau)$ and q^μ . In general, we may write

$$s(\tau) = \int^\tau dt \sqrt{-x_t^2} \quad . \quad (2.3)$$

We thus have a correct classical description of the motion of the particle, but are able to say nothing about its internal symmetries. In order to describe a particle which carries internal symmetry indices, it is clear that we need more variables in addition to the position $x^\mu(\tau)$. We begin by introducing functions $\phi_\alpha(\tau)$, $\alpha = 1, \dots, n$, which form a basis for the spinor representation of $U(n)$. As described in the introduction, this $U(n)$ symmetry refers only to observable symmetries of the quark at the end of the string rather than hidden color symmetry. The ϕ_α are also taken to be scalars under Lorentz transformations and τ - reparametrizations. The simplest action for a massive point particle which is invariant under $U(n)$, Poincare transformations and τ - reparametrizations is

$$S = \int_{\tau_1}^{\tau_2} d\tau L_0 (\phi_\alpha(\tau), \partial_\tau \phi_\alpha(\tau))$$

where the point-particle Lagrangian is

$$L_0 = \sum_\alpha \left[\frac{\partial_\tau \phi_\alpha^\dagger \partial_\tau \phi_\alpha}{\sqrt{-x_\tau^2}} - m^2 \sqrt{-x_\tau^2} \phi_\alpha^\dagger \phi_\alpha \right] \quad (2.4)$$

Hereafter, sums over α will be implicit.

This Lagrangian is closely related to the standard field theoretic description of a free spinless particle with internal symmetry.⁽¹⁵⁾ To see this, consider the spacetime Lagrangian density for a free $U(n)$ Klein-Gordon particle:

$$\mathcal{L} = -\partial_\mu \phi^\dagger(x) \partial^\mu \phi(x) - m^2 \phi^\dagger(x) \phi(x) \quad . \quad (2.5)$$

To restrict the field to a world-line, we require that x^μ be replaced by $x^\mu(\tau)$, where τ parametrizes the world-line. The Cartesian Minkowski metric $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ can be decomposed at any point on the world-line into a complete set of vectors consisting of the timelike tangent to the world-line, $x^\mu_\tau = \partial x^\mu / \partial \tau$, and all of the spacelike normals $n^\mu_i(\tau)$. Thus we have

$$\eta^{\mu\nu} = \frac{x^\mu_\tau x^\nu_\tau}{x_\tau^2} + n^\mu_i n^\nu_i \quad (2.6a)$$

where a sum over the index i is implied. Note that

$$\begin{aligned} x^\mu_\tau \eta_{\mu\nu} x^\nu_\tau &= x_\tau^2 < 0 \\ n^\mu_i \eta_{\mu\nu} x^\nu_\tau &= 0 \\ n^\mu_i \eta_{\mu\nu} n^\nu_j &= \delta_{ij} \quad . \end{aligned} \quad (2.6b)$$

The metric $\eta^{\mu\nu}$ is τ -reparametrization invariant and raises or lowers indices in Minkowski space as usual. Using eq. (2.6a) we can write

$$\eta^{\mu\nu} \partial_\mu \phi^\dagger(x) \partial_\nu \phi(x) = \frac{x^\mu_\tau \partial_\mu \phi^\dagger x^\nu_\tau \partial_\nu \phi}{x_\tau^2} + n^\mu_i \partial_\mu \phi^\dagger n^\nu_i \partial_\nu \phi \quad (2.7)$$

If the field $\phi(x)$ is not to leave the world line, we cannot allow any nonvanishing normal derivatives. Thus we take

$$n^\mu_i \partial_\mu \phi(x) = 0 \quad . \quad (2.8)$$

Furthermore, we note that by the chain rule of differentiation

$$x_{\tau}^{\mu} \partial_{\mu} \phi(x(\tau)) = \partial_{\tau} \phi(\tau) \quad . \quad (2.9)$$

Therefore, we may now consider ϕ to be effectively a function of τ , and substitute eqs. (2.7, 2.8, 2.9) into eq. (2.5). Multiplying by $\sqrt{-x_{\tau}^2}$ (which effectively is a Jacobian), we obtain eq. (2.4). The general technique for restricting an arbitrary field theory to a subspace of arbitrary dimension such as a world-sheet instead of a world-line is discussed in the Appendix.

The equations of motion are obtained by varying the action in the standard way with respect to both $\phi_{\alpha}(\tau)$ and $x^{\mu}(\tau)$. We find the canonical momenta

$$\frac{\partial L_0}{\partial(\partial_{\tau} \phi_{\alpha}^{\dagger})} = \frac{\partial_{\tau} \phi_{\alpha}}{\sqrt{-x_{\tau}^2}} \equiv \Pi_{\alpha} \quad (2.10)$$

$$\frac{\partial L_0}{\partial(\partial_{\tau} x_{\mu})} = [\Pi^{\dagger} \Pi + m^2 \phi^{\dagger} \phi] \frac{x_{\tau}^{\mu}}{\sqrt{-x_{\tau}^2}} \equiv p^{\mu} \quad (2.11)$$

and the Euler equations

$$\partial_{\tau} \Pi_{\alpha} + \sqrt{-x_{\tau}^2} m^2 \phi_{\alpha} = 0 \quad (2.12)$$

$$\partial_{\tau} p^{\mu} = 0 \quad . \quad (2.13)$$

The constants of motion of this system are the total momentum p^{μ} , the Lorentz transformation generators

$$M^{\mu\nu} = x^{\mu}(\tau) p^{\nu}(\tau) - x^{\nu}(\tau) p^{\mu}(\tau)$$

and

$$\mu = \Pi^{\dagger}(\tau) \Pi(\tau) + m^2 \phi^{\dagger}(\tau) \phi(\tau) \quad . \quad (2.14)$$

We also identify the following constraint, which results from τ - reparametrization invariance:

$$p^2 + \mu^2 = 0 \quad . \quad (2.15)$$

For arbitrary $x^\mu(\tau)$, we can solve eq. (2.12) for $\phi_\alpha(x(\tau))$ using the parameter $s(\tau)$ defined by eq. (2.3). The result is

$$\sqrt{2m} \phi_\alpha(\tau) = a_\alpha e^{-ims} + b_\alpha^* e^{ims} + c_\alpha + msd_\alpha \quad , \quad (2.16)$$

where a_α , b_α^* , c_α , and d_α are dimensionless complex constants. Replacing (2.16) in (2.14), we find that the only way to satisfy the constraint for all τ when $m \neq 0$ is to set

$$c_\alpha = 0 = d_\alpha \quad (2.17a)$$

$$\mu = \sqrt{-p^2} = m(a_\alpha^* a_\alpha + b_\alpha^* b_\alpha) \quad . \quad (2.17b)$$

Furthermore, using (2.11) and (2.13) we may solve for $x^\mu(\tau)$:

$$x^\mu(\tau) = q^\mu + p^\mu s(\tau) / \sqrt{-p^2} \quad . \quad (2.18)$$

This is the same as eq. (2.2).

Finally, we note that if $U(n)$ is broken by assigning different masses to the components of ϕ_α ,

$$m = \begin{bmatrix} m_1 & 0 & 0 \dots \\ 0 & m_2 & 0 \dots \\ 0 & 0 & m_3 \dots \\ \vdots & \vdots & \vdots \end{bmatrix} \quad (2.19)$$

Eq. (2.17b) becomes

$$\mu = \sqrt{-p^2} = a^* m a + b^* m b \quad . \quad (2.20)$$

It is straightforward to quantize the theory in a given gauge such as $x^0 = \tau p^0 / \sqrt{-p^2}$ by assuming standard commutation rules for \underline{p} and \underline{q} and taking

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta} = [b_\alpha, b_\beta^\dagger] \quad . \quad (2.21)$$

and all other commutators zero. This is then consistent with the canonical commutation rules $[\phi_\alpha, \Pi_\beta^\dagger] = i \delta_{\alpha\beta}$ etc.

The interpretation of the solution (2.18) and (2.20) is that the center of mass \underline{x} moves like a free particle, while the effective mass of the system, $\sqrt{-p^2}$ is equal to the sum of the number of quanta at the point \underline{x} times the appropriate free mass m_i . The more quanta we put at the point \underline{x} , the heavier the system becomes. The effective mass of the system also depends on the kind of quanta we put in if the $U(n)$ symmetry is broken as in eq. (2.19). This is a satisfactory description of free scalar particles with (broken) internal symmetry.

Although we have restricted ourselves to a $U(n)$ multiplet for the purposes of illustration, it is clear that the treatment can be extended to any representation of any internal symmetry group. It is also clear that our approach could be generalized to interacting theories such as ϕ^4 etc., which would change the solution (2.16) as well as the spectrum of (2.20). We will not attempt to treat these matters here.

B. Dirac Particles with Internal Symmetry

Since it is believed that free quarks would obey the Dirac equation, we now proceed to derive the Lagrangian for a Dirac particle restricted to a world-line using the methods of the previous subsection. We begin with the spacetime Lagrangian density for $U(n)$ quarks,

$$\mathcal{L} = \sum_{\alpha, \beta} \left(-\frac{1}{2} \bar{\psi}_\alpha(x) \not{\partial} \psi_\alpha(x) - \bar{\psi}_\alpha(x) m_{\alpha\beta} \psi_\beta(x) \right) \quad . \quad (2.22)$$

Hereafter sums over the indices α will be implicit. Recall that the indices α refer only to observable symmetries, not to color. Our γ -matrix conventions are, e.g., those of Weinberg⁽¹⁶⁾.

We now restrict $x^\mu(\tau)$ to a world-line parametrized by τ and forbid $\psi(x(\tau))$ to leave the world-line by imposing the condition

$$n_1^\mu(\tau) \partial_\mu \psi(x(\tau)) = 0 \quad . \quad (2.23)$$

Then when we replace the metric $\eta^{\mu\nu}$ in eq. (2.22) by the expression (2.6a), we find the following Lagrangian for a classical pointlike Dirac particle:

$$L_0 = \frac{x_\tau^\mu}{2\sqrt{-x_\tau^2}} \bar{\psi}(\tau) \gamma_\mu \overleftrightarrow{\partial}_\tau \psi(\tau) - \sqrt{-x_\tau^2} \bar{\psi}(\tau) m \psi(\tau) \quad . \quad (2.24)$$

The canonical momenta may now be identified as

$$\begin{aligned} p^\mu &= \frac{x_\tau^\mu}{\sqrt{-x_\tau^2}} \left[-\frac{x_\tau^\nu}{2x_\tau^2} \bar{\psi} \gamma_\nu \overleftrightarrow{\partial}_\tau \psi + \bar{\psi} m \psi \right] \\ &\quad + \frac{1}{2\sqrt{-x_\tau^2}} \bar{\psi} \gamma^\mu \overleftrightarrow{\partial}_\tau \psi \quad (2.25) \\ x &= \frac{\partial L_0}{\partial(\partial_\tau \bar{\psi})} = -\frac{x_\tau}{2\sqrt{-x_\tau^2}} \psi; \quad \bar{x} = \frac{\partial L_0}{\partial(\partial_\tau \psi)} = \bar{\psi} \frac{x_\tau}{2\sqrt{-x_\tau^2}} \quad . \end{aligned}$$

To simplify our expressions, we now define

$$\begin{aligned} \dot{x}^\mu &= \frac{dx^\mu}{ds} = \frac{x_\tau^\mu}{\sqrt{-x_\tau^2}} \quad (\dot{x}^2 = -1) \\ \dot{\psi} &= \frac{d\psi}{ds} = \frac{\psi_\tau}{\sqrt{-x_\tau^2}} \quad (2.26) \end{aligned}$$

where $s(\tau)$ is the parameter (2.3) defined earlier. The Euler equations then become

$$\dot{p}^\mu = 0$$

$$\begin{aligned} \dot{\psi} - \left(\frac{1}{2} \dot{x} \ddot{x} - m \dot{x}\right) \psi &= 0 \\ \dot{\bar{\psi}} + \bar{\psi} \left(\frac{1}{2} \dot{x} \ddot{x} - m \dot{x}\right) &= 0 \end{aligned} \quad (2.27)$$

We may thus derive a number of constants of motion, including the total momentum p^μ and

$$\begin{aligned} \bar{\psi} \lambda_\alpha \psi \\ \bar{\psi} \lambda_\alpha (\gamma_5 \gamma^\mu \dot{x} + \dot{x} \gamma_5 \gamma^\mu) \psi \end{aligned} \quad (2.28)$$

where λ_α is any $U(n)$ matrix which commutes with the mass matrix

$$[\lambda_\alpha, m] = 0$$

It is convenient to define the constant mass parameter

$$\mu = \bar{\psi} m \psi = -p \cdot \dot{x} \quad (2.29)$$

Integrating this equation we find

$$p \cdot x = -\mu s + d$$

where d is a constant.

The generators of the Lorentz group are also constants of the motion. To derive an expression for them, we note that x^μ , ψ and $\bar{\psi}$ transform under Lorentz transformations as

$$\begin{aligned} \delta x^\mu &= \omega^{\mu\nu} x_\nu \\ \delta \psi_a &= \frac{1}{4} \omega_{\mu\nu} \sigma_{ab}^{\mu\nu} \psi_b, & \delta \bar{\psi}_a &= -\frac{1}{4} \omega_{\mu\nu} \bar{\psi}_b \sigma_{ba}^{\mu\nu} \end{aligned}$$

where

$$\sigma^{\mu\nu} = \frac{1}{2i} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \quad (2.30)$$

Noether's theorem then implies that

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu + S^{\mu\nu} \quad (2.31)$$

is the constant generator of Lorentz transformations and the spin matrix is

defined by

$$\begin{aligned} S^{\mu\nu} &= -\frac{1}{2} \bar{\psi} \sigma^{\mu\nu} \chi + \frac{1}{2} \bar{\chi} \sigma^{\mu\nu} \psi \\ &= \frac{1}{4} \bar{\psi} (\sigma^{\mu\nu} \dot{\chi} + \dot{\chi} \sigma^{\mu\nu}) \psi . \end{aligned} \quad (2.32)$$

We note the total momentum can be rewritten as

$$p^\mu = \mu \dot{x}^\mu + S^{\mu\nu} \ddot{x}_\nu \quad (2.33)$$

From eq. (2.32), we see that

$$S^{\mu\nu} \dot{x}_\nu = 0 \quad (2.34)$$

where we have used the fact that

$$\begin{aligned} \sigma^{\mu\nu} \gamma^\lambda + \gamma^\lambda \sigma^{\mu\nu} &= \sigma^{\lambda\mu} \gamma^\nu + \gamma^\nu \sigma^{\lambda\mu} \\ &= \sigma^{\nu\lambda} \gamma^\mu + \gamma^\mu \sigma^{\nu\lambda} . \end{aligned}$$

Equations (2.31), (2.33) and (2.34) are recognizable as the basis of Frenkel's theory of spinning relativistic particles⁽¹⁷⁾. However, our theory differs substantially from that of Frenkel because our μ is a dynamical variable defined by (2.29). We also have additional equations of motion (2.27) which determine ψ , and hence the properties of μ . The variables $\psi_\alpha(\tau)$ are absent in Frenkel's theory⁽¹⁸⁾.

We also find the following relation among our variables

$$\frac{1}{2} p^2 S^{\mu\nu} S_{\mu\nu} + p_\mu S^{\mu\nu} S_{\nu\lambda} p^\lambda = -\frac{1}{2} (S^{\mu\nu} S_{\mu\nu}) (\bar{\psi} m \psi)^2 . \quad (2.35)$$

Defining the Pauli-Lubanski tensor (in four dimensions)

$$W^\mu = \frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} M_{\nu\lambda} p_\sigma \equiv M^{*\mu\nu} p_\nu \quad (2.36)$$

we find that eq. (2.35) may also be written

$$W^2 = \frac{1}{2} (S^{\mu\nu} S_{\mu\nu}) (\bar{\psi} m \psi)^2 . \quad (2.37)$$

We next find it convenient to separate $x^\mu(s)$ into two parts,

$$x^\mu(s) = q^\mu(s) + r^\mu(s) \quad (2.38)$$

where

$$r^\mu(s) = -S^{\mu\nu} p_\nu / p^2 \quad (2.39)$$

so that

$$p \cdot q = p \cdot x, \quad p \cdot r = 0 \quad (2.40)$$

Examining $M^{\mu\nu}$, we find

$$q^\mu = (M^{\mu\nu} p_\nu + p^\mu p \cdot x) / p^2 \quad (2.41a)$$

$$\dot{q}^\mu = p^\mu p \cdot \dot{x} / p^2 = -u p^\mu / p^2 \quad (2.41b)$$

Now we define

$$\begin{aligned} R^{\mu\nu} &= M^{\mu\nu} - q^\mu p^\nu + q^\nu p^\mu \\ &= r^\mu p^\nu - r^\nu p^\mu + S^{\mu\nu} \end{aligned} \quad (2.42)$$

where

$$R^{\mu\nu} p_\nu = 0 \quad (2.43)$$

and

$$R^{\mu\nu} R_{\mu\nu} = 2W^2 / p^2 = S^{\mu\nu} S_{\mu\nu} - 2r^2 p^2 \quad (2.44)$$

An analysis of the Poisson brackets of the elements of $R^{\mu\nu}$ indicates that they generate the little group which leaves the momentum p^μ invariant. Thus the constraint equation (2.44) relates the ratio of the two Casimir operators of the Poincare group W^2 and p^2 to the Casimir operator $(R^{\mu\nu})^2$ of the little group. Using eqs. (2.34) and (2.39), we find

$$0 = p_\mu S^{\mu\nu} \dot{x}_\nu / p^2 = r \cdot \dot{x} \quad . \quad (2.45)$$

We may now find an equation of motion for r^μ by examining

$$\begin{aligned} 0 &= S^{\mu\nu} \dot{x}_\nu = (R^{\mu\nu} - r^\mu p^\nu + r^\nu p^\mu) \dot{x}_\nu \\ &= R^{\mu\nu} \dot{x}_\nu + \mu r^\mu \quad , \end{aligned} \quad (2.46)$$

where we have used eqs. (2.29) and (2.45). Since eqs. (2.41b) and (2.43) imply $R^{\mu\nu} \dot{q}_\nu = 0$, we finally obtain

$$R^{\mu\nu} \dot{r}_\nu + \mu r^\mu = 0 \quad . \quad (2.47)$$

Examining the expression

$$\begin{aligned} S^{\mu\nu} S_{\nu\lambda} p^\lambda &= S^{\mu\nu} S_{\nu\lambda} S^{\lambda\sigma} \ddot{x}_\sigma \\ &= -\frac{1}{2} (S^{\alpha\beta} S_{\alpha\beta}) S^{\mu\nu} \ddot{x}_\nu \\ &= -\frac{1}{2} (S^{\alpha\beta} S_{\alpha\beta}) (p^\mu - \mu \dot{x}^\mu) \end{aligned} \quad (2.48)$$

which follows from eqs. (2.33) and (2.34), we may use eq. (2.39) to express \ddot{x}^μ as

$$\mu \dot{x}^\mu = p^\mu - S^{\mu\nu} r_\nu p^2 / \left(\frac{1}{2} S^{\alpha\beta} S_{\alpha\beta} \right) \quad . \quad (2.49)$$

Since $p \cdot \dot{x} = -\mu$ is a constant, eq. (2.41b) implies $\ddot{q}^\mu = 0$. Thus if we differentiate eq. (2.49), we find

$$\mu \ddot{x}^\mu = \mu \ddot{r}^\mu = -S^{\mu\nu} \dot{r}_\nu p^2 / \left(\frac{1}{2} S^{\alpha\beta} S_{\alpha\beta} \right) \quad ,$$

where we used (2.40), (2.42) and (2.45). But eq. (2.47) indicates $r \cdot \dot{r} = 0$ and eq. (2.40) makes $p \cdot \dot{r} = 0$. Thus from eq. (2.42),

$$S^{\mu\nu} \dot{r}_\nu = R^{\mu\nu} \dot{r}_\nu \quad .$$

This information combined with equation (2.47) allows us to write

$$\ddot{r}^\mu + \gamma^2 r^\mu = 0 \quad , \quad (2.50)$$

where

$$\begin{aligned}\gamma^2 &= 2p^2 / (S^{\alpha\beta} S_{\alpha\beta}) = p^2 \mu^2 / W^2 \\ &= 2\mu^2 / (R^{\alpha\beta} R_{\alpha\beta})\end{aligned}\quad (2.51)$$

The variable $x^\mu(s)$ thus executes harmonic motion with angular frequency γ , so

$$r^\mu = (a^\mu e^{i\gamma s} + a^{*\mu} e^{-i\gamma s}) / \sqrt{-2p^2} \quad (2.52)$$

From the definition (2.36) for W^μ and the expression (2.34) for r^μ , we see that

$$\begin{aligned}p \cdot r &= 0 \\ W \cdot r &= -p_\mu S^{\mu\nu} S_{\nu\lambda} p^\lambda / p^2 \\ &= -\frac{1}{4} S^{\mu\nu} S_{\mu\nu}^* = 0\end{aligned}\quad (2.53)$$

where $S^{\mu\nu} S_{\mu\nu}^* = 0$ follows from $S^{\mu\nu} x_\nu = 0$. Thus r^μ moves in a plane perpendicular to both p^μ and W^μ . From eqs. (2.47) and (2.53), we see that a^μ and W^μ are eigenvectors of $R^{\mu\nu}$:

$$\begin{aligned}i R^{\mu\nu} a_\nu + a^\mu \left(\frac{1}{2} R^{\alpha\beta} R_{\alpha\beta}\right)^{1/2} &= 0 \\ i R^{\mu\nu} W_\nu &= 0\end{aligned}\quad (2.54)$$

Furthermore,

$$\begin{aligned}a^2 &= 0 \\ p \cdot a &= 0 \\ W \cdot a &= 0\end{aligned}\quad (2.55)$$

Here a^μ , $a^{*\mu}$ and W^μ are analogous to the $m = +1$, $m = -1$ and $m = 0$ components of a spin one vector. p^μ is invariant under rotations by $R^{\mu\nu}$, so $R^{\mu\nu}$ generates the little group of p^μ . This becomes clear from an analysis of the Poisson brackets of $R^{\mu\nu}$. Thus we may finally write

$$\begin{aligned}x^\mu(s) &= [M^{\mu\nu} p_\nu + p^\mu(d - \mu s)] / p^2 \\ &+ (a^\mu e^{i\gamma s} + a^{*\mu} e^{-i\gamma s}) / \sqrt{-2p^2}\end{aligned}\quad (2.56)$$

so $\dot{x}^\mu(s)$ consists of periodic circular motion in a plane superimposed upon a pure translation. The overall helical motion is identifiable as the effective classical Zitterbewegung resulting from the quantum mechanical interference of positive and negative frequency components in the Dirac equation. The Zitterbewegung is a familiar consequence of attempting to localize a Dirac particle, as may be seen explicitly from an appropriate wave packet construction⁽¹⁹⁾.

Now we turn to the solution of the equations of motion (2.27) for ψ . Equation (2.54) can be used to show that

$$[e^{\theta R}]^{\mu\nu} a_\nu = a^\mu \exp[i\theta\{\frac{1}{2} R^{\alpha\beta} R_{\alpha\beta}\}^{1/2}]. \quad (2.57)$$

Next we define the Lorentz transformation in the space of Dirac matrices as

$$U(\theta) = \exp[\frac{1}{4} \theta \sigma_{\mu\nu} R^{\mu\nu} / \{\frac{1}{2} R^{\alpha\beta} R_{\alpha\beta}\}^{1/2}] \quad (2.58)$$

Thus

$$\begin{aligned} U(\theta) \not\propto U^\dagger(\theta) &= e^{i\theta} \not\propto \\ U(\theta) \not\propto U^\dagger(\theta) &= \not\propto \end{aligned} \quad (2.59)$$

We can therefore write

$$-\frac{1}{2} \dot{x}(s) \ddot{x}(s) + m \dot{x}(s) = U(\gamma s) [-\frac{1}{2} \dot{x}(0) \ddot{x}(0) + m \dot{x}(0)] U^\dagger(\gamma s) \quad (2.60)$$

The equation of motion for ψ can now be written

$$\dot{\tilde{\psi}} + M \tilde{\psi} = 0$$

where the s-independent matrix M is given by

$$M = \frac{1}{4} \sigma^{\mu\nu} R_{\mu\nu} + [-\frac{1}{2} \dot{x}(0) \ddot{x}(0) + m \dot{x}(0)] \quad (2.61)$$

and

$$\tilde{\psi}(s) = U^\dagger(\gamma s) \psi(s) \quad .$$

This equation can now be solved directly by quadratures to give

$$\psi(s) = U(\gamma s) e^{-isM} \lambda \quad (2.62)$$

where λ is an s -independent spinor.

The solution of our classical spin 1/2 Dirac particle problem is now complete. The quantum theory, however, is nontrivial and will be deferred to a later investigation.

C. Supersymmetric Spin 1/2 Particle Without Zitterbewegung

The spin 1/2 Dirac particle discussed in the previous section possessed a classical Zitterbewegung with the result that the particle's velocity did not vanish in the frame where $\underline{p} = 0$, as seen from eqs. (2.49) and (2.50). One might therefore ask if there exists a spin 1/2 particle Lagrangian which gives the particle's momentum proportional to its velocity; such a particle would correspond more closely to the traditional picture of a positive energy classical spinning particle.

A sufficient condition to make a particle's momentum and velocity proportional is the constraint

$$S^{\mu\nu} p_\nu = 0 \quad (2.63)$$

where $S^{\mu\nu}$ is the spin part of the Lorentz group generator⁽²⁰⁾. This condition guarantees that the particle's spin degrees of freedom consist only of spatial rotations in the rest frame. That p^μ is parallel to \dot{x}^μ can be seen directly from eq. (2.63) by writing

$$M^{\mu\nu} p_\nu = (x^\mu p^\nu - x^\nu p^\mu + S^{\mu\nu}) p_\nu = x^\mu p^2 - p^\mu x \cdot p$$

and taking a derivative to yield

$$p^\mu = \dot{x}^\mu (p^2 / p \cdot \dot{x})$$

One way to ensure that $S^{\mu\nu} p_\nu = 0$ is to search for a Lagrangian implying that

$$S^{\mu\nu} = \frac{1}{4} \bar{\psi} [\sigma^{\mu\nu} \frac{\not{\partial}}{m} + \frac{\not{\partial}}{m} \sigma^{\mu\nu}] \psi \quad (2.64)$$

instead of eq. (2.32). This can be achieved if our canonical momenta conjugate to $\bar{\psi}$ and ψ take the form

$$\begin{aligned} \chi &= - \frac{\not{\partial}}{2m} \\ \bar{\chi} &= \bar{\psi} \frac{\not{\partial}}{2m} \end{aligned} \quad (2.65)$$

instead of eq. (2.25). We have constructed a Lagrangian with all the required invariance properties; it is given by the expression

$$\begin{aligned} L_0 &= - \frac{1}{2} [(\bar{\psi} \gamma^\mu m \psi)^2]^{1/2} [-(x_\tau^\mu - \bar{\psi} \frac{1}{2m} \overleftrightarrow{\partial}_\tau \gamma^\mu \psi)^2]^{1/2} \\ &\quad - \frac{1}{2} (i \bar{\psi} \gamma_\mu m \psi) (x_\tau^\mu - \bar{\psi} \frac{1}{2m} \overleftrightarrow{\partial}_\tau \gamma^\mu \psi) . \end{aligned} \quad (2.66)$$

The canonical momentum conjugate to x^μ is

$$p^\mu = + \frac{1}{2} \frac{\sqrt{-V^2}}{\sqrt{-T^2}} T^\mu - \frac{1}{2} V^\mu \quad (2.67)$$

where

$$\begin{aligned} V^\mu &= i \bar{\psi} \gamma^\mu m \psi \\ T^\mu &= x_\tau^\mu - \bar{\psi} \frac{1}{2m} \overleftrightarrow{\partial}_\tau \gamma^\mu \psi \end{aligned} \quad (2.68)$$

The equations of motion are

$$\begin{aligned} \partial_\tau p^\mu &= 0 \\ i \partial_\tau \psi - m^2 \frac{\sqrt{-T^2}}{\sqrt{-V^2}} \psi &= 0 \end{aligned} \quad (2.69)$$

$$-i \partial_\tau \bar{\psi} - m^2 \frac{\sqrt{-T^2}}{\sqrt{-V^2}} \bar{\psi} = 0 \quad .$$

We see that

$$\mu^2 \equiv i (\bar{\chi} m^2 \psi + \bar{\psi} m^2 \chi) = i \bar{\psi} \not{p} m \psi \quad (2.70)$$

is a constant of the motion.

The constraint associated with τ -reparametrization can now be written

$$p^2 + \mu^2 = 0 \quad . \quad (2.71)$$

After some calculations using the equations of motion and the constraints, we find first that

$$T^\mu + x_\tau^\mu - V^\mu \sqrt{-T^2} / \sqrt{-V^2} = 0 \quad (2.72)$$

so that eq. (2.67) may be written

$$p^\mu = x_\tau^\mu \sqrt{-V^2} / 2 \sqrt{-T^2} \quad .$$

Squaring this equation, we finally obtain

$$\frac{1}{2} \sqrt{-V^2} / \sqrt{-T^2} = \sqrt{-p^2} / \sqrt{-x_\tau^2} = \mu / \sqrt{-x_\tau^2}$$

so

$$p^\mu = \mu x_\tau^\mu / \sqrt{-x_\tau^2} \quad . \quad (2.73)$$

The equations for ψ can now also be simplified:

$$i \partial_\tau \psi - \frac{m^2 \sqrt{-x_\tau^2}}{2\mu} \psi = 0 \quad (2.74)$$

$$-i\partial_\tau \bar{\psi} - \sqrt{-x_\tau^2} \bar{\psi} \frac{m^2}{2\mu} = 0 \quad .$$

We now find the explicit solutions for $\psi(s)$ in the form

$$\psi(s) = e^{-\frac{1m^2s}{2\mu}} \lambda \quad (2.75)$$

where s is the parameter defined by eq. (2.3), and λ is a constant spinor. The coordinate $x^\mu(s)$ consists of a pure translation,

$$x^\mu = q^\mu + \frac{p^\mu}{\sqrt{-p^2}} s \quad . \quad (2.76)$$

The quantum theory of this system is straightforward. We find in the $x^0 = p^0 \tau / \mu$ gauge that the canonical commutation rules are satisfied provided

$$[q^i, p^j] = i \delta^{ij} \quad (2.77a)$$

$$\{\lambda_a^\alpha, \bar{\lambda}_b^\beta\} = i \frac{2m_{\alpha\beta} \not{p}_{ab}}{p^2} \quad . \quad (2.77b)$$

Furthermore, p^μ commutes with λ_a^α and $\bar{\lambda}_b^\beta$, but q^i does not. However, both x^i and p^i commute with ψ and $\bar{\psi}$ in accordance with the canonical commutation rules.

We note the similarity of eq. (2.77b) to the supersymmetry commutation relations⁽²¹⁾. This is why we call ψ the "supersymmetric" quark. Here, however, the λ_a^α , are simply canonical variables. We remark that the \not{p} on the right hand side of eq. (2.77b) is essential to obtain only positive norm states. This is best seen in the rest frame ($\underline{p} = 0$) where we calculate the norm to be

$$\begin{aligned} \langle 0 | \lambda_a^\alpha \lambda_b^{\beta\dagger} | 0 \rangle &= + \frac{2m_{\alpha\beta}}{p^0 p_0} p_0 (\gamma^0 \gamma_0) \\ &= \frac{2m_{\alpha\beta}}{p^0} > 0 \quad . \end{aligned}$$

(Remember that $\bar{\psi} = \psi^\dagger \gamma_4$, $\gamma_4 = i\gamma^0$). A similar ghost killing factor was previously discussed in connection with Chan-Paton type spin factors in the context of dual models⁽¹⁴⁾.

III. Review of String with Point Masses

Before proceeding to attach quark fields to world-lines connected by strings, we review the closely related problem of attaching structureless point masses⁽¹³⁾. For simple quarks, like those in sections II A and II C, the dynamics of the two systems are nearly equivalent⁽²²⁾. Some of the results of this section will therefore be directly applicable to our quark models for mesons and baryons to be given in section IV. In subsections A1 and A2 we study meson-like cases and in section B we analyze a baryon-like system. The structureless point masses which appear in this section correspond to those of the quarks attached to the ends of the string, as will be demonstrated in the next section.

A. Two Mass Case (Mesons)

We first consider the action for two masses, at the points $x^\mu(0) = x^\mu(\tau, \sigma = 0)$ and $x^\mu(\pi) = x^\mu(\tau, \sigma = \pi)$, joined by a relativistic string,

$$S = \int_{\tau_1}^{\tau_2} d\tau \left\{ -\mu_0 \sqrt{-x_\tau^2(0)} - \mu_\pi \sqrt{-x_\tau^2(\pi)} - \gamma \int_0^\pi d\sigma \sqrt{-g} \right\} \quad (3.1)$$

Here

$$x_\sigma^\mu = \partial x^\mu(\sigma, \tau) / \partial \sigma, \quad x_\tau^\mu = \partial x^\mu(\sigma, \tau) / \partial \tau$$

and

$$-\gamma \sqrt{-g} \equiv -\gamma \left[(x_\tau \cdot x_\sigma)^2 - x_\tau^2 x_\sigma^2 \right]^{1/2}$$

is the Nambu Lagrangian density for a free relativistic string. The string contribution to the action is effectively a relativistic potential exerting a force on the two mass points. This will become clearer below.

The action (3.1) has been extensively analyzed in ref.(13) so we will give here only an outline of the main results. The equations of motion are

$$\partial_{\tau} K^{\mu}(\sigma, \tau) + \partial_{\sigma} N^{\mu}(\sigma, \tau) = 0 ; \quad 0 < \sigma < \pi \quad (3.2a)$$

$$\partial_{\tau} p^{\mu}(0) - \gamma N^{\mu}(\sigma=0, \tau) = 0 ; \quad \sigma = 0 \quad (3.2b)$$

$$\partial_{\tau} p^{\mu}(\pi) + \gamma N^{\mu}(\sigma=\pi, \tau) = 0 ; \quad \sigma = \pi ,$$

where

$$K^{\mu} = (x_{\tau}^{\mu} x_{\sigma}^2 - x_{\sigma}^{\mu} x_{\sigma} \cdot x_{\tau}) / \sqrt{-g} \quad (3.2c)$$

$$N^{\mu} = (x_{\sigma}^{\mu} x_{\tau}^2 - x_{\tau}^{\mu} x_{\sigma} \cdot x_{\tau}) / \sqrt{-g}$$

$$p^{\mu}(0) = \mu_0 x_{\tau}^{\mu}(0) / \sqrt{-x_{\tau}^2(0)}$$

$$p^{\mu}(\pi) = \mu_{\pi} x_{\tau}^{\mu}(\pi) / \sqrt{-x_{\tau}^2(\pi)} \quad (3.2d)$$

1. Timelike Gauge

We examine the simplest possible longitudinal motions of this system, which occur when the string lies always along a single line. We may then choose a timelike gauge such that

$$x^0(\tau, \sigma) = \tau \quad (3.3a)$$

and x_{σ} is independent of σ ,

$$x(\tau, \sigma) = x(\tau, 0) + \frac{\sigma}{\pi} [x(\tau, \pi) - x(\tau, 0)] \quad (3.3b)$$

All other components of $x^{\mu}(\tau, \sigma)$ are assumed to vanish. The Hamiltonian in **this** gauge is

$$H = \sqrt{p^2(0) + \mu_0^2} + \sqrt{p^2(\pi) + \mu_{\pi}^2} + \gamma |x(\pi) - x(0)| \quad (3.4)$$

In the center of mass where $p(0) + p(\pi) = 0$, H reduces to the invariant mass

$$M = \sqrt{k^2 + \mu_0^2} + \sqrt{k^2 + \mu_\pi^2} + \gamma|r| \quad (3.5)$$

where k and r are canonically conjugate relative variables.

The periodic classical motions and the Bohr-Sommerfeld approximation to the quantum theory follow from the analysis of the k - r phase-space diagram for fixed M in Fig. (3.1).

The action variable J is

$$J = \oint k \, dr = - \oint r \, dk$$

and may be computed to be

$$\begin{aligned} \gamma J = & \Delta^{1/2} - 2\mu_0^2 \ln \left[\frac{1}{2M\mu_0} \{M^2 + \mu_0^2 - \mu_\pi^2 + \Delta^{1/2}\} \right] \\ & - 2\mu_\pi^2 \ln \left[\frac{1}{2M\mu_\pi} \{M^2 + \mu_\pi^2 - \mu_0^2 + \Delta^{1/2}\} \right] \end{aligned} \quad (3.6)$$

where

$$\Delta(M, \mu_0, \mu_\pi) = [M^2 - (\mu_0 + \mu_\pi)^2][M^2 - (\mu_0 - \mu_\pi)^2] \quad (3.7)$$

The Bohr-Sommerfeld quantization rule is then

$$J = 2\pi\hbar (n + \text{const.}); \quad n = 0, 1, 2, \dots \quad (3.8)$$

The exact quantum spectrum of this equation can be found by solving the nonlocal Schrodinger equation

$$\theta = \int_{-\infty}^{\infty} \gamma \, dk' \, G(k, k') \, \psi(k') + [\sqrt{k^2 + \mu_0^2} + \sqrt{k^2 + \mu_\pi^2} - M] \psi(k) \quad (3.9)$$

where

$$G(k, k') = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_0^{\infty} dr r \cos [r(k - k')] e^{-\epsilon r}$$

$$= P \left\{ \frac{1}{\pi(k-k')^2} \right\} \quad (3.10)$$

and P denotes the principal value. Exact solutions of this equation are not known.

2. Lightlike Gauge

We now repeat the analysis of the previous subsection in the lightlike gauge

$$x^+ \equiv x^0 + x = \tau$$

$$x^- \equiv x^0 - x = x^-(0) + \frac{\sigma}{\pi} [x^-(\pi) - x^-(0)]. \quad (3.11)$$

The Hamiltonian in this gauge is

$$p^- \equiv p^0 - p$$

$$= \frac{\mu_0^2}{2p^+(0)} + \frac{\mu_\pi^2}{2p^+(\pi)} + \gamma |x^-(\pi) - x^-(0)| \quad (3.12)$$

and the total (+) momentum is

$$p^+ = p^+(0) + p^+(\pi) \quad (3.13)$$

We then make the canonical transformation

$$\kappa = (p^+(\pi) - p^+(0))/2p^+$$

$$\rho = p^+(x^-(\pi) - x^-(0)) \quad (3.14)$$

so that the invariant mass-squared can be written

$$M^2 = 2p^+p^- = \frac{\mu_0^2}{\frac{1}{2} - \kappa} + \frac{\mu_\pi^2}{\frac{1}{2} + \kappa} + \gamma |\rho| \quad (3.15)$$

The Bohr-Sommerfeld quantization procedure yields the same results as before, eqs. (3.6-3.8), while the exact quantum spectrum follows from the Schrodinger equation

$$\gamma \int d\kappa' G(\kappa, \kappa') \psi(\kappa') + \left[\frac{\mu_0^2}{\frac{1}{2} - \kappa} + \frac{\mu_\pi^2}{\frac{1}{2} + \kappa} - M^2 \right] \psi(\kappa) = 0, \quad (3.16)$$

where $G(\kappa, \kappa')$ is given in eq. (3.10). 't Hooft⁽⁹⁾ has derived this equation using a color gauge theory in two spacetime dimensions. We find it remarkable that such similar results arise from such different origins. Exact solutions of this equation are unknown.

B. Three Masses (Baryons)

Now we join three masses, at the points $x^\mu(0) = x^\mu(\sigma = 0)$, $x^\mu(1) = x^\mu(\sigma = \sigma_1)$, $x^\mu(\pi) = x^\mu(\sigma = \pi)$, with two strings. We take the action to be

$$S = \int_{\tau_1}^{\tau_2} d\tau \left\{ -\mu_0 \sqrt{-x_\tau^2(0)} - \mu_1 \sqrt{-x_\tau^2(1)} - \mu_\pi \sqrt{-x_\tau^2(\pi)} \right. \\ \left. - \gamma \int_{\sigma_0}^{\sigma_1} d\sigma \sqrt{-g} - \gamma \int_0^\pi d\sigma \sqrt{-g} \right\}. \quad (3.17)$$

We now restrict ourselves to longitudinal motions of the string lying on a straight line: We use the timelike gauge

$$x^0(\sigma, \tau) = \tau \quad (3.18)$$

and choose the σ -gauge so that x_σ is independent of σ between masses

$$0 < \sigma < \sigma_1: x(\tau, \sigma) = x(\tau, 0) + \frac{\sigma}{\sigma_1} [x(\tau, \sigma_1) - x(\tau, 0)] \\ \sigma_1 < \sigma < \pi: x(\tau, \sigma) = x(\tau, \sigma_1) + \frac{\sigma - \sigma_1}{\pi - \sigma_1} [x(\tau, \pi) - x(\tau, \sigma_1)] \quad (3.19)$$

All other components of $x^\mu(\tau, \sigma)$ are taken to vanish. The Hamiltonian then

becomes

$$H = \sqrt{p^2(0) + \mu_0^2} + \sqrt{p^2(1) + \mu_1^2} + \sqrt{p^2(\pi) + \mu_\pi^2} \\ + \gamma |x(1) - x(0)| + \gamma |x(\pi) - x(1)| \quad (3.20)$$

Going to the center of mass and choosing appropriate canonical pairs of relative coordinates, H becomes the invariant mass

$$M = \sqrt{\frac{1}{4}(k_1 + k_2)^2 + \mu_0^2} + \sqrt{\frac{1}{4}(k_1 - k_2)^2 + \mu_\pi^2} \\ + \sqrt{k_2^2 + \mu_1^2} + \gamma |r_1 + r_2| + \gamma |r_1 - r_2| \quad (3.21)$$

It is simpler to examine the motion of this system in the zero-mass limit. A typical motion is plotted in Fig. 3.2.

Analysis of the action variables for vanishing masses gives the result

$$M^2 = \gamma(J_1 + J_2); \quad J_1 = n, \quad J_2 = 2m \quad n, m = \text{integers} \quad (3.22)$$

The action variables J_1 and J_2 correspond to the normal modes of the system. When the initial conditions are such that $J_2 = 0$, the motion described by J_1 is plotted in Fig. 3.3a. When $J_1 = 0$, the motion is that of Fig. 3.3b. We see that J_1 describes an oscillation with a "diquark" remaining on one edge of the system, while J_2 gives an oscillation with a "diquark" periodically crossing the middle of the system. The lowest modes are then purely of these two types.

The experimental data on baryon spectra favor the 56 and 70 representations of $SU(6)$. Such representations have a natural interpretation⁽²³⁾ in terms of diquark correlations inside the baryons. The string model discussed here then seems to give the dynamics required for these correlations to occur.

IV. Quarks on the Ends of Strings

We are now ready to apply the methods of section II to attach quark quantum numbers to world-lines joined by the relativistic string potential. These systems constitute our proposed model for hadrons.

Mesons will be represented by a quark and an antiquark attached to opposite ends of the string, as shown in Fig. 4.1. Our model Lagrangian for mesons gives a σ - and τ - reparametrization invariant action with the form

$$S = \int_{\tau_1}^{\tau_2} d\tau \{ L_0(\sigma=0) + L_0(\sigma=\pi) - \gamma \int_0^\pi \sqrt{-g} \} \quad (4.1)$$

where L_0 is one of the point quark Lagrangians examined in section II and $-\gamma\sqrt{-g}$ is the string Lagrangian density treated in section III.

Our baryon model consists of three quarks connected by strings. Of three possible configurations shown in Fig. 4.2 a, b, c, the simplest is probably that of Fig. 4.2a, with all quarks lying on the string. The motions exhibited in Fig. 3.2 apply to this case and show that each quark spends some time at the edges as well as the middle. The configuration of Fig. 4.2b is intuitively an excited state with respect to the one in Fig. 4.2a, thus its low energy spectrum is probably included in this latter case. The action corresponding to Fig. 4.2a takes the form

$$S = \int_{\tau_1}^{\tau_2} d\tau \{ L_0(\sigma=0) + L_0(\sigma=\sigma_1) + L_0(\sigma=\pi) - \gamma \left(\int_0^{\sigma_1} + \int_{\sigma_1}^\pi \right) d\sigma \sqrt{-g} \} \quad (4.2)$$

where L_0 is again any point quark Lagrangian. The three point Lagrangians are functions of the coordinates $x^\mu(\tau, \sigma=0)$, $x^\mu(\tau, \sigma=\sigma_1)$ and $x^\mu(\tau, \sigma=\pi)$, respectively. We will not study the other possible models for baryons in this paper.

We will impose good triality upon our systems from the outset as a phenomenological principle; only one quark will be assigned to each world-line,

and we will allow only strings attached to three quarks, or to one quark and one antiquark. This principle is ordinarily dictated by color symmetry arguments⁽²⁾ which play an implicit rather than an explicit role in our treatment, as described in the introduction.

Clearly the most desirable course at this point is to analyze the quantum spectrum of each of our models for mesons and baryons using the spin 1/2 point quark Lagrangians in four spacetime dimensions. As one might expect, this analysis becomes exceedingly complex. Since our main goal here is to introduce the basic ideas of our method, we will be content to develop a feeling for the implications by analyzing the simpler but less realistic models. In the rest of this section, we will treat mainly the spinless quark model for mesons with motion restricted to a two-dimensional subspace of spacetime. The model for baryons will be outlined at the end.

A. A Model for Mesons

We begin our investigation of meson models by examining the action (4.1) with L_0 taken as

$$L_0(x(0)) = \partial_\tau \phi_0^\dagger \partial_\tau \phi_0 [-x_\tau^2(0)]^{-1/2} - \phi_0^\dagger m^2 \phi_0 [-x_\tau^2(0)]^{1/2} \quad (4.3)$$

and similarly for $L_0(x(\pi))$ with $\sigma = \pi$ quantities substituted for $\sigma = 0$ quantities. Our action principle is defined by requiring the variations

$$\delta \phi_0(\tau), \quad \delta \phi_\pi(\tau), \quad \delta x^\mu(\tau, 0), \quad \delta x^\mu(\tau, \pi),$$

$$\delta x^\mu(\tau, \sigma), \quad 0 < \sigma < \pi, \quad ,$$

to be arbitrary except at $\tau = \tau_1, \tau_2$, where they vanish. We thus obtain the equations of motion

$$0 = \partial_\tau K^\mu(\tau, \sigma) + \partial_\sigma N^\mu(\tau, \sigma) \quad ; \quad 0 < \sigma < \pi \quad (4.4)$$

$$\begin{aligned} \partial_\tau \left[\frac{x_\tau^\mu(0)}{\sqrt{-x_\tau^2(0)}} (\Pi_0^\dagger \Pi_0 + \phi_0^\dagger m^2 \phi_0) \right] + \gamma N^\mu(\tau, \sigma=0) &= 0 \\ \partial_\tau \left[\frac{x_\tau^\mu(\pi)}{\sqrt{-x_\tau^2(\pi)}} (\Pi_\pi^\dagger \Pi_\pi + \phi_\pi^\dagger m^2 \phi_\pi) \right] - \gamma N^\mu(\tau, \sigma=\pi) &= 0 \end{aligned} \quad , \quad (4.5)$$

where K^μ and N^μ are defined by eq. (3.2c). Introducing the s -parameter of eq. (2.3), the ϕ equations of motion, essentially the same as eq. (2.12), may be written

$$\begin{aligned} \frac{d^2 \phi_0}{ds^2} + m^2 \phi_0 &= 0 ; \quad \sigma = 0 \\ \frac{d^2 \phi_\pi}{ds^2} + m^2 \phi_\pi &= 0 ; \quad \sigma = \pi \end{aligned} \quad . \quad (4.6)$$

Thus we see that ϕ_0 possesses the solution

$$\phi_{0\alpha}(s) = \left[\frac{e^{-ims}}{\sqrt{2m}} \right]_{\alpha\beta} a_\beta(0) + \left[\frac{e^{ims}}{\sqrt{2m}} \right]_{\alpha\beta} b_\beta^*(0) \quad , \quad (4.7)$$

with ϕ_π having a similar expression

Equations (4.6) also imply that

$$\mu_0 = \partial_s \phi_0^\dagger \partial_s \phi_0 + \phi_0^\dagger m^2 \phi_0 \quad (4.8)$$

$$\mu_\pi = \partial_s \phi_\pi^\dagger \partial_s \phi_\pi + \phi_\pi^\dagger m^2 \phi_\pi$$

are constants of motion, so that eqs. (4.5) may be reexpressed in exactly the same form as eqs. (3.2b), (3.2d). The other constants of motion are the Poincare group generators p^μ and $M^{\mu\nu}$ and the $U(n)$ symmetry group generators

$$Q_{\alpha\beta} = \phi_{0\alpha}^\dagger \overleftrightarrow{\partial}_s \phi_{0\beta} + \phi_{\pi\alpha}^\dagger \overleftrightarrow{\partial}_s \phi_{\pi\beta} \quad . \quad (4.9)$$

which commute with the mass term in the Lagrangian L_0 .

B. Longitudinal Motions and Their Spectrum

Restricting ourselves to motion in two spacetime dimensions and choosing the timelike gauge analogous to eqs. (3.3), we find the Hamiltonian

$$H = \sqrt{p^2(0) + \mu_0^2} + \sqrt{p^2(\pi) + \mu_\pi^2} + \gamma |x(\pi) - x(0)| \quad (4.10)$$

Here $p(0)$, $p(\pi)$ are identical in form to eq. (3.2d), except that μ_0 and μ_π are the nontrivial canonical variables (4.8).

While the quantum theory of the variables p and x is nontrivial, as we saw in section III, we may quantize the fields ϕ_0 and ϕ_π in a straightforward manner. From eq. (4.7), we deduce that for $\sigma = 0$

$$[a_\alpha(0), a_\beta^\dagger(0)] = \delta_{\alpha\beta} = [b_\alpha(0), b_\beta^\dagger(0)] \quad (4.11)$$

are acceptable commutation relations implying

$$i[\Pi_{0\alpha}, \phi_{0\beta}^\dagger] = \delta_{\alpha\beta}, \quad (4.12)$$

where $\Pi_{0\alpha}$ is the canonical momentum, e.g., (2.10). Similar equations hold at $\sigma = \pi$. Thus eq. (4.8) may be written

$$\mu_0 = a_\alpha^\dagger(0) m_{\alpha\beta} a_\beta(0) + b_\alpha^\dagger(0) m_{\alpha\beta} b_\beta(0) + \text{const.} \quad (4.13)$$

and similarly for μ_π . It is possible to cancel all or part of the normal-ordering constant in eq. (4.13) by adding an extra term to the Lagrangian proportional to $\sqrt{-x_\tau^2(0)}$ (and $\sqrt{-x_\tau^2(\pi)}$). We will set the normal ordering constant equal to zero in the analysis which follows.

We see that our meson system has been reduced effectively to that of section III, except that the mass variables μ_0 , μ_π are now operators which take on different values depending on the different masses of the quarks in the multiplet. Symmetry breaking thus appears in a natural way, and the masses at the ends of the string are now identified directly with the masses of the quarks making up a given meson.

For example, the π^+ meson will be a string with \bar{u} and d quark masses on the ends, while for a K^+ meson, the \bar{u} and s masses appear. The internal symmetry content of mesons is described by the states $a_\alpha^+(0) b_\beta^+(\pi) | 0 \rangle$ where $a_\alpha^+(0)$ creates a quark of type α at $x(0)$ while $b_\beta^+(\pi)$ creates an antiquark of type β at $x(\pi)$. The particles π^+ , K^+ etc. and their excitations are described by the states, e.g.,

$$|K^+\rangle = a_d^+(0) b_u^+(\pi) |0\rangle \quad (4.14)$$

The spectrum, e.g., of the K^+ family, is then given by

$$\begin{aligned} H_{K^+} &= \langle K^+ | H | K^+ \rangle \\ &= \sqrt{p^2(0) + m_d^2} + \sqrt{p^2(\pi) + m_u^2} + \gamma |x(\pi) - x(0)| \end{aligned} \quad (4.15)$$

We see from the form of eq. (4.15) that

- (a) The system becomes heavier for larger quark separations.
- (b) The lowest mass occurs for a shrunk string ($x(0) \approx x(\pi)$) and corresponds to the ground state of the standard quark model where the quarks are approximately at the same spacetime point. The quark masses determine the intercept of the trajectories, which are therefore in general nondegenerate.

We can calculate approximately the quantized radial excitation spectrum of the π^+ , K^+ etc. families from our longitudinal mode Hamiltonian (4.10). This spectrum would correspond to the mass states with fixed spin in a Chew-Frautchi plot. To obtain the angular excitations one would have to include the transverse modes of the string as well, which we have not yet done. We now apply the semi-classical Bohr-Sommerfeld quantization procedure as in section III and ref. (13). Proceeding as for eq. (4.15), we find for a meson with quark masses m_d and m_u the effective Hamiltonian

$$H = \sqrt{p^2(0) + m_0^2} + \sqrt{p^2(\pi) + m_\pi^2} + \gamma |x(\pi) - x(0)|. \quad (4.16)$$

The resulting spectrum has the following properties

(a) if $m_0 = m_\pi = 0$ (e.g. pion, with massless quarks)

$$2\pi\hbar \gamma n = M_n^2, \quad n = 0, 1, 2, \dots \quad (4.17a)$$

(b) if $m_0 = 0, m_\pi = m \neq 0$ (e.g. kaon)

$$2\pi\hbar \gamma n = M_n^2 - m^2 \ln(M_n^2/m^2) - m^2 \quad (4.17b)$$

(c) If $m_0 = m_\pi = m$

$$2\pi\hbar \gamma n = M_n \sqrt{M_n^2 - 4m^2} - 4m^2 \ln \left[\frac{M_n}{2m} + \sqrt{\frac{M_n^2}{4m^2} - 1} \right] \quad (4.17c)$$

(d) If $m_0 \neq m_\pi \neq 0$

$$\begin{aligned} 2\pi\hbar \gamma n = & \Delta^{1/2}(M_n, m_0, m_\pi) \\ & - 2m_0^2 \ln \left\{ \frac{M_n^2 + m_0^2 - m_\pi^2}{2m_0 M_n} + \frac{\Delta^{1/2}(M_n, m_0, m_\pi)}{2m_0 M_n} \right\} \\ & - 2m_\pi^2 \ln \left\{ \frac{M_n^2 + m_\pi^2 - m_0^2}{2m_\pi M_n} + \frac{\Delta^{1/2}(M_n, m_0, m_\pi)}{2m_\pi M_n} \right\}, \end{aligned} \quad (4.17d)$$

where Δ is defined in eq. (3.7). This shows that with massive quarks the spectrum is not linear. However, for mesons containing small quark masses, $m_0^2, m_\pi^2 \leq 0.3 \text{ GeV}^2$, the deviation from linearity is very small and the spectrum quickly becomes essentially linear. On the other hand, if the quark mass is large, such as the conjectured charmed quark with $m_c \approx 2 \text{ GeV}$, then the curvature is substantial. This may be a welcomed feature if the new resonances⁽²⁴⁾ are interpreted as charmonium states⁽²⁵⁾.

We remark that since we have included neither spin nor spin-spin interactions at this stage, our present results are not necessarily realistic.

If we assume, however, that the above formulas are applicable to the octet of observed pseudoscalar mesons, then taking, e.g.,

$$\begin{aligned} |\pi^+ \rangle &= a_{\rho}^{\dagger}(0) b_{\pi}^{\dagger}(\pi) |0\rangle \\ |K^+ \rangle &= a_{\rho}^{\dagger}(0) b_{\lambda}^{\dagger}(\pi) |0\rangle \\ |n \rangle &= \frac{1}{\sqrt{3}} [a_{\rho}^{\dagger}(0) b_{\rho}^{\dagger}(\pi) + a_{\eta}^{\dagger}(0) b_{\pi}^{\dagger}(\pi) - 2a_{\lambda}^{\dagger}(0) b_{\lambda}^{\dagger}(\pi)] |0\rangle \end{aligned} \quad (4.18)$$

and putting $n = 0$ in eq. (4.17), we obtain the following masses for the ground states:

$$\begin{aligned} m_{\pi^+} &= m_{\rho} + m_{\pi} \\ m_{K^+} &= m_{\rho} + m_{\lambda} \\ m_n &= \frac{1}{3}(m_{\rho} + m_{\pi} + 4m_{\lambda}) \end{aligned} \quad (4.19)$$

For $m_{\rho} \approx m_{\pi}$, eq. (4.19) leads to the linear mass formula

$$m_{\pi} + 3m_n = 4m_K \quad (4.20)$$

which is in reasonable agreement with experiment. Given our crude model, we consider this result encouraging. The coefficients in eq. (4.20) are the same as those in Gell-Mann and Okubo's quadratic mass formula.

We will not attempt to treat baryons in detail here. The basic procedure would be to construct baryonic states analogous to eq. (4.18) and take matrix elements of the Hamiltonian following from an action like eq. (4.2). In this case the Hamiltonian would take the form of eq. (3.20). The Bohr-Sommerfeld spectrum for massless quarks would then be given by eq. (3.22), while for massive quarks the trajectories would in general be nonlinear.

V. Weak and Electromagnetic Interactions

The close connection between the present model and field theory suggests a compelling approach for dealing with weak and electromagnetic interactions. We propose to extend the procedure of section II to include quark fields coupled minimally to a set of vector mesons, where the gauge group may in general be non-Abelian. The string itself will not be coupled to these vector mesons because, as discussed in the introduction, the string is assumed to have the same properties as color glue. Provided the meson-quark coupling is small, as in the unified theories of weak and electromagnetic interactions⁽²⁶⁾, we may treat the interaction perturbatively. Thus the successes of gauge theories in their application to weak and electromagnetic interactions would be expected to persist in our model. The picture that emerges is one in which the strong interactions mediated by the string are solved in the absence of weak forces, which are then considered as small perturbations on the system.

For the purposes of illustration, let us consider the model of section IV with spinless quarks. The point quark Lagrangians $L_0(x(0))$ and $L_0(x(\pi))$ will be modified in the presence of interactions. The new form of L_0 at each point follows from examining the gauge-invariant spacetime Lagrangian density

$$\mathcal{L}(x) = -(D^\mu \phi(x))^\dagger (D_\mu \phi(x)) - \phi^\dagger(x) m^2 \phi(x) \quad (5.1)$$

where D^μ is the covariant derivative defined by

$$D^\mu \phi(x) = \partial^\mu \phi(x) - \frac{1}{2} g \lambda_\alpha A_\alpha^\mu(x) \phi(x) \quad (5.2)$$

Following the same procedure as in section IIA, we find the modified point particle Lagrangian

$$\begin{aligned} L_0(x) = & \frac{1}{\sqrt{-x^2}} (\phi_\tau - \frac{1}{2} g \lambda_\alpha A_\alpha^\mu x_{\tau\mu} \phi)^\dagger (\phi_\tau - \frac{1}{2} g \lambda_\alpha A_\alpha^\mu x_{\tau\mu} \phi) \\ & + \frac{g^2}{4\sqrt{-x_\tau^2}} \phi^\dagger \lambda_\alpha \lambda_\beta \phi \sum_1 (n_1^\mu A_{\alpha\mu} n_1^\nu A_{\beta\nu}) - \sqrt{-x_\tau^2} \phi^\dagger m^2 \phi, \end{aligned} \quad (5.3)$$

where $A_{\alpha}^{\mu}(x(\tau))$ may be considered as an external field. The Lagrangian (5.3) is invariant under the infinitesimal gauge transformation restricted to the world-line:

$$\lambda \cdot \delta A^{\mu}(x) = i[\lambda \cdot A(x), \lambda \cdot A^{\mu}(x)] + \partial^{\mu}(\lambda \cdot A(x)) \quad (5.4)$$

$$\delta \phi(\tau) = i \lambda \cdot A(x) \phi(\tau)$$

where

$$n_1^{\mu}(\tau) \partial_{\mu} A(x) = 0 \quad (5.5)$$

This latter condition is necessary to keep $\phi(\tau)$ on the same world-line following the gauge transformation. The normal components of A_{α}^{μ} , namely $n_1^{\mu} A_{\mu}^{\alpha}(x)$, cannot be gauge-transformed away in general.

We observe that the above procedure for coupling gauge fields to the ends of the string is quite different from that of Ademollo et al.⁽²⁶⁾, who did not use field theory as a starting point.

The current that couples to the gauge fields is located only on the world-lines of each quark, where $x^{\mu} = x^{\mu}(\tau, \sigma = 0)$ or $x^{\mu} = x^{\mu}(\tau, \sigma = \pi)$. The local current at $x^{\mu} = x^{\mu}(0)$ is then given by

$$\begin{aligned} J_{\alpha}^{\mu}(x(0)) &= \frac{1}{g} \frac{\partial L_0(x(0))}{\partial A_{\mu}^{\alpha}(x(0))} \\ &= \left[\phi_0^{\dagger} \frac{1}{2} \frac{\lambda^{\alpha}}{\tau} \overleftrightarrow{\partial}_{\tau} \phi_0 + \frac{g}{4} \phi_0^{\dagger} \{ \lambda^{\alpha}, \lambda^{\beta} \} \phi_0 A_{\beta}^{\nu}(x(0)) \partial_{\tau} x_{\nu}(0) \right] \frac{x_{\tau}^{\mu}(0)}{\sqrt{-x_{\tau}^2(0)}} \\ &\quad + \sum_1 \left[\frac{g}{4} \phi_0^{\dagger} \{ \lambda^{\alpha}, \lambda^{\beta} \} \phi_0 A_{\beta}^{\nu}(x(0)) n_{1\nu}(x(0)) \right] n_1^{\mu}(x(0)) \end{aligned} \quad (5.6)$$

and a similar expression gives the current at $x^\mu = x^\mu(\pi)$. The coupling scheme described here allows us in principle to calculate hadron form factors. We have not yet carried out this program.

VI. Conclusion

Motivated by the many parallels between the dual-string picture of hadrons and the quark-gluon field theories of hadron dynamics, we have sought a method of attaching quark quantum numbers to world-lines joined by relativistic strings. We began by developing techniques for restricting classical quark fields, with any desired measurable quantum numbers, to a world-line. Very simple point-particle theories resulted when we considered spinless quarks and spin 1/2 quarks without Zitterbewegung. The latter spin-1/2 model contains an algebra reminiscent of supersymmetry. A much more complex and interesting theory possessing Zitterbewegung arose when we restricted a classical Dirac field to a world-line.

By attaching structureless masses to the string as in ref. (13), we developed an intuitive picture for the dynamics of the simple longitudinal string oscillations. The string with two masses on each end gave a roughly linear mass-squared spectrum as expected of a meson-like system. In a lightlike gauge, this system is described by an integral equation found also by 't Hooft in a totally different context. A baryon-like system resulted from placing a third mass in the middle of the string; for small masses, the normal modes of this system simulate diquarks oscillating against a third single quark.

Next, we analyzed the longitudinal spectrum of a model for mesons consisting of spinless SU(3) quarks on the ends of the string. Systematic deviations from a linear spectrum were found in the Bohr-Sommerfeld approximation to the quantum theory, while symmetry breaking appeared in a natural way. The SU(3) pseudoscalar meson masses were found to obey a formula similar to that of Gell-Mann and Okubo, but with masses replacing squared-masses. Our technique for including internal symmetries therefore has nontrivial implications.

Finally, we observed that we could couple external fields to quarks on the ends of the string in a straightforward way. The essence of the technique consisted of examining the spacetime field-theoretic Lagrangian for a quark

coupled to a vector gauge field and restricting the quark fields to a world-line. The gauge fields were then interpretable as external field potentials and the system was invariant under a restricted class of gauge transformations. Furthermore, the point quark Lagrangian permitted us to clearly identify the physical currents.

Only the simplest aspects of our proposed models have been worked out in detail here. There are clearly many other facets which would be interesting to explore. The baryon spectrum needs to be investigated more thoroughly, as do the problems of using Dirac quarks for both mesons and baryons. Understanding the quantum mechanics of these models will surely be challenging. Many additional effects will occur when one allows arbitrary motions of the string, instead of considering only longitudinal motions as we did here. Calculating form factors for models with Dirac-like point quarks will give stringent conditions on the phenomenological validity of our proposals.

It is also possible to replace or supplement the relativistic string potential by more complicated interactions. For example, an Iwasaki-Kikkawa spinning string⁽²⁸⁾, corresponding to the Neveu-Schwartz model, would be expected to generate spin-spin interactions between the quarks. One could also conceive of stringlike potentials that would generate more complicated spin-spin interactions, or even isospin-isospin interactions; such "strings" would then have a close phenomenological correspondence to the effects of field-theoretic quark binding.

Acknowledgements

We are pleased to acknowledge much stimulating contact with W.A. Bardeen and R.D. Peccei while this work was being developed.

Appendix -- Fields on a World-Line, a World Sheet, etc.

Following a suggestion of Giles and Tye⁽¹²⁾ we treat a field on a space smaller than physical spacetime by using the induced metric on the smaller subspace to generate Poincare-invariant Lagrangians. Parametrizing the subspace by the variables τ^a , we write the spacetime position of any point in the subspace as $x^\mu(\tau^a)$. The induced metric is defined in terms of the tangents $\frac{\partial x^\mu}{\partial \tau^a}$ as

$$g_{ab} = \frac{\partial x^\mu}{\partial \tau^a} \eta_{\mu\nu} \frac{\partial x^\nu}{\partial \tau^b} \equiv x_a \cdot x_b, \quad (A.1)$$

where the x^μ -space metric $\eta^{\mu\nu}$ is taken to be flat, but could depend on x^μ , e.g., in polar coordinates. In Cartesian coordinates, $\eta^{\mu\nu} = \text{diag. } (-1, 1, 1, 1)$. Writing

$$g = \det(g_{ab}), \quad (A.2)$$

we find the invariant volume element in τ^a -space to be

$$(d\omega) = (d\tau) \sqrt{-g}. \quad (A.3)$$

If we now define g^{ab} as the inverse of g_{ab} , we can examine the metric $\eta^{\mu\nu}$ written in terms of a complete set of vectors

$$\eta^{\mu\nu} = x_a^\mu g^{ab} x_b^\nu + \sum_i n_i^\mu n_i^\nu. \quad (A.4)$$

$\eta_{\mu\nu}$ becomes the inverse of $\eta^{\mu\nu}$ provided the n_i^μ are an independent set of normals to the subspace,

$$\begin{aligned} n_i^\mu \eta_{\mu\nu} n_j^\nu &= \delta_{ij} \\ n_i^\mu \eta_{\mu\nu} x_a^\nu &= 0 \\ x_a^\mu \eta_{\mu\nu} x_b^\nu &= g_{ab}. \end{aligned} \quad (A.5)$$

The metric g_{ab} has one timelike direction, while n_i^2 is always spacelike.

We confirm that

$$\begin{aligned} \eta^{\mu\nu} \eta_{\nu\lambda} x_c^\lambda &= x_c^\mu \\ \eta^{\mu\nu} \eta_{\nu\lambda} n_1^\lambda &= n_1^\mu \end{aligned} \quad (A.6)$$

Now let us consider a field $\phi(x)$ and restrict it to live only on the τ^a subspace, so that its normal derivatives out of the subspace vanish,

$$n_1^\mu \partial_\mu \phi(x(\tau^a)) = 0 \quad (A.7)$$

Then we take ϕ to depend on τ^a through the variables $x^\mu(\tau^a)$. This gives effectively $\phi = \phi(\tau^a)$, where each point τ^a corresponds to a particular point on the subspace embedded in x^μ space.

The result is that derivatives of ϕ with respect to x_μ are replaced by

$$\begin{aligned} \partial^\mu \phi &= \eta^{\mu\nu} \partial_\nu \phi = (x_a^\mu g^{ab} x_b^\nu + \sum_i n_i^\mu n_i^\nu) \frac{\partial \phi}{\partial x_\nu} \\ &= x_a^\mu g^{ab} \frac{\partial \phi(\tau^a)}{\partial \tau^a} \end{aligned} \quad (A.8)$$

In a one-dimensional timelike τ - space corresponding to a point particle's world-line, we find

$$\begin{aligned} g_{ab} &= g = x_\tau^\mu \eta_{\mu\nu} x_\tau^\nu \equiv x_\tau^2 \\ g^{ab} &= g^{-1} = 1/x_\tau^2 \\ \eta^{\mu\nu} \frac{\partial \phi}{\partial x^\nu} &= \frac{x_\tau^\mu}{x_\tau^2} \frac{\partial \phi(\tau)}{\partial \tau} \end{aligned} \quad (A.9)$$

The invariant line element is simply

$$ds = d\tau \sqrt{-g} = d\tau \sqrt{-x_\tau^2} \quad (A.10)$$

References and Footnotes

- (1) For a review and a comprehensive list of references, see S. Mandelstam, Physics Reports 13C, 260 (1974).
- (2) For a discussion of the advantages of the colored quark model, see H. Fritzsch, M. Gell-Mann and H. Leutwyler, Phys. Lett. 47B, 365 (1973). Confinement with non-Abelian gauge bosons is discussed by D. Gross and F. Wilczek, Phys. Rev. D8, 3633 (1973) and S. Weinberg, Phys. Rev. Lett. 31, 494 (1973).
- (3) S. Mandelstam, Nucl. Phys. B64, 205 (1973); B69, 77 (1974).
- (4) Y. Nambu, Proc. Int. Conf. on Symmetries and Quark Models, Wayne State University, 1969.
H. Nielsen, 15th Int. Conf. on High Energy Physics, Kiev, 1970.
L. Susskind, Nuovo Cimento 69, 210 (1970).
L.N. Chang and F. Mansouri, Phys. Rev. D5, 2535 (1972).
P. Goddard, J. Goldstone, C. Rebbi and C.B. Thorn, Nucl. Phys. B61, 45 (1973).
- (5) H. Harari, Phys. Rev. Lett. 22, 562 (1969).
J.L. Rosner, Phys. Rev. Lett. 22, 689 (1969).
- (6) H.B. Nielsen and P. Clesen, Nucl. Phys. B61, 47 (1973).
- (7) K. Wilson, Phys. Rev. D10, 2445 (1974).
- (8) J. Kogut and L. Susskind, Phys. Rev. D11, 395 (1975).
- (9) G. 't Hooft, Nucl. Phys. B75, 461 (1974).
We thank C. Rebbi for bringing this reference to our attention.
- (10) A. Neveu and J.H. Schwarz, Nucl. Phys. B31, 86 (1971).
K. Bardakci and M.B. Halpern, Phys. Rev. D3, 2493 (1971).
- (11) M.B. Halpern, Quantum "Solitons" which are SU(n) Fermions, Berkeley preprint, 1975 (to be published).
- (12) R.C. Giles, Stanford University Thesis, 1975, and
R.C. Giles and S.-H.H. Tye, SLAC preprint (to be published).

L.N. Chang and F. Mansouri, in Workshop on Current Problems in High Energy Physics, Johns Hopkins University (1974).

See also our Appendix.

- (13) W.A. Bardeen, I. Bars, A.J. Hanson and R.D. Peccei, Stanford preprint (to be published). Massive end-points have been briefly considered also by A. Chodos and C. Thorn, Nucl. Phys. B72, 509 (1974).
- (14) I. Bars, Nuclear Phys. B31, 15 (1971).
- (15) For a rudimentary attempt in this direction, see S.K. Wong, Nuovo Cimento 65A, 689 (1970).

We would in general obtain a different set of equations.

- (16) S. Weinberg, Phys. Rev. D7, 1068 (1973).
- (17) J. Frenkel, Zeitsch. f. Phys. 37, 273 (1926);
J. Weyssenhoff, Acta Phys. Pol. 9, 1 (1947).
"
 - (18) See however, F. Gursey, Nuovo Cimento 5, 784 (1957).

The spin-1/2 field in this reference, satisfies a different equation than our two models. We thank F. Gursey for pointing out to us this work.

- (19) K. Huang, Am. J. of Phys. 20, 479 (1952).
- (20) See, e.g., A.J. Hanson and T. Regge, Ann. Phys. (N.Y.) 87, 498 (1974).
- (21) J. Wess and B. Zumino, Nucl. Phys. 70B, 39 (1974);
A. Salam and J. Strathdee, Nucl. Phys. 76B, 477 (1974);
D. Volkov and P. Akulov, Phys. Lett. 46B, 109 (1974).
- (22) In these two cases we can show without using the free Euler equation $\partial_{\tau} p^{\mu} = 0$, that the interacting quark-string system reduces effectively to the system considered in section III. The only change is the replacement of the parameters μ_0 and μ_{π} by time-independent mass operators, as discussed in section IV.
- (23) J.D. Bjorken, private communication.
- (24) J.J. Aubert et al., Phys. Rev. Lett. 33, 1404 (1974);
C. Bacci et al., Phys. Rev. Lett. 33, 1408 (1974);
G.S. Abrams et al., Phys. Rev. Lett. 33, 1453 (1974).

- (25) T. Appelquist and E.D. Politzer, Phys. Rev. Lett. 34, 43 (1975).
- (26) S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967);
A. Salam, in Elementary Particle Theory, ed. N. Svartholm,
(Wiley, New York 1969).
- (27) Ademollo et al., Nuovo Cimento 21A, 77 (1974).
- (28) Y. Iwasaki and K. Kikkawa, Phys. Rev. D8, 441 (1973);
L.N. Chang, K. Macrae and F. Mansouri, Phys. Lett. 57B, 59
(1975).

Figure Captions

- Fig. 1.1 Duality diagram for the s-t contribution to meson-meson scattering.
The dotted area represents the surface swept out by the string.
- Fig. 3.1 Phase-space diagram following from eq. (3.5).
- Fig. 3.2 A typical motion resulting from the Hamiltonian (3.21) in the zero
mass limit.
- Fig. 3.3 (a) Pure J_1 mode, (b) pure J_2 mode, indicating diquark
correlations inside low mass baryons.
- Fig. 4.1 Model for meson consisting at a given time of a point quark and
a point anti-quark connected by a string.
- Fig. 4.2 Three possible configurations for the quarks on the string, giving
model baryons.

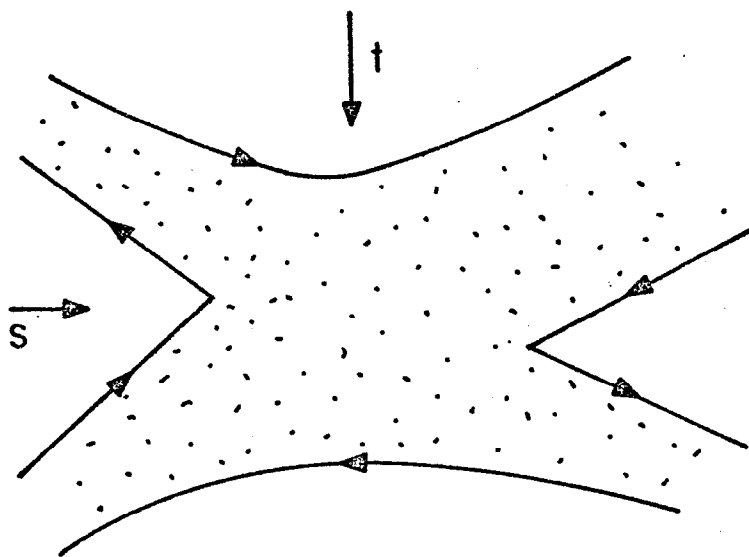


Fig. 1.1

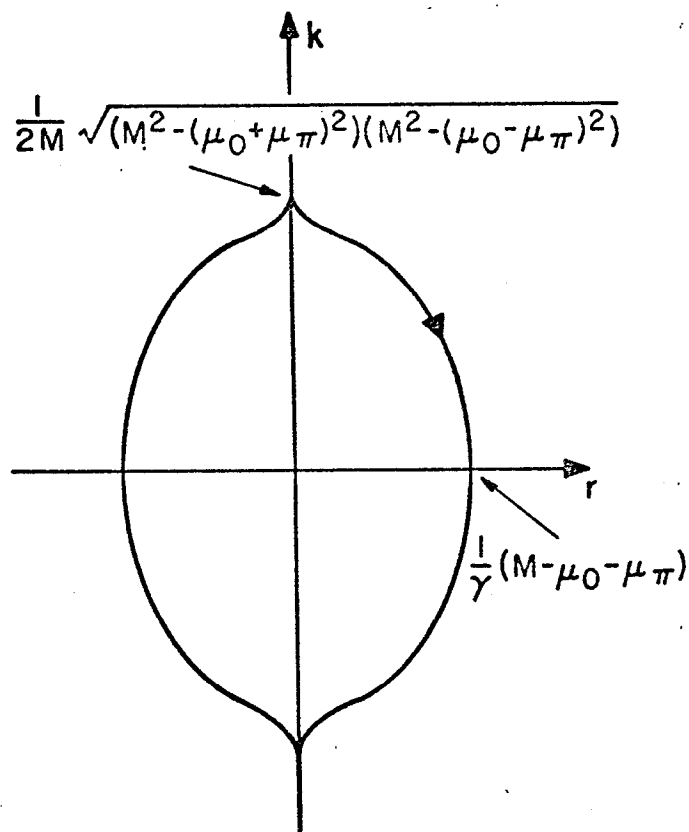


Fig. 3.1

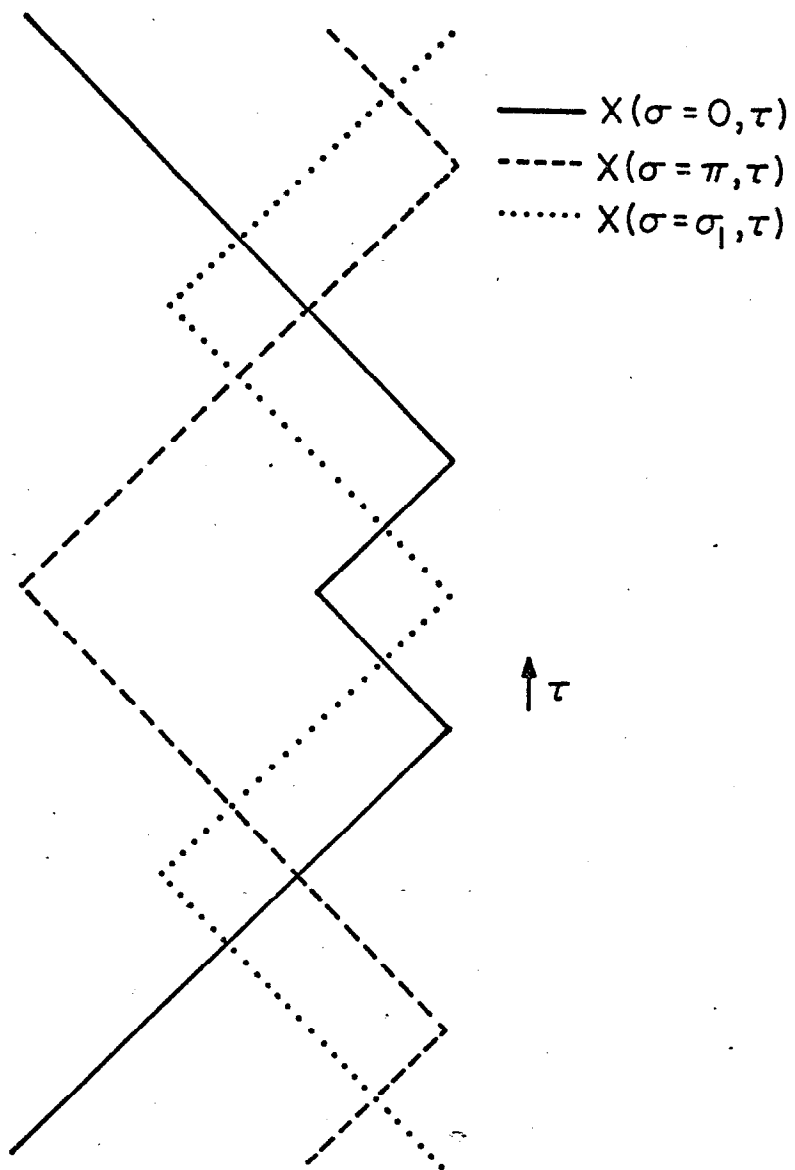


Fig. 3.2

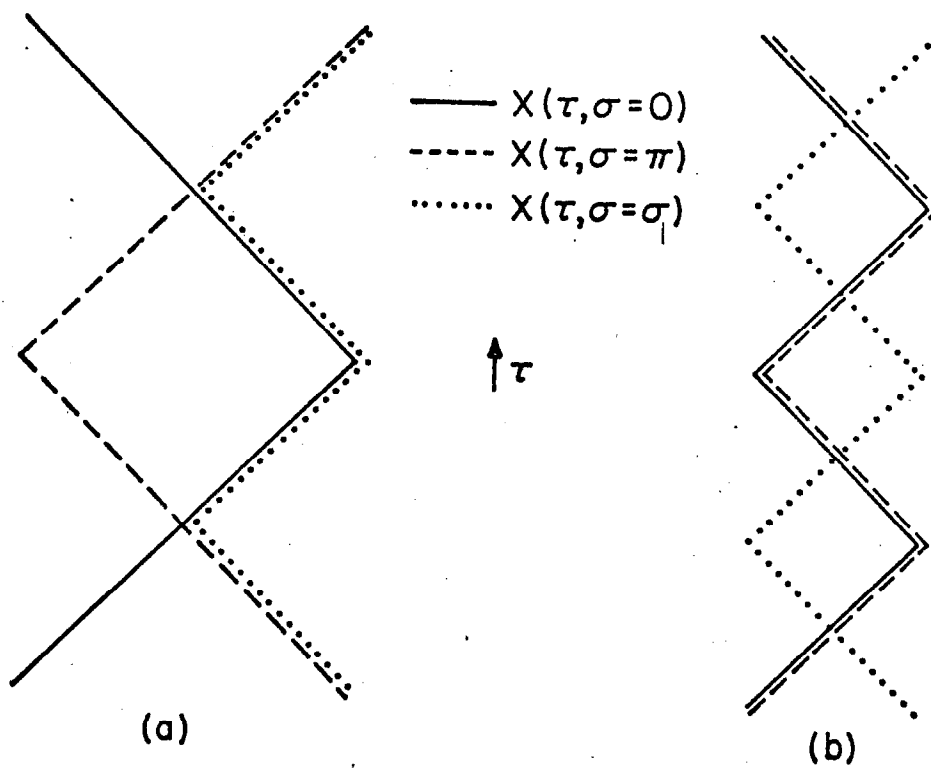


Fig. 3.3



Fig. 4.1

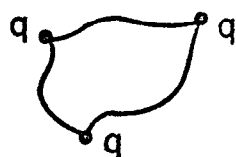
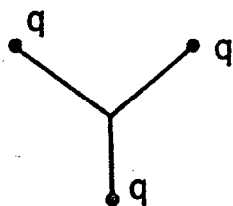


Fig. 4.2