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SINGULAR CORES IN THE THREE-BODY PROBLEM

III. UNIQUE EQUATIONS*

D. D. Brayshaw
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

Three-particle scattering theory is reexamined for interactions which contain singular cores in a finite number of two-body partial-waves. It is shown that the Faddeev equations do not possess a unique solution for the corresponding t matrix, and hence cannot be used directly to examine "realistic" singular core models of the N-N interaction. For the special case of a pure boundary condition model, alternative one-dimensional equations are derived based on the Schrödinger and Faddeev representations. These are shown to be completely equivalent, and to uniquely specify the three-body observables. The Faddeev version is shown to be a special case of the author's boundary condition formalism. Equations suitable for "realistic" models which include potential tails are presented.

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I. INTRODUCTION

The hard core and its generalization, the boundary-condition model (BCM), provide a useful abstraction in describing the short-range behavior of the nucleon-nucleon (N-N) interaction. Both have been employed to construct models of the N-N potential which are "realistic" in the sense of providing an excellent fit to N-N scattering data and the static properties of the deuteron.[1] In order to assess the effectiveness of this description (e.g., in comparison to soft-core models) one must probe the off-shell structure by studying systems of three or more nucleons. However, special problems arise in performing three- (or more) body calculations in the presence of singular cores. This was first noted by the present author almost five years ago,[2] and further explored in two subsequent articles.[3]

In the first of these papers (SC I), it was shown that if the three-body wave function is taken to vanish identically whenever any pair of particles is within their core radius, then the Faddeev equations do not possess a unique solution; this assumption is equivalent to requiring a singular core to be present in <u>each</u> two-particle partial-wave. It was further shown that for a "pure" BCM (no external potential), the Faddeev equations could be reduced to a one-dimensional form (since the BCM t matrix is not separable, this is a unique feature of the model). However, due to the absence of information in the interior (core) region, this reduction does not lead to a unique solution, and it was necessary to supplement the BCM by an auxiliary boundary-condition. The choice of this additional constraint is limited by analyticity and three-particle unitarity, and the form proposed in SC I is essentially the simplest consistent with these requirements. Unfortunately, this procedure introduced an arbitrary parameter (W_0) into the problem which was aesthetically unsatisfying, although subsequent numerical work in SC II demonstrated a complete lack of sensitivity to this parameter.

On the other hand, the treatment in SC I depended on some special operator relations and did not apply to the case where a singular core is present in only a finite number of partial-waves, and there were indications that this could change the character of the problem. Thus V. Efimov had shown that the Schrödinger equation for three identical bosons interacting only via s-wave hard cores could be reduced to a unique one-dimensional form,[4] and Kim and Tubis had solved the (two-dimensional) Faddeev equations numerically for an s-wave Herzfeld potential (hard core plus square well).[5] However, these results raised some questions in that Efimov's derivation did not address the problem of three-body unitarity, and hence might have corresponded to a trivial (and parameterless) auxiliary condition rejected in SC I. Similarly, it can be shown that the Faddeev kernel is not square-integrable (L_2) even for an s-wave core, and thus it was surprising that Kim and Tubis could achieve a stable numerical result with their technique. In addition, it was not clear how the ambiguity demonstrated in SC I could arise in going from a finite to an infinite number of partial-wave cores, whether it could be eliminated by following this limiting process, and whether numerical results would be stable as cores were added to p, d, ... waves.

Recent results by V. N. Efimov (a different Efimov than the author of Ref. 4) have added to the confusion surrounding this subject.[6] Thus, by extending the method of Ref. 5, he was able to derive a unique one-dimensional equation for three identical bosons interacting via the s-wave BCM (this differs from the hard core in that the logarithmic derivative parameter λ_0 is finite). This distinction enables one to include an attraction, and hence the system can bind for suitable values of λ_0. If one takes the (average) nucleon mass, and requires the two-body system to have the deuteron effective-range parameters, the core radius r_c and λ_0 are determined and the three-body problem is uniquely specified.

Numerical results for this model have recently been obtained by V. N. Efimov and H. Schulz, who quote a binding energy of 7.7 MeV.[7] This is to be compared with results of 18.4 MeV and 12.7 MeV obtained by this author and Kim and Tubis, respectively, using the same values of r_c and λ_0. Although the assumptions of SCI (and the resultant equations) differ from those of Efimov in the sense noted above, it is nevertheless surprising that results as different as 18.4 and 7.7 can be obtained with models which embody essentially the same physics. Moreover, the model studied by Kim and Tubis is exactly identical to that of Efimov and Schulz, and hence a discrepancy of 5 MeV (compared to errors of a few percent) is quite disturbing. Inasmuch as all of these authors have taken some pains to verify their numerics, it is clear that our understanding of this problem is far from satisfactory.

The aim of the present article is to clarify this situation by relating the various approaches taken to this problem. We begin in Section II by considering the method employed by the Efimovs.[4,6] As noted above, this approach has only been carried through in the very special case of identical particles and s-wave cores. However, by utilizing the powerful operator techniques developed in SCI, we are able to derive a one-dimensional equation for the pure BCM which is valid in the most general case (no restrictions except to a finite number of two-particle partial-waves). Moreover, we demonstrate that this equation provides a unique solution, and that uniqueness of the wave function does not require the special symmetry for identical particles relied on by Efimov. The derivation is also more general in that it is applied to the scattering state, rather than the bound state problem, and one can apply the methods of SCI to explicitly verify three-particle unitarity.

We next present a completely independent derivation in Section III based on the Faddeev equations; this again leads to a unique (but different) one-dimensional equation. In the process we prove that a unique solution to the Faddeev equations does not exist, even if the singular cores are restricted to a finite number of partial-waves. This clearly casts some doubt on the numerical work by Kim and Tubis. The precise relationship between the different approaches is then examined in Section IV, and the following results are obtained: (1) the three-particle scattering amplitudes T_s and T_f defined by the derived integral equations of Sections II and III are related by $T_f = T_s^T$, and hence are identical; (2) as the number of partial-waves is taken to infinity the latter equation is identical to the result of SC I (and the same ambiguity reappears); (3) as is the case for conventional potentials, there exists suitable input to the author's general boundary condition formalism (BCF) which exactly reproduces the equation of Section III.[9] The distinction between the equation derived in SC I and the new equations can thus be expressed as a more general type of "auxiliary condition" in the sense of SC I.

The implications of these results are discussed in Section V. In particular, one consequence of Section III is that the Faddeev equations are never suitable for three-particle calculations if any singular cores are present. This presents an apparent impediment to the consideration of "realistic" singular core models (i.e., with external potentials) in the three-nucleon problem. However, we demonstrate that a trivial extension of the derivation given in Section III yields suitable integral equations for this problem. It should be stressed that the considerations of this article do not pertain to the applicability of the BCF as a general approach to the three-body problem. In fact, it is the only formalism sufficiently general to incorporate both conventional potentials and singular cores.

II. SCHRÖDINGER DERIVATION

In this section we derive a one-dimensional equation for the pure BCM using the method devised by V. Efimov[4] (and extended by V. N. Efimov[6]). This requires that we begin with the Schrödinger equation for a model with finite repulsion, and subsequently go to the hard core limit. In doing so we shall rely heavily on the operator techniques developed in SC I. This permits us to paraphrase the Efimov derivation in a more elegant fashion, while at the same time extending it to a much more general situation. Although we ignore spin and isospin degrees of freedom, these can be included by a simple extension of the basis states, and thus our result is quite general.

We denote the mass of particle α by m_α and the total c.m. energy by W. Three-particle states are described by the usual Jacobi variables $\vec{p}_\alpha$, $\vec{q}_\alpha$, which correspond to the relative momentum of particles β and γ, and the momentum of α relative to the $\beta\gamma$ c.m., respectively. The conjugate variables in the coordinate representation are taken to be $\vec{x}_\alpha, \vec{y}_\alpha$. Introducing the corresponding reduced masses μ_α, M_α, these definitions imply that the quantities

$$\begin{aligned} \bar{P}^2 &= p_\alpha^2/2\mu_\alpha + q_\alpha^2/2M_\alpha \quad , \\ \bar{R}^2 &= x_\alpha^2 + (M_\alpha/\mu_\alpha)\, y_\alpha^2 \quad , \end{aligned} \tag{1}$$

are independent of the index α. The condition for a physical scattering state (on-shell condition) is that $\bar{P}^2 = W$.

In order to rid ourselves of the channel indices, and to consequently simplify the required manipulations, we follow the method of SC I and introduce a Hilbert space of states $|\alpha\vec{p}\,\vec{q}>$ with the normalization

$$<\alpha\vec{p}\,'\vec{q}\,'|\beta\vec{p}\,\vec{q}> = \delta_{\alpha\beta}\,\delta(\vec{p}\,' - \vec{p})\;\delta(\vec{q}\,' - \vec{q}) \quad ,$$

$$\sum_{\alpha} \int d\vec{p}\; d\vec{q}\; |\alpha\, \vec{p}\; \vec{q}\rangle \langle \alpha\, \vec{p}\; \vec{q}\,| = 1 \quad . \tag{2}$$

Here the index α implies that $\vec{p}, \vec{q}$ are the numerical values of $\vec{p}_\alpha, \vec{q}_\alpha$, and the completeness relation automatically performs the sum over channels in an operator product. Although only two of the six vectors $\vec{p}_\beta, \vec{q}_\beta$ (or $\vec{x}_\beta, \vec{y}_\beta$) are linearly independent, it is convenient to retain all of them in order to most simply express the corresponding two-particle operators. The connections between them can be expressed via an operator I which "interconnects" the Faddeev channels. Thus

$$\begin{aligned} \langle \alpha\, \vec{p}'\, \vec{q}'\, |I| \beta\, \vec{p}\; \vec{q}\rangle &= -\delta(\vec{p} - \vec{p}'_\beta)\, \delta(\vec{q} - \vec{q}'_\beta) \quad , \\ \langle \alpha\, \vec{x}'\, \vec{y}'\, |I| \beta\, \vec{x}\; \vec{y}\rangle &= -\delta(\vec{x} - \vec{x}'_\beta)\, \delta(\vec{y} - \vec{y}'_\beta) \quad , \end{aligned} \tag{3}$$

where $\vec{p}'_\beta, \vec{q}'_\beta\, (\vec{x}'_\beta, \vec{y}'_\beta)$ are the appropriate linear combinations of $\vec{p}'\,, \vec{q}'\;\; (\vec{x}'\,, \vec{y}')$, and the diagonal matrix elements of I vanish. One can easily verify that I is a symmetric operator ($I = I^T$) with the following properties[10]:

$$\begin{aligned} I^{-1} &= (1+I)/2 \quad , \\ (1-I)^2 &= 3(1-I) \quad , \\ (1-I)\,(2+I) &= 0 \quad . \end{aligned} \tag{4}$$

As an illustration, we note that the three-body state vector is $|\Psi\rangle$, with the Faddeev channel decomposition $|\Psi\rangle = \sum_\alpha |\psi_\alpha\rangle$. With the above definitions

$$\begin{aligned} \Psi(\vec{x}_\alpha, \vec{y}_\alpha) &= \langle \alpha\, \vec{x}\; \vec{y}\, |\Psi\rangle \\ &= \langle \alpha\, \vec{x}\; \vec{y}\, |(1-I)\, |\psi\rangle \\ &= \sum_\beta \psi_\beta(\vec{x}_\beta, \vec{y}_\beta) \quad , \end{aligned} \tag{5}$$

where

$$\langle \alpha\, \vec{x}\; \vec{y}\, |\psi\rangle = \psi_\alpha(\vec{x}_\alpha, \vec{y}_\alpha) \quad . \tag{6}$$

In general, any observable quantity A (not artificially decomposed into channels) has the structure

$$\begin{aligned} A &= (1-I)\,\bar{A}(1-I) \\ &= \frac{1}{3}(1-I)\,A \\ &= \frac{1}{3}A(1-I) \quad . \end{aligned} \tag{7}$$

We may now define the free Green's function (G_0) and the two-particle t matrix (t) as operators on this space; thus

$$\begin{aligned} &\langle \alpha \vec{p}' \vec{q}' | t | \beta \vec{p}\,\vec{q} \rangle = \delta_{\alpha\beta}\,\delta(\vec{q}' - \vec{q})\; t_\alpha(\vec{p}', \vec{p}; s_\alpha) \quad , \\ &\langle \alpha \vec{p}'\vec{q}' | G_0 | \beta \vec{p}\,\vec{q} \rangle = \frac{\delta_{\alpha\beta}\,\delta(\vec{p}' - \vec{p})\;\delta(\vec{q}' - \vec{q})}{p^2/2\mu_\alpha + q^2/2M_\alpha - W - i\epsilon} \end{aligned} \tag{8}$$

where $s_\alpha = W - q^2/2M_\alpha$. Similarly, defining the potential operator V such that

$$\langle \alpha \vec{x}' \vec{y}' | V | \beta \vec{x}\,\vec{y} \rangle = \delta_{\alpha\beta}\,\delta(\vec{y} - \vec{y}')\; V_\alpha(\vec{x}', \vec{x}) \quad , \tag{9}$$

t satisfies the Lippmann-Schwinger equation

$$\begin{aligned} t &= V - V G_0 t \\ &= V - t\, G_0 V \quad , \end{aligned} \tag{10}$$

written on this Hilbert space. We may now state the Schrödinger equation as an operator equation on our basis in the form

$$\left[H_0 - W + (1-I)V\right] |\Psi\rangle = 0 \quad , \tag{11}$$

with $G_0 = (H_0 - W - i\epsilon)^{-1}$. Substitution of the channel representation leads to the relations (using Eq. (10))

$$|\psi\rangle = |\phi\rangle - G_0 V |\Psi\rangle \quad , \tag{12a}$$

$$|\psi\rangle = (1 - G_0 t)\,|\phi\rangle + G_0 t\, I |\psi\rangle \quad , \tag{12b}$$

$$|\Psi> = (1-G_0 t)\left[|\phi> - I|\psi>\right] . \tag{12c}$$

Here $|\phi>$ is an arbitrary incoming (plane-wave) state in our basis satisfying $(H_0-W)|\phi> = 0$; Eq. (12b) is the wave function version of the Faddeev equations. Furthermore, by writing Eq. (11) in the form

$$(H_0 + V - W)|\Psi> = IV|\Psi> \tag{13}$$

and employing Eq. (10), we deduce that

$$|\Psi> = (1-G_0 t)\left[(1-I)|\phi> + G_0\, IV|\Psi>\right] , \tag{14}$$

and hence that

$$V|\Psi> = t(1-I)\,|\phi> + tG_0\, IV|\Psi> . \tag{15}$$

All of the above equations assume that V is a typical potential with "normal" convergence properties. We now specialize somewhat to potentials of finite range such that $V_\alpha(\vec{x},\vec{x}')$ vanishes for $x > a_\alpha$ (or $x' > a_\alpha$). We embody this constraint into our formal development by introducing an operator $\tilde{V}$ such that

$$<\alpha\,\vec{x}'\vec{y}'|\tilde{V}|\beta\,\vec{x}\,\vec{y}> = \delta_{\alpha\beta}\,\delta(\vec{y}-\vec{y}')\,\delta(\vec{x}-\vec{x}')\,\theta(a_\alpha - x) , \tag{16}$$

where θ is the unit step function (this notation is consistent with SC I). The finite range restriction can then be expressed as

$$V = \tilde{V}V = V\tilde{V} , \tag{17}$$

(clearly $\tilde{V}$ is a projection operator). A general property of such models is that

$$tG_0 = tG_0\,\tilde{V} + t'G_0(1-\tilde{V}) , \tag{18}$$

where t' has the operator structure of t in Eq. (8), with

$$t'_\alpha(\vec{p}',\vec{p};s_\alpha) = \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) P_\ell(\hat{p}\cdot\hat{p}')\, t_{\alpha\ell}(p',\kappa_\alpha;s_\alpha)\, G_{\alpha\ell}(p,s_\alpha). \tag{19}$$

Here $t_{\alpha\ell}(p',p;s_\alpha)$ is the partial-wave projection of $t_\alpha(\vec{p}',\vec{p};s_\alpha)$, and κ_α is the on-shell momentum value; $\kappa_\alpha = (2\mu_\alpha s_\alpha)^{1/2}$. The function $G_{\alpha\ell}$ is arbitrary except

for the requirements that it be entire as a function of p (for fixed s_α), and satisfy

$$G_{\alpha\ell}(\kappa_\alpha, s_\alpha) = 1 \quad ,$$

$$\int_0^\infty \frac{dp\, p^2\, G_{\alpha\ell}(p, s_\alpha)\, j_\ell(xp)}{p^2 - \kappa_\alpha^2 - i\epsilon} = \frac{\pi i \kappa_\alpha}{2} h_\ell(x\kappa_\alpha) \tag{20}$$

for $x > a_\alpha$ (a particular choice is discussed following Eq. (24) of SC I). Physically, Eq. (18) is just the statement that the wave function of the two-particle system takes its asymptotic form for $x > a_\alpha$ (note that the two-body state vector is given by $|\psi_2\rangle = (1-G_0 t)|\phi\rangle$, and use the symmetry of G_0 and t). In practice, the precise form of $G_{\alpha\ell}$ never matters, since its integrated property (Eq. (20)) is all that is required.

Returning to Eq. (15), it follows that <u>for finite range potentials</u>,

$$V|\Psi\rangle = \left[t(1-I) + \tilde{V}\mathscr{D} + gY\right]|\phi\rangle \quad , \tag{21}$$

where

$$\begin{aligned} \mathscr{D}|\phi\rangle &\equiv t\, G_0 \tilde{V}\, I\, V|\Psi\rangle \quad , \\ gY|\phi\rangle &\equiv t' G_0 (1-\tilde{V}) I\, V|\Psi\rangle \quad , \end{aligned} \tag{22}$$

and g has the form

$$\langle \alpha \vec{p}' \vec{q}' | g | \beta \vec{p}\, \vec{q}\rangle = \delta_{\alpha\beta} \delta(\vec{q} - \vec{q}') \frac{\delta(p-p')}{p^2} \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) P_\ell(\hat{p}\cdot\hat{p}')\, t_{\alpha\ell}(p', \kappa_\alpha; s_\alpha) \quad . \tag{23}$$

With this definition $\langle \alpha \vec{p}\, \vec{q} | Y$ is independent of the value of p (note that Y and $\mathscr{D}$ are <u>different</u> quantities than those that appear in SC I; the notation here is that of V. N. Efimov). Consequently, we deduce that

$$\tilde{V}(1-I)\tilde{V}\mathscr{D}\,|\phi\rangle = \tilde{V}(1-I)V|\Psi\rangle - \tilde{V}(1-I)\left[t(1-I)+gY\right]|\phi\rangle \quad , \tag{24}$$

and that

$$|\psi> = \left[1 - G_0 t(1-I) - G_0 \tilde{V}\mathscr{D} - G_0 gY\right]|\phi> \quad , \tag{25}$$

using Eq. (12a). Equivalently, defining the channel t matrix τ such that

$$|\psi> = (1 - G_0\tau)\,|\phi> \quad , \tag{26}$$

we have

$$\tau = t(1-I) + \tilde{V}\mathscr{D} + gY \quad . \tag{27}$$

Up until this point we have said nothing about singular cores, and there are no delicacies involved in the above manipulations. Our motivation has been to derive certain operator relations (and in particular Eqs. (24) and (25)) which are valid for arbitrary finite range models (which may include finite repulsive cores). The basic idea of the Efimov approach is to take these equations to the hard core (or BCM) limit. In doing so, it should be noted that the potential operator appears only in the combination $V|\Psi>$ in Eq. (24). All reference to V can be eliminated by noting that

$$\lim_{(BCM)} \tilde{V}(1-I)V|\Psi> = 0 \quad . \tag{28}$$

This may be deduced by considering the Schrödinger equation (Eq. (11)). Inside the core region, that equation can only be satisfied in the limit of infinite repulsion if $|\Psi> \to 0$. Except on the boundary, this implies that $(H_0 - W)|\Psi> \to 0$, and the overall operator $\tilde{V}$ in Eq. (28) projects onto the interior. In taking this limit, it is important to note that possible difficulties are avoided due to the separation achieved in Eq. (21). Thus, although operator products involving t may (and do) become delicate after the limit is taken, the multiplicative nature of g implies that no problems arise in taking $t_{\alpha\ell}(p', \kappa_\alpha; s_\alpha)$ to $t^{BC}_{\alpha\ell}(p', \kappa_\alpha; s_\alpha)$ in Eq. (23). Also, $\mathscr{D}$ is taken to be the limit of the product given in Eq. (22), and

hence one expects Eq. (24) to remain valid in that limit. We thus obtain

$$\tilde{V}(1-I)\,\tilde{V}\mathcal{D} = -\tilde{V}(1-I)\left[t(1-I) + g\,Y\right] \tag{29}$$

in the BCM limit.

To this condition we must add the basic statement of the boundary-condition in order to obtain an equation for the unknown quantity Y, which via Eqs. (27) and (29) is sufficient to determine τ (and hence $|\Psi>$). We first recall from SCI that[11]

$$\begin{aligned} t^{BC}_{\alpha\ell}(p,\kappa_\alpha;s_\alpha) &= N_{\alpha\ell}(p)/D_{\alpha\ell}(\kappa_\alpha) \quad , \\ N_{\alpha\ell}(p) &= (a_\alpha\lambda_{\alpha\ell} - \ell)\, j_\ell(a_\alpha p) + a_\alpha p\, j_{\ell+1}(a_\alpha p) \quad , \\ D_{\alpha\ell}(\kappa_\alpha) &= i\pi\mu_\alpha\kappa_\alpha\left[(a_\alpha\lambda_{\alpha\ell} - \ell)\, h_\ell(a_\alpha\kappa_\alpha) + a_\alpha\kappa_\alpha\, h_{\ell+1}(a_\alpha\kappa_\alpha)\right] \quad . \end{aligned} \tag{30}$$

Here $\lambda_{\alpha\ell}$ is the (constant) boundary-condition parameter such that

$$\lim_{\epsilon\to 0}\ \psi'_{\alpha\ell}(a_\alpha+\epsilon)/\psi_{\alpha\ell}(a_\alpha+\epsilon) = \lambda_{\alpha\ell} \quad . \tag{31}$$

Furthermore, if $\hat{N}_{\alpha\ell}(x)$ is the Fourier transform of $N_{\alpha\ell}(p)$, one can show that

$$\frac{1}{a_\alpha}\int_0^\infty dx\, x^2\, \hat{N}_{\alpha\ell}(x)\, f_\ell(x) = \lambda_{\alpha\ell} f_\ell(a_\alpha) - f'_\ell(a_\alpha) \tag{32}$$

for an arbitrary function $f_\ell(x)$. Therefore, defining an operator b such that

$$<\alpha\,\vec{p}'\,\vec{q}'\,|b|\beta\,\vec{p}\,\vec{q}> = \delta_{\alpha\beta}\,\delta(\vec{q}-\vec{q}')\sum_\ell\left(\frac{2\ell+1}{4\pi}\right) P_\ell(\hat{p}\cdot\hat{p}')\, N_{\alpha\ell}(p)/N_{\alpha\ell}(\kappa_\alpha) \quad , \tag{33}$$

the boundary condition can be represented formally as

$$b\,|\Psi> = 0 \quad . \tag{34}$$

As a consequence of Eqs. (23) and (33), it follows that

$$b\,G_0 g A = A \quad , \tag{35}$$

providing A is an operator such as Y with no p-dependence (see remark following Eq. (23)). Also, the nature of $N_{\alpha\ell}$ involves the implicit limit $x \to a_\alpha\,(+)$, and

hence

$$b\tilde{V} = 0 \quad ,$$
$$\tilde{V}t' = 0 \quad , \tag{36}$$
$$\tilde{V}g = 0 \quad .$$

Finally, we observe that

$$\begin{aligned} bG_0t &= b\left[\tilde{V}G_0t + (1-\tilde{V})\, G_0\tilde{t}^{BC}\right] \\ &= b\, G_0\tilde{t}^{BC} \\ &= b \end{aligned} \tag{37}$$

using $(t'^{BC})^T \equiv \tilde{t}^{BC}$ (same notation as SC I); the last step requires the explicit formulas of Eq. (30), and is implied by the two-particle boundary-condition

$$b\,|\psi_2\rangle = b(1 - G_0t)\,|\phi_2\rangle = 0 \quad . \tag{38}$$

With this background we can complete the derivation by substituting $|\Psi\rangle = (1-I)\,|\psi\rangle$ into Eq. (34), with $|\psi\rangle$ given by Eq. (25). We thus obtain

$$b(1-I)\left[1 - G_0t(1-I) - G_0\tilde{V}\mathscr{D} - G_0gY\right] = 0 \quad , \tag{39}$$

since $|\phi\rangle$ is arbitrary; or

$$Y = bIG_0t(1-I) - b(1-I)\, G_0\tilde{V}\mathscr{D} + bIG_0gY \quad , \tag{40}$$

using Eqs. (34) - (37). This will clearly become an integral equation for Y (and reduce to one-dimension in a partial-wave decomposition) providing one can uniquely solve Eq. (29) in order to eliminate $\mathscr{D}$ in favor of Y. Technically, Eq. (29) does <u>not</u> possess a unique solution, but it turns out that the ambiguity resides in a subspace which is not required for our purposes.

In order to investigate this point we must be more precise in defining our model. In the above we have implicitly assumed that the partial-wave sums are truncated to values $\ell \leq \ell_\alpha$. We may incorporate this restriction explicitly

by defining a diagonal projection operator Λ such that

$$\begin{aligned} \Lambda_\alpha &= 1 \qquad \text{for } \ell \leq \ell_\alpha \quad , \\ &= 0 \qquad\quad \ell > \ell_\alpha \quad ; \end{aligned} \tag{41}$$

Λ commutes with all of our basic operators except I. The precise statement of Eq. (29) is then

$$\Lambda\tilde{V}(1-I)\,\Lambda\tilde{V}\mathscr{D} = -\Lambda\tilde{V}(1-I)\left[t(1-I)+gY\right] \quad . \tag{42}$$

In order to solve for $\mathscr{D}$ it is tempting to define an inverse χ such that

$$\Lambda\tilde{V}(1-I)\;\Lambda\tilde{V}\chi = \Lambda\tilde{V} \quad , \tag{43}$$

but one can explicitly show that this equation has no solution. Specifically, if we define a diagonal projection operator R in the coordinate representation such that

$$\begin{aligned} R_\alpha &= 1 \qquad \text{if } \bar{R} < \text{Min}\,(a_\beta) \quad , \\ &= 0 \qquad \text{otherwise} \quad , \end{aligned} \tag{44}$$

it can easily be shown that

$$\Lambda\tilde{V}(1-I)\;\Lambda\tilde{V}\;R\,\chi_0 = 0 \tag{45}$$

for a nontrivial subspace χ_0 (note that Eq. (1) implies that R commutes with I, $\tilde{V}$ and Λ). However, it is possible to define an operator B such that

$$\Lambda\tilde{V}(1-I)\;\Lambda\tilde{V}B = (1-R)\;\Lambda\tilde{V} \quad . \tag{46}$$

Thus, although RB is not uniquely determined by Eq. (46), one can show that (1-R)B <u>is</u> uniquely specified. Here we observe that after a partial-wave decomposition Eq. (46) reduces to a set of coupled integral equations in the variables x,y. However, by making the change of variables

$$\begin{aligned} x &= \bar{R}\,\cos\phi \quad , \\ y &= (\mu_\alpha/M_\alpha)^{1/2}\,\bar{R}\,\sin\phi \quad , \end{aligned} \tag{47}$$

and recalling that I, $\tilde{V}$, Λ conserve $\bar{R}$, these can be reduced to a set of one-dimensional equations in the variable ϕ. Furthermore, these equations can be solved analytically in the special case of identical particles and s-wave cores, as was first shown by V. Efimov.[4]

We therefore define B by Eq. (46), and write the solution of Eq. (42) as[12]

$$\mathscr{D} = -B\tilde{V}(1-I)\left[t(1-I) + gY\right] \quad . \tag{48}$$

This is possible because R only has support in the region interior to all the cores, and hence

$$R = \tilde{V}R = R\tilde{V} \quad . \tag{49}$$

Also, Eq. (18) implies

$$\begin{aligned} t|\phi\rangle &= \left[t G_0 \tilde{V} + t' G_0(1-\tilde{V})\right] G_0^{-1}|\phi\rangle \\ &= t'|\phi\rangle \quad , \end{aligned} \tag{50}$$

for an on-shell state $|\phi\rangle$. Therefore, by Eq. (36) we have

$$\begin{aligned} \left[t(1-I)+gY\right]|\phi\rangle &= (1-\tilde{V})\left[t(1-I)+gY\right]|\phi\rangle \\ &= (1-R)\left[t(1-I)+gY\right]|\phi\rangle \quad , \end{aligned} \tag{51}$$

and since our derivation requires that $\mathscr{D}$ is ultimately to act on $|\phi\rangle$, we find that Eq. (48) requires only the unique (1-R) portion of B. It is useful to define an operator $\bar{Q}$ such that

$$\bar{Q} = 1 - \Lambda\tilde{V}B(1-I) \quad ; \tag{52}$$

we can then write Eq. (40) in the form

$$Y = \bar{\Omega} + \bar{K}Y \quad , \tag{53}$$

where

$$\begin{aligned} \bar{\Omega} &= bG_0\left[1 - (1-I)\,\bar{Q}\right]t(1-I) \quad , \\ \bar{K} &= bG_0\left[1 - (1-I)\,\bar{Q}\right]g \quad . \end{aligned} \tag{54}$$

Due to the nature of b, Eq. (53) reduces to a set of coupled integral equations in a single continuous variable (q).

We have thus derived a one-variable equation for the operator Y. Using Eqs. (27), (48) and (52) we find that the channel t matrix is given by

$$\tau = \bar{Q}\left[t(1-I) + gY\right] . \tag{55}$$

However, the complete three-body t matrix is given by

$$T = (1-I)\tau \quad , \tag{56}$$

and the three-body wave function is

$$\langle \alpha \vec{x}\, \vec{y} | \Psi \rangle = \langle \alpha \vec{x}\, \vec{y} | 1-G_0 T | \phi \rangle \quad . \tag{57}$$

We thus conclude that Y is completely sufficient to specify a unique solution to the problem. In this context we note that Eq. (51) again implies that the ambiguous part of B (or $\bar{Q}$) is annihilated in constructing τ from Eq. (55). This observation eliminates the need for the symmetry argument relied on by Efimov in the identical boson problem.

III. FADDEEV DERIVATION

We consider in this section a completely independent derivation based on the Faddeev equations. In the process we demonstrate that the Faddeev equations as they stand do not possess a unique solution in the presence of singular cores, but that the full three-particle t matrix is nevertheless uniquely determined by a reduced equation in one dimension. Again for simplicity we consider only the pure BCM (no exterior potentials).

The Faddeev equations can be stated as an operator relation on our Hilbert space by substitution of Eq. (26) into Eq. (12b). We thus obtain

$$\tau = t(1-I) + t\, I G_0 \tau \quad . \tag{58}$$

As in SCI, we assume that this equation holds when t and τ are replaced by their respective BCM limits (due to the presence of G_0 in the product $tIG_0\tau$ there is enough convergence to justify this assumption). However, unlike SCI we regard t as being restricted to the truncated angular-momentum space defined

by Λ, and thus Eq. (58) is to hold with $t \rightarrow \Lambda t = t\Lambda$. The three-particle state vector is then

$$\begin{aligned} |\Psi> &= (1-I)(1-G_0\tau)|\phi> \\ &= (1-I)(1-G_0\Lambda\tau)|\phi> \quad . \end{aligned} \tag{59}$$

Using the explicit form of the BCM t matrix we have

$$\tilde{V}G_0 t = t G_0 \tilde{V} = \tilde{V} \quad , \tag{60}$$

and thus we obtain

$$\Lambda\tilde{V}|\Psi> = 0 \tag{61}$$

by applying $\Lambda\tilde{V}G_0$ to Eq. (58). Physically, Eq. (61) implies that the three-particle wave function vanishes if any two particles are within their core, providing that a core is present in the corresponding channel wave function.[13] In this context we observe that the projection $(1-\Lambda)|\Psi>$ is nontrivial, since scattering can occur in those states due to the two other Faddeev channels (recall that Λ and I do not commute). Defining $M = 1-G_0\tau$, we have

$$\begin{aligned} |\Psi> &= (1-I)M|\phi> \quad , \\ \Lambda\tilde{V}(1-I)M &= 0 \quad . \end{aligned} \tag{62}$$

As in the previous section, we require the operator $\bar{Q}$ defined in Eq. (52). Using Eq. (62), we deduce that

$$\bar{Q}M = M \tag{63}$$

(Eq. (46) implies that $B = B\Lambda\tilde{V}$). Furthermore, Eq. (46) can be used to demonstrate that $B = B^T$ (this is required for (1-R)B and can be imposed on RB). By combining Eqs. (46) and (52), one can easily prove the following useful properties of the $\bar{Q}$ operator:

$$\begin{aligned} \Lambda\tilde{V}(1-I)\bar{Q} &= \Lambda\tilde{V}(1-I)R \quad , \\ \bar{Q}\bar{Q}(1-R) &= \bar{Q}(1-R) \quad , \\ [(1-I)\bar{Q}]^T &= (1-I)\bar{Q} \quad , \\ \bar{Q}\tilde{V}\Lambda &= \tilde{V}\Lambda R. \end{aligned} \tag{64}$$

We also recall the following property of the BCM t matrix from SCI, [14]

$$G_0 t = \tilde{V} + (1-\tilde{V})\, G_0 \tilde{t} \quad ,$$
$$\tilde{t}\tilde{V} = 0 \quad . \tag{65}$$

This can be established from the above by using Eq. (60) and taking the transpose of Eq. (18). We thus identify

$$\tilde{t} = (t'^{BC})^T \tag{66}$$

in terms of the operator t' defined in Eq. (19). Noting that

$$\tilde{t}R = \tilde{t}\tilde{V}R = 0 \quad , \tag{67}$$

we may apply the above properties to show that

$$\begin{aligned} \tilde{t}I\tilde{Q}\tilde{V}\Lambda &= \tilde{t}I\tilde{V}\Lambda R \\ &= \tilde{t}RI\tilde{V}\Lambda \\ &= 0 \quad . \end{aligned} \tag{68}$$

With this result we easily prove our assertion concerning the nonuniqueness of the Faddeev equations, which are equivalent to the relation

$$M = 1 - G_0 t + G_0 t\, IM \quad . \tag{69}$$

Let χ_0 be a nontrivial solution of Eq. (45), then

$$\begin{aligned} (1 - G_0 tI)\Lambda R\chi_0 &= \left[1 - \Lambda\tilde{V}I - (1-\tilde{V})\, G_0 \tilde{t}I\right]\tilde{V}\Lambda R\chi_0 \\ &= \Lambda\tilde{V}(1-I)\ \Lambda\tilde{V}R\,\chi_0 \\ &= 0 \quad . \end{aligned} \tag{70}$$

Therefore, if M is a solution of Eq. (69), $M' = M + \Lambda R\chi_0$ is also a solution. We note that the corresponding contribution to $|\Psi>$ is

$$\begin{aligned} \Delta|\Psi> &= (1-I)\ \Lambda R\chi_0 \\ &= \tilde{V}(1-I)\ \Lambda\tilde{V}R\chi_0 \quad , \end{aligned} \tag{71}$$

and hence this produces no change in $\Lambda|\Psi>$. To prove uniqueness for $(1-\Lambda)|\Psi>$ is more difficult; however, the on-shell value of T (not just ΛT) does not require the ambiguous part of M.

This leads us to suspect that a unique equation does exist from which to determine T, and in fact it is quite simple to derive. We define

$$\bar{X} = \tilde{t} I M \quad , \tag{72}$$

which implies that

$$\begin{aligned} \bar{X} &= \tilde{t} I \bar{Q} M \\ &= \tilde{t} I \bar{Q} \left[1 - G_0 t + G_0 t I M \right] \\ &= \tilde{t} I \bar{Q} \left[1 - G_0 \tilde{t} + G_0 \bar{X} \right] \quad , \end{aligned} \tag{73}$$

using Eqs. (65), (68) and (69). Due to the separable form of $\tilde{t}$, this provides a one-dimensional equation from which to determine $\bar{X}$. Similarly, we have

$$\begin{aligned} M &= 1 - \Lambda + \Lambda M \quad , \\ \Lambda M &= \Lambda \tilde{V} M + \Lambda (1 - \tilde{V}) \left[1 - G_0 \tilde{t} + G_0 \bar{X} \right] \quad , \end{aligned} \tag{74}$$

and hence

$$(1-I) M = (1-I) \bar{Q} (1-R) \left[1 - G_0 \tilde{t} + G_0 \bar{X} \right] + (1-I) \bar{Q} R M \quad . \tag{75}$$

We thus observe that $\bar{X}$ is sufficient to determine everything except $\Lambda \tilde{V} M$. Also,

$$\begin{aligned} \Lambda (1-I) \bar{Q} R M &= \Lambda \tilde{V} (1-I) M \\ &= 0 \quad , \end{aligned} \tag{76}$$

so that $\bar{X}$ uniquely determines $\Lambda |\Psi>$.

In order to relate $\bar{X}$ to τ we adopt the procedure (and notation) of SC I and define operators $\hat{t}, F$ such that

$$<\alpha \vec{p}' \vec{q}' | \hat{t} | \beta \vec{p} \vec{q}> = \delta_{\alpha\beta} \delta(\vec{q}' - \vec{q}) \sum_{\ell} \left(\frac{2\ell+1}{4\pi} \right) P_{\ell}(\hat{p} \cdot \hat{p}') N_{\alpha\ell}(p) \quad , \tag{77}$$

$$<\alpha \vec{p}' \vec{q}' | F | \beta \vec{p} \vec{q}> = \delta_{\alpha\beta} \delta(\vec{q}' - \vec{q}) \frac{\delta(p' - p)}{p^2} \sum_{\ell} \left(\frac{2\ell+1}{4\pi} \right) P_{\ell}(\hat{p} \cdot \hat{p}') \, G_{\alpha\ell}(p', s_{\alpha}) \, D^{-1}_{\alpha\ell}(\kappa_{\alpha}) \, .$$

Thus

$$\begin{aligned} \hat{t} \tilde{V} &= 0 \quad , \\ \tilde{t} &= F \hat{t} \quad , \\ \hat{t} G_0 F \hat{t} &= \hat{t} \quad , \end{aligned} \tag{78}$$

the last relation being similar to Eq. (35). Given the form of Eq. (73), we may then define a new operator X such that

$$\bar{X} = F(\hat{t} + X) \quad . \tag{79}$$

It follows that X satisfies

$$X = \hat{t}(I\bar{Q}-1) + \hat{t}I\bar{Q}\, G_0 F\, X \quad , \tag{80}$$

or

$$\hat{t}(1-I)\bar{Q}\,(1 + G_0 F\, X) = 0 \quad . \tag{81}$$

Moreover, from Eq. (74) we deduce that

$$\begin{aligned}(1-\tilde{V})\, G_0 \tau &= (1-\tilde{V})\Lambda\,(1-M) \\ &= -(1-\tilde{V})\, G_0 F\, X \quad ,\end{aligned} \tag{82}$$

and thus

$$\hat{t} G_0 \tau = -X \quad . \tag{83}$$

It is therefore clear that if we go on-shell from the left, $\tau \rightarrow -FX$; i.e.,

$$\langle \phi | \tau = -\langle \phi | FX \quad . \tag{84}$$

This can be made more explicit if we introduce a partial-wave decomposition, coupling $\vec{\ell}(\vec{p})$ and $\vec{\lambda}(\vec{q})$ to form a state of total angular momentum L. Our basis states then become $|\alpha LM\ell\lambda pq\rangle$, and Eq. (83) is equivalent to the relation

$$\langle \alpha LM\ell\lambda pq | X = -D_{\alpha\ell}(\kappa_\alpha) \langle \alpha LM\ell\lambda \kappa_\alpha q | \tau \quad . \tag{85}$$

Thus X is essentially the on-shell value of τ; note, however, that Eq. (85) is valid for all q, not just in the physical region (κ_α^2 negative as well as positive).

We have thus demonstrated that although the Faddeev equations do not possess a unique solution for the BCM t matrix, a unique one-variable equation for the quantity X can be deduced from them (Eq. (80)). Given X, there is sufficient information to construct both the <u>on-shell</u> three-body t matrix (T), using Eqs. (56) and (85), and the Λ projection of the wave function (for comparison

with SC I we note that X was denoted by Y in that reference). Actually, we have nowhere used the fact that $|\phi>$ is an on-shell state in this section, and hence X is sufficient to determine T half-on-shell (from the left), and ΛT fully off-shell.[15] By symmetry we can then construct $|\Psi>$ uniquely, as in Section II. We thus arrive at an alternative formulation of the problem, having obtained two rather different looking integral equations (Eqs. (53) and (80)). In the next section we demonstrate that these two equations are in fact equivalent.

IV. COMPARISON OF BCM FORMALISMS

In the preceding section it was shown that the Faddeev equation is uniquely related to the three-particle observables via the solution of a reduced (one-dimensional) equation. The latter is quite different in form from the equation derived in Section II, and we begin below by first establishing their equivalence. We next demonstrate that as $\Lambda \to 1$, the equations are identical to the result of SC I, and the same ambiguity reappears. In order to relate singular core models to the general boundary condition formalism (BCF) proposed by this author,[9] we explicitly exhibit the input to the BCF which is required to exactly duplicate the T operator defined in the preceding sections. Finally, we consider an alternative equation which may be derived by the techniques of Section II, and show that it is precisely identical to Eq. (53).

It will be convenient in what follows to employ the angular momentum decomposition noted above, and to thus employ the $|\alpha LM\ell\lambda pq>$ basis states. In order to simplify the notation, we will consider the conserved quantities L, M held fixed and not state them explicitly, and will also eliminate reference to the superfluous p-variable in our one-dimensional equations. We therefore introduce the basis $|\alpha\ell\lambda q>$ with the normalization

$$<\alpha\ell\lambda q|\beta\ell'\lambda' q'> = \delta_{\alpha\beta}\delta_{\ell\ell'}\delta_{\lambda\lambda'}\frac{\delta(q-q')}{q^2} \quad . \tag{86}$$

We will retain the notation $\bar{K}$, $\bar{\Omega}$ of Eq. (53), but now regard this as an equation on the simplified basis. We therefore have

$$\langle\alpha\ell'\lambda'q'|\bar{K}|\beta\ell\lambda q\rangle = N^{-1}_{\alpha\ell'}(\kappa'_\alpha)\langle\alpha LM\ell'\lambda'\kappa'_\alpha q'|\hat{t}G_0[1-(1-I)\bar{Q}]\hat{t}^T|\beta LM\ell\lambda\kappa_\beta q\rangle * \\ * D^{-1}_{\beta\ell}(\kappa_\beta) \quad , \tag{87}$$

and take

$$\bar{\Omega}|\phi\rangle = \bar{K}\Omega_0|\phi\rangle \quad , \\ \langle\alpha\ell'\lambda'q'|\Omega_0|\phi\rangle = \int_0^\infty dp'p'^2\langle\alpha LM\ell'\lambda'p'q'|(1-I)|\phi\rangle \quad . \tag{88}$$

In obtaining Eq. (88) we have used the fact that $|\phi\rangle$ puts p' on-shell in simplifying $t(1-I)|\phi\rangle$. Defining

$$\bar{Z} = (1-\bar{K})^{-1} \tag{89}$$

on the reduced basis, we find that

$$Y|\phi\rangle = (\bar{Z}-1)\Omega_0|\phi\rangle \quad . \tag{90}$$

Similarly, we define Ω^T such that

$$\langle\alpha LM\ell'\lambda'p'q'|\Omega^T|\beta\ell\lambda q\rangle = -\langle\alpha LM\ell'\lambda'p'q'|(1-I)\bar{Q}\hat{t}^T|\beta LM\ell\lambda\kappa_\beta q\rangle \quad . \tag{91}$$

Recalling Eqs. (55) and (56), it follows that

$$T^r = -\Omega^T D^{-1}(Y+\Omega_0) \\ = -\Omega^T D^{-1}\bar{Z}\Omega_0 \quad , \tag{92}$$

where the notation "r" means that T^r produces the half-on-shell t matrix when acting to the right on an on-shell state; i.e., $T^r|\phi\rangle = T|\phi\rangle$. Here it is understood that T^r is to be sandwiched between states of the full basis, whereas intermediate states in evaluating Eq. (92) are taken in the reduced $|\alpha\ell\lambda q\rangle$ basis. We have also used the notation D to represent the multiplicative factor $D_{\alpha\ell}(\kappa_\alpha)$ as a diagonal operator.

We now consider the transpose of T^r. Applying Eq. (64), we first obtain

$$\langle\alpha\ell'\lambda'q'|\bar{K}^T|\beta\ell\lambda q\rangle = D^{-1}_{\alpha\ell'}(\kappa'_\alpha)\langle\alpha LM\ell'\lambda'\kappa'_\alpha q'|\hat{t}I\bar{Q}G_0\hat{t}^T|\beta LM\ell\lambda\kappa_\beta q\rangle N^{-1}_{\beta\ell}(\kappa_\beta) \quad , \tag{93}$$

where we have used

$$\hat{t}(1-\bar{Q}) = \hat{t}\tilde{V}(1-\bar{Q}) = 0 \tag{94}$$

to simplify the product. Employing Eq. (68), which also holds for $\hat{t}$ in view of Eq. (78), we note that

$$\hat{t}I\bar{Q}G_0\hat{t}^T = tI\bar{Q}(1-\tilde{V})G_0\hat{t}^T \quad . \tag{95}$$

However, it is straightforward to show that

$$(1-\tilde{V})G_0\hat{t}^T|\beta LM\ell\lambda\kappa_\beta q\rangle = \int_0^\infty dp\, p^2(1-\tilde{V})G_0F|\beta LM\ell\lambda pq\rangle N_{\beta\ell}(\kappa_\beta)\, D_{\beta\ell}(\kappa_\beta)$$

$$= (1-\tilde{V})\, G_0\tilde{G}|\beta LM\ell\lambda\kappa_\beta q\rangle\, N_{\beta\ell}(\kappa_\beta) \quad ; \tag{96}$$

i.e., the intermediate integration in $(1-\tilde{V})G_0\hat{t}^T$ just puts $N_{\beta\ell}(p)$ on-shell. Here we have defined $\tilde{G}$ such that

$$\langle\alpha LM\ell'\lambda'p'q'|\tilde{G}|\beta LM\ell\lambda pq\rangle = \delta_{\alpha\beta}\delta_{\ell'\ell}\delta_{\lambda'\lambda}\,\frac{\delta(q'-q)}{q^2}\,G_{\alpha\ell}(p', s_\alpha) \quad . \tag{97}$$

We may now define a new kernel K by the relation

$$K = D\bar{K}^T D^{-1} \quad ; \tag{98}$$

it follows that

$$\langle\alpha\ell'\lambda'q'|K|\beta\ell\lambda q\rangle = \langle\alpha LM\ell'\lambda'\kappa'_\alpha q'|\hat{t}I\bar{Q}G_0\tilde{G}|\beta LM\ell\lambda\kappa_\beta q\rangle\, D^{-1}_{\beta\ell}(\kappa_\beta) \quad . \tag{99}$$

Defining $Z = (1-K)^{-1}$, it is straightforward to obtain

$$(T^r)^T = -\Omega_0^T D^{-1} Z\Omega \quad . \tag{100}$$

On the other hand, Eq. (84) implies that

$$\begin{aligned} \langle\phi|T &= -\langle\phi|(1-I)FX \\ &= -\langle\phi|\Omega_0^T D^{-1}X \quad , \end{aligned} \tag{101}$$

and Eq. (80) can be stated in the reduced basis as

$$X = \Omega + KX \quad . \tag{102}$$

We therefore conclude that the Faddeev derivation produces the operator

$$T^{\ell} = -\Omega_0^T D^{-1} Z \Omega \quad , \tag{103}$$

where the "ℓ" signifies that T^{ℓ} is identical to T when taken on-shell from the left. Comparing Eqs. (100) and (103), we have established that

$$(T^r)^T = T^{\ell} \quad . \tag{104}$$

If we denote the operators produced by the two derivations as T_s and T_f, respectively, the content of Eq. (104) is that

$$\langle\phi|T_f|A\rangle = \langle A|T_s|\phi\rangle \tag{105}$$

for an arbitrary state $|A\rangle$. However, we have derived our singular core formalism as the limit of a theory which is invariant under time-reversal, and hence we expect T to be symmetric. One can infer this, for example, from Eq. (58). We therefore conclude that both versions of the formalism are entirely equivalent in producing the half-on-shell three-particle t matrix. In addition, of course, the Faddeev version also produces the Λ projection of the fully off-shell t matrix (ΛT).

If we now compare the Faddeev derivation to the development given in SC I, we observe that they differ only in the use of $\bar{Q}$ rather than the operator Q. That is, in contrast to the properties given in Eq. (64), it was shown in SC I that (in

the limit $\Lambda \to 1$) an operator Q exists such that

$$\tilde{V}(1-I)Q = (1-I)\,Q\tilde{V} = 0 \quad ,$$
$$QQ = Q \quad , \tag{106}$$
$$\left[(1-I)Q\right]^T = (1-I)\,Q \quad ,$$
$$Q\tilde{V} = \tilde{V}Q\tilde{V} \quad .$$

In fact, Q has the same form as Eq. (52) with a suitable choice of B, which in this case can be obtained in closed form (see the Appendix of SCI). Since Q is a projection operator it is clear that the form obtained is unique, and hence we can formally set R=0.

We therefore again obtain Eqs. (80) - (85), but with one important difference: Eq. (80) no longer uniquely determines X. This is due to the fact that as $\Lambda \to 1$ (and consequently $\bar{Q} \to Q$) the kernel is no longer compact, and there are non-trivial solutions of the homogeneous equation. Specifically, in the $(\alpha \vec{x}\,\vec{y})$ representation with $x=a_\alpha$, there is in general a maximum displacement y_α^o such that at least one of the pairs $(\alpha\beta)$, $(\alpha\lambda)$ are within their core irrespective of $\hat{x}\cdot\hat{y}$ if $y < y_\alpha^o$. For identical particles and core radii $a_\alpha = a$, one finds $y_\alpha^o = \sqrt{3}\,a/2$. This inner region may be represented by the projection operator θ which corresponds to the unit step function $\theta(y_\alpha^o - y)$ on the reduced $|\alpha\ell\lambda q\rangle$ basis. It can easily be shown that

$$\theta\,(1-K) = 0 \quad , \tag{107}$$

if K is defined as in Eq. (99) but with $\bar{Q} \to Q$.

It is thus clear that the limiting case $\Lambda \to 1$ considered in SCI is quite consistent with the present results in that limit, and that the ambiguity which was noted is unavoidable. One thus must pay a price for the simplicity of Q (the $\bar{Q}$ operator is quite complex even in the simplest case). In this context we observe that if we denote the kernel and driving term of the SCI result by K_1 and Ω_1,

respectively, then

$$(1-\theta)K_1 = (1-\theta)K \quad ,$$
$$(1-\theta)\Omega_1 = (1-\theta)\,\Omega \quad ; \tag{108}$$

i.e., the equations differ only in the overlap region. This may be established by noting that if the three core projection operators are denoted by $\tilde{V}_i$, then

$$\begin{aligned}(1-\theta)\hat{t}\,I\bar{Q} &= \hat{t}(1-\theta)(1-\tilde{V})I\bar{Q} \\ &= (1-\theta)\hat{t}\,I(1-\tilde{V}_i)(1-\tilde{V}_j)\bar{Q} \quad ,\end{aligned} \tag{109}$$

where the indices i, j depend on the label α of the intermediate state acting on $\bar{Q}$. Thus $\bar{Q}$ is projected onto the region where only one pair can be within their core; and the only nontrivial case is when $\tilde{V}_i \neq \tilde{V}_j \neq \tilde{V}$. In this region $B \equiv \tilde{V}$ regardless of Λ, and hence Q and $\bar{Q}$ coincide.

During the past several years, this author has developed the boundary condition idea into an alternative general formalism for treating three-particle systems. By generalizing the approach (e.g., taking $\hat{t}\,|\Psi> \neq 0$) it is possible to eliminate any specific reference to the BCM itself. This procedure modifies the one-dimensional equation obtained by the addition of two input functions $\hat{B}$ and $\hat{C}$ which summarize the off-shell structure (if V_α vanishes for $x > a_\alpha$ then $\hat{B}=0$). Recently, an explicit connection was derived relating this formalism to the Faddeev approach, and it was shown that suitable functions $\hat{B}$, $\hat{C}$ may always be chosen so as to exactly reproduce the Faddeev amplitudes given any combination of two- and three-particle potentials.[16] We now employ the same technique to show that this is also true for the restricted BCM ($\Lambda \neq 1$).

The BCM and BCF equations may be written in the form

$$\begin{aligned}X &= \hat{t}(I-1)\bar{Q} + \hat{K}D^{-1}X \quad , \\ X_1 &= (1-\theta)\,\hat{t}(I-1)\bar{Q} + \hat{K}_1 D^{-1}X_1 \quad ,\end{aligned} \tag{110}$$

respectively, where

$$\hat{K} = KD \quad ,$$
$$\hat{K}_1 = (1-\theta)\hat{K} + \theta\bar{C} \quad . \tag{111}$$

Here $\bar{C}$ is related to $\hat{C}$ by the equation

$$\bar{C}D^{-1} = \hat{R} + \hat{C}(1-\hat{R}) \quad , \tag{112}$$

where $\hat{R}$ is a diagonal operator corresponding to the factor

$$R_{\alpha\ell}(q) = 1 - D_{\alpha\ell}(\overline{\kappa_\alpha})/D_{\alpha\ell}(\kappa_\alpha) \quad , \tag{113}$$

and $\overline{\kappa_\alpha}$ corresponds to κ_α with W replaced by the (negative) energy parameter W_0. As noted in the Introduction, the case $\hat{C}=0$ is essentially the simplest consistent with unitarity, convergence and analyticity (note that $\hat{C}$ is required to be real-valued).

For energies below the threshold for breakup ($W<0$) the distinction between the driving terms is unimportant (both effectively go to $-\hat{t}$),[17] and one may guarantee equivalence by setting $\hat{K}=\hat{K}_1$, thus determining $\hat{C}$. However, this is inadequate for $W>0$ since it would require $\hat{C}$ to be complex (and the driving terms would differ for an incoming state of three free particles). We therefore define a diagonal projection operator $\mathscr{P}$ on the reduced basis such that

$$\mathscr{P}_\alpha(q) = \theta(Q_\alpha - q) \quad ,$$
$$Q_\alpha = (2M_\alpha W)^{1/2} \quad ; \tag{114}$$

thus $\mathscr{P}$ is unity acting on a physical on-shell state ($\kappa_\alpha^2 > 0$). In order to obtain equivalent <u>physical</u> results, it is sufficient to require that $X\mathscr{P}=X_1\mathscr{P}$. Following the procedure of Ref. 16, we note that operators γ, γ_1 exist such that

$$\hat{t}(I-1)\bar{Q} = -(1-\gamma\mathscr{P})\, N\Omega_0 \quad ,$$
$$(1-\theta)\,\hat{t}(I-1)\bar{Q} = -(1-\gamma_1\mathscr{P})\, N\Omega_0 \quad , \tag{115}$$

when acting on a physical state $\mathscr{P}|\phi>$. Here we have used N to represent the factor $N_{\alpha\ell}(\kappa_\alpha)$ as a diagonal operator; N, γ, γ_1 are taken to act on the reduced basis. Explicitly,

$$<\alpha\ell\lambda q|\gamma|\beta\ell'\lambda' q'> = \frac{1}{3}<\alpha LM\ell\lambda\kappa_\alpha q|\hat{t}(I-1)(\bar{Q}-1)|\beta LM\ell'\lambda'\kappa'_\beta q'> N^{-1}_{\beta\ell'}(\kappa'_\beta) \quad , \tag{116}$$

and

$$\gamma_1 = (1-\theta)\gamma + \theta \quad . \tag{117}$$

We next define a real operator U such that

$$\begin{aligned} (1-U)^{-1}(1-\gamma\mathscr{P}) &= 1-\gamma_1\mathscr{P} \quad , \\ 1-U &= (1-\gamma\mathscr{P})(1-\gamma_1\mathscr{P})^{-1} \quad . \end{aligned} \tag{118}$$

The existence of these inverse operators (and U) is a consequence of the fact that $\gamma\mathscr{P}$, $\gamma_1\mathscr{P}$ are bounded kernels on the finite subspace $0 \le q \le Q_\alpha$ (this is the reason for introducing $\mathscr{P}$). Defining V such that

$$\hat{K} = UD + V \quad , \tag{119}$$

the first line of Eq. (110) becomes

$$X\mathscr{P} = (1-\theta)\hat{t}(I-1)\bar{Q}\mathscr{P} + (1-U)^{-1} VD^{-1} X\mathscr{P} \quad . \tag{120}$$

A comparison with the X_1 equation then implies that $X\mathscr{P}$ and $X_1\mathscr{P}$ will be identical providing that

$$\begin{aligned} \hat{K}_1 &= (1-U)^{-1} V \\ &= D + (1-U)^{-1}(\hat{K} - D) \quad . \end{aligned} \tag{121}$$

We note that Eqs. (117) and (118) imply that

$$\begin{aligned} (1-\theta)(1-U)^{-1} &= (1-\theta) \quad , \\ U &= \theta U \quad , \end{aligned} \tag{122}$$

so that the $(1-\theta)$ projection of Eq. (121) reduces to $(1-\theta)\hat{K}_1 = (1-\theta)\hat{K}$, as it should. Applying θ we obtain

$$\begin{aligned} \theta\hat{K}_1 &= \theta\bar{C} \\ &= \theta D + \theta\Delta \quad , \end{aligned} \tag{123}$$

where

$$\begin{aligned} \Delta &= \theta(1-U)^{-1}(\hat{K}-D) \\ &= \theta(1-\mathscr{P})(1-\gamma\mathscr{P})^{-1}(\hat{K}-D) \end{aligned} \tag{124}$$

using Eqs. (111) and (118). Given $\bar{C}$ we can compute $\hat{C}$ from Eq. (112); one may verify that the result is a real-valued operator with the same convergence properties as $\theta\hat{K}$.

We have therefore verified that a suitable input function $\hat{C}$ exists such that the BCF reproduces the observable consequences of the BCM in three-particle scattering. Together with the results of Ref. 16, this implies that the BCF provides a suitable framework in which to investigate both soft and singular core models of the N-N interaction. In the sense of SCI, Eq. (123) can be regarded as an "auxiliary boundary condition" pertaining to the case $\Lambda \neq 1$, since it implies that

$$\theta\Delta D^{-1} X_1 = 0 \quad , \tag{125}$$

which is of the same form as Eq. (92) of SCI.

In concluding this section we briefly sketch the derivation of an apparently distinct equation and prove its equivalence to Eq. (53). We first note that Eqs. (19), (23) and (97) imply that

$$t' = g\tilde{G}^T \quad . \tag{126}$$

One may directly verify that

$$\begin{aligned} \tilde{G}^T G_0(1-\tilde{V}) &= bG_0(1-\tilde{V}) \\ &\equiv fG_0 \quad , \end{aligned} \tag{127}$$

where f has the same form as $\hat{t}$ (see Eq. (77)) except that $N_{\alpha\ell}(p)$ is replaced by

$$f_{\alpha\ell}(p,\kappa_\alpha) = ia_\alpha\kappa_\alpha\left[a_\alpha\kappa_\alpha h_{\ell+1}(a_\alpha\kappa_\alpha)\ j_\ell(a_\alpha p) - h_\ell(a_\alpha\kappa_\alpha)\ a_\alpha p\, j_{\ell+1}(a_\alpha p)\right] \ ; \quad (128)$$

this implies that $f_{\alpha\ell}(\kappa_\alpha,\kappa_\alpha) = 1$. If we now substitute $V|\Psi\rangle$ as given in Eq. (21) into the expression for $gY|\phi\rangle$ in Eq. (22), we deduce that

$$\begin{aligned} Y &= bG_0gY \\ &= \tilde{G}^T G_0(1-\tilde{V})\, I[t(1-I) + \tilde{V}\mathscr{D} + gY] \\ &= fG_0\, I\bar{Q}[t(1-I) + gY] \quad , \end{aligned} \quad (129)$$

which is an apparent alternative to Eq. (53). However, using

$$\begin{aligned} (1-I)\bar{Q}g &= \left[(1-\Lambda\tilde{V})(1-I)\bar{Q} + \Lambda\tilde{V}(1-I)R\right] g \\ &= (1-\Lambda\tilde{V})(1-I)\bar{Q}g \end{aligned} \quad (130)$$

by Eqs. (64) and (36), we have that

$$\begin{aligned} \bar{K} &= bG_0(1-\tilde{V})\left[(1-\bar{Q}) + I\bar{Q}\right]g \\ &= bG_0(1-\tilde{V})I\bar{Q}\, g \\ &= fG_0\, I\bar{Q}g \quad . \end{aligned} \quad (131)$$

Similarly, one may show that

$$\bar{\Omega}|\phi\rangle = fG_0\, I\bar{Q}\, t(1-I)\,|\phi\rangle \quad , \quad (132)$$

and hence Eqs. (53) and (129) are identical. Within the context of their particular model both forms were derived by Efimov and Schulz, who demonstrated their numerical equivalence.[7]

V. DISCUSSION

In the preceding sections we have derived a number of results which bear on the questions raised in the Introduction. In the first place, we have shown that for models in which singular cores are restricted to a finite number of partial-waves ($\Lambda \neq 1$), both the Schrödinger and Faddeev approaches lead to unique one-dimensional equations which are in fact equivalent. In the limit $\Lambda \to 1$

we recover the equations of SC I and the same ambiguity reappears. Thus, the nonuniqueness noted earlier in SC I is an inescapable consequence of requiring $|\Psi>$ to be identically zero inside the cores (implying $\Lambda=1$), and is not related to the method of derivation. The distinction between the $\Lambda\neq1$ and $\Lambda=1$ models may be regarded as an "auxiliary boundary condition" in the sense of SC I, which we have explicitly exhibited.

With respect to the various numerical calculations based on the s-wave BCM, it is therefore clear that the equations employed by this author and by Efimov and Schulz are quite different, and the numerical "discrepancy" in the three-particle binding energy (E_B) is not surprising. In this context it should be noted that the equation solved numerically in SC II is actually an approximation to the $\Lambda=1$ problem, in that the Q operator was used but t was truncated to $\ell=0$ only. We expect the numerical results to converge rapidly as $\Lambda\hat{t}\rightarrow\hat{t}$ for the same reasons which apply in the case of potentials; the behavior of the kernel as a function of ℓ,λ (for fixed L) is very similar to that of the Faddeev kernel with conventional off-shell t matrices. The part of the BCM t matrix which is poorly behaved in this limit corresponds to $\tilde{V}G_0t=\tilde{V}$ (which becomes local), and this acts only on the interior region, producing the already noted ambiguity. Having chosen an <u>ad hoc</u> auxiliary condition to compensate for the implied nonuniqueness, the resulting equation is stable as $\Lambda\rightarrow1$.[18] The difference between this s-wave <u>approximation</u> and the true s-wave BCM corresponds to a different choice of the auxiliary condition, and the corresponding addition to the kernel ($\hat{C}$) is apparently sufficiently repulsive to account for the difference in E_B (18.4 MeV vs. 7.7 MeV).

The situation is quite different with regard to the numerical results of Kim and Tubis.[8] We have shown above that the s-wave BCM Faddeev equation does

not possess a unique solution, and hence the usual procedure of numerically evaluating the determinant $|1-tIG_0|$ in order to search for a zero at $W=-E_B$ should not produce a stable result. Similarly, methods which detect an eigenvalue by locating values of W at which the iterated equation fails to converge are inapplicable. In such cases one may define a discrete eigenvalue spectrum only on a subspace, and this is effectively accomplished by the reduction to one-dimensional form. It is therefore unlikely that a straightforward application of standard numerical procedures to the two-dimensional (Faddeev) form could lead to a unique result. We have also shown that the proper result should coincide with that of Efimov and Schulz,[7] who have tested their numerical procedures by applying two different techniques to the two versions of the (Schrödinger) one-dimensional equation (Eqs. (53) and (129)). The 12.7 MeV result of Kim and Tubis thus appears rather suspect. On the other hand, it should be noted that they previously tested their techniques on the Herzfeld potential (hard core plus square-well), and found the result to be compatible with the limit of a finite repulsive core.[5] It is therefore possible that the problem lies with this particular model, since the BCM is slightly less convergent than the hard core (a special case), and perhaps this can be checked along the lines of Ref. 5. Nevertheless, on balance our results (and additional arguments presented by Efimov and Schulz) indicate that the 7.7 MeV result is to be preferred.

An important corollary of our result is that one cannot directly solve the Faddeev equations for "realistic" singular core models such as the Hamada-Johnston hard core, or the BCM of Feshbach and Lomon.[1] One must first eliminate the noncompactness of the kernel by reducing to an appropriate subspace. For this purpose we note that the development in Section III remains valid if t is taken to include the effect of potentials external to the core. Thus,

if V_e is such a potential, we showed previously that t satisfies the equation[11]

$$t = t^{BC} + \left(1-t^{BC} G_0\right) V_e (1-G_0 t) \quad , \tag{133}$$

which is convergent for reasonable potentials V_e (e.g., bounded by a Yukawa potential). Here t^{BC} is the off-shell t matrix for the pure BCM (what we called t in Section III). It is easy to verify that if $\tilde{t}$ is defined by

$$\tilde{t} = \tilde{t}^{BC} + \left(1-\tilde{t}^{BC} G_0\right) V_e (1-G_0 t) \quad , \tag{134}$$

then Eqs. (58) - (76) remain valid. We may thus calculate the half-on-shell t matrix T^{ℓ} via

$$T^{\ell} = (1-I)(\tilde{t}-\bar{X}) \quad , \tag{135}$$

where $\bar{X}$ satisfies Eq. (73). We note that the latter does not in general reduce to one-dimensional form for $V_e \neq 0$ (the exception being if V_e is separable).

We have thus provided a practical framework in which to probe the off-shell consequences of N-N interaction models which employ singular cores. Although such information would nicely complement our present knowledge of soft core models, the insensitivity of the trinucleon system is such that one cannot be overly optimistic regarding our ability to distinguish between such approaches experimentally. Furthermore, the applicability of singular core models to systems of three or more particles depends to a large extent on one's physical interpretation. Thus, if we regard the BCM as simply a mathematical abstraction representing a large finite repulsion plus a strong surface attraction, the formalism of Sections II and III is certainly appropriate and has been shown both theoretically and numerically (via the calculations of Efimov and Schulz) to be consistent with the limit of such a picture.

On the other hand, it may be argued that the BCM is a phenomenological device which simulates the net effect of an interior region in which ordinary

potential theory is invalid. Thus, Feshbach and Lomon justified it on the basis of quantum field theory, which implies that the N-N interaction becomes highly nonlocal at distances $x \lesssim (2m_\pi)^{-1}$, whereas the interaction energy is so huge that there is virtually no sensitivity to the asymptotic energy $(\kappa_\alpha^2/2\mu_\alpha)$.[1] Another point of view has been advanced by this author, who noted that such an empirical effect is also to be expected on the basis of Pauli exclusion between composite nucleons (for which there is steadily accumulating experimental evidence).[19] Either interpretation would imply that the boundary condition must be modified if another hadron is present and within core range of either nucleon. This viewpoint would thus invalidate the potential theory arguments as applied in the interior region where cores overlap, and would consequently reintroduce a (physically motivated) ambiguity into the three-body problem. From the standpoint of three-nucleon phenomenology this would appear as a nonnegligible three-body force.

Formally, the problem so posed is precisely that considered in SCI, and one could in principle relate the "auxiliary condition" to the generalized boundary condition required in the overlap region. In the absence of such specific information, one might argue that the ad hoc condition proposed in SCI is physically as well motivated as the particular form derived for the $\Lambda \neq 1$ problem above (as well as being much simpler). This argument is especially cogent in the relativistic problem, in which the overlap region is a complete enigma. Thus, as noted recently by this author, a straightforward covariant generalization of the SCI formalism provides a natural first approximation for describing relativistic three-body systems.[20] This description may then be supplemented by introducing phenomenological terms $(\hat{B}, \hat{C})$ to correct the off-shell structure, as in the nonrelativistic BCF. Recent calculations using correct two-particle

shifts (energy-dependent $\lambda_{\alpha\ell}$) and $\hat{B}=\hat{C}=0$ have produced quite interesting results for relativistic π-d scattering[21] and the basic nuclear force problem.[22]

Finally, we consider the implications of our results with respect to the BCF. This author has previously shown that the boundary condition technique provides an alternative general description of three-particle systems.[9,20] In particular, for arbitrary combinations of two- and three-particle potentials, real input operators $\hat{B}, \hat{C}$ can always be chosen so as to exactly reproduce the Faddeev amplitudes.[16] The present work extends this statement to models with singular cores. As an immediate consequence, we observe that eventual calculations with "realistic" singular core models cannot alter conclusions previously reached concerning the insensitivity of n-d elastic and breakup differential cross sections to the off-shell structure.[23] Thus, the off-shell content of these models can be expressed in terms of the values obtained for the n-d doublet scattering length, the triton binding energy, and the static properties of the triton wave function. Values of these parameters for currently known models are systematically in conflict with experiment, and hence it would be interesting to see if the singular core predictions are significantly different.[24] Once the triton computation is complete, effective values for the $\hat{B}, \hat{C}$ operators (which are weakly dependent on W) can be computed for input into the BCF. In this way the consequences of such models for the scattering states can be readily explored.

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12. Explicit formulas for $\mathscr{D}$ are given in Ref. 6.
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18. In contrast, our results imply that the equations derived in this article will not produce numerically stable results as cores are added to additional partial-waves.

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