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SOLITARY WAVES IN SINE-GORDON AND POLYNOMIAL FIELD THEORIES

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ERRATUM

Equation (12) in this paper should be changed to read:

$$\frac{\lambda}{2\check{k}^2} = -\frac{M^2}{\check{k}^2}$$

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ABSTRACT

Solitary wave solutions of the sine-Gordon equation are shown to be special cases of solitary wave solutions of polynomial field theories with interaction Lagrange densities $\mathscr{L}_I = \alpha \phi^{2p+2}/(2p+2) + \lambda \phi^{4p+2}/(4p+2)$, $p \neq 0, -\frac{1}{2}, -1$.

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Self interacting quantum field theories lead to nonlinear, dispersive field equations. Several of these field equations have exact solutions of a type known variously in classical physics as cnoidal waves, solitary waves or solitons. These excitations, referred to here generically as solitary waves, have received considerable attention recently since they lead to results which are independent of perturbation theory [1-5]. In this paper the solitary wave solutions of the sine-Gordon equation[2,3], an equation with an infinite sequence of self interactions, are shown to be special cases of the solitary wave solutions of the field equations arising from the polynomial Lagrange density $\mathscr{L}_I = \alpha\,\phi^{2p+2}/(2p+2) + \lambda\,\phi^{4p+2}/(4p+2)$.

The sine-Gordon equation may be written

$$\partial_\mu \partial^\mu \eta + (M^2/g)\sin(g\eta) = 0 . \qquad (1)$$

It is evident from the series expansion of $\sin(g\eta)$ that this field equation is the Klein-Gordon equation plus an infinite sequence of self interaction terms with a single coupling constant g. The solitary wave solutions of this equation are fields $\eta(\chi)$, where

$$\chi = \check{k}\cdot\check{x} = k_0 x_0 - \vec{k}\cdot\vec{x} \qquad (2)$$

and $\check{k}^2 \neq 0$.

Defining

$$\psi = g\eta/2 \qquad (3)$$

and using Eq. (2), Eq. (1) becomes

$$\check{k}^2 \frac{d^2\psi}{d\chi^2} + \frac{M^2}{2}\sin 2\psi = \check{k}^2 \frac{d^2\psi}{d\chi^2} + M^2 \sin\psi\cos\psi = 0 . \qquad (4)$$

The first integral of this equation is, for $\check{k}^2 \neq 0$

$$\frac{1}{2}\left(\frac{d\psi}{d\chi}\right)^2 + \frac{M^2}{2\check{k}^2}\sin^2\psi = E/2\ , \tag{5}$$

where E is an arbitrary constant. Eq. (5) may be separated to give

$$\chi + \chi' = \int d\psi \left[E - (M^2/\check{k}^2)\sin^2\psi\right]^{-\frac{1}{2}} \tag{6}$$

where χ' is a constant. Now, let

$$\sin\psi = \theta \tag{7}$$

to obtain

$$\chi + \chi' = \int d\theta \left[E - (M^2/\check{k}^2 + E)\,\theta^2 + (M^2/\check{k}^2)\,\theta^4\right]^{-\frac{1}{2}} . \tag{8}$$

The corresponding integral for the polynomial self interaction is (6) (In reference [6] the special case $\check{k}^2 = m^2$ was discussed. The generalization to arbitrary $\check{k}^2 \neq 0$ is easily obtained.)

$$\chi + \chi' = \int d\phi \left[B - m^2\phi^2/\check{k}^2 - \alpha\,\phi^{2p+2}/(p+1)\check{k}^2 - \lambda\,\phi^{4p+2}/(2p+1)\check{k}^2\right]^{-\frac{1}{2}} . \tag{9}$$

Comparing Eqs. (8) and (9) it is evident that for $\alpha = 0$, $p = \frac{1}{2}$, the solutions, related to Jacobi elliptic functions, coincide for the special choice of constants

$$B = E\ , \tag{10}$$

$$m^2/\check{k}^2 = E + M^2/\check{k}^2\ , \tag{11}$$

$$\lambda/2 = M^2/\check{k}^2\ . \tag{12}$$

Thus, the solitary wave solutions of the sine-Gordon equation are special cases of the solitary wave solutions of the $\lambda\phi^4$ theory. The integral in Eq. (9) leads to both non-localized and localized (soliton) solutions, depending on the choice of sign of $\check{k}^2$.

For the case $B = 0$ the integral in Eq. (9) can be performed in terms of elementary functions. For $\check{k}^2 = m^2$ one set of particular solutions is (6), for a system of volume V,

$$\phi^{(\pm)} = \frac{A_{\vec{k}}^{(\pm)}}{D(\omega V)^{\frac{1}{2}}}\, e^{\mp i\check{k}\cdot\check{x}} \left(1 - \frac{\lambda A_{\vec{k}}^{(\pm)2}\, e^{\mp 2i\check{k}\cdot\check{x}}}{8m^2 \omega V D^2}\right)^{-1}, \tag{13}$$

$$\omega^2 = \vec{k}^2 + m^2\,, \tag{14}$$

where $A_{\vec{k}}^{(\pm)}$ are annihilation or creation operators respectively of the linear field theory and D is an arbitrary constant. These are persistent solutions of the interacting field equations — containing the coupling constant λ for all times rather than reducing to in or out fields for large times. Using the definition of η and Eqs. (10)-(12) the corresponding particular solutions of the sine-Gordon equation are

$$\eta^{(\pm)} = \frac{2}{g}\sin^{-1}\left[\frac{A_{\vec{k}}^{(\pm)}}{D(\omega V)^{\frac{1}{2}}}\, e^{\mp i\check{k}\cdot\check{x}}\left(1 + \frac{A_{\vec{k}}^{(\pm)2}\, e^{\mp 2i\check{k}\cdot\check{x}}}{4D^2\omega V}\right)^{-1}\right]. \tag{15}$$

Other solutions, both local and non-local, may be obtained using Ref. [6].

Since the constant D is arbitrary, it may be chosen to make $\eta^{(\pm)}$ regular at $g = 0$. However, this choice is not necessary. Hence, solitary wave solutions of the sine-Gordon equation exist for which an expansion in the coupling constant is not possible. This result is also obtained in polynomial field theories[9-10].

The connection between sine-Gordon and $\lambda\phi^4$ solitary waves established here opens an interesting avenue of study for the solutions of the latter theory. In space time of dimension 1+1, "new" solutions of field equations such as the sine-Gordon and Korteweg-de Vries equations may be related to "old" solutions by means of Bäcklund transformations[7, 8]. Thus, this technique can be useful in generating families of solutions of the $\lambda\phi^4$ theory and, perhaps, of more general polynomial and nonpolynomial field theories.

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