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MATRIX INVERSION METHOD FOR INTEGRAL EQUATIONS WITH KERNELS CONTAINING POLES*

Harold Cohen

Physics Department, California State University, Los Angeles
Los Angeles, California 90032

and

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

An interpolative technique is presented which yields accurate numerical solutions to various types of linear integral equations, the kernels of which contain poles. It is also shown that two techniques which are successful when the kernel is weakly singular are unsatisfactory when the singularities are poles. The approaches discussed for the various equations are illustrated by example.

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I. Introduction

Linear integral equations with kernels that contain pole singularities arise in various physical problems. In this paper, three such representative equations are considered to illustrate a method of obtaining a numerical solution to such equations by matrix inversion.

In general, nonsingular linear integral equations can be inverted by approximating the integral by a sum over a suitable set of quadrature points. That is, if the kernel, K, is nonsingular,

$$\psi(z) = \psi_0(z) + \lambda \int_a^b K(z,z')\psi(z')\,dz' \tag{1}$$

is written approximately as

$$\psi(z_i) = \psi_0(z_i) + \lambda \sum_{j=1}^{N} w_j K(z_i,z_j)\psi(z_j) \tag{2}$$

where z_j and w_j are the abscissas and weights of the quadrature used. The solution by matrix inversion is then achieved.

$$\psi(z_i) = \sum_{j=1}^{N} [1-\lambda M]^{-1}_{ij}\,\psi_0(z_j) \tag{3}$$

with $M_{ij} = w_j K(z_i,z_j)$.

If $K(z,z')$ is singular, such a straightforward scheme cannot, in general, be used. If, for example,

$$\lim_{z \to z'} K(z,z') = \infty$$

the diagonal elements of M_{ij} are infinite.

In an earlier paper by Ickovic and myself [1], integral equations which had weakly singular kernels were considered. A weakly singular kernel is defined by

$$\lim_{z \to z'} K(z,z') = \infty, \quad \lim_{z \to z'} (z - z')K(z,z') = 0 \tag{4}$$

The method presented in ref. 1 is outlined briefly at this point. Referring to equation (1), we interpolated $\psi(z')$ over the range $z' \in [a,b]$ as

$$\psi(z') = \sum_{j=1}^{N} F_j(z')\psi(z_j) \tag{5a}$$

where $F_j(z')$ is a known function which must satisfy $F_j(z_k) = \delta_{jk}$. As before, z_k belongs to a set of quadrature abscissas. In ref. 1, we showed that if $K(z,z') = |z-z'|^{-1/2}$, it was sufficient to take

$$F_j(z) = \begin{cases} 1 & \text{for } \theta_j \le z \le \theta_{j+1} \\ 0 & \text{otherwise} \end{cases} \tag{5b}$$

where θ_j is the midpoint between z_{j-1} and z_j. Thus, equation (1) becomes

$$\psi(z_i) = \psi_0(z_i) + \lambda \sum_{j=1}^{N} \psi(z_j) \int_{\theta_j}^{\theta_{j+1}} K(z_i, z')\, dz'$$

The integrated kernel is no longer infinite and matrix inversion can be applied.

In ref. 1, we compared this approach to two other techniques which also handle weakly singular equations successfully. A scheme introduced by Ullman [2] treats the equation as if the kernel were nonsingular, as in equation (2). However, the singular (diagonal) term is replaced by

$$\psi(z_i) \int_{z_i - w_i/2}^{z_i + w_i/2} K(z_i, z')dz' \tag{6}$$

which is no longer infinite. Thus, the integral equation can be inverted.

The other successful approach discussed in ref. 1 was a well-known simple subtraction scheme which Schlitt [3] had used to solve the weakly singular equation used as an example in ref. 1. In this approach, equation (1) is written

$$\psi(z) = \psi_0(z) + \lambda \int_a^b K(z,z')[\psi(z') - \psi(z)]dz' + \lambda\psi(z) \int_a^b K(z,z')dz' \tag{7}$$

Assuming $\psi(z')$ has a Taylor series expansion around $z' = z$, the first integrand is no longer infinite (in light of equation (4)), and so the first integral can be approximated by a sum over quadratures. The second integral is evaluated analytically, the integrated kernel being finite for all z since the singularity is weak.

This paper is concerned with integral equations with kernels that have pole singularities. Three types of linear equations are considered and examples are presented to illustrate that the proposed method works well for these equations. In section II, the type of equation considered is one for which the kernel is of the form

$$K(z,z') = \frac{T(z,z')}{(z-z')_P} \tag{8}$$

where T is a bounded, continuous, well-behaved function and P implies the Cauchy principle value. The proposed method of solution is presented in this section.

In section II, an equation with the kernel described in equation (8) is treated, considering both the inhomogeneous and homogeneous forms. Section III is devoted to applying the proposed approach to the Lippmann-Schwinger equation, the kernel of which is of the form

$$K(z,z';E) = \frac{V(z,z')}{(z'-E)_P} \tag{9}$$

where E is a fixed (energy) parameter.

In section IV, the approach is applied to nonlinear equations of the type that arise from dispersion relations. The numerical problems encountered in attempting to invert the resulting algebraic system are described.

Two appendices are included in the paper. In the first, integral equations containing infinite limits are considered. No illustrative examples are considered.

The second appendix deals with the generation of inhomogeneous equations like those discussed in section II which have known solutions.

II. Linear Equations with Cauchy Singularities

The first type of equation considered is one containing the kernel of equation (8); namely

$$\psi(z) = \psi_0(z) + \lambda \int_a^b \frac{T(z,z')}{(z-z')_P} \psi(z')\, dz' \tag{10}$$

where λ is a known constant, and ψ_0 and T are well-behaved, bounded functions.

It is quite straightforward to show [1] that if either a or b is finite, equation (10) can be transformed into an equation of the form

$$\phi(x) = f(x) + \lambda \int_{-1}^{1} \frac{U(x,y)}{(x-y)_P} \phi(y)\, dy \tag{11}$$

It is in this form that a solution to equation (10) is studied.

An example of such an equation is the Omnes equation [4] which arises in the theory of low energy scattering processes involving the pion and nucleon in the initial or final state. The simplest form of the Omnes equation which contains the pole singularity is

$$\psi(z) = F(z) + \frac{1}{\pi} \int_1^{\infty} \frac{h^*(z')}{(z'-z-i\epsilon)} \psi(z')\, dz' \tag{12}$$

where $h(z) = \exp\,[i\delta(z)] \sin \delta(z)$. Writing $(z'-z-i\epsilon)^{-1} = (z'-z)_P^{-1} + i\pi\delta(z'-z)$, and noting that $1 - ih^*(z) = \exp\,[-i\delta(z)]\cos \delta(z)$, equation (12) can be written as

$$G(z) = F(z) + \frac{1}{\pi} \int_1^{\infty} \frac{\tan \delta(z')G(z')}{(z'-z)_P}\, dz' \tag{13}$$

where $G(z) \equiv \exp\,[-i\delta(z)] \cos \delta(z)\psi(z)$. Using the transformations

$z = 2/(1+x)$, $z' = 2/(1+y)$, $\phi(x) = G(z(x))/(1+x)$, $f(x) = F(z(x))/(1+x)$

equation (13) becomes

$$\phi(x) = f(x) + \frac{1}{\pi} \int_{-1}^{1} \frac{\tan \delta[2/(1+y)]}{(x-y)_P} \phi(y)\, dy \tag{14}$$

which is in the form of equation (11).

At first, I attempted to solve equation (14) using the approach of ref. 1, where $\phi(y)$ is interpolated over the interval [-1,1] as in equation (5). It was found that this simple approach, which approximates equation (11) as

$$\phi(x_i) = f(x_i) + \lambda \sum_{j=1}^{N} \phi(x_j) \int_{\theta_j}^{\theta_{j+1}} \frac{U(x_i,y)dy}{x_i-y} \tag{15}$$

is unsatisfactory. Approximating integrals which contain poles by equation (5) is very inaccurate. To illustrate this inaccuracy, I compared the value of the integral

$$I_M(x_i) = \int_{-1}^{1} \frac{y^M dy}{(x_i-y)_P} \tag{16a}$$

as approximated by equations (5)

$$\int_{-1}^{1} \frac{y^M dy}{(x_i-y)_P} \simeq \sum_{j=1}^{N} x_j^M \int_{\theta_j}^{\theta_{j+1}} (x_i-y)_P^{-1}\, dy = -\sum_{j=1}^{N} x_j^M \log|(\theta_{j+1}-x_i)| \tag{16b}$$

to the analytic form

$$\int_{-1}^{1} \frac{y^M dy}{(x_i-y)_P} = x_i^M \log[1+x_i)/(1-x_i)] - \sum_{k=1}^{M} x_i^{(M-k)}[1-(-1)^k]/k \tag{16c}$$

I also investigated the accuracy of Ullman's scheme to this problem by comparing equation (16c) to

$$\int_{-1}^{1} \frac{y^M dy}{(x_i-y)_P} \simeq \sum_{\substack{j=1\\ j\neq i}}^{N} w_j x_j^M (x_i-x_j)^{-1} \tag{16d}$$

The term corresponding to $j = i$ is

$$\int_{x_i - w_i/2}^{x_i + w_i/2} (x_i - y)_P^{-1} \, dy = 0$$

Table I contains the results of these comparisons for M = 50 at representative values of x_i, in the 20, 40, and 80 point Gauss-Legendre quadrature sets. As can be seen, the inaccuracies arise at points near the end points of the integral, where I_M is singular.

The approach used by Schlitt was also considered in ref. 1. This approach involves writing equation (11) in the form

$$\phi(x) = f(x) + \lambda \int_{-1}^{1} \frac{U(x,y)[\phi(y) - \phi(x)]}{(x-y)_P} \, dy + \lambda\phi(x) \int_{-1}^{1} \frac{U(x,y)}{(x-y)_P} \, dy \tag{17}$$

The second integral can be evaluated analytically, and the first integral, since it no longer contains a singularity, can be approximated by a sum; namely, with $x = x_i$, the first integral is

$$\sum_{j=1}^{N} w_j U(x_i, x_j)[\phi(x_j) - \phi(x_i)]/(x_i - x_j) \tag{18a}$$

The diagonal term in this sum is

$$-w_i \; U(x_i, x_i) \left.\frac{\partial\phi}{\partial y}\right|_{y=x_i} \tag{18b}$$

This scheme, therefore, introduces the unknown derivative $\partial\phi/\partial y$ in addition to the unknown ϕ. It is, of course, straightforward to relate $\partial\phi/\partial y$ to ϕ. For example, as was done in equation (5a), interpolating ϕ by

$$\phi(y) = \sum_j F_j(y)\phi_j \tag{19a}$$

yields

$$\left.\frac{\partial\phi}{\partial y}\right|_{x_i} = \sum_j F_j'(x_i)\phi_j \tag{19b}$$

This leads to the equation

$$\phi(x_i) = f(x_i) + \lambda \sum_{j\neq 1} U(x_i, x_j)[\phi(x_j) - \phi(x_i)](x_i - x_j)^{-1}$$

$$- \lambda w_i U(x_i, x_i) \sum_j F_j'(x_i)\phi(x_j) + \lambda\phi(x_i) \int_{-1}^{1} U(x_i, y)(x_i - y)_P^{-1}\, dy \tag{19c}$$

This equation is in satisfactory form for inverting the integral equation. However, it is somewhat cumbersome, and the remaining integral in equation (19c) might require numerical evaluation.

In what follows, I propose a slightly different manipulation of equation (11), and a particular interpolative approximation to ϕ. It yields a noticeably less cumbersome equation than (19c) and, as will be illustrated by example, highly accurate results.

To begin, equation (11) is rewritten

$$\phi(x) = f(x) + \lambda \int_{-1}^{1} \frac{[U(x,y) - U(x,x)]}{(x-y)} \phi(y)\, dy + \lambda U(x,x) \int_{-1}^{1} \frac{\phi(y)\, dy}{(x-y)_P} \tag{20}$$

It is immediately noted that the first integrand is no longer singular, and the first integral is thus approximated by a sum. With $x = x_i$, the first integral is approximately

$$\sum_{j=1}^{N} w_j [U(x_i, x_j) - U(x_i, x_i)] \phi(x_j)(x_i - x_j)^{-1} \tag{21a}$$

where the $j = i$ term

$$-w_i \left. \frac{\partial U(x_i, y)}{\partial y} \right|_{y=x_i} \phi(x_i) \tag{21b}$$

does not contain the derivative of ϕ.

In order that the matrix inversion be straightforward, the second integral of equation (20) must be approximated (accurately) by a sum in which ϕ is evaluated at the same quadrature points $\{x_i\}$ as used in equation (21a) for the approximation to the first integral. To accomplish this, note that the quadrature abscissas, $\{x_i\}$, are the zeros of some polynomial $A_N(x)$ (for an N-point quadrature rule). That is, $A_N(x_i) = 0$. Thus, ϕ is interpolated over the interval $[-1,1]$ as

$$\phi(y) = \sum_{j=1}^{N} \frac{A_N(y)\phi(x_j)}{(y-x_j)A'_N(x_j)} \tag{22}$$

Because the interval is $[-1,1]$, it is very convenient (though not necessary [5]) to take $A_N(y)$ to be the Legendre polynomial $P_N(y)$. With equation (22), the second integral becomes

$$\sum_{j=1}^{N} \frac{\phi(x_j)}{P'_N(x_j)} \int_{-1}^{1} P_N(y)(x_i-y)^{-1}(y-x_j)^{-1}dy \tag{23}$$

Using the integral representation of the Legendre function of the second kind,

$$Q_N(x) = \frac{1}{2} \int_{-1}^{1} P_N(y)(x-y)_P^{-1}dy$$

the second integral becomes

$$2\sum_{j=1}^{N} \frac{\phi(x_j)}{P'_N(x_j)} \frac{[Q_N(x_i)-Q_N(x_j)]}{(x_i-x_j)} \tag{24a}$$

where the $j = i$ term is

$$2\phi(x_i)\frac{Q'_N(x_i)}{P'_N(x_i)} \tag{24b}$$

Thus, with equations (21) and (24), the matrix equation for ϕ is

$$\phi(x_i) = f(x_i) + \lambda \sum_{j=1}^{N} M_{ij}\phi(x_j) \qquad (25a)$$

where

$$M_{ij} = \left\{ w_j [U(x_i,x_j)-U(x_i,x_i)] + 2\,\frac{U(x_i,x_i)}{P_N'(x_j)}[Q_N(x_i)-Q_N(x_j)] \right\}(x_i-x_j)^{-1} \qquad (25b)$$

the diagonal terms being given by equations (21b) and (24b). The matrix elements are all finite, and relatively simple to generate, and equation (25a) can be inverted.

At this point, it is noted that when the limits of the integral in equation (10) are finite, transforming to the interval [-1,1] and using the Legendre polynomial interpolation seems to be the most convenient and natural interpolation. When the limits are semi-infinite or infinite, interpolations using other polynomials are possible. In an appendix to this paper, an interpolative approach like the one above is discussed for equations in which the limits are $[a,\infty]$, which can be transformed to [-1,1], and for those with limits $[-\infty,\infty]$, which cannot be cast into the form of equation (11).

To illustrate the accuracy of a method, it is customary to solve an equation, and compare the results to a known solution. In general, since the kernel of equation (11) is singular, both the homogeneous and nonhomogeneous equations have nontrivial solutions. Thus, the total solution is the particular solution to the inhomogeneous equation, added to a linear combination of all independent solutions of the homogeneous equation [6,7]. The inversion method will only yield a solution to the inhomogeneous equation; that is, only a particular solution. However, if the inhomogeneous term, f(x), is generated by first choosing a function $\phi(x)$ as solution, the particular solution obtained by matrix inversion can be compared with the chosen analytic form. That is, if ϕ is chosen to be some known function $\phi_0(x)$, and if f(x) is generated from ϕ_0 by

$$f(x) = \phi_0(x) - \lambda \int_{-1}^{1} U(x,y)\phi_0(y)(x-y)_P^{-1}\, dy \tag{26}$$

then the solution to equation (11) obtained by matrix inversion is then $\phi_0(x)$. Generating an equation with a known solution is discussed in greater detail in Appendix B.

To illustrate the matrix inversion approach of this paper, I solved the Omnes equation in the form of equation (14). The function f(x) is generated from (and, therefore, yields the solution)

$$\phi(x) = (1+x) \exp[-x^2](9-x^2)^{-1} \tag{27}$$

The phase shift in equation (14) is parametrized by

$$\delta(z) = \frac{(z-1)}{z^2}\pi \qquad 1 \leq z \leq \infty$$

or

$$\delta[2/(1+y)] = (1-y^2)\pi/4 \qquad -1 \leq y \leq 1 \tag{28}$$

The function f(x) is then generated using equation (27) for ϕ_0 with $U(x,y) = \tan[(1-y^2)\pi/4]$ and $\lambda = 1/\pi$. The resulting f(x) is a rather complicated looking series involving a double sum. The identity [8]

$$\tan[(1-y^2)\pi/4] = \sec[y^2\pi/2] - \tan[y^2\pi/2] = \frac{4}{\pi}\left(\sum_{\substack{k=1 \\ k\text{-odd}}}^{\infty} [(2k-1)+y^2]^{-1} - \sum_{\substack{k=1 \\ k\text{-even}}}^{\infty} [(2k-1)-y^2]^{-1} \right)$$

was used.

In Table II, the result of inverting the Omnes equation is compared with the analytic form of equation (27). As can be seen, the technique is quite accurate. Using a 10-point quadrature rule yields six decimal-place accuracy. A 20-point rule yields results which are essentially limited by the accuracy of the computer (10 decimal places in this single precision computation). Thus, this straightforward scheme appears to be very satisfactory for equations typified by equation (10).

As mentioned above, the total solution to the integral equation (10) also requires the solution of the homogeneous counterpart. When the interpolative approach described by equations (20) through (25) is applied to the homogeneous form, an equation identical to equation (25) with $f(x_i) = 0$ results. Thus, solution of the homogeneous integral equation becomes solution of the homogeneous algebraic set of equations

$$\phi(x_i) = \lambda \sum_{j=1}^{N} M_{ij}\phi(x_j) \tag{29}$$

where M_{ij} is given by equation (25b).

An example of such an equation, and a method of solution, is considered by Bareiss and Neumann [9]. To solve this set of equations, the idea is to normalize the unknown function so that, for example, $\phi(x_N) = 1$. That is, equation (29) is divided by $\phi(x_N)$ and becomes

$$1 = \lambda M_{NN} + \lambda \sum_{j=1}^{N-1} M_{Nj}\psi_j \tag{30a}$$

$$\psi_i = \lambda M_{iN} + \lambda \sum_{j=1}^{N-1} M_{ij}\psi_j \tag{30b}$$

where $\psi_i = \phi(x_i)/\phi(x_N)$. The N-1 dimensional subset of equations in (30b) are solved by matrix inversion, with equation (30a) serving as a consistency check on the solution.

To illustrate the accuracy of the proposed approach for homogeneous equations, I solved the Milne equation for the radiative flux (the problem treated in section IV of ref. 9); a problem well known in the theory of radiative transfer and neutron transport [10].

$$[1 - \frac{c}{2} z \log \frac{(1+z)}{(1-z)}] \psi(z) = \frac{c}{2} \int_{-1}^{0} z'\psi(z')(z'-z)_P^{-1} \, dz' \tag{31}$$

To compare the results of inverting equation (31) by the technique proposed here with those obtained in refs. 9 and 10, I used a 20-point quadrature, and inverted the set of equations corresponding to equation (30b). I then used a Lagrange interpolation to obtain $\psi(z)$ at the points reported in refs. 9 and 10. The comparison is presented in table III. As will be noted, the 20-point quadrature rule with the present approach is closer to Chandrasekhar's values than the 50-point scheme used in ref. 9. As can be seen in table IV, the consistency of the calculation, suggested by equation (30a) is also quite good.

It therefore appears that an integral equation, the kernel of which contains a Cauchy singularity, can be inverted in a straightforward way using a relatively small matrix.

III. Kernels with Fixed Poles: the Lippmann-Schwinger Equation

In this section, the technique described for inverting equations with Cauchy-like kernels is applied to equations with kernels containing a fixed pole. To describe the scheme for such a kernel, a Lippmann-Schwinger equation will be used as an illustrative example.

The partial wave Lippmann-Schwinger equation, which is the momentum space integral equation corresponding to the Schroedinger equation, has the form

$$\psi_\ell(p^2,p'^2;k_0^2) = V_\ell(p^2,p'^2) - \frac{2}{\pi}\int_0^\infty k^2\, dk\, V_\ell(p^2,k^2)\psi_\ell(k^2,p'^2;k_0^2)(k^2-k_0^2-i\epsilon)^{-1} \quad (32)$$

where, in this form, the on-shell amplitude is related to the phase shift by

$$\psi_\ell(k_0^2,k_0^2;k_0^2) = -\exp[i\delta_\ell(k_0^2)]\sin\delta_\ell(k_0^2)/k_0 \quad (33)$$

The first step, as expected, is to project out the principal value integral in equation (32). Suppressing the angular momentum dependence, this becomes

$$\psi(p^2,p'^2;k_0^2) = V(p^2,p'^2) - \frac{2}{\pi}\int_0^\infty V(p^2,k^2)\psi(k^2,p'^2;k_0^2)(k^2-k_0^2)_P^{-1}k^2\, dk$$

$$- ik_0\, V(p^2,k_0^2)\psi(k_0^2,p'^2;k_0^2) \quad (34)$$

Setting $p^2 = k_0^2$ yields an equation for $\psi(k_0^2,p'^2;k_0^2)$. Inserting this into equation (34) yields the desired equation, containing only the principal value integral.

$$\psi(p^2,p'^2;k_0^2) = V(p^2,p'^2) - ik_0V(p^2,k_0^2)V(k_0^2,p'^2)[1+ik_0V(k_0^2,k_0^2)]^{-1}$$

$$- \frac{2}{\pi}\int_0^\infty \left\{V(p^2,k^2)-ik_0V(p^2,k_0^2)V(k_0^2,k^2)[1+k_0V(k_0^2,k_0^2)]^{-1}\right\}\frac{\psi(k^2,p'^2;k_0^2)}{(k^2-k_0^2)_P}k^2\, dk$$

$$(35)$$

It is now straightforward that one can proceed to use the technique of section II to invert equation (35). First, a transformation is made from (p,k,k_0) which $\epsilon\,[0,\infty]$, to variables (x,y,E) which $\epsilon\,[-1,1]$, where $E = (k_0^2-1)/(k_0^2+1)$. The subtraction is now made at $y = E$ $(k = k_0)$ rather than at $y = x$ $(k = p)$ as in section II. Transformation to the (x,y,E) variables yields the matrix equation

$$\phi(x_i,x';E) = f(x_i,x';E) - \sum_{j=1}^{N} M_{ij}\phi(x_j,x';E) \tag{36a}$$

with

$$M_{ij} = \left\{w_j[U(x_i,x_j;E)-U(x_i,E;E)] + 2\,\frac{U(x_i,E;E)}{P_N'(x_j)}[Q_N(x_j)-Q_N(E)]\right\}(x_j-E)^{-1} \tag{36b}$$

where $U(x,y;E)$ results from the transformation of the bracketed terms in the integral of equation (35) and $f(x,x';E)$ comes from the transformation of the inhomogeneous term. As before, all elements of M_{ij} are finite, the difference being replaced by derivatives if x_j happens to equal the chosen value of E.

To illustrate the scheme's value in solving this kind of equation, I consider the Lippmann-Schwinger equation treated by Osborn [11]. Equation (32) has been normalized to yield equation (33) on-shell, as in ref. 11. In ref. 11, Osborn presents a method whereby he interpolates ψ using the eigensolutions of the free Green's function equation. He solves equation (32) for the Yamaguchi potential

$$V(p^2,p'^2) = \lambda(p^2+\beta^2)^{-1}(p'^2+\beta^2)^{-1} \tag{37}$$

λ and β being chosen as -8.110 and 1.444 respectively so that the triplet n-p bound state energy and scattering length are correctly given by equation (37). This potential has the nice feature that because it is separable, the kernel of equation (32) is degenerate. Thus, it is quite straightforward to find the analytic solution. It is

$$\psi(p^2, p'^2; k_0^2) = (p^2 + \beta^2)^{-1}(p'^2 + \beta^2)^{-1}\left\{1 + \frac{\lambda}{2\beta(\beta - ik_0)^2}\right\}^{-1} \tag{38}$$

I compare the result of inverting equations (36) to this analytic form setting $p' = k_0$. The transformation $p^2 = (1-x)/(1+x)$ takes the interval $[0,\infty]$ to $[-1,1]$. These comparative results at representative values of x are presented in Table V. As can be seen, a 20-point quadrature rule yields accuracy to three decimal places. A 30-point scheme increases the accuracy to at least four decimal places, indicating the viability of this approach for solving equations with kernels containing fixed poles.

As a final comment, it will be noted that the solution to this fixed pole equation is less accurate for a given number of quadrature points than the corresponding problem containing a Cauchy-like kernel. This may be due to the fact that the example chosen here with a fixed pole has a complex solution, and the inaccuracy may be the result of using complex arithmetic here, whereas real arithmetic was used in the examples of section II.

IV. Non-linear Integral Equations Arising from Dispersion Relations

In this section, integral equations which come about from dispersion relations for scattering amplitudes are studied. Such equations are of the form

$$\mathrm{Re}F(x) = F_0(s) + \frac{1}{\pi}\int_a^\infty V(s,s')\,\mathrm{Im}\,F(s')\,(s'-s)_P^{-1}\,ds' \qquad (39)$$

where $V(s,s')$ is specified by the type of subtraction made. Equations arising from dispersion relations are non-linear, the non-linearity coming from the unitarity constraint

$$\mathrm{Im}\,F(s) = \rho(s)\,|F(s)|^2 \qquad (40)$$

To handle the singular part of the kernel in Eq. (39), it is quite reasonable to use the interpolative approach introduced in this paper. Transforming the interval to $[-1,1]$, and then applying the interpolative technique, Eq. (39) becomes

$$\alpha(x_i) = F_0(x_i) + \frac{1}{\pi}\sum_{j=1}^{N} C_{ij}\,\beta(x_j) \qquad (41)$$

where

$$F(s) \equiv \alpha(s) + i\beta(s) \qquad (42a)$$

$$s = \begin{cases} -1 + 2a/x & a \neq 0 \\ (1+x)/(1-x) & a = 0 \end{cases} \qquad (42b)$$

and

$$C_{ij} = \left(\left[V_1(x_i,x_j) - V_1(x_i,x_j)\right]w_j + 2V_1(x_i,x_i)\left[Q_N(x_i) - Q_N(x_j)\right]/P_N'(x_j)\right)(x_i - x_j)^{-1} \qquad (42c)$$

Here, $V_1(x_i,x_j) = V[s(x_i), s(x_j)] \times$ (a factor coming from transforming the range of integration).

Equation (40) can now be written as

$$\beta(x_i) = \rho(x_i)\left[\alpha^2(x_i) + \beta^2(x_i)\right] \tag{43}$$

Putting Eq. (41) into Eq. (43) yields

$$\beta(x_i) = \rho(x_i) F_0^2(x_i) + \frac{2}{\pi} F_0(x_i)\rho(x_i) \sum_{j=1}^{N} C_{ij}\beta(x_j)$$

$$+ \rho(x_i)\frac{1}{\pi^2}\sum_{j,k}^{N} C_{ij} C_{ik}\beta(x_j)\beta(x_k) + \rho(x_i)\beta^2(x_i) \tag{44}$$

With

$$E_i \equiv \sum_{j=1}^{N}\left\{\delta_{ij} - \frac{2}{\pi} F_0(x_i)\rho(x_i) C_{ij}\right\}^{-1}\rho(x_j) F_0^2(x_j) \tag{45a}$$

and

$$V_{ijk} \equiv \sum_{\ell=1}^{N}\left\{\delta_{i\ell} - \frac{2}{\pi} F_0(x_i)\rho(x_i) C_{i\ell}\right\}^{-1}\frac{\rho(x_\ell)}{\pi^2}\left\{\delta_{j\ell}\delta_{k\ell} + \frac{4}{\pi^2} C_{\ell j} C_{\ell k}\right\} \tag{45b}$$

Eq. (44) becomes

$$\beta(x_i) = E_i + \sum_{j,k} V_{ijk}\,\beta(x_j)\beta(x_k) \tag{46}$$

In order to illustrate the viability of the proposed approach to this type of problem, I considered, as an example, an equation introduced by Blankenbecler, Goldberger, Khuri, and Treiman [12] with which they tested the validity of approximating the left-hand cut in potential scattering by the contribution from the Born term. The equation is

$$F(s) = -2g/(1+4s) + \frac{1}{\pi}\int_0^\infty ds'\, \pi\, \mathrm{Im}\, F(s')\,(s' - s - i\epsilon)^{-1} \tag{47a}$$

with

$$\mathrm{Im}\, F(s) = \sqrt{s}\; |F(s)|^2 \quad . \tag{47b}$$

The analytic solution to this equation is

$$F(s) = \frac{-2g/(1+4s)}{1 + \frac{2g(1+4s)}{\pi}\displaystyle\int_0^\infty \frac{\sqrt{s'}\, ds'}{(1+4s')^2\,(s' - s\, i\epsilon)}} \tag{48}$$

The simplest approach to a solution of Eq. (46) would be an iterative one,

$$\beta_i^{(N)} = E_i + \sum_{j,k} V_{ijk}\, \beta_j^{(N-1)}\, \beta_k^{(N-1)} \tag{49}$$

with $\beta_i = \beta(x_i)$, and $\beta_i^{(0)} = E_i$. I find that such an approach diverges quite rapidly, and thus is not suited to solving this type of problem.

I also tried another iterative approach. Rewriting Eq. (46) as

$$\beta_i = E_i + \frac{1}{2}\sum_{j,k} (V_{ijk} + V_{ikj})\, \beta_j \beta_k \tag{50a}$$

I then considered the iteration

$$\beta_i^{(N)} = E_i + \frac{1}{2}\sum_{j,k} (V_{ijk} + V_{ikj})\, \beta_j^{(N-1)}\, \beta_k^{(N)} \tag{50b}$$

again using $\beta_i^{(0)} = E_i$. This is a linear approximation to equation (46) which is solved for $\beta_i^{(N)}$ by matrix inversion. This procedure does not seem to diverge, but oscillates over a fairly wide range, and does not converge to any solution. Thus, it appears that the numerical problem of obtaining solutions to non-linear algebraic systems of equations (including the question of uniqueness of solution) is a formidable one, and will be left for a later investigation by other methods [13]. However, at this time, there is no reason to doubt that once a satisfactory method of solving the algebraic system is found, the interpolative approach proposed here will prove suitable for dispersion integral equations as well as linear ones.

Appendix A

Interpolative Approach to Linear Integral Equations with Infinite or Semi-Infinite Limits

In the body of this paper, all integrals were transformed to the interval $[-1, 1]$, including equations in which the parameters ranged over the interval $[0, \infty)$. The range $[-1, 1]$ is found to be very convenient for this approach since the integral

$$\int_{-1}^{1} \phi(y)\,(x-y)_P^{-1}\,dy$$

is easily evaluated when $\phi(y)$ is interpolated over the Legendre polynomials. Quite a bit of information about the second Legendre functions $Q_N(x)$ is readily available. Equations involving the interval $(-\infty, \infty)$ were not considered because there is no simple transformation which will take

$$\int_{-\infty}^{\infty} f(x')\,(x'-x)_P^{-1}\,dx' \rightarrow \int_{-1}^{1} f(y')\,T(y')\,(y-y')_P^{-1}\,dy'$$

where $T(y')$ is defined for a transformation by

$$dx' = T(y')\,dy'$$

It is possible to deal directly with semi-infinite or infinite intervals using the interpolative approach of this paper. To illustrate this, consider first an integral equation of the form

$$\phi(x) = f(x) + \lambda \int_0^{\infty} \frac{[U(x,y) - U(x,x)]}{(x-y)}\,\phi(y)\,dy + \lambda U(x,x) \int_0^{\infty} \frac{\phi(y)}{(x-y)_P}\,dy \qquad \text{(A1)}$$

and define

$$I(x) \equiv \int_0^\infty \frac{\phi(y)}{(x-y)_P}\, dy \tag{A2}$$

It is necessary to interpolate $\phi(y)$ in this integral over the interval $[0, \infty]$ in such a way that the integral of the interpolating functions, with the pole, can be evaluated easily, as with

$$Q_N(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_N(y)}{(x-y)_P}\, dy \tag{A3}$$

For the interval $[0, \infty]$, it is most natural to approximate the first integral in Eq. (A1) by a sum over Laguerre quadratures [14]. Therefore, the interpolating function for the second integral, I(x), should involve Laguerre polynomials, which satisfy the recurrence relation

$$N L_N(x) = (2N-1-x) L_{N-1}(x) - (N-1) L_{N-2}(x) \qquad N \geq 2 \tag{A4}$$

Define

$$\Lambda_N(x) \equiv \int_0^\infty \exp[-y]\; L_N(y)\; (x-y)_P^{-1}\, dy \tag{A5}$$

Using Eq. (A4), it is easy to show that $\Lambda_N(x)$ also satisfies the recurrence relation

$$N \Lambda_N(x) = (2N-1-x)\Lambda_{N-1}(x) - (N-1)\Lambda_{N-2}(x) \qquad N \geq 2 \tag{A6}$$

where

$$\int_0^\infty \exp[-y]\, L_{N-1}(y)\, dy = 0 \qquad N \geq 2$$

by the orthogonality of L_N. The Λ_N can thus be generated from

$$\Lambda_0(x) = \exp[-x]\,\mathrm{Ei}(x)\,, \qquad \Lambda_1(x) = 1 + (1-x)\Lambda_0(x)$$

Thus, in I(x) in Eq. (A2), $\phi(y)$ can be interpolated over the interval $[0, \infty]$ as

$$\phi(y) = \sum_{j=1}^{N} \frac{\phi(x_j)\exp[-y]\,L_N(y)}{(y-x_j)\exp[-x_j]\,L'_N(x_j)} \tag{A7}$$

where the points x_j are the zeros of $L_N(x)$. Then, $I(x_i)$ becomes

$$I(x_i) = \sum_{j=1}^{N} \frac{\phi(x_j)\,[\Lambda_N(x_i) - \Lambda_N(x_j)]}{\exp[-x_j]\,L'_N(x_j)\,(x_i - x_j)} \tag{A8}$$

To evaluate the terms in Eq. (A8), $L'_N(x_j)$ is needed. It can be generated from

$$L'_N(x_j) = N[L_N(x_j) - L_{N-1}(x_j)]/x_j = -N\,L_{N-1}(x_j)/x_j \tag{A9}$$

The $j=i$ term in Eq. (A8) contains $\Lambda'_N(x_j)$, which can be generated from

$$\Lambda'_N(x) = \frac{1}{x}L_N(0) - \Lambda_N(x) + \frac{N}{x}\left\{[\Lambda_N(x) - \Lambda_{N-1}(x)] - [\Lambda_N(0) - \Lambda_{N-1}(0)]\right\} \tag{A10}$$

With $\Lambda_N(x)$ completely defined, the integral equation of Eq. (A1) can now be approximated by

$$\phi_i = f_i + \lambda \sum_{j=1}^{N} \left\{ w_j[U_{ij} - U_{ii}] + \frac{\lambda U_{ii}[\Lambda_{Ni} - \Lambda_{Nj}]}{\exp[-x_j]\,L'_{Nj}} \right\} (x_i - x_j)^{-1}\,\phi_j \tag{A11}$$

and can be inverted.

However, as pointed out earlier, the $[0,\infty]$ type equation can be transformed to the $[-1, 1]$ form, the integrand still retaining the singularity in the form $(x-y)^{-1}_P$.

It may, in fact, be desirable to make the transformation. My experience has been that unless the integrand contains the factor $\exp[-y]$ naturally, the integral

$$\int_0^\infty f(y)\, dy = \int_0^\infty \exp[-y]\, (f(y) \exp[y])\, dy$$

is poorly approximated by a Laguerre quadrature rule $\sum_j w_j \exp[x_j] f(x_j)$ [14]. Thus, the first integral in Eq. (A1) may be very badly approximated by such a sum, unless $U(x,y) \sim \exp[-y]$. Transforming $[0, \infty]$ to $[-1, 1]$ and using a Legendre quadrature rule is generally fairly accurate.

Equations involving the interval $[-\infty, \infty]$ cannot be transformed to $[-1, 1]$ with the integrand of the $[-1, 1]$ integral still containing the pole in the form $(x-y)_P^{-1}$. Thus, an approach similar to that described above for the $[0, \infty]$ type equation is essential.

To begin, a quadrature over the Hermite polynomials is most natural for a $[-\infty, \infty]$ integral [15]. The Hermite polynomials obey the recurrence relation

$$H_{N+1}(x) = 2x H_N(x) - 2N H_{N-1}(x) \qquad N \geq 1 \tag{A12}$$

Defining

$$\eta_N(x) \equiv \int_{-\infty}^{\infty} \exp[-y^2]\, H_N(y)\, (x-y)_P^{-1}\, dy \tag{A13}$$

$\eta_N(x)$ satisfies

$$\eta_{N+1}(x) = 2x\, \eta_N(x) - 2N\, \eta_{N-1}(x) \qquad N \geq 1 \tag{A14}$$

where

$$\int_{-\infty}^{\infty} \exp[-y^2] H_N(y)\, dy = 0 \qquad N \geq 1$$

by the orthogonality of H_N. It is straightforward to show that

$$\eta_0 = \int_{-\infty}^{\infty} \exp[-y^2] dy = \sqrt{\pi}\,, \qquad \eta_1 = 2\sqrt{\pi}\,(x-1)$$

Thus, for an integral equation of the form

$$\phi(x) = f(x) + \lambda \int_{-\infty}^{\infty} \frac{[U(x,y) - U(x,x)]}{(x-y)} \phi(y)\, dy + \lambda U(x,x) \int_{-\infty}^{\infty} \frac{\phi(y)}{(x-y)_P} dy \tag{A15}$$

the second integral can be evaluated by interpolating $\phi(y)$ as

$$\phi(y) = \sum_{j=1}^{N} \frac{\phi(x_j) \exp[-y^2] H_N(y)}{\exp\left[-x_j^2\right] H_N'(x_j)\; (y - x_j)} \tag{A16}$$

The second integral at $x = x_i$ becomes

$$\sum_j \frac{\phi(x_j)\, [\eta_N(x_i) - \eta_N(x_j)]}{\exp\left[-x_j^2\right] H_N'(x_j)\, (x_i - x_j)} \tag{A17}$$

Using

$$H_N'(x) = 2N\, H_{N-1}(x)$$

it can easily be shown that

$$\eta_N'(x) = -\eta_{N+1}(x) \tag{A18}$$

Thus, the diagonal (j = i) term of Eq. (A17) is well defined and the interpolative approach of this paper can be applied to equations with infinite intervals. However, as with the $[0, \infty]$ integral equation, caution must be exercised. Unless $U(x, y)$ contains the factor e^{-y^2}, the first integral in Eq. (A15) may be poorly approximated by a sum over a Hermite quadrature.

As a final note, it is obvious that non-linear equations with infinite or semi-infinite limits can also be dealt with straightforwardly using the approach outlined in this appendix. The problem becomes one of finding solutions to the non-linear algebraic system, as in section IV.

Appendix B

The Generation of Inhomogeneous Cauchy-like Linear Integral Equations with Known Solutions

In the text of Section II, it was noted that, in general, the solution to Eq. (11) is determinable only up to arbitrary additions of solutions to the homogeneous equation. However, it seems to be possible to generate an equation which has a known solution by generating the known inhomogeneous term, f (x), from an a priori chosen solution $\phi_0(x)$ as in Eq. (26). If the approach discussed in connection with Eq. (26) is valid, the numerical solution obtained will be $\phi(x) = \phi_0(x)$.

Starting with Eq. (11),

$$\phi(x) = f(x) + \lambda \int_{-1}^{1} \frac{U(x,y)}{(x-y)_P} \phi(y)\, dy \tag{B1}$$

with

$$f(x) = \phi_0(x) - \lambda \int_{-1}^{1} \frac{U(x,y)\, \phi_0(y)}{(x-y)_P}\, dy \tag{B2}$$

Thus the difference $\Delta(x) \equiv \phi(x) - \phi_0(x)$ satisfies the homogeneous equation

$$\Delta(x) = \lambda \int_{-1}^{1} \frac{U(x,y)}{(x-y)_P} \Delta(y)\, dy \tag{B3}$$

As discussed in Section II, such a homogeneous equation generally has a nontrivial solution. Thus, one would suspect that $\phi(x) \neq \phi_0(x)$ in general. However, the numerical results presented for the Omnes equation indicate that

$\Delta(x) = 0$. I have been unable to find a proof that such an equality must obtain [15].

I have investigated the possibility that this equality exists because of some particular properties of the Omnes equation. For example, $\phi(x) = \phi_0(x)$ may be due to the fact that the Omnes equation has the property that

$$U(x, y) = U(y) \qquad x, y \in [-1, 1] \tag{B4}$$

and, furthermore, the parameterization used for the phase shift gives

$$U(\pm 1) = 0 \tag{B5}$$

To test whether $\phi(x) = \phi_0(x)$ is due to some properties unique to the Omnes equation, I have generated some other examples which have no physical significance, but with which I attempted to test whether the equality holds. Two such examples are given below. The first is

$$\phi(x) = f(x) + \lambda \int_{-1}^{1} \frac{(y - x^3)\, e^{-(x^2 + y^2)}}{(x - y)_P}\, \phi(y)\, dy \tag{B6}$$

with

$$\phi_0(x) = e^{x^2}.$$

This yields

$$f(x) = \phi_0(x) - \lambda \int_{-1}^{1} \frac{(y - x^3) e^{-(x^2 + y^2)}}{(x - y)_P}\, \phi_0(y)\, dy = e^{x^2} + \lambda e^{-x^2} \left\{ 2 - (x - x^3) \log\left(\frac{1+x}{1-y}\right) \right\} \tag{B7}$$

The factor $y - x^3$ was included to eliminate the effect the inherent singularities at $x = \pm 1$ might have on numerical accuracy. The second example considered was

$$\phi(x) = f(x) + \lambda \int_{-1}^{1} \frac{e^{-(x^2+y^2)}}{(x-y)_P} \phi(y)\, dy \tag{B8}$$

again with $\phi_0(x) = e^{x^2}$, and, thus,

$$f(x) = e^{x^2} - \lambda e^{-x^2} \log\left[(1+x)/(1-x) \right] \tag{B9}$$

This was considered to see if the absence of the "singularity killing" factor had much effect on the accuracy of the numerical approach. As can be seen in Table VI, the method is accurate with and without this factor.

However, the point to be made here is that in these examples the numerical solution matches $\phi_0(x)$, as can be seen from Table VI. Thus, although I cannot prove that the equality should hold, it appears, in general, that the described method of generating an inhomogeneous term for an equation like (B1) does yield an analytic function $\phi_0(x)$, to which a numerical solution $\phi(x)$ can be compared, and, therefore, is a useful tool for testing a proposed numerical approach.

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Table I

Values of $I_M(x)$ of Eq. (16a) using the Cohen-Ickovic and Ullman approximations, for various quadrature rules.

	x	Ullman	Cohen-Ickovic	Analytic
20 point	.9931	.2402	.7640	.0784
	.0765	-.0031	-.0036	-.0031
40 point	.9982	1.2146	1.8613	1.6121
	.0388	-.0016	-.0016	-.0017
80 point	.9996	2.5415	3.2071	3.1322
	.0195	-.0008	-.0008	-.0008

Table II

Comparison of solution of Omnes equation with analytic form, using 10- and 20-point quadrature rules

	x	Calculated	Analytic
10-point	.9739	9.495625E-2	9.495630E-2
	.4334	1.348063E-1	1.348059E-1
	-.8651	7.737198E-3	7.737322E-3
20-point	.9931	9.27589695E-2	9.27589695E-2
	.3737	1.34831485E-1	1.34831485E-1
	-.9931	3.19790732E-4	3.19790733E-4

Table III

Comparison of radiative flux as found by present method, method of Ref. 9, and Chandrasekhar results

$-\mu$	Bareiss, Neumann, 50 points	Present Method	Chandrasekhar
.05	.4027	.4031	.4032
.35	.6154	.6159	.6159
.65	.8097	.8102	.8102
.95	1.000	1.000	1.000

Table IV

Check on consistency of solution of Eq. (30b) by substitution of solution into Eq. (30a)

$\lambda \sum_{j=1}^{N-1} M_{Nj}$	$1 - \lambda M_{NN}$	% Difference
1.14393	1.14402	.008%

Table V

Comparison of solution to Lippmann-Schwinger equation with Yamaguchi potential to analytic solution

x	Calculated	Analytic
	$k_0^2 = 0.1$	
20 points		
.9931	.006721 - .021557i	.006722 - .021556i
.3737	.4589 - 1.4720i	.4590 - 1.4719i
-.7463	.8803 - 2.8236i	.8805 - 2.8235i
-.9931	.9401 - 3.0153i	.9403 - 3.0152i
30 points		
.9969	.0030451 - .0097653i	.0030453 - .0097652i
.3527	.47038 - 1.50844i	.47041 - 1.50843i
-.7678	.88599 - 2.84125i	.88604 - 2.84121i
-.9969	.94111 - 3.017998i	.94116 - 3.017959i
	$k_0^2 = 10$	
20 points		
.9931	-.00194843 - .00029696i	-.00194839 - .00029694i
.3737	-.133043 - .020277i	-.133040 - .020276i
-.7463	-.255214 - .038897i	-.255208 - .038895i
-.9931	-.272543 - .041538i	-.272536 - .041536i
30 points		
.9969	-.000882656 - .000134552i	-.000882650 - .000134519i
.3527	-.136343 - .0207793i	-.136342 - .0207790i
-.7678	-.256810 - .0391391i	-.256808 - .0391386i
-.9969	-.272785 - .0415739i	-.272783 - .0415733i
	$k_0^2 = .22$ (E = -.6361)	
20 points		
.9931	-.00041236 - .01673687i	-.00041238 - .01673687i
.3737	-.028157 - 1.142829i	-.028156 - 1.142829i
-.6361	-.052206 - 2.118930i	-.052204 - 2.118930i
-.9931	-.057680 - 2.341118i	-.057678 - 2.341118i

Table VI

Comparison of calculated and analytic solutions to Eq. (B6) and (B8). ϕ_6 and ϕ_8 are, respectively, the solutions to these two equations, presented at representative values of x and λ.

	x	λ	ϕ_6	ϕ_8	Analytic ϕ ($\exp[x^2]$)
10 points					
	.9739	.5	2.5818186	2.5818230	2.5818183
		20	2.5816087	2.5815079	
		200	2.5816501	2.5814394	
	.1480	.5	1.0224163	1.0224232	1.0224110
		20	1.0224905	1.0223935	
		200	1.0224327	1.0223866	
20 points					
	.9931	.5	2.68130714	2.68130714	2.68130714
		20	2.68130713	2.68130720	
		200	2.68130726	2.68130720	
	.0765	.5	1.00587349	1.00587349	1.00587349
		20	1.00587350	1.00587349	
		200	1.00587348	1.00587349	