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EXTENSIONS OF THE DERIVATIVE DISPERSION RELATIONS FOR AMPLITUDE ANALYSES*

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ABSTRACT

It is shown that analyticity properties of scattering amplitudes as expressed by the derivative analyticity relations can be combined with measured cross section and polarization data to extract the full scattering amplitudes in a simple and practical way. The method should be particularly useful in problems involving a coupled amplitude analysis. The measurement of a complete set of spin parameters, or a phase shift analysis at only one energy, is sufficient to eliminate ambiguities in the extraction at all other energies. Our procedure is illustrated by considering both simple mathematical examples and real data.

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I. INTRODUCTION

If only experimental information is used, it is widely known that a complete set of 2N-1 measurements are needed in order to determine N complex scattering amplitudes at each value of the kinematic variables (e.g., s,t) up to an overall phase. However, if scattering amplitudes satisfy dispersion relations, then their real and imaginary parts are related, and in principle only half as many measurements are required (but over a range of energies), and the s-dependent phase ambiguity is not present. Considerable effort has been expended by many people in applying this method to data.[1] It is quite difficult to apply in practice because one needs to know or to make assumptions about the imaginary parts of the amplitude at all energies above the threshold for each value of t. It is, at best, only feasible to apply this method near the forward direction in a few reactions. A quick and flexible method to extract physics under more general conditions would be very useful. It has recently been shown[2,3] that the analyticity properties of amplitudes which are embodied in (global) dispersion relations can be equally well implemented by writing derivative analyticity relations (DAR). These relations for even and odd signature amplitudes typically[4] read:

$$\begin{aligned} \mathrm{Re}\, M_+(s,t) &= s \tan\left[\frac{\pi}{2}\frac{d}{dy}\right] (\mathrm{Im}\, M_+(s,t)/s) \\ \mathrm{Re}\, M_-(s,t) &= \tan\left[\frac{\pi}{2}\frac{d}{dy}\right] (\mathrm{Im}\, M_-(s,t)) \quad , \end{aligned} \tag{1}$$

where $y = \ell n\, s$. At energies above the resonance region, where the amplitudes vary adiabatically with energy, it is an adequate and controllable approximation to retain only one derivative. Then local relations

$$\begin{aligned} \mathrm{Re}\, M_+ &\cong \frac{\pi}{2}\, s \frac{d}{dy} (\mathrm{Im}\, M_+/s) \\ \mathrm{Re}\, M_- &\cong \frac{\pi}{2} \frac{d}{dy} (\mathrm{Im}\, M_-) \end{aligned} \tag{2}$$

should approximately hold. Modifications of the analysis useful when there are rapid variations will be discussed below. If $d\sigma/dt$ is measured for a spinless reaction in a region of energy such that the derivatives with respect to $y = \ell n\, s$ can be computed (at a given t value), the phase of the amplitude can be computed by solving Eq. (2) together with

$$s^2 \frac{d\sigma}{dt} = |\, \mathrm{Re} M \,|^2 + |\, \mathrm{Im} M \,|^2 . \tag{3}$$

In the following, we shall demonstrate that it feasible to extend this approach, for example, to reactions involving spin at any momentum transfer. After a short discussion of the formalism, the practicality of the approach is illustrated by constructing several model amplitudes and their resultant "data" and then retrieving the amplitudes via the DAR. In addition to demonstrating the validity of the method, these examples will help develop our intuition, techniques for handling real data, and illustrate possible difficulties and how to avoid them. The method will finally be applied to a brief discussion of the available data on the reaction $K_L^0 p \rightarrow K_S^0 p$. Compton scattering and a brief discussion of πp and pp scattering at the lower energies.

We are optimistic that this method will lead to significant progress in the art of extracting physics from data because of its simplicity.

II. FORMALISM

In Ref. 3 it was shown that for an even signatured amplitude the dispersion relation

$$\mathrm{Re}\, M_+(s,t) = \frac{2}{\pi} s^2 P \int \frac{ds'}{s'(s'^2 - s^2)} \mathrm{Im}\, M_+(s',t) \qquad (3)$$

can be written in the equivalent form

$$\mathrm{Re}\, M^+(s,t) = s^a \tan\left[\frac{\pi}{2}(a - 1 + \frac{d}{dy})\right](s^{-a} \mathrm{Im}\, M_+(s,t)) , \qquad (4)$$

where a is an arbitrary number ($0 < a < 2$ in the present case) which can be chosen to make the derivative of the last factor as small as possible. A similar result holds for odd signatured amplitudes.

A

For our present purposes, it is somewhat more convenient, although not necessary, to use an equivalent result that connects the magnitude and the phase of the amplitude. If the amplitude has no zeroes, its phase and logarithm satisfy a simple dispersion relation[5]. The equivalent phase-magnitude DAR is

$$\phi_+(s,t) = - \tan\left[\frac{\pi}{2} \frac{d}{dy}\right] \ell n\, F_+ , \qquad (5)$$

where

$$M_+(s,t) \equiv F_+(s,t)\, e^{i\phi_+(s,t)} \qquad (6)$$

If $\ell n\, F_+$ varies slowly with energy then

$$\phi_+(s,t) \sim - \frac{\pi}{2} \frac{d}{dy} \ell n\, F_+ , \qquad (7)$$

or equivalently

$$\sim - \frac{\pi}{2} a - \frac{\pi}{2} \frac{d}{dy} \ell n(s^{-a} F_+) .$$

For the case of odd signature, one finds

$$\phi_-(s,t) \sim \frac{\pi}{2} - \frac{\pi}{2}\frac{d}{dy}\ell n F_- \qquad (8)$$
$$\sim \frac{\pi}{2}(1+b) - \frac{\pi}{2}\frac{d}{dy}\ell n (s^{-b}F_-) .$$

When zeroes are present (for example, a complex zero in s at a fixed), the phase gets an extra contribution and this can be easily handled (an example will be present later), but in general it is always correct to use the amplitude written in terms of its real and imaginary parts rather than the phase and amplitude. These zeroes can be a reflection of CDD zeroes in the amplitude and are interesting to study.

One complexity that is present (and it will be ignored after this remark) is that one should work with analytic functions, e.g. the invariant amplitudes, for each reaction rather than the convenient and more physical functions, e.g., the s-channel helicity amplitudes, which may have extra kinematic singularities. It will take some experience with the method to develop the most efficient way to proceed but there are no fundamental problems in this regard.

B

In the single amplitude case, in our normalization, we have for positive signature

$$F_+(s,t)/s = R_+ = (d\sigma/dt)^{\frac{1}{2}}$$

and

$$\phi_+(s,t) = -\frac{\pi}{2} - \frac{\pi}{2}\frac{d}{dy}\ell n R_+ . \qquad (9)$$

These simple equations have an important physical consequence for elastic processes at sufficiently high energy — one expects the differential cross sections to increase in the forward direction and the t-slope to also increase. This means that there is a finite value of t at which the function R_+ is essentially

constant. This "crossover" in the same singatured amplitude at different energies signals that the phase is zero at the cross over point or that the real part of the amplitude has a zero at this t value.[3]

C

Let us now turn to the two amplitude case, such as meson-nucleon scattering, where both the differential cross section and polarization are known. It is always possible to combine the different physical quantities so that the amplitudes of interest have a given signature. Therefore, consider two even signatured amplitudes f_{++} and f_{+-}, respectively the nonflip and the flip amplitude where $s^{-1} f_{\lambda'\lambda} = R_{\lambda'\lambda} \exp(iQ_{\lambda'\lambda})$, and the data is given as

$$\frac{d\sigma}{dt} \equiv A^2 = R_{++}^2 + R_{+-}^2 \tag{10}$$

and

$$A^2 P = 2 R_{++} R_{+-} \sin(\phi_{+-} - \phi_{++}) .$$

Using the DAR for the phase, as given in Eq. (7) or (8), the quantity R_{+-} can be eliminated and the result is a differential equation for R_{++}, written in the form

$$\frac{\pi}{2} \frac{d}{dy} \ell n z = -(1-z^2) \sin^{-1}\left[\tfrac{1}{2} P/z(1-z^2)^{\frac{1}{2}}\right] ,$$

where

$$z \equiv R_{++}/A .$$

Given the data A(s,t) and P(s,t), this can be solved numerically at each t value, and the amplitudes constructed. Note that since this is a differential equation, there is an arbitrary integration constant which depends only on t and must be determined at one energy value from other information such as spin-rotation parameters. An alternative and perhaps more attractive approach is to extend

the analysis down to energies where complete phase-shift solutions exist and amplitudes can be fully reconstructed. This boundary condition will then determine the integration constant as well.

It is possible to rewrite this equation in an integral form which is very convenient for solving by iteration and explicitly shown the nature of the integration constant. One such form can be motivated by defining

$$R_{++} = A\cos\theta$$

$$R_{+-} = A\sin\theta$$

which automatically insures the cross section constraint, and then $\theta(y,t)$ must satisfy the equation

$$\pi \frac{d}{dy}\,\theta(y,t) = -\sin\Big[2\theta(y,t)\Big]\sin^{-1}\Big[P/\sin 2\theta(y,t)\Big]. \tag{11}$$

In general one must solve this equation numerically. The phases of the amplitudes are given by

$$\phi_{++} = -\frac{\pi}{2} - \frac{\pi}{2}\frac{d}{dy}\ell n A - \sin^2\theta \sin^{-1}\Big[P/\sin 2\theta\Big]$$

and

$$\phi_{+-} = -\frac{\pi}{2} - \frac{\pi}{2}\frac{d}{dy}\ell n A + \cos^2\theta \sin^{-1}\Big[P/\sin 2\theta\Big].$$

If the polarization is small (one amplitude is small or they both have the same energy dependence) in the sense that $\sin^{-1}\left(P/\sin 2\theta\right) \sim P/\sin 2\theta$, then one immediately has the approximate solution

$$\theta(y,t) = \theta_0(t) - \frac{1}{\pi}\int_{y_0}^{y} dz\, P(z)\,, \tag{12}$$

where $\theta_0(t)$ is an energy independent integration constant. The phases of the amplitudes are then given by

$$\phi_{++} \simeq -\frac{\pi}{2}\left(1+\frac{A'}{A}\right) - \tfrac{1}{2} P\tan\theta$$

$$\phi_{+-} \simeq -\frac{\pi}{2}\left(1+\frac{A'}{A}\right) + \tfrac{1}{2} P\,\mathrm{ctn}\,\theta \ ,$$

where $A' = dA/dy$. These phases must be increased by $\pi/2$ for odd signature. The arbitrary integration constant $\theta_0(t)$ must be determined at $y = y_0(t)$ from additional input at each t value.

One interesting feature of these equations is that while a knowledge of the energy derivative of the differential cross section is required to evaluate the phase, no such derivatives are required of the polarization. Furthermore, Eq. (11) is in a convenient form for solution by iteration by starting off with $\theta_0 = 0$, and then simply integrating as in Eq. (12). The data is required only between the values of y_0 and y. The approximation leading to Eq. (12) will be valid whenever P is small because the amplitudes are approximately in phase or one of them is small.

D

In this paragraph, we will consider a simplified example of line-reversed reactions. Spin effects will be neglected so that, for example, reactions such as $\pi N \to KY$ and $\bar{K}N \to \pi Y$ are described by two measured cross sections

$$\frac{d\sigma}{dt}(\pi N) = A^2 = |R_+ e^{i\phi_+} + R_- e^{i\phi_-}|^2$$

$$\frac{d\sigma}{dt}(\bar{K}N) = B^2 = |R_+ e^{i\phi_+} - R_- e^{i\phi_-}|^2$$

where $R_\pm e^{i\phi_\pm}$ are the even and odd signatured amplitudes. Introducing the parametrization

$$R_+ = S\cos\theta$$

$$R_- = S\sin\theta \ ,$$

one finds

$$S^2 = \tfrac{1}{2}(A^2 + B^2)$$

and

$$(A^2 - B^2)/(A^2 + B^2) \equiv T^2 = \sin 2\theta \cos(\phi_- - \phi_+) \ . \qquad (13)$$

Using the DAR relations

$$\phi_+ = -\frac{\pi}{2} - \frac{\pi}{2}\frac{d}{dy}\ell n R_+$$

$$\phi_- = -\frac{\pi}{2}\frac{d}{dy}\ell n R_- \ ,$$

one finds the differential equation

$$\pi\theta' = \sin 2\theta \, \sin^{-1}\left[T^2/\sin 2\theta\right] , \qquad (14)$$

an equation that can be handled as in the case of polarization.

It is also possible to treat the case of spin in these coupled reactions where the input information now contains the two measured polarizations. The resulting coupled equations are not amenable to a simple analytic treatment, but they can be directly integrated numerically.

E

In this section we will discuss techniques that allow the DAR's to be applied in the low energy region where resonances and inelastic thresholds cause the amplitudes to vary quite rapidly. The procedure here is to rewrite the amplitudes so that only first derivatives are required. Two different procedures will be described below, a difference method and a ratio method.

While it is obvious that one cannot apply the DAR directly to reactions with a rapidly varying energy dependence, the analyticity property of the amplitudes allows a perturbation theory to be developed which permits such applications. It also shows the proper relation between the DAR's and conventional dispersion relations. Suppose that data on Im M(s,t) indicates that it is rapidly varying in some region of s. Define a new function $M_1(s,t)$ by

$$M_1(s,t) = M(s,t) - s\,G(s,t) ,$$

where G is an explicit analytic function chosen so that Im M_1 is not rapidly varying. Then (for positive signature)

$$\text{Re}\,M(s,t) = s\,\text{Re}\,G(s,t) + \frac{\pi}{2}\, s\frac{d}{dy}\left(s^{-1}\,\text{Im}\,M_1(s,t)\right) , \tag{15}$$

where Re G(s,t) can be constructed explicitly. A convenient choice for G in the low energy region is a sum of signatured Breit-Wigner forms,

$$G(s,t) = \sum_{i=1}^{N} G_i(s,t)$$

where

$$G_i(s,t) = H_i\Gamma_i\left[\left(s_i - s - \Gamma_i\left((\bar{s}-s)/s_i\right)^{\frac{1}{2}}\right)^{-1} \pm (s \to -s)^{-1}\right] , \tag{16}$$

where $\bar{s}$ is an assumed threshold value, and H_i, Γ_i and s_i can all depend on t. Numerically, the contribution from the crossed terms are usually small. The procedure is to choose these parameters by inspection and to compute Re M by Eq. (15). The accuracy of the method can be gauged by the dependence of the predicted value of Re M on the precise value of these parameters. Of course, if one includes an arbitrarily large number of terms and carefully fits all the parameters, M_1 can be neglected. This is then mathematically equivalent to an

evaluation of the exact dispersion relations. In fact, the optimum choice for G is the numerically constructed analytic function given by the exact dispersion relation. The use of the DAR approach can allow a considerable simplification in this procedure, especially in the analysis of reactions involving several amplitudes, such as a case with polarization. Let us now turn to an alternative expansion method which has some definite advantages in practical applications.

In this ratio method, it is convenient to proceed as follows. Define

$$Q_+(s,t) \equiv s \prod_{i=1}^{N} (1 + G_i) = |Q_+| e^{iq_+}$$

where the G_i are chosen as before, and then use the DAR phase-amplitude relation on the ratio $F(s,t)/Q_+(s,t)$. The result for the phase is

$$\phi_+(s,t) = q_+ - \frac{\pi}{2} - \frac{\pi}{2} \frac{d}{dy} \ell n |M_+/Q_+| ,$$

which can be quite accurate if the parameters in the G_i are chosen so that the higher derivatives of the logarithmic ratio are small.

III. MATHEMATICAL EXAMPLES

In this section, the method will be illustrated by applying it to several explicit examples.

A

The simplest example is a simple Regge pole with even signature,

$$M_+ = \beta\, s^{\alpha} e^{-i\pi\alpha/2} .$$

The magnitude $R_+ = \beta\, s^{\alpha}$ when used in Eq. (5) immediately gives the correct phase

$$\phi_+ = -\frac{\pi}{2}\frac{d}{dy}\ell n R_+ = -\frac{\pi}{2}\alpha .$$

Another example which illustrates how to handle zeroes is

$$M_+ = a + \tfrac{1}{2} b\left[\ell n(s_0 - s) + \ell n(s_0 + s)\right],$$

where a and b can be functions of t. The correct phase is

$$\tan\phi_+ = -\frac{\pi}{2} b\left[a + \tfrac{1}{2} b\, \ell n(s+s_0)(s-s_0)\right]^{-1} ,$$

whereas the phase-magnitude DAR relation gives

$$\phi_+ = -\frac{\pi}{2} b\left(s^2/(s^2 - s_0^2)\right)\left(a + \frac{b}{2}\ell n(s^2 - s_0^2)\right)\Big/\left[\left(a + \frac{b}{2}\ell n(s^2 - s_0^2)\right)^2 + \frac{\pi^2}{4} b^2\right].$$

These are obviously not the same — the reason is that M_+ can have zeroes when s is such that

$$\ell n(s^2 - s_0^2)^{\frac{1}{2}} = -\frac{a}{b} + i\frac{\pi}{2} ,$$

and these singularities in $\ell n R_+$ were not taken into account in the derivation. For energies far away from the zero, there is no problem. Near the zero, the

approximate phase varies rapidly in energy. In the examples we have treated, a rapid variation in the phase signals the possibility of this problem and it can be checked. Note that the DAR between the real and imaginary parts of M_+ doesn't suffer from this problem and is quite accurate far from the threshold ($s \gg s_0$).

B

For our next spinless example, consider the difference of two Regge poles:

$$M_+ = \beta_1 s^{\alpha_1} e^{-i\pi\alpha_1/2} - \beta_2 s^{\alpha_2} e^{-i\pi\alpha_2/2} . \tag{17}$$

The analytic comparison is not very illuminating. A numerical comparison of the exact phase with the approximate one is shown in Fig. 1 for the values $\beta_1 = 1$, $\beta_2 = 0.5$, $\alpha_1 = 0.9+t$, $\alpha_2 = 0.5 + 0.6\,t$. The zeroes of the amplitude are at

$$\ell n\, s = (\alpha_1 - \alpha_2)^{-1} \ell n\, \beta_2/\beta_1 + i\frac{\pi}{2} ,$$

and one sees that the error in the phase increases through $\pi/2$ near the position of the zeroes as might be expected. However, away from the zeroes, the procedure is quite accurate.

C

An interesting example is that provided by an absorbed Regge pole. In the case of the Pomeron trajectory, a suitable form is

$$M_+ = s\, e^{-i\pi/2} \left[e^{Bt} - \frac{A}{A+B} e^{ABt/(A+B)} \right] , \tag{18}$$

where

$$B = \alpha' (\ell n\, s - i\pi/2)$$

and we choose $A = 4$, and $\alpha = 1 + 0.5t$. This amplitude is predominantly imaginary and the differential cross section following from (18) provides somewhat realistic "data." The simple phase method gives good results except where M_+ has zeroes. In Fig. 2 the results of a numerical analysis is presented. Since $\mathrm{Re}\, M_+$ is quite small, it is sensitive to details of the procedure and is reconstructed quite well except for the point where $\mathrm{Im}\, M_+$ and the differential cross section have a dip and vary rapidly.

D

For the final mathematical model, a case with spin will be considered. The flip and nonflip amplitudes will be introduced as

$$R_{+-} = \epsilon(t)\, e^{(\alpha_1 - 1)y}, \quad \phi' = -\frac{\pi}{2}\alpha_1$$
$$R_{++} = e^{(\alpha_0 - 1)y}, \quad \phi = -\frac{\pi}{2}\alpha_0 \,. \tag{19}$$

which lead to "data" of the form

$$A = e^{(\alpha_0 - 1)y}\,(1 + \epsilon^2 e^{2\Delta y})^{\frac{1}{2}}$$

and

$$P = 2\epsilon \sin\frac{\pi}{2}\Delta\, e^{-\Delta y}\,(1 + \epsilon^2 e^{-2\Delta y})^{-1}$$

where $\Delta = \alpha_0 - \alpha_1$.

Applying the approximate result given by Eq. (12), one finds

$$\theta = \theta_0 + H(y)\,,$$

where

$$H(y) = \frac{2}{\pi\Delta}\sin\frac{\pi\Delta}{2}\tan^{-1}\left[\epsilon\, e^{-\Delta y}\right]$$

and $\theta_0 = \theta\ (y = \infty)$. If one retains only linear powers of H in the answer, the result for the reconstructed amplitudes $\bar{R}_{++}$ and $\bar{R}_{+-}$ are

$$\bar{R}_{++} = A\cos\theta_0 - A\sin\theta_0 H(y)$$

$$\bar{R}_{+-} = A\sin\theta_0 + A\cos\theta_0 H(y) \ .$$

A comparison with the input amplitudes of Eq. (19) clearly shows the significance of θ_0 as a phase rotation in the amplitude. If we choose $\theta_0 = 0$, then to order ϵ

$$\bar{R}_{+-} \simeq AH(y) = \epsilon\, e^{(\alpha_1 - 1)y}\left(\frac{2}{\pi\Delta}\sin\frac{\pi\Delta}{2}\right)$$

$$\bar{R}_{++} \simeq A = e^{(\alpha_0 - 1)y} \ .$$

The flip amplitude is not given exactly because of our original expansion of the tangent derivative operator. If Δ is small, which this expansion demands, then the error is small. Even if $\Delta \sim \frac{1}{2}$, as expected from Pomeron-Regge interference, for example, the reconstructed and input data differ by only 10%. This expansion can clearly be improved by various methods, such as keeping higher order terms.

The above examples have been discussed to show that the method is workable, and particularly to help develop an intuition about how to proceed with cases of real data.

Let us now finally turn to the analysis of some physical examples involving real data.

IV. APPLICATIONS AND COMMENTS

The DAR method and its extensions discussed here can be applied to determine applications in many reactions. Below we briefly discuss $K_2 p \to K_1 p$ and Compton scattering. Other reactions are hypercharge exchange, where the appropriate line reversed pairs generally exist to deal with both signatures, πN elastic and charge-exchange scattering; reactions such as $\pi N \to \rho N$ (and $KN \to K^* N$) where in special situations (assuming a good s-p wave separation) one has only one amplitude determining an observable such as $\rho_{11}^{s} \frac{d\sigma}{dt}$ at $t = 0$, or $\rho_{00}^{s} \frac{d\sigma}{dt}$ to leading order in s; information of pp amplitudes from $\bar{p}p$ data, etc. Sufficient data may exist to determine the backward πN amplitudes in a similar manner.

From a different point of view, it appears to us that this method will be a powerful way to extend phase shift analysis from low to intermediate energies. A determination of a set of amplitudes directly by this method has two advantages: (a) the amplitudes are analytic, and (b) many fewer parameters are required than for a direct phase shift analysis. Having found amplitudes, one can partial wave expand them and determine a set of phase shifts.

In general there should be little difficulty in applying this technique when the input data is rapidly varying. Since we are dealing with analytic functions, one simply chooses an analytic function which varies like the data and divides it out as described in the text so that the remainder is slowly varying, applying the method to the remainder. Since one has divided out an analytic function which satisfies the dispersion relation, the full phase is that of the known function plus that determined from the remainder as explained in the text. Further, one can always improve the results in a careful application by retaining the third derivative in the expansion of $\tan\left(\frac{\pi}{2} \frac{d}{dy}\right)$.

A. $K_L^0 p \to K_S^0 p$

The amplitude for this process has odd signature only and therefore it is straightforward to use our method. In general the amplitude will have helicity-flip and helicity non-slip parts, respectively dominated by ρ and ω exchange and, since no polarization data are available, one cannot perform a general amplitude analysis. However the helicity-flip contribution vanishes at $t = 0$ and presumably at $t \simeq -0.5$ and therefore we can hope to measure the phase of the helicity non-flip amplitude at these t values.

At $t = 0$ there are actually two independent ways of measuring the phase: (i) from the s-dependence of $\left(\frac{d\sigma}{dt}\right)_{t=0}$:

$$\phi_-^0 = -\frac{\pi}{4}\frac{d}{dy}\left[\ell n \left(\frac{d\sigma}{dt}\right)_{t=0}\right]$$

or (ii) from the s-dependence of the imaginary part of the amplitude as given by the optical theorem:

$$\frac{8\pi}{R}\,\mathrm{Im}\,F_-^0 = \sigma_T(K^+n) - \sigma_T(K^-n) < 0 \ .$$

It is remarkable that experimentally both $\left(\frac{d\sigma}{dt}\right)_{t=0}$ and $\mathrm{Im}\,F_-^0$ are power-behaved from a few GeV/c to 60 GeV/c,[6,7] and therefore the phase can be obtained most easily. The results from methods (i) and (ii) are shown in Fig. 3 and are in good agreement with independent measurements using K_L^0-K_S^0 interference[7] or optical point extrapolations[6].

At $t = -0.5$, we have[6]

$$\phi_-^0 = -\frac{\pi}{4}\frac{d}{dy}\left(\ell n \frac{d\sigma}{dt}\right) = \frac{\pi}{2}\,(1.02 \pm 0.22)$$

indicating a very small real part. From the trend of $\frac{d}{dy}\left(\ell n \frac{d\sigma}{dt}\right)$ around $t = -0.5$, it is quite probable that $\mathrm{Re}\, F_-^0$ has a zero in this neighborhood: this is in qualitative agreement with the behavior of $\mathrm{Re}\, F_-^0$ in πN scattering (ρ exchange) at 6 GeV/c.

B. $\gamma p \rightarrow \gamma p$

Compton scattering is a nice example with an even signature amplitude. The helicity non-flip amplitude is large and dominated by P and f exchange, while the flip amplitudes are smaller. In the forward direction I = 1 exchange (A_2, therefore mostly flip) has been measured to be small by comparing γp and γd Compton scattering. There is no direct experimental information on the helicity structure of I = 0 exchange, however we know from πN scattering that it is helicity non-flip to a good approximation and we expect γp to exhibit the same character. We therefore neglect helicity-flip contributions, and assume the phase we obtain from $d\sigma/dt$ is that of the dominant helicity non-flip amplitude using the data of Anderson et al.[8] for $0.1 < -t < 0.8$. The data obtained by Boyarski et al.[9] at smaller t ($0.025 < -t < 0.17$) have also been analyzed, and they do not show a significant deviation in phase from t = 0 determinations. Results shown that between t = 0 and t = -0.8 the phase ϕ_+^0 does not vary very much, changing from 102° to 110°; the separation into Re F and Im F is shown in Fig. 4. We have performed the same analysis on the data of Buschorn et al.[10] between 2 and 7 GeV to extract the real part at 4 GeV and the results are shown in Fig. 5. There is a strong energy dependence of the γp real part between 4 and 10 GeV indicating the dominance of Regge effects (f exchange) over Pomeron exchange in the real part of the Compton amplitude in this region.

C. Real Parts at Low Energy (t = 0)

Following the method outlined in Section II.E we have computed the real parts of $\pi^{\pm}p$, pp and $\bar{p}p$ elastic scattering in the resonance region from total cross section data. We started by taking any adequate fit to the high energy data[11] and using it to obtain a smooth curve through the low energy data. Then the difference between the data and the high energy extrapolation was approximated by a sum of Breit-Wigner forms with positions, heights, signs and widths fixed by inspection rather than detailed fitting. An example of the parametrization used is presented in Table 1 for the $\pi^- p$ reaction. The results of the calculations are shown in Figs. 6, 7 and 8 for $\pi^{\pm}p$ and pp real parts. There is good agreement with both the conventional integral dispersion relations[12] and with data whenever available.[13,14]

We have not explicitly included pole terms or threshold effects in our calculations and we put the lowest threshold to s = 0 in all applications. The fact that the results are still good at very low energies illustrates the power of using analytic functions to construct an "effective" amplitude and using the DAR's as a perturbation theory.

It should be emphasized that, although the presentation we are using is strictly local, its successful implementation at low energy (when large variations of the amplitude occur) requires input data over a reasonably large range of energies. Indeed, the use of a contribution extrapolated from high energies was rather essential to get a reasonable result. By using more local data (for example, $3 < s < 4$ GeV^2) we were able to reproduce the relative variation of the real parts, but not its actual values. In that sense our relations are <u>not</u> strictly local in content, but they certainly constitute a much easier method to compute real parts than conventional integral dispersion relations.

V. CONCLUSIONS

In this paper, we have investigated the usefulness of the DAR's in both mathematical and physical examples involving experimental data. The new extensions to nonforward directions, lower energies, coupled channels, and polarization problems was discussed in some detail. Basically, instead of a complete set of measurements at a particular energy, which may be hard, one needs about half as many kinds of measurements but at several nearby energies.

In conclusion, we would like to clarify[15] the usefulness of the derivative analyticity relation approach. In a strict sense, the DAR's are equivalent to conventional dispersion relations, but they can be used to simplify their evaluation by using the difference or ratio perturbation theory described in the text. The second and perhaps most important use of the derivative approach is in simplifying the amplitude analysis in the case of spin or coupled channels. The method described in the text for such cases can also be extended down into the low energy or resonance region by using the perturbation theory described in Section II.E. We are not trying to replace standard dispersion relation techniques when they can be easily used, but to extend the region of applicability to new regions of t and to new reactions, and to simplify the process of combining analyticity and data to learn new physics.

While it will take some experience with real data to uncover the shortcomings of the DAR's and to develop efficient methods to overcome them, it does appear that this approach offers a simple, quick, yet powerful way to extract amplitudes from data.

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FIGURE CAPTIONS

Fig. 1a: Comparison of exact (solid lines) with first derivative calculation (dashed lines) of the imaginary part of f for different t values. The parameters of the Regge poles are given in the text.

Fig. 1b: Same comparison as in Fig. 1a, but for the real part of f.

Fig. 2: Absorbed Regge pole amplitude compared with the approximate values (dashed lines) given by the DAR.

Fig. 3: A comparison of the phase of the amplitude as measured for $K_\ell p \to K_s p$ and as predicted by the DAR (the shaded bands in the figure).

Fig. 4: The real and imaginary parts of the amplitude for Compton scattering at 10 GeV as given by the DAR.

Fig. 5: A comparison of the real parts of the amplitude as given by the DAR for 10 and 4 GeV. A systematic and strong energy dependence is evident.

Fig. 6: The ratio of real to imaginary part of the amplitude for $\pi^+ p$ elastic scattering as determined by the DAR, dispersion relations and experiment.

Fig. 7: Same as Fig. 6, but for $\pi^- p$ elastic scattering.

Fig. 8: Same as Fig. 6, but for pp elastic scattering.

TABLE 1

H	$s_0(GeV^2)$	$\Gamma(GeV)$
-48	0.9	0.4
60	1.53	0.06
-10	1.87	0.20
11	2.28	0.075
- 1.5	2.5	0.05
27.5	2.8	0.15
3.0	3.5	0.15
4.5	5.0	0.6
2	7.0	0.4

The function G of Eq. (15) is a sum of terms of the form of Eq. (16), with parameters as shown for $\pi^- p$. One term is added below threshold ($s_0 = 0.9$), because one needs a function well determined over 1-2 units of $\ell n\, s$ to obtain its derivative in the middle of the range. Thus to calculate the real part at a few hundred MeV/c one needs σ_T well described down to threshold. In the actual calculations we set $s_{th} = 0$ since we have not included threshold corrections anywhere.

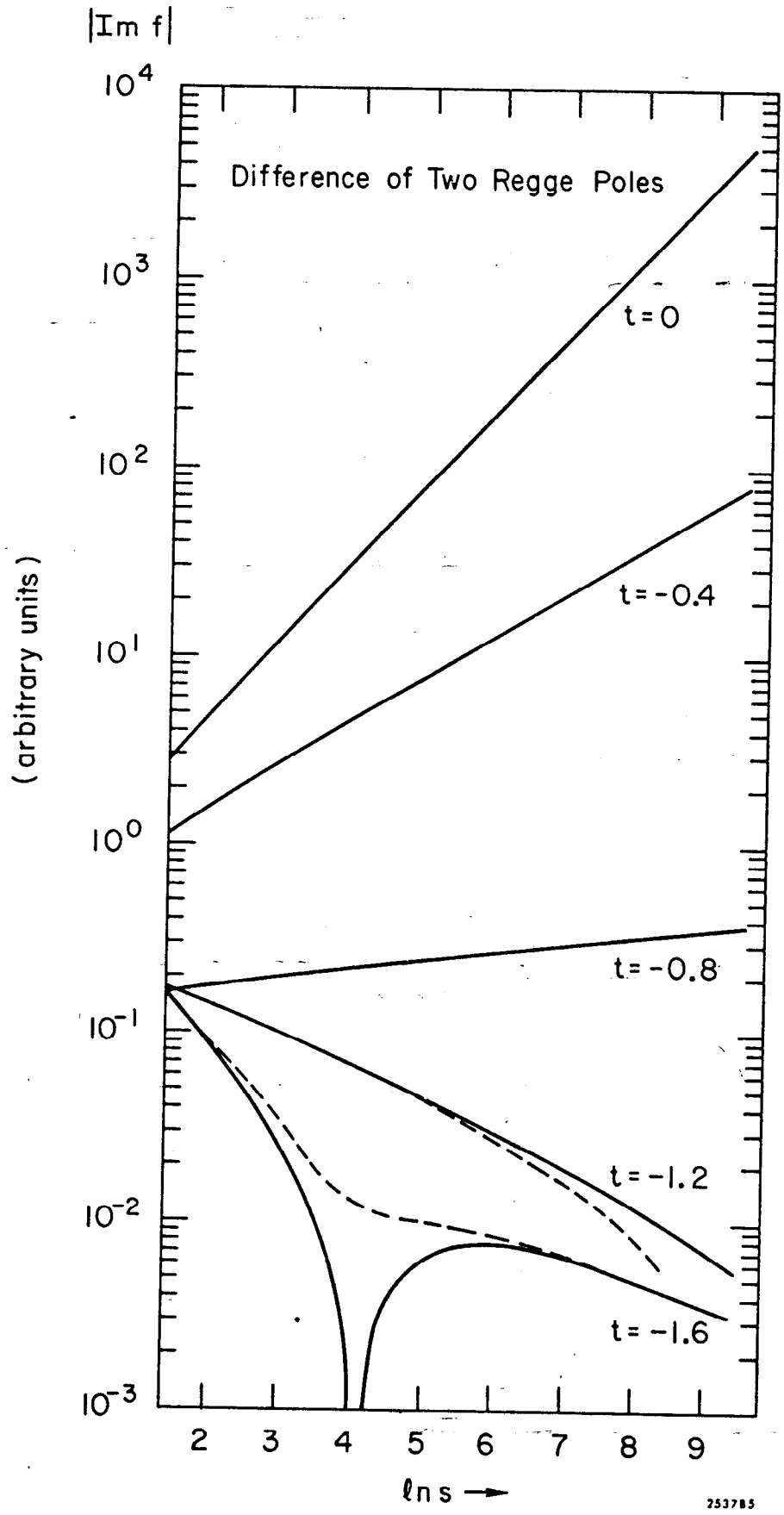
|Im f|
Difference of Two Regge Poles
t = 0
t = -0.4
t = -0.8
t = -1.2
t = -1.6
(arbitrary units)
ln s
253785

Fig. 1a

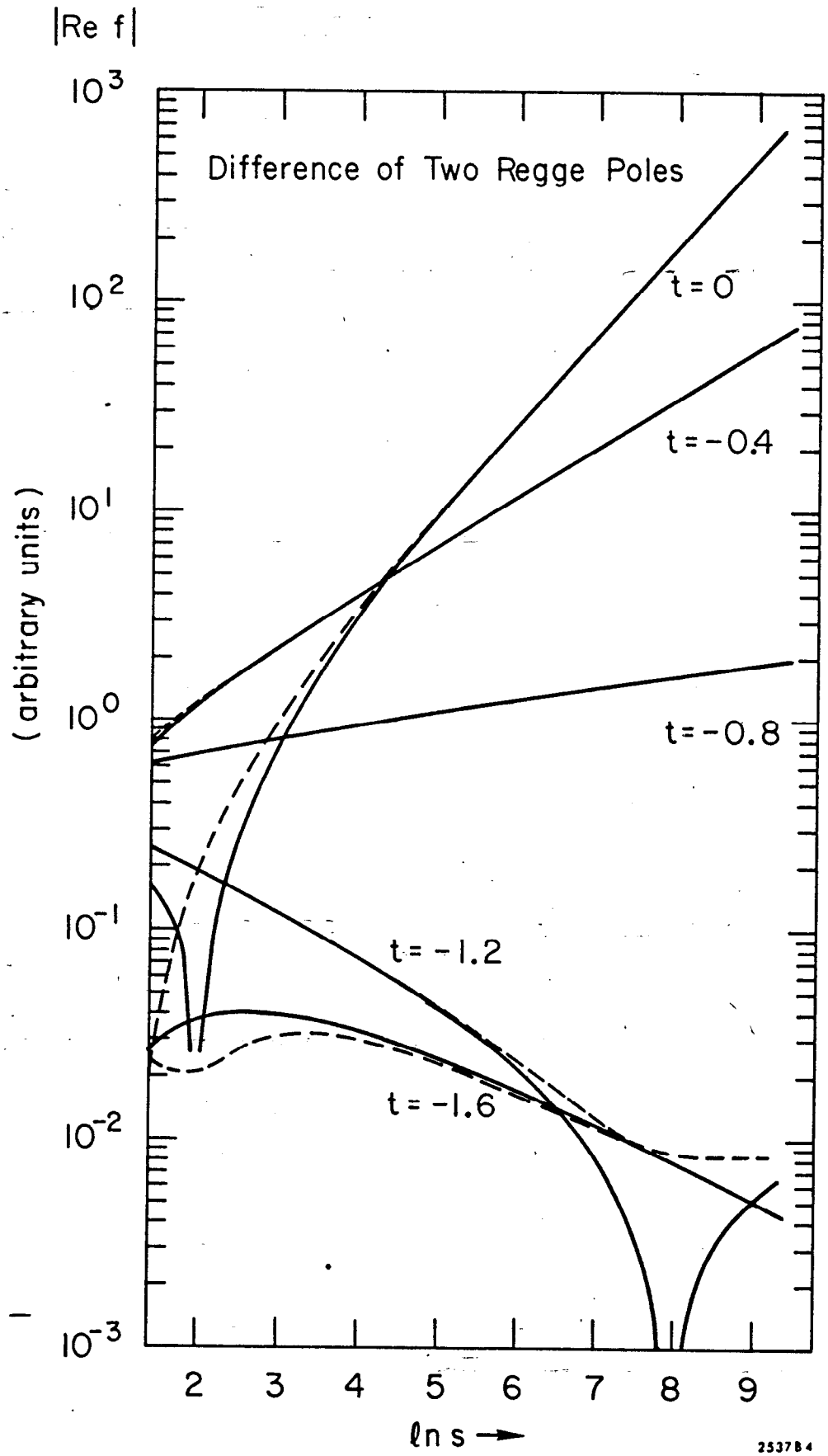
|Re f|
Difference of Two Regge Poles
t = 0
t = -0.4
t = -0.8
t = -1.2
t = -1.6
(arbitrary units)
10^3
10^2
10^1
10^0
10^-1
10^-2
10^-3
2
3
4
5
6
7
8
9
ℓn s →
2537B4

Fig. 1b

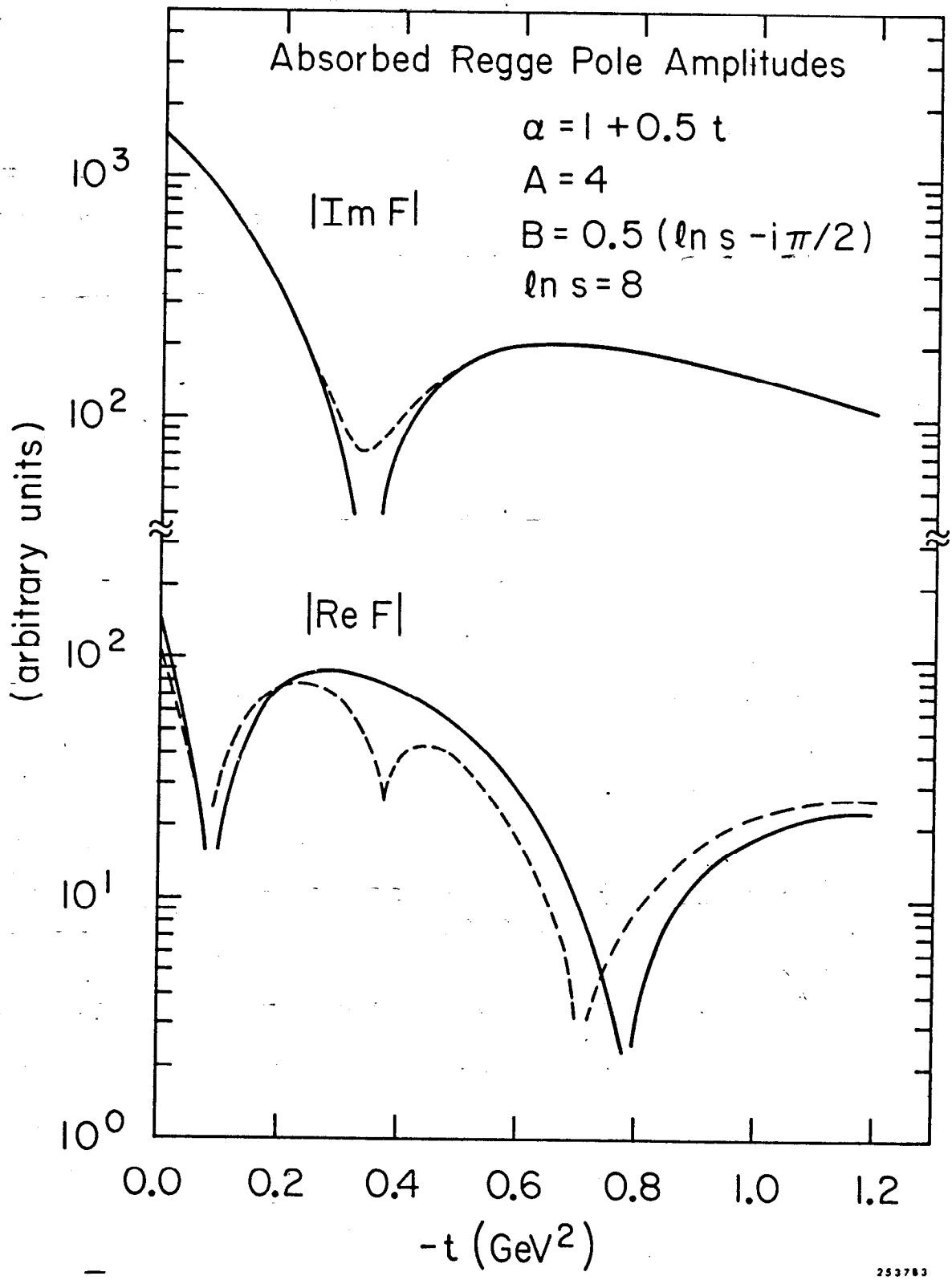
Absorbed Regge Pole Amplitudes
α = 1 + 0.5 t
A = 4
B = 0.5 (ln s - iπ/2)
ln s = 8
|Im F|
|Re F|
(arbitrary units)
10^3
10^2
10^2
10^1
10^0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
-t (GeV^2)
253783

Fig. 2

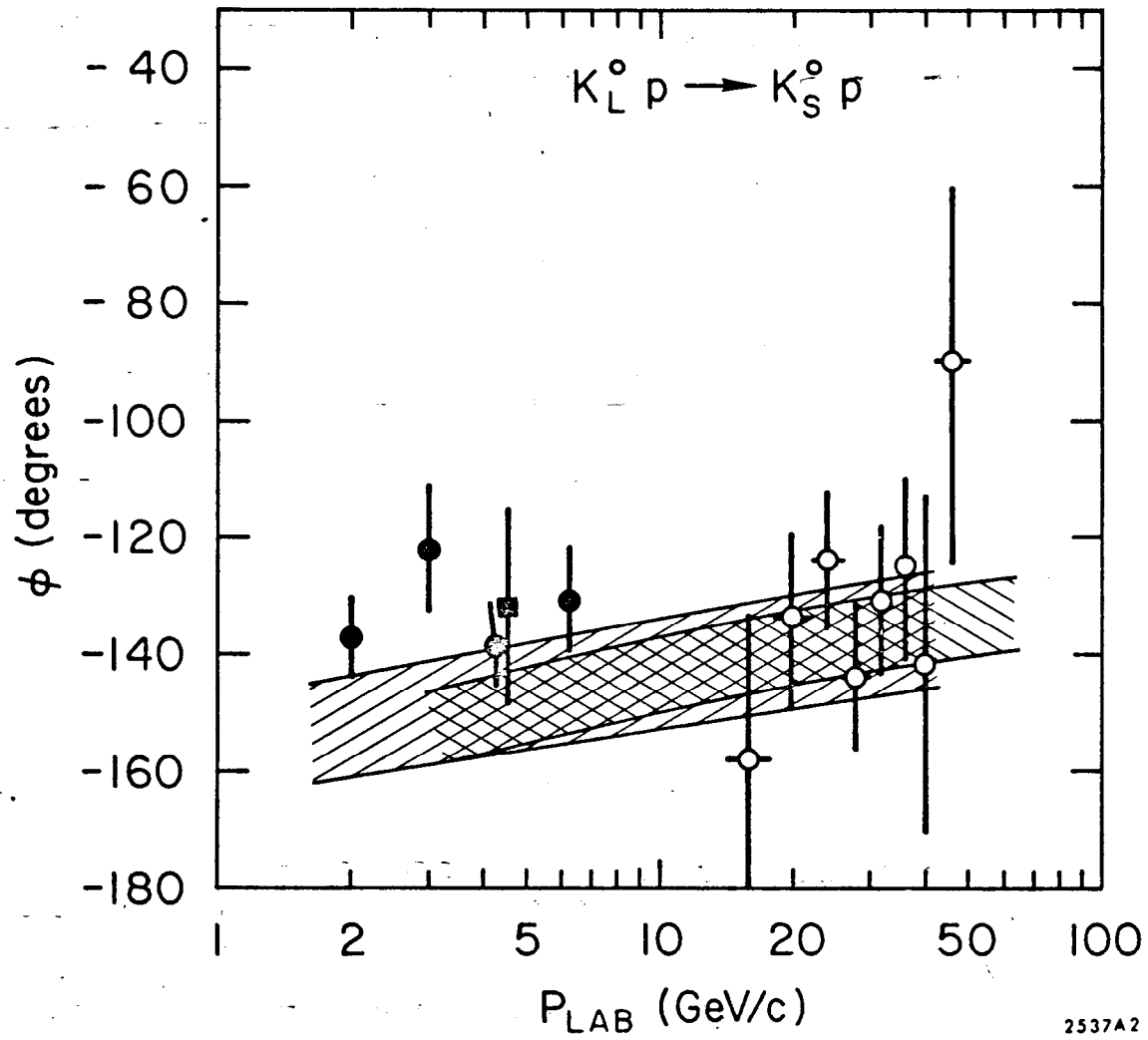
$K_L^0 p \rightarrow K_S^0 p$
- 40
- 60
- 80
-100
-120
-140
-160
-180
ϕ (degrees)
1
2
5
10
20
50
100
P_{LAB} (GeV/c)
2537A2

Fig. 3

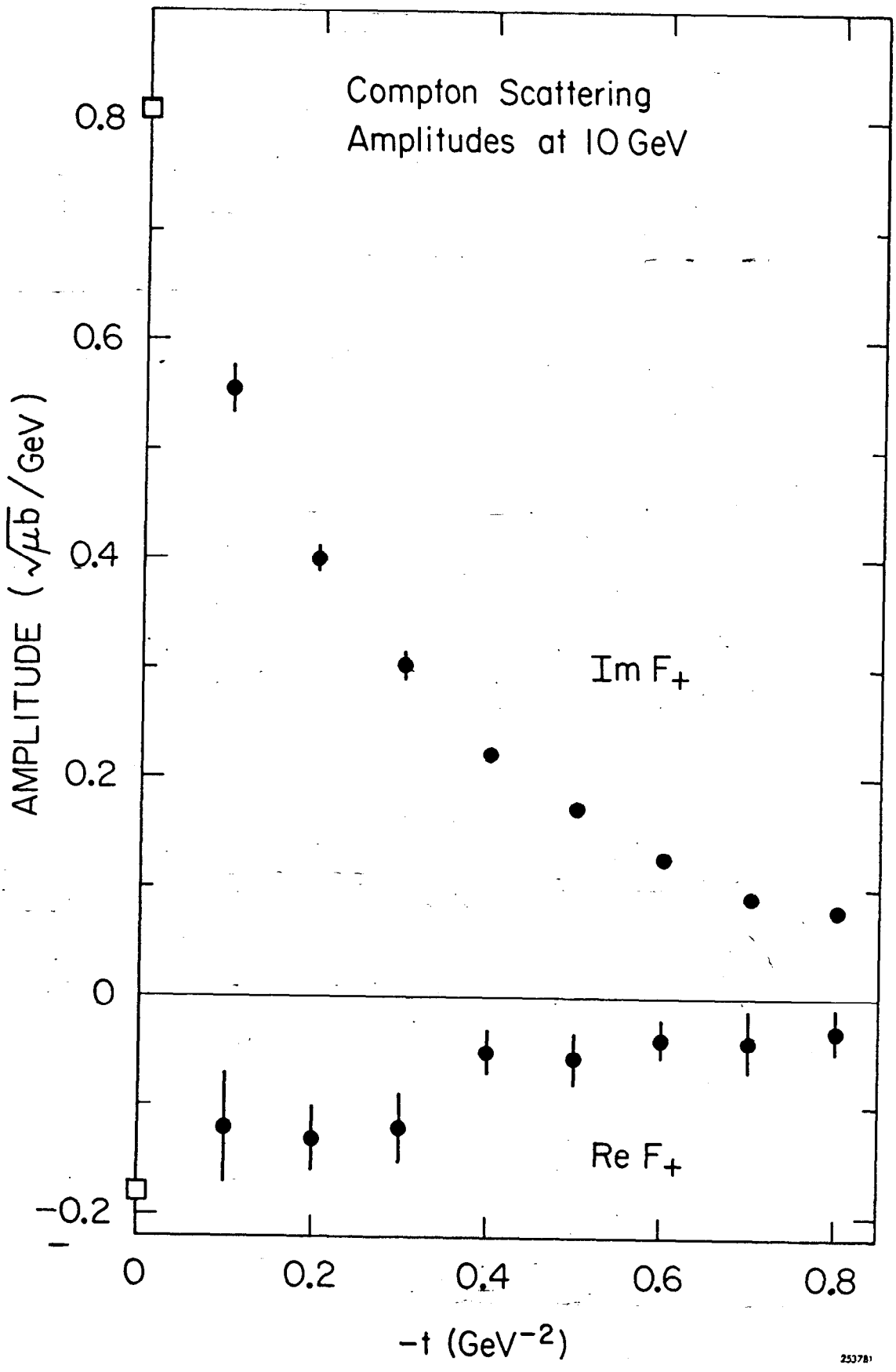
Compton Scattering
Amplitudes at 10 GeV
AMPLITUDE ($\sqrt{\mu b}$/GeV)
Im F_+
Re F_+
0.8
0.6
0.4
0.2
0
−0.2
0
0.2
0.4
0.6
0.8
−t (GeV^{-2})
2537B1

Fig. 4

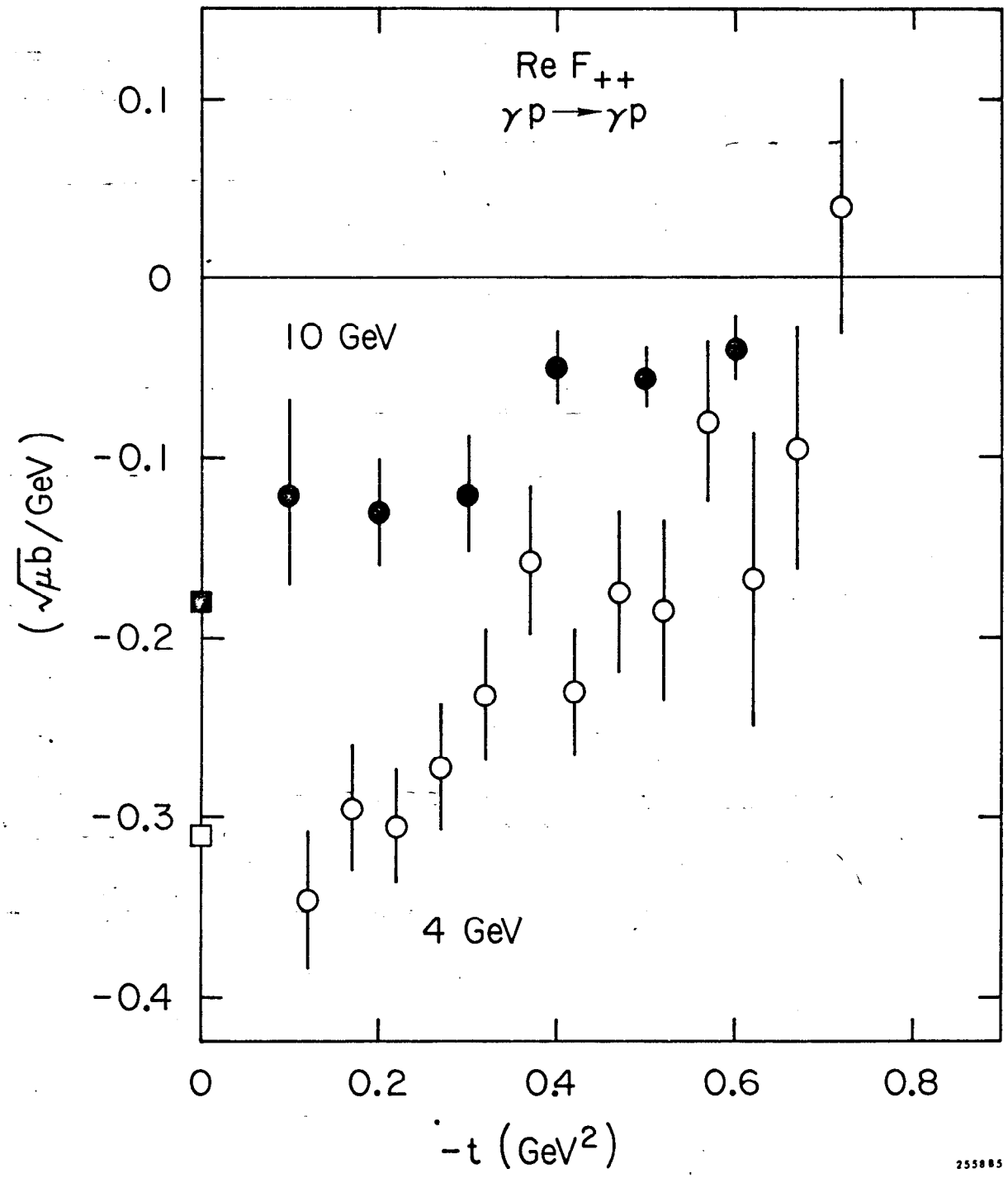
Re F_{++}
$\gamma p \rightarrow \gamma p$
0.1
0
-0.1
-0.2
-0.3
-0.4
10 GeV
4 GeV
$(\sqrt{\mu b}/GeV)$
0
0.2
0.4
0.6
0.8
$-t\ (GeV^2)$
255885

Fig. 5

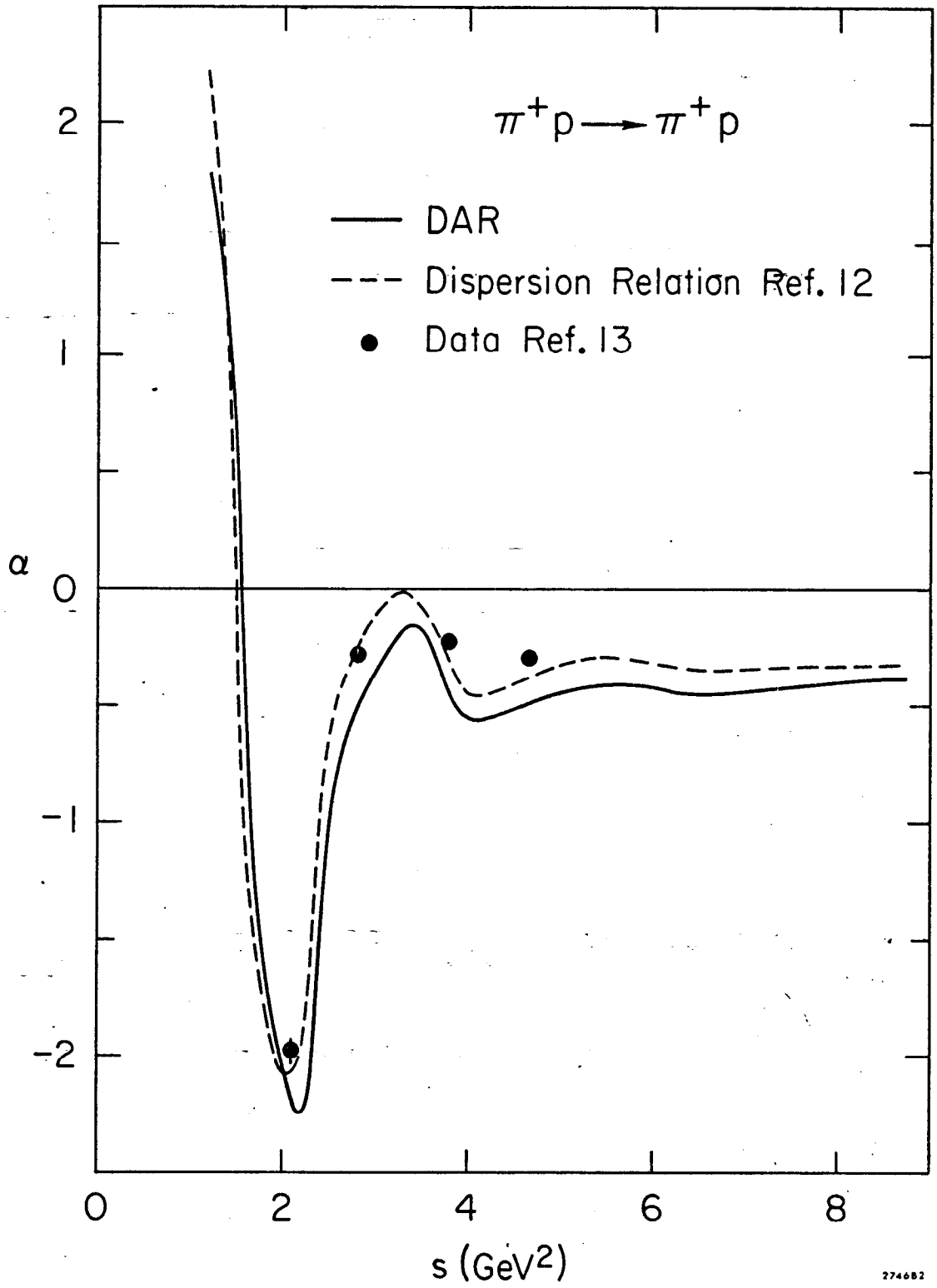
$\pi^+ p \longrightarrow \pi^+ p$
DAR
Dispersion Relation Ref. 12
Data Ref. 13
α
2
1
0
-1
-2
0
2
4
6
8
s (GeV²)
274682

Fig. 6

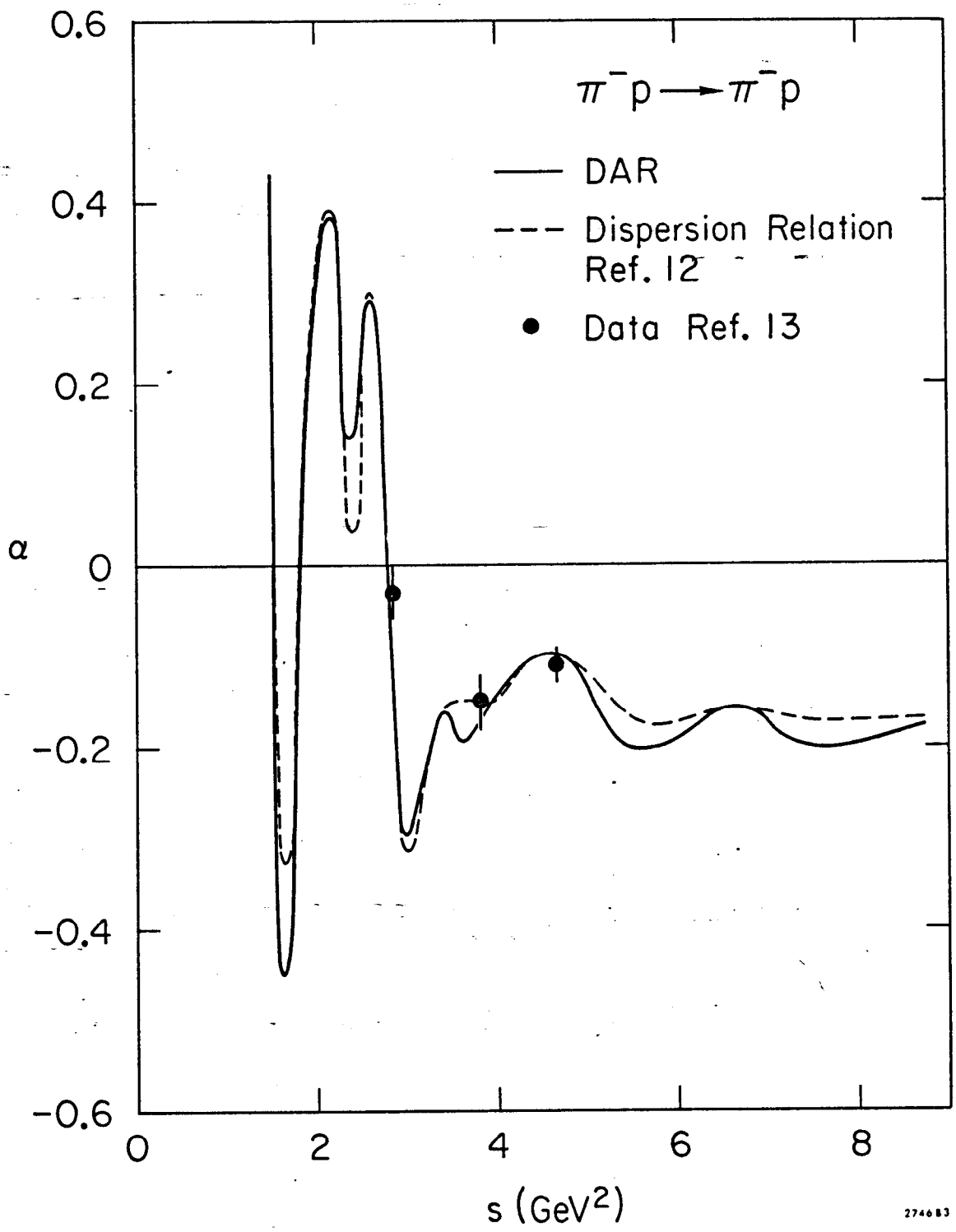
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
α
0
2
4
6
8
s (GeV²)
π⁻p → π⁻p
DAR
Dispersion Relation
Ref. 12
Data Ref. 13
274683

Fig. 7

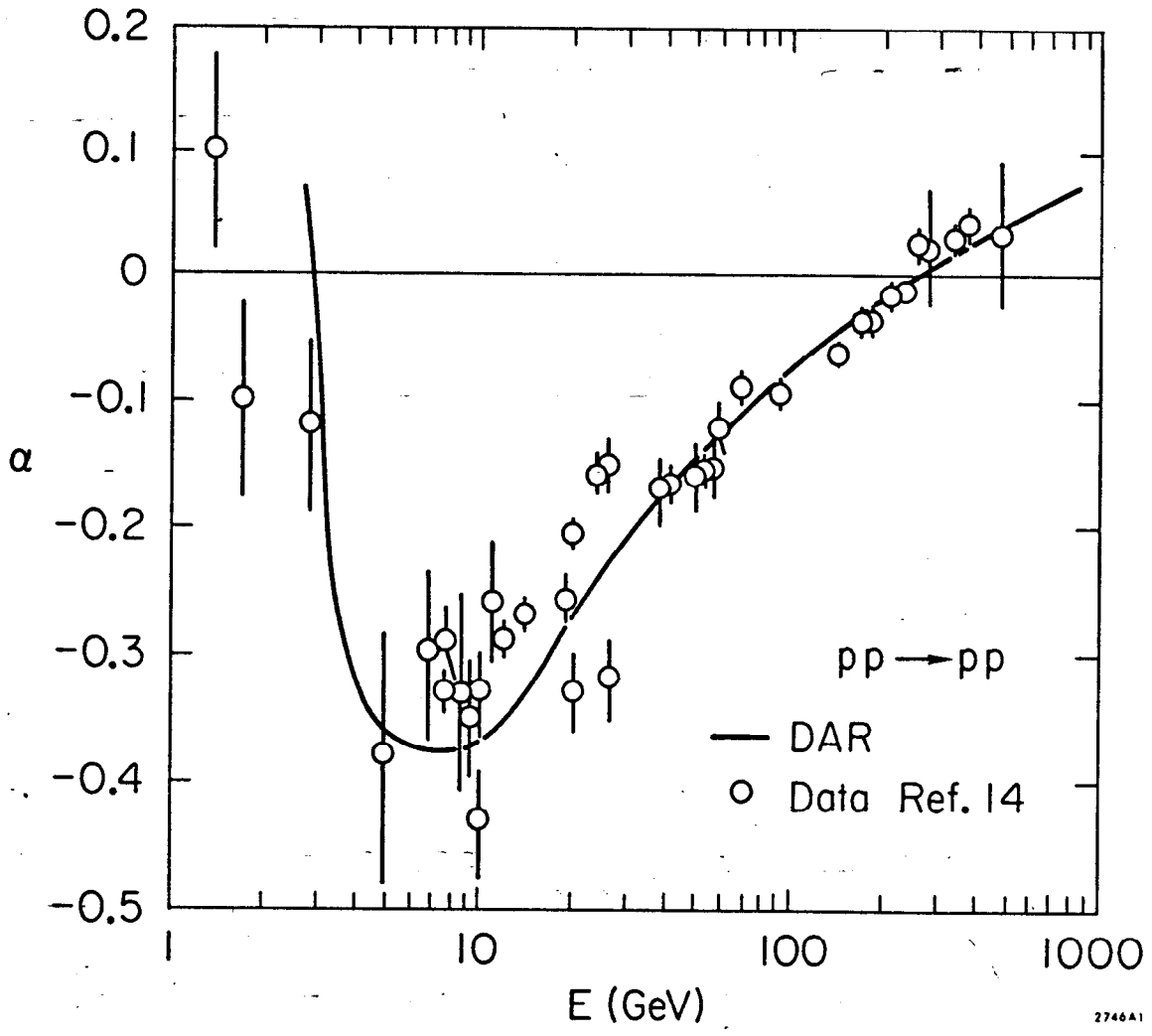
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
α
1
10
100
1000
E (GeV)
pp → pp
DAR
Data Ref. 14
2746A1

Fig. 8