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REPLY TO THE COMMENTS BY NASH ABOUT THE VIOLATION OF DIMENSIONAL ANALYSIS*

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ABSTRACT

We reply to the comments by Nash about our approach to the partial differential equations of renormalizable quantum field theory. By a simple example, the lack of content of his remarks is made more manifest.

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In his paper[1] on our approach[2] to the partial differential equations of quantum field theory, Nash has completely misrepresented the violation of dimensional analysis. For, in Eq. (8) of Ref. 2, the physical mass parameter m^2 in the argument of the θ-function in that equation is most certainly measured in terms of the intrinsic scale μ. (If this were not so, the operator $\lambda \frac{\partial}{\partial\lambda} + \mu \frac{\partial}{\partial\mu}$ would not be a dimensional analysis operator, as it would not contain all of the fundamental scales in the respective Green's functions.) Therefore, in this Eq. (8) of Ref. 2, for example, the argument of the θ-function, $\lambda^2(\Sigma p_j)^2 - m^2$, satisfies

$$(\lambda \frac{\partial}{\partial\lambda} + \mu \frac{\partial}{\partial\mu})(\lambda^2(\Sigma p_j)^2 - m^2) = 2(\lambda^2(\Sigma p_j)^2 - m^2) \tag{1}$$

since m^2 must be quadratic in μ. In the more general case where there are several intrinsic mass parameters $\{\mu_j\}$, the arguments of the respective θ-functions would satisfy the analogous relation

$$(\lambda \frac{\partial}{\partial\lambda} + \sum_j \mu_j \frac{\partial}{\partial\mu_j})(\lambda^2(\Sigma p_j)^2 - m^2) = 2(\lambda^2(\Sigma p_j)^2 - m^2) \quad , \tag{2}$$

since, again, m^2 must be a quadratic function of $\{\mu_j\}$. Thus, the term $R\Gamma^{(n)}$ in Eq. (11) of Ref. 2 arises because the Green's functions can be singular at threshold so that (referring to Eqs. (7), (8), and (11) of this reference (2))

$$\begin{aligned}(\mu \frac{\partial}{\partial\mu} + \lambda \frac{\partial}{\partial\lambda})\rho(\lambda\rho_j/\mu)\theta(\lambda^2(\Sigma p_j)^2 - m^2) &= \rho(\lambda p_j/\mu)\delta(\lambda^2(\Sigma p_j)^2 - m^2)(\mu \frac{\partial}{\partial\mu} + \lambda \frac{\partial}{\partial\lambda}) \\ &\qquad (\lambda^2(\Sigma p_j)^2 - m^2) \\ &= 2\delta(\lambda^2(\Sigma p_j)^2 - m^2)(\lambda^2(\Sigma p_j)^2 - m^2)\rho(\lambda p_j/\mu) \\ &\equiv 2\bar{\rho}\,\mu^2\delta(\lambda^2(\Sigma p_j)^2 - m^2) \quad ,\end{aligned} \tag{3}$$

where

$$\bar{\rho}\mu^2 \equiv \lim_{(\lambda^2(\Sigma p_j)^2 - m^2) \to 0} (\lambda^2(\Sigma p_j)^2 - m^2)\rho(\lambda p_j/\mu) \tag{4}$$

We have shown in footnote 6 of our Ref. 2 that in general this last limit is not zero to finite orders in perturbation theory. (We shall return to this point below.) Nowhere in Ref. 2 is it intended that a violation of dimensional analysis occurs simply because the respective physical mass thresholds m^2 are not functions of λ and the intrinsic scales $\{\mu_j\}$. Nash's Eqs. (12) and (18), the basis of his remarks, are therefore completely incorrect descriptions of the terms $R\Gamma^{(n)}$ which we introduced in Ref. 2. These remarks by Nash are therefore without a foundation.

As we stated above, we have established the nontriviality of the limit (4) in finite orders of perturbation theory in footnote 6 of Ref. 2 in the case of quantum electrodynamics. In a recent work,[3] we have explicitly discussed this limit in perturbation theory in general field theories as well as in non-perturbative situations. Again, the limit can easily be verified to be meaningful by explicit calculation in the general calculable case. For a complete discussion, we refer the reader to Ref. 3. Here, in order to illustrate why it is sufficient for our purposes to establish the nontriviality of (4) in calculable situations, let us consider a rather simple example.

Namely, consider the theory of the massless scalar field with the quartic self-coupling $-g\phi^4/4!$. It has been shown by Callan[4] and Symanzik[5] that the 1PI Green's functions $\{\Gamma^{(n)}\}$ of this theory satisfy

$$(\mu \frac{\partial}{\partial\mu} + \beta \frac{\partial}{\partial g} - n\gamma)\Gamma^{(n)}(\lambda p_j, g, \mu) = 0 \tag{5}$$

to each order in renormalized perturbation theory. Here, μ is the normalization point, and β and γ have their usual meanings.[4,5] In particular, the six-point 1PI function must satisfy this Eq. (5) to each order in perturbation theory.

Let's work to order g^3. Then, to this order,

$$\mu \frac{\partial}{\partial \mu} \Gamma^{(6)} = 0 + 0(g^4) \tag{6a}$$

$$\beta \frac{\partial}{\partial g} \Gamma^{(6)} = 0 + 0(g^4) \tag{6b}$$

and

$$\gamma \Gamma^{(6)} = 0 + 0(g^5) \tag{6c}$$

so that indeed (5) is true to this order, as it should be.

On the other hand, in applications, it is desired to solve (5) for $\Gamma^{(6)}$ as a function of the external scale λ. It has been customary, therefore, to convert the operator in (5) to an operator involving this scale λ by using dimensional analysis. Since the function $\Gamma^{(6)}$ has engineering dimension $4-6 = -2$, it has become a common practice to attempt to apply Euler's theorem to conclude

$$(\mu \frac{\partial}{\partial \mu} + \lambda \frac{\partial}{\partial \lambda})\Gamma^{(6)} \stackrel{?}{=} -2\Gamma^{(6)} \tag{7}$$

so that (5) would become the analogue of Nash's Eq. (4):

$$(-\lambda \frac{\partial}{\partial \lambda} + \beta \frac{\partial}{\partial g} - 6\gamma - 2)\Gamma^{(6)}(\lambda p_j, g, \mu) \stackrel{?}{=} 0 \quad . \tag{8}$$

Eq. (8) would then be the desired equation for the dependence on the external scale λ.

Our central observation in Ref. 2 is that because the limit (4) above is nonzero in general, <u>Eq. (7) is not true so that Eq. (8) must be modified.</u> Indeed, since we are working to order g^3, let's now check to see if the limit (4) is nonzero to this order. The relevant contribution to $\Gamma^{(6)}$ is the one-loop graph shown in Fig. 1. It's sufficient to study $\mathrm{Im}\Gamma^{(6)}$, since the operators in (7) and (8) are real. By explicit calculation[6] we find, for example, if $\lambda^2 r^2 = \lambda^2 t^2$,

$$\mathrm{Disc}\ \Gamma^{(6)} = \frac{-g^3 \theta(\lambda^2 s^2)}{8\pi\lambda^2 s^2 \sqrt{1-4r^2/s^2}} \log\left|\frac{\sqrt{s^2(s^2-4r^2)} - s^2 + 2r^2}{\sqrt{s^2(s^2-4r^2)} + s^2 - 2r^2}\right| + 0(g^4) \tag{9}$$

Thus, to order g^3, the limit in (4) is

$$\bar{\rho}\mu^2 = -\frac{\mu^2 g^3}{16\pi\sqrt{1-4r^2/s^2}} \log\left|\left(\sqrt{s^2(s^2-4r^2)}-s^2+2r^2\right)/\left(\sqrt{s^2(s^2-4r^2)}+s^2-2r^2\right)\right| \quad (10a)$$

and

$$\left(\mu\frac{\partial}{\partial\mu}+\lambda\frac{\partial}{\partial\lambda}\right)\mathrm{Im}\,\Gamma^{(6)} = -2\,\mathrm{Im}\,\Gamma^{(6)} - \frac{g^3\delta(\lambda^2 s^2)}{8\pi\sqrt{1-4r^2/s^2}}$$

$$\log\left|\left(\sqrt{s^2(s^2-4r^2)}-s^2+2r^2\right)/\left(\sqrt{s^2(s^2-4r^2)}+s^2-2r^2\right)\right| \quad (10b)$$

so that Eq. (7) above is not true to finite orders in perturbation theory. In this simple example, our violation $R\Gamma^{(n)}$ of dimensional analysis is

$$\mathrm{Im}R\Gamma^{(6)} = \frac{g^3\delta(\lambda^2 s^2)}{8\pi\sqrt{1-4r^2/s^2}}\log\left|\frac{\sqrt{s^2(s^2-4r^2)}-s^2+2r^2}{\sqrt{s^2(s^2-4r^2)}+s^2-2r^2}\right| , \quad (11)$$

in agreement with Eq. (13) of Ref. 2. Combining (11) with (5) yields

$$\left(-\lambda\frac{\partial}{\partial\lambda}+\beta\frac{\partial}{\partial g}-6\gamma-2\right)\mathrm{Im}\Gamma^{(6)} = \frac{g^3\delta(\lambda^2 s^2)}{8\pi\sqrt{1-4r^2/s^2}}\log\left|\frac{\sqrt{s^2(s^2-4r^2)}-s^2+2r^2}{\sqrt{s^2(s^2-4r^2)}+s^2-2r^2}\right| , \quad (12)$$

which is now the correct equation to order g^3.

Our new dimensional analysis violating term $R\Gamma^{(n)}$ (see Eq. (11) of Ref. 2) has therefore been shown to be present to finite orders in perturbation theory by explicit calculation in the simple example treated here. To repeat, the general calculable case is discussed in Refs. 2 and 3.

To each order in coupling we have correct equations of the type (12) <u>provided we include our violation term</u>. These can then be solved with confidence that the error made is of higher order than the last term kept and, for asymptotic regions, less significant since, in general, higher orders are more singular at threshold and therefore less significant in our[3] particular integral for equations of the type (12). This is completely analogous to computing β and γ only to finite order. Nash's bold speculation about the absence of very

singular thresholds when perturbation theory is summed through all orders is thus completely irrelevant. We shall not be concerned further with these remarks by Nash.

REFERENCES

1. C. Nash, preceding paper.
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3. B. F. L. Ward, "Differential Dispersive Approach to Large Momentum Transfer Processes: I," SLAC-PUB-1565 (March 1975).
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5. K. Symanzik, Commun. Math. Phys. 18, 227 (1970).
6. We suppress permutations of external momenta.

FIGURE CAPTION

1. Order g^3 contribution to $\Gamma^{(6)}$ in scalar field theory.

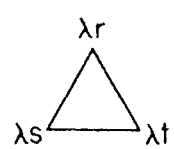

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Fig. 1