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COMMENT ON CHARMONIUM BOUND STATE CALCULATIONS USING POTENTIALS*

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ABSTRACT

We show that using the Dirac equation to calculate quark bound states can give rise to significantly different eigenspectra than those obtained by using the same potential in the Schrœdinger equation, even when $\langle v^2/c^2 \rangle$ is small. The origin of this effect is identified and discussed.

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The discovery[1] of the new particles $\psi(3.1)$, $\psi'(3.7)$ and possible $\psi''(4.15)$ has brought renewed vigor to attempts to understand hadron spectroscopy by means of effective quark- (anti-) quark interaction potentials.[2,3] As suggested by Appelquist and Politzer,[2] the structure of such a potential may be obtained from a field theoretic starting point by examining the properties of the β-function In particular, the conjecture of "infra-red slavery" leads to a potential that confines quarks. Since quarks are fermions, one might naturally expect to use this potential in Dirac equations for the quarks. Due to well known difficulties,[4] it is usually preferable to go to the nonrelativistic limit where one obtains a reduced mass Schroedinger equation for the two-body system with the same potential. This approach is usually justified a posteriori by showing that $\langle v^2/c^2 \rangle \ll 1$. However, in the case of quark-confining potentials, and for states far from the two-constituent threshold, we shall show below that satisfying this condition is not sufficient to guarantee that the eigenvalues so obtained are accurate up to corrections of order $\langle v^2/c^2 \rangle$. Using previously suggested potentials[3] in the Dirac equation, we find significantly different ($\sim 10\%$) eigenspectra which do not match the mass splittings of the observed states. Differences from experimentally observed mass splittings are $\sim 40\%$ (smaller), versus the few percent accuracy suggested by the Schrœdinger calculations.

Conceptually, the problem of describing mesons as quark-antiquark bound states starts with a set of coupled Dirac equations for the interacting quark operator fields. Then, as an approximation, these are simplified by taking the one-quark (antiquark) matrix elements and by replacing all source terms with an effective, static potential. (This is the potential inferred from the β-function.) This procedure gives the Dirac wave function equation for a particle in a static potential. The usual next approximation involves replacing the Dirac equation

by a reduced mass Schrœdinger equation. However, by using the Dirac equation itself, one takes naturally into account some of the corrections to the simple nonrelativistic Schrœdinger equation; for example, spin-orbit couplings, which have been added by hand in some of the Schrœdinger calculations.[5]

Following the example of positronium, we have solved numerically a reduced mass Dirac equation, using some of the potentials current in the literature.[3] (If we do the calculation in the spirit of the shell model, i.e., separately introducing the (full mass) quark and antiquark into a fixed potential, very similar results are obtained.[4]) The results are given in Table I, where the mass of the n^{th} radially excited state is given by

$$m_n = 2m_c + \left(E_n - \frac{m_c}{2}\right) \tag{1}$$

where m_c is the (charm) quark mass, and E_n is the n^{th} radial excitation eigenvalue obtained by solving the Dirac equation with a reduced mass $m_c/2$.

The values in Table I differ from the results obtained by previous calculations[3] using the same potentials in the Schroedinger equation (as shown in Table II) by more than the relativistically induced spin orbit coupling corrections estimated[6] by $\langle v^2/c^2 \rangle \sim 1\text{-}4\%$. Especially important are the much reduced (~40%) mass splittings, which are considerably smaller than those observed for $\psi'\text{-}\psi$ and $\psi''\text{-}\psi'$. We shall now show that these discrepancies arise from a combination of the large values that a confining potential must attain in order to confine the quarks and the technique of nonrelativistic reduction.

For a central, scalar potential, V(r), the Dirac equation can be rewritten as a second order differential equation for the "large" component (g(r)) of the

fermion radial wave function[7]

$$\left[\frac{1}{2m}\left(\frac{-1}{r}\frac{d^2}{dr^2}r + \frac{\kappa(\kappa+1)}{r^2}\right) - \frac{E^2-(m+V)^2}{2m}\right] g(r) = -\frac{dV}{dr}\frac{\left(\frac{dg}{dr} - \frac{\kappa}{r}g\right)}{E+m+V} \quad . \tag{2}$$

The spinor wave function is defined as

$$\psi(r,\theta,\phi) = \begin{pmatrix} g(r)\ Y^{j_3}_{j\ell_A}(\theta,\phi) \\ \\ if(r)\ Y^{j_3}_{j\phi_B}(\theta,\phi) \end{pmatrix} \tag{3}$$

with $\ell_A = |\kappa| - \theta(-\kappa) = j \pm 1/2$, $\ell_B = |\kappa| - \theta(\kappa)$, $|\kappa| = j+1/2$. The "small" component, f(r), is determined from g(r) by

$$f(r) = \frac{1}{E+m+V}\left[\frac{dg}{dr} + \frac{1+\kappa}{r}g\right] \tag{4}$$

Note that $\kappa(\kappa+1) = \ell_A(\ell_A+1)$, and that $m \rightarrow m/2$ for the reduced mass case.

By comparison, the Schrœdinger equation for the radial wave function R(r) is

$$\left[\frac{1}{2m}\left(-\frac{1}{r}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2}\right) - (E_{sch} - V_{sch})\right] R(r) = 0 \quad . \tag{5}$$

(Again, $m \rightarrow m/2$ for the reduced mass case.) Equations (2) and (5) do yield very similar eigenspectra if:

a) $\langle v^2/c^2 \rangle \ll 1$ so that the r.h.s. of Eq. (2) is small ($dV/(mdr) \ll V$), and

b) the following equation is satisfied, where $E_{sch} \equiv E-m$:

$$E^2 - (m+V)^2 = 2m(E_{sch} - V_{sch}) \quad . \tag{6}$$

If

$$E_{sch} \ll m \qquad \text{and} \qquad |V| \ll m \tag{7}$$

over a sufficiently large range of r where the wave function is appreciable, then Eqs. (6) and (7) imply

$$V \approx V_{sch} \tag{8}$$

over that range of r, and Eqs. (2) and (5) are indeed equivalent to a good approximation.

This is clearly the case in the picture suggested by Appelquist and Politzer[2] for the lowest state ψ, where $E_{sch} = E - m_{reduced} \sim .1$ GeV, $m_{reduced} \sim .8$ GeV and $|V| \lesssim .1$ GeV in the region of the (Bohr) radius of the so-called ground state of charmonium.[8] However, when

$$E - m \gtrsim m \qquad \text{and} \qquad |V| \gtrsim m \tag{9}$$

for an appreciable interval of r where the wave function is appreciable, then it is no longer clear that the eigenspectra of Eqs. (2) and (5) should be very similar. Furthermore, the conditions in Eq. (9) are necessarily satisfied when the potential acts to confine quarks. The reasoning is as follows: the confinement is achieved by exponential damping of the quark wave function outside the classical radius of the state (turning point). Aside from possible nodes, there is no reason for the wave function to become very small before this point, and in fact, it is typically not small. As $V \sim E - m$ near a turning point, and $E - m \gtrsim m$ holds for excited states far from the two quark threshold (as is the case here), it follows that Eqs. (9) do apply over a finite range of r where the wave function is nonnegligible.

To demonstrate that this is indeed the origin of the differences between the results given in Tables I and II,[9] we have solved Eq. (6) for V using each V_{sch} in turn to obtain

$$V(r, E) = \sqrt{(E-m)^2 + m^2 + 2m V_{sch}} \quad . \tag{10}$$

We then solved the Dirac equation as before, using this energy dependent potential, and obtained the results listed in Table III. (Note that the results are not exactly comparable to Table II as they are calculated in a specific spin state (j=1) which the Schrœdinger equation can not distinguish from other spin states.)

The appearance of an energy-dependent effective potential may not be so unreasonable as it first seems, as the effective coupling constant in the underlying field theory depends on energy as well as on three-momenta:

$$\frac{d\bar{g}}{dt} = \beta(\bar{g}(t)) \quad , \qquad t = \ell n(s/s_0) \quad . \tag{11}$$

If we denote the three-dimensional Fourier transform of $\bar{g}(t)$ by $g(r, E)$, a conceivable approximate expression for the "effective potential" is

$$V = \frac{g^2(r, E)}{r} \quad . \tag{12}$$

By assuming a specific form for the function β (in Eq. (11)) which is in agreement with asymptotic freedom, one may obtain a potential which behaves as $1/r\,\ell n(r)$ at small distances and as r^p $(p>0)$ for large quark-quark (antiquark) separations.

However, in order to use this potential directly in the Schrœdinger equation one must now not only conjecture that Eq. (12) occurs, but also that this gives rise to the specific energy dependence of the potential in Eq. (10). We are not trying here to cast doubt on the possibility that confinement potentials may be generated by the so-called "infra-red slavery" mechanism. Rather, we are suggesting that such potentials, guessed from the β-function, should be applied, if at all,[11] in the Dirac equation, resulting usually in more complicated and energy dependent potentials in the reduced mass Schrœdinger equation version of these problems.[12]

Note that this procedure does not have too large an effect on the absolute scale of masses for the resulting bound states,[13] but does have a very large effect on the mass differences between various excitations. The latter is a most striking feature of the difference between using the usual (guessed) potentials in Eq. (2) or in Eq. (5).

These smaller mass differences are also more in agreement with some calculations[14] we have done using the MIT bag model[10]: we were unable to find a consistent and reasonable set of parameters which gave both an overall mass scale of ~ 3 GeV and mass differences of $\gtrsim .4$ GeV between excited states. Along this line, we also note that Vinciarelli[15] has claimed, within the framework of field theory bag models, that the bound state mass differences are $\approx .2$ GeV, for quarks of mass $\sim 1 - 2$ GeV.

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FOOTNOTES AND REFERENCES

1. J.-E. Augustin et al., Phys. Rev. Letters 33, 1406 (1974);
J. J. Aubert et al., Phys. Rev. Letters 33, 1404 (1974);
G. S. Abrams et al., Phys. Rev. Letters 33, 1453 (1974).
2. T. Appelquist and H. D. Politzer, Phys. Rev. Letters 34, 43 (1975).
3. B. J. Harrington et al., Phys. Rev. Letters 34, 168 (1975);
E. Eichten et al., Phys. Rev. Letters 34, 369 (1975).
4. Spurious states appear corresponding to center-of-mass motion of the bound particles.
5. For example, see J. F. Gunion and R. S. Willey, "Hadronic Spectroscopy for a Linear Quark Containment Potential," PRINT-75-0114, U. of Pittsburgh preprint (Feb. 1975).
6. See especially the first paper in Ref. 3.
7. We use the notation of Advanced Quantum Mechanics, by J. J. Sakurai (Addison-Wesley, Reading, Mass., 1967); pp. 122-125.
8. While it is true that $|V| \to \infty$ as $r \to 0$ in the picture of Ref. 2, the volume in which this occurs is small—of order r^3, so that there is little effect on the eigenvalue near the two particle threshold. However, if the binding were stronger, so as to concentrate most of the wave function at small r (e.g., for a deeply bound ground state), then this analysis suggests that solving Eq. (5) might give a poor estimate of the true eigenvalue.
9. One must always beware of the possibility that such differences arise from the numerical techniques used. In addition to the check discussed in the text, we have tested our computer program by correctly calculating several eigenstates for each of the following: hydrogen atom, ortho- and para-positronium, and the MIT bag model (see Ref. 10) treated as a fixed radius,

square well potential. Unfortunately, we could not compare our results in detail with the analytic results of C. L. Critchfield, "Harmonic Potentials in Dirac's Equations," LA-UR-74-1813, Los Alamos Scientific Laboratory preprint (Oct. 1974) due to the choice of quantum numbers and range of parameters. However, a good qualitative agreement is apparent.

10. A. Chodos et al., Phys. Rev. D 8, 2599 (1974).

11. One would be on firmer theoretical ground if one searches for these bound states by using the Bethe-Salpeter equation, with a scattering kernel inferred from the properties of the β-function directly in momentum space. This point has been stressed to one of us by T. Appelquist (private communication).

12. Of course, one could always give up trying to link these potentials to an underlying field theory and revert to the earlier approach, namely to use some guessed potential as an ansatz in the Schrœdinger equation.

13. A difference in overall mass scale should not be taken too seriously, as it depends strongly on m_c, whereas the differences of energy eigenvalues vary slowly in the neighborhood of a given value of m_c. In fact, it can be argued that the quark mass term ($2m_c$) in Eq. (2) should really be an additional free parameter. This is because there are no asymptotic quark states in the underlying field theory (by construction or assumption) and so there is no longer a well-defined connection between the measure of the kinetic energy of a quark in the potential and the "free quark mass" (position of the pole in the quark propagator). See the second paper in Ref. 3.

14. These calculations were done with S.-H. H. Tye.

15. P. Vinciarelli, "Radial Recurrences in Field Theoretical Bag Models (and the Spectrum of Charmonium)," Ref. TH. 1981-CERN preprint (Jan. 1975).

TABLE I

Orthocharmonium Radial Eigenspectra Obtained by Numerical Calculation Using the Reduced Mass Dirac Equation

j=1	E(GeV)	m(GeV)
(a) Potential of Harrington et al. (Ref. 3), $m_c = 1.16$ GeV, $V = V_a = (.211r - .120)$ GeV, $[r] = GeV^{-1}$		
n=1	1.21 ± .01	2.95 ± .02
2	1.59 ± .03	3.33 ± .04
3	1.91 ± .06	3.65 ± .07
(b) Potential of Eichten et al. (Ref. 3), $m_c = 1.6$ GeV (kinetic only),[a] $V = V_b = (-.200r^{-1} + .194r)$ GeV, $[r] = GeV^{-1}$		
n=1	1.39 ± .01	3.10 ± .02 [a]
2	1.77 ± .01	3.48 ± .02
3	2.04 ± .01	3.75 ± .02

[a] $2m_c = 1.71$ GeV is used as a free parameter to fit n=1 state.

TABLE II

Orthocharmonium Radial Eigenspectra Obtained by Others
Using the Reduced Mass Schrœdinger Equation (Ref. 3)

j=undetermined	$E = \frac{m_c}{2} + E_{sch}$(GeV)	m(GeV)
(a) Harrington et al.		
n=1	1.36 ± .01	3.105 ± .005 [a]
2	1.95 ± .01	3.695 ± .005 [a]
3	2.44 ± .01	4.18 ± .01
(b) Eichten et al.		
n=1	1.40 ± .01 [b]	3.105 ± .005 [a]
2	1.99 ± .01 [b]	3.695 ± .005 [a]
3	2.38 ± .01 [b]	4.18 ± .01

[a] Input

[b] E. Eichten, private communication. The eigenvalues (E_{sch}) themselves, did not appear in the second paper in Ref. 3.

TABLE III

Orthocharmonium Radial Eigenspectra Obtained by Numerical Calculation Using the Reduced Mass Dirac Equation with Energy Dependent Potentials (see Eq. (10))

j=1	E(GeV)	m(GeV)
(a) $V_{sch} = V_a$ (see Table I)		
n=1	1.38 ± .01	3.12 ± .02
2	1.94 ± .06	3.68 ± .07
3	2.50 ± .05	4.24 ± .06
(b) $V_{sch} = V_b$ (see Table I)		
n=1	1.51 ± .01	3.095 ± .005 [a]
2	2.01 ± .02	3.60 ± .03
3	2.42 ± .01	3.93 ± .02

[a] Input to determine $2m_c$ in Eq. (1) treated as a free parameter.