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STRONGLY COUPLED FIELDS: I. GREEN'S FUNCTIONS*

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ABSTRACT

We formulate a general theory of the strong coupling limit of renormalizable interacting quantum fields. For definiteness, our ideas are explicitly illustrated in the case of the scalar field with quartic self-coupling, the usual testing ground for new ideas in field theory. More precisely, the generating functional for connected Green's functions is explicitly constructed for this latter theory in the limit of large coupling. The problem of regularization is treated in detail. For purposes of illustration, the first two orders in the large coupling limit of the Fourier transform of the connected two point function are computed. It is found that the Fourier transform for 4-momentum p approaches its large-(p^2) limit essentially exponentially. The applicability of our approach to all renormalizable theories is thereby made more manifest.

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I. INTRODUCTION

This is the first of what will presumably be a series of reports on the behavior of renormalizable quantum field theory in the limit of large coupling. The motivation for these reports is the obvious, the empirical observation of many strongly coupled particles in nature.

In this first report, we shall give a general formulation of the Green's functions of renormalizable field theory in the limit of large coupling. The subsequent works[1] will deal with various interesting realistic applications and additional technical details.

We shall, however, illustrate our ideas here in the case of scalar field theory with quartic self-coupling, the simplest of renormalizable situations. This we do in the interest of completeness and clarity. Indeed, this theory would appear to embody already all of the additional complexities of large coupling in comparison with weak coupling. Thus, we shall be able to illustrate all of the necessary machinery for handling these complexities.

This first report is intended to be pedagogic. It is organized as follows: In Section II, we give our general theory of large coupling. In Section III, we illustrate our ideas in the case of scalar field theory. Appendix II contains a comparison of the regularization procedure used in Section III with the more conventional regularizations. And Section IV contains some concluding remarks. (The remaining appendices contain relevant technical details.)

II. STRONG COUPLING THEORY

In this section we shall formulate the strong coupling limit of the Green's functions of renormalizable interacting field theory. We shall do this by the use of Feynman path integrals.[2] The analogous discussion in terms of the Tomonaga-Schwinger[3,4] approach is not attempted. The physical equivalence between these two approaches to field theory is certainly well accepted by now.

The path integral provides a very convenient representation of the connected Green's functions of a theory, for the generating functional Z of these functions is just

$$e^{iZ(\{J_i\})} = \int \mathscr{D}\{\phi_i\} \exp i\int d^4x \left[\mathscr{L}\{\phi_i\} + \sum_i J_i\phi_i\right] \tag{2.1}$$

where $\mathscr{L}\{\phi_i\}$ is the Lagrangian and $\{J_i\}$ are external sources. The Lagrangian $\mathscr{L}$ has the form

$$\mathscr{L} = \frac{1}{2}(\sum_i \partial_\mu\phi_i\partial^\mu\phi_i - \sum_{i,j} M_{ij}\phi_i\phi_j) - \sum_{i,j,k} g_{ijk}\phi_i\phi_j\phi_k - \sum_{i,j,k,\ell} g_{ijk\ell}\,\phi_i\phi_j\phi_k\phi_\ell + \ldots \tag{2.2}$$

where ... includes other possible (renormalizable) terms involving other kinds of fields such as those of spin 1/2, for example. The constants $\{g\}$ are here referred to as couplings. The matrix M_{ij} is usually regarded as the mass matrix.

As is well known, we have

$$i^n < 0|T^*(\phi_{i_1}(x_1)\ldots\phi_{i_n}(x_n))|0>_c = \frac{\delta^n iZ}{\delta J_{i_1}(x_1)\ldots\delta J_{i_n}(x_n)} \quad . \tag{2.3}$$

Thus it is sufficient for our purposes to learn the large coupling behavior of Z, for then from (2.3) we shall readily obtain the same limit of all connected Green's functions.

To determine the behavior of Z, it is instructive to recall that, for small couplings, Z and its functional derivatives have a Taylor series expansion

$$Z = \sum_{n=0}^{\infty} g^n G_n \quad . \tag{2.4}$$

The notation g^n is symbolic when there are more than one coupling. Thus, we ask, "If we know that

$$f(x) = \sum_{n \geq 0} a_n x^n \qquad \text{for } x \sim 0 \ , \tag{2.5}$$

what can be said about f(x) for $x \to \infty$?" One can clearly distinguish three cases:

I. f(x) has no limit as $x \to \infty$.

II. $|f(x)| \xrightarrow[x \to \infty]{} L < \infty$ but

$$f(x) \neq \sum_{n \geq 0} x^{-n} b_n \quad \text{as } x \to \infty \ .$$

III. $|f(x)| \xrightarrow[x \to \infty]{} L < \infty$ and

$$f(x) = \sum_{n \geq 0} x^{-n} b_n \quad \text{as} \quad x \to \infty \ .$$

An example of class I is e^x. Class II is exemplified by tanh x. And, as an example of class III we note $x/(1+x)$. The following claim is easily established.

Claim: The functional derivatives of Z belong to class III.

Proof: The proof is elementary. Let $\{f_i\}$ be the set of couplings which are large. Scale all fields by an appropriate factor so that only nonpositive powers of $\{f_i\}$ appear in the resulting action integral, I', in (2.1). The effect of such scalings on $\mathscr{D}\{\phi_i\}$ is to multiply it by an unknown J independent function of $\{f_i\}$. Such a multiplication clearly has no effect on the J-dependent part of the logarithm of the RHS of (2.1). Thus, the connected Green's functions may also be obtained as

$$\frac{\delta^n iZ}{\delta J_{i_1}(x_1)\ldots\delta J_{i_n}(x_n)} = \frac{\delta^n}{\delta J_{i_1}(x_1)\ldots\delta J_{i_n}(x_n)} \log \int \mathcal{D}\{\phi\} \exp i I' \quad . \tag{2.6}$$

But, since I' has only terms with coefficients which are either independent of coupling or small, it may clearly be expanded in an ordinary Taylor series in its small[5] coefficients. This gives an expansion of type III for the respective Green's functions, since these small coefficients are just negative powers of $\{f_i\}$. This completes the proof. Q.E.D.

Hence, the connected Green's functions of quantum field theory possess discussible limits as the coupling tends to ∞. It is not apparent that we are restricted to renormalizable interactions. However, only in this latter case do we expect to obtain a finite theory in the large coupling limit, for in this case the usual power counting assures that the only infinities which may occur in the evaluation of the Green's functions will at worst be interpretable as renormalizations to a finite number of parameters. However, we do not have a proof that we do not obtain a finite theory in the large coupling limit of unrenormalizable interactions. We shall now illustrate the ideas of this section in the case of the scalar field with quartic self-coupling.

III. SCALAR FIELD THEORY

We consider here the behavior of the Green's functions of scalar field theory in the limit of large coupling. We shall first construct explicitly the unrenormalized functions. After having done this, we shall then illustrate a regularization procedure for the respective functions. The ideas in the preceding section will thereby be explicitly illustrated.

A. Unrenormalized Green's Functions

The Lagrangian (plus source term) is

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \varphi_u \partial^\mu \varphi_u - m_b^2 \varphi_u^2) - g_b \varphi_u^4 + J\varphi_u \tag{3.1}$$

where J is at present an arbitrary function and the subscripts u and b indicate "unrenormalized" and "bare", respectively. The generating functional Z for connected functions is hence

$$e^{iZ(J)} = \int \mathscr{D}\varphi \exp i \int d^4x \left[\frac{1}{2}(\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2) - g\varphi^4 + J\varphi\right] \tag{3.2}$$

where we have dropped the subscripts u and b, for we shall discuss the question of renormalization in detail below. We wish to implement the results of the previous section. Some insight must be exercised in doing this, for otherwise one will produce the usual Taylor series in g, a series presumably only valid for $g \to 0$.

In order to proceed, we have found the following physical equivalences helpful (see Appendix I):

$$\exp -i\int d^4x\, b\varphi^{2n} \equiv \int \mathscr{D}\sigma \exp i\int d^4x \left[\sigma^2 + 2\sqrt{b}\, \varphi^n \sigma\right] \tag{3.3a}$$

$$\exp i\int d^4x\, F(\varphi) \equiv \int \mathscr{D}\rho \mathscr{D}\pi \exp i\int d^4x \left[F(\rho) + \pi(\rho - \varphi)\right] \quad , \tag{3.3b}$$

where F is essentially arbitrary. By "physical equivalence" we mean that the two sides of these equations are equal up to, perhaps, an unimportant constant factor. In view of these relations, we have the physical equivalence

$$e^{iZ} \equiv \int \mathscr{D}\kappa \mathscr{D}\pi \mathscr{D}\varphi \mathscr{D}\sigma \exp i\int d^4x \Big[\tfrac{1}{2}(\partial_\mu \kappa\, \partial^\mu \kappa - m^2 \kappa^2)$$

$$+ J\kappa + \pi(\kappa - \varphi) + \sigma^2 + 2\sqrt{g}\,\varphi^2\sigma\Big] \quad . \tag{3.4}$$

By Fubini's theorem these integrals may be done in any order in which they exist. Hence, doing the κ integral gives

$$e^{iZ} \equiv \int \mathscr{D}\pi \mathscr{D}\varphi \mathscr{D}\sigma \exp i\int d^4x \left[-\tfrac{1}{2}\int d^4y\,(J+\pi)(x)\Delta_F(x-y)(J+\pi)(y) - \pi\varphi + \sigma^2 + 2\sqrt{g}\,\varphi^2\sigma\right] \tag{3.5}$$

where Δ_F is Feynman's solution of

$$(\Box_x + m^2)\Delta_F = -\delta(x-y) \quad . \tag{3.6}$$

We next eliminate the π field. This will be done in two steps. First, we remove the explicit coupling of φ and π by the shift

$$\varphi \to \varphi + \frac{\pi}{4\sqrt{g}\,\sigma} \quad . \tag{3.7}$$

There results

$$e^{iZ} = \int \mathscr{D}\pi \mathscr{D}\varphi \mathscr{D}\sigma \exp i\int d^4x \left[-\tfrac{1}{2}\int d^4y\,(J+\pi)(x)\Delta_F(x-y)(J+\pi)(y) + \sigma^2 + 2\sqrt{g}\,\varphi^2\sigma - \frac{\pi^2}{8\sqrt{g}\,\sigma}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \prod_{j=1}^{n} \left[\int d^4x_j \frac{d\eta_j \eta_j^2}{8\sqrt{g}} \frac{d\rho_j}{2\pi} \frac{d\beta_j}{\beta_j} \frac{d\alpha_j}{2\pi}\right] \int \mathscr{D}\pi \mathscr{D}\varphi \mathscr{D}\sigma \exp\Big\{ i\sum_j (\alpha_j \beta_j$$

$$+ \eta_j \rho_j) - i\sum_j (\rho_j \pi(x_j) + \alpha_j \sigma(x_j)) + i\int d^4x \Big[-\tfrac{1}{2}\int d^4y\,(J+\pi)(x)\Delta_F(x-y)(J+\pi)(y)$$

$$+ \sigma^2 + 2\sqrt{g}\,\varphi^2\sigma\Big]\Big\} \quad . \tag{3.8}$$

In making this last step, we have used

$$\frac{\pi^2}{\sigma} = \int d\eta\, \eta^2 \frac{d\rho}{2\pi} \int \frac{d\beta}{\beta} \frac{d\alpha}{2\pi} \exp\left[i\alpha(\beta - \sigma) + i\rho(\eta - \pi)\right] \quad . \tag{3.9}$$

We next effect the π integral as follows: First, recall

$$\pi(x_j) = \int \frac{d^4k}{(2\pi)^4}(\pi_1(k)\cos k\cdot x_j + \pi_2(k)\sin k\cdot x_j) \tag{3.10a}$$

where

$$\pi_1(k) + i\pi_2(k) \equiv \int d^4x\, e^{+ik\cdot x}\pi(x) \quad . \tag{3.10b}$$

Hence, the shift $\pi \to \pi - J$ gives

$$\int \mathscr{D}\pi \exp i\int \frac{d^4k}{(2\pi)^4}\left[\frac{-(J+\pi)(k)(J+\pi)(-k)}{2(k^2-m^2)} - \sum_j \rho_j\pi_1(k)\cos k\cdot x_j - \sum_j \rho_j\pi_2(k)\sin k\cdot x_j\right]$$

$$\equiv \exp i\int \frac{d^4k}{(2\pi)^4}\left\{\left[\sum_j \rho_j(J_1(k)\cos k\cdot x_j + J_2(k)\sin k\cdot x_j)\right] + \frac{k^2-m^2}{2}\left[(\sum_j \rho_j \cos k\cdot x_j)^2 + (\sum_j \rho_j \sin k\cdot x_j)^2\right]\right\}$$

$$= \exp\left\{i\sum_{j=1}^{n} \rho_j J(x_j) + i\int \frac{d^4k}{(2\pi)^4}\frac{(k^2-m^2)}{2}\sum_{i,j}^{n}\rho_i\rho_j \cos k\cdot(x_i-x_j)\right\} \quad . \tag{3.11}$$

We now have

$$e^{iZ} = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\prod_{j=1}^{n}\left[\int d^4x_j \frac{d\eta_j \eta_j^2}{8\sqrt{g}}\int \frac{d\rho_j}{2\pi}\frac{d\beta_j}{\beta_j}\frac{d\alpha_j}{2\pi}\right]\exp\left[i\sum_j(\alpha_j\beta_j + \eta_j\rho_j + \rho_j J(x_j))\right.$$

$$\left. + \frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\sum_{i,j}(k^2-m^2)\rho_i\rho_j \cos k\cdot(x_i-x_j)\right]\int \mathscr{D}\varphi\mathscr{D}\sigma \exp i\int d^4x\left[\sigma^2 + 2\sqrt{g}\,\varphi^2\sigma\right.$$

$$\left. - \sum_j \delta(x-x_j)\alpha_j\sigma(x)\right] \quad . \tag{3.12}$$

We must next study the remaining functional integral in this last equation:

$$I(\{\alpha_j\},\{x_j\}) \equiv \int \mathscr{D}\varphi\mathscr{D}\sigma \exp i\int d^4x\left[\sigma^2 + 2\sqrt{g}\,\varphi^2\sigma - \sum_j \alpha_j\delta(x-x_j)\sigma(x)\right] \tag{3.13}$$.

A simple shift removes the σ field:

$$I(\{\alpha_j\},\{x_j\}) = \int \mathscr{D}\varphi \exp i\left[\int d^4x\left[-g\varphi^4\right] + \sqrt{g}\sum_j \alpha_j\varphi^2(x_j) - \frac{1}{4}\sum_{i,j}\alpha_i\alpha_j\delta(x_i-x_j)\right] \quad . \tag{3.14}$$

This last integral over φ is clearly delicate. In order to extract its physical content, it is sufficient to study it relative to its value at $\{\alpha_i\} = 0$: We therefore need to determine

$$R = \frac{\int \mathcal{D}\varphi \exp i\left[\int d^4x\left[-g\varphi^4\right]+\sqrt{g}\sum_j \alpha_j\varphi^2(x_j)\right]}{\int \mathcal{D}\varphi \exp i\int d^4x[-g\varphi^4]} \qquad (3.15)$$

We proceed as follows: First, recall

$$\int \mathcal{D}\varphi \equiv \lim_{N\to\infty} \prod_{j=1}^{N} \int d\varphi(y_j) \qquad (3.16)$$

where the $\{y_i\}$ may correspond to a covering of space-time of uniform measure ΔX, i.e., each y_i is at the center of one and only one of the sets in this covering, except perhaps at ∞, and each set in the covering has measure ΔX. Whenever we may take $\{x_j\}\subset\{y_j\}$, (3.15) is readily evaluated as (see Appendix III)

$$R = \prod_{j=1}^{n} \frac{\int_{-\infty}^{\infty} d\varphi_j \exp i\,(-g\,\Delta X\,\varphi_j^4 + \sqrt{g}\,\alpha_j\varphi_j^2)}{\int_{-\infty}^{\infty} d\varphi_j \exp i\,(-g\,\varphi_j^4)\,\Delta X} \to \prod_{j=1}^{n} \frac{2(\Delta X)^{1/4}\sqrt{\pi/\alpha_j}}{\Gamma(1/4)i^{1/4}} \quad \text{for } \Delta X \downarrow 0, \qquad (3.17)$$

, presuming $x_i \neq x_j$ for $i \neq j$. We call this the normal case.

In the exceptional case that an x_j cannot appear in $\{y_i\}$, it must correspond to a boundary value at $t = \pm\infty$. Our only constraint is that $\varphi \to {}^{\text{in}}_{\text{out}}$ field as $t \to \mp\infty$. But, in and out fields are well-behaved functions of x. Hence, from the regions near $t = \pm\infty$, we get, for example, if all x_j are at the boundary,

$$R = \exp(i\sqrt{g}\sum_j \alpha_j\varphi^2(x_j)) \qquad . \qquad (3.18)$$

Finally, when $\{x_j\}\subset\{y_j\}$ but two x_j coincide, another exceptional case, we get, for $x_i = x_{j_0}$ and $(\alpha_i, \alpha_{j_0}) \leftarrow (\alpha_i - \alpha_{j_0}, \alpha_{j_0} + \alpha_i)/\sqrt{2}$, where $i \neq j_o$,

$$R = \frac{1}{\sqrt[4]{2}} \prod_{j \neq i}^{n} \frac{2(\Delta X)^{1/4}}{\Gamma(1/4) i^{1/4}} \sqrt{\pi/\alpha_j} \tag{3.19}$$

with the analogous expression for $x_1 = x_2 = x_3$, etc. Any other exceptional case can be seen to be at most a combination of the two types of cases represented by (3.18) and (3.19), possibly several times.

Now, how shall we weight these contributions? Well, the region where R is given by (3.18) is of size

$$\prod_j \frac{(\Delta X)^{1/4} \int d^3\vec{x}_j}{\int d^4 x_j}$$

relative to $\prod_j \int d^4 x_j$; the analogous estimate holds for the similar exceptional cases. The region where R is given by (3.19) and the analogous expressions is of size

$$\frac{\Delta X \int d^4 x_{j_o}}{\int d^4 x_i \int d^4 x_{j_o}}$$

relative to $\pi_j \int d^4 x_j$. It might therefore appear quite simple to resolve this weighting problem. However, R occurs in (3.12) multiplied by the factor

$$\exp i \left\{ \sum_j (\alpha_j \beta_j + \eta_j \rho_j + \rho_j J(x_j)) \right.$$

$$\left. + \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \sum_{i,j} (k^2 - m^2) \rho_i \rho_j \cos k \cdot (x_i - x_j) - \frac{1}{4} \sum_{i,j} \alpha_i \alpha_j \delta(x_i - x_j) \right\} \tag{3.20}$$

This factor (3.20) involves several infinite expressions. We should not, therefore, expect to be able to decide what to do until we "renormalize" these infinities.

Of course, these infinities should have been expected, since our field theory is not finite, so that the products of operators at short distances are in general too singular in their unrenormalized form. We can already make the following observation, however: from the structure of (3.20) it is clear that any systematic procedure which renders the normal case (R given by (3.17)) finite renders the exceptional cases negligible. Hence, we may write

$$e^{iZ} = \sum_{n=0}^{\infty}\left(\frac{-i}{4\sqrt{g}}\right)^n \frac{1}{n!}\left[\prod_{j=1}^{n}\int d^4x_j d\eta_j \eta_j^2 \frac{d\rho_j}{2\pi}\frac{d\beta_j}{\beta_j}\frac{d\alpha_j}{2\pi}\frac{(\Delta X)^{1/4}\sqrt{\pi/\alpha_j}}{\Gamma(1/4)i^{1/4}}\right]$$
$$\exp i\left[\sum_j(\alpha_j\beta_j + \rho_j\eta_j + \rho_j J(x_j)) + \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\sum_{i,j}(k^2-m^2)\rho_i\rho_j \cos k\cdot(x_i-x_j)\right.$$
$$\left.-\frac{1}{4}\sum_{i,j}\alpha_i\alpha_j\delta(x_i-x_j)\right] \tag{3.21}$$

This is an explicit (formal) representation of the large coupling limit of the unrenormalized connected Green's functions of scalar field theory with quartic self-coupling.

We shall therefore have a complete theory of the strong coupling limit of the Lagrangian (3.1) provided we can consistently interpret the infinities in the expression (3.21). We shall next turn to this issue.

B. Regularization

We shall now discuss a method for interpreting the infinities in (3.21). Let us first recall, again, that,since this theory is renormalizable, there can be at most a finite number of parameters. These parameters depend, in general, on the cutoff, i.e., on ΔX. The objective is to show that these parameters can be chosen so that the dependence on ΔX disappears.

The formula (3.21) contains the expressions

$$\Delta_F^{-1}(x_i-x_j) \equiv \int\frac{d^4k}{(2\pi)^4}(k^2-m^2) \quad \cos k\cdot(x_i-x_j) \tag{3.22a}$$

and

$$\delta(x_i - x_j) = \int \frac{d^4k e^{-ik\cdot(x_i - x_j)}}{(2\pi)^4} . \tag{3.22b}$$

These expressions diverge for $x_i = x_j$. We choose to cut them off in such a manner that

$$\Delta X |\delta(0)| = 1 \tag{3.23}$$

as it should. This means (from (3.22b)) that an appropriate expression for the regularization of $\delta(x)$ is that suggested by continuation to n dimensions:[6]

$$\delta(x) = \lim_{\lambda\downarrow 0} \delta^{\lambda}(x) = \lim_{\lambda\downarrow 0} \frac{e^{-ix^2/4\lambda^2}}{(2\lambda\sqrt{\pi})^4\sqrt{i^2}} \tag{3.24a}$$

with

$$\frac{\Delta X}{(2\lambda\sqrt{\pi})^4} = 1 . \tag{3.24b}$$

The distribution (3.22a) is then readily interpreted by

$$\Delta_F^{-1}(x) = \lim_{\lambda\downarrow 0} \Delta_F^{-1}(x;\lambda) \equiv \lim_{\lambda\downarrow 0} \left(\frac{-\partial^2}{\partial x_\mu \partial x^\mu} - m^2\right)\delta^{\lambda}(x) = \lim_{\lambda\downarrow 0}\left\{\frac{2i}{\lambda^2} + \frac{x^2}{4\lambda^4} - m^2\right\}\frac{e^{-ix^2/4\lambda^2}}{(2\lambda\sqrt{\pi})^4\sqrt{i^2}} . \tag{3.25}$$

With these regularizations in mind, we introduce

$$V_0^{\lambda}(x) = \frac{\delta^{\lambda}(x)}{|\delta^{\lambda}(0)|} = \frac{e^{-ix^2/4\lambda^2}}{\sqrt{i^2}} \tag{3.26}$$

$$V_2^{\lambda}(x) = \lambda^2(2\lambda\sqrt{\pi})^4 \Delta_F^{-1}(x;\lambda) = \left\{2i + \frac{x^2}{4\lambda^2} - \lambda^2 m^2\right\}\frac{e^{-ix^2/4\lambda^2}}{\sqrt{i^2}} . \tag{3.27}$$

We then obtain, upon rescaling

$$\alpha_j \rightarrow \alpha_j/\sqrt{|\delta^{\lambda}(0)|}$$

$$
\begin{aligned}
\rho_j &\rightarrow \rho_j \lambda (2\lambda\sqrt{\pi})^2 \\
\beta_j &\rightarrow \beta_j \sqrt{|\delta^\lambda(0)|} \\
\eta_j &\rightarrow \eta_j / \lambda (2\lambda\sqrt{\pi})^2 \\
x_j &\rightarrow 2m_R \lambda x_j \quad ,
\end{aligned}
\tag{3.28}
$$

$$
e^{iZ(J)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i}{4\sqrt{g}}\right)^n \prod_{j=1}^{n} \Bigg[\int d^4x_j \left\{ \frac{(2\lambda m_R)^4 (\Delta X)^{1/4} |\delta^\lambda(0)|^{-1/4}}{\lambda^2 (2\lambda\sqrt{\pi})^4} \right\}
$$

$$
\int d\eta_j \eta_j^2 \int \frac{d\rho_j}{2\pi} \int \frac{d\beta_j}{\beta_j} \int \frac{d\alpha_j}{2\pi} \frac{\sqrt{\pi/\alpha_j}}{i^{1/4}\Gamma(1/4)} \Bigg] \exp i \Bigg[\sum_j (\alpha_j \beta_j + \rho_j \eta_j +
$$

$$
\rho_j \lambda (2\lambda\sqrt{\pi})^2 J(2x_j m_R \lambda)) + \frac{1}{2}\sum_{i,j} \rho_i \rho_j V_2(x_i - x_j) - \frac{1}{4}\sum_{i,j} \alpha_i \alpha_j V_0(x_i - x_j) \Bigg] \tag{3.29}
$$

where we have defined

$$
V_0(x_i - x_j) = \frac{e^{-i(x_i - x_j)^2 m_R^2}}{\sqrt{i^2}}
$$

$$
V_2(x_i - x_j) = \left\{ 2i + m_R^2 (x_i - x_j)^2 - \lambda^2 m^2 \right\} \frac{e^{-i(x_i - x_j)^2 m_R^2}}{\sqrt{i^2}} \tag{3.30}
$$

and m_R may be identified with an appropriate mass parameter (cutoff parameter).

The factor

$$
\frac{(2\lambda m_R)^4 (\Delta X)^{1/4} |\delta^\lambda(0)|^{-1/4}}{\lambda^2 (2\lambda\sqrt{\pi})^4}
$$

in (3.29) is, by (3.24b), equal to $4m_R^4/\pi$. Further, clearly without loss of physical content, we may absorb the factor $\lambda(2\lambda\sqrt{\pi})^2$ multiplying $J(2m_R\lambda x_j)$ into the wave function renormalization constant for the field φ. This factor may thus be omitted with this understanding.

The only remaining dependence on λ is therefore through the $m_R\lambda$ in the argument of the source J and the $\lambda^2 m^2$ in $V_2(x)$. In fact, for sources J such that

$$J(ax) = a^{-3} J(x) \tag{3.31}$$

the only dependence on λ is through $\lambda^2 m^2$, if the factor $\lambda(2\lambda\sqrt{\pi})^2$ is not absorbed into the wave function renormalization constant. It should be intuitively clear from this last fact that our theory is finite for an appropriate definition of a finite number of parameters. We shall not restrict, initially, our attention to such sources as (3.31), however. We therefore record our basic result, for the moment, as

$$e^{iZ(J)} \equiv \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i}{\sqrt{g}}\right)^n \prod_{j=1}^{n} \left[\int d^4x_j \frac{m_R^4}{\pi} \int d\eta_j \eta_j^2 \int \frac{d\rho_j}{2\pi} \int \frac{d\beta_j}{\beta_j} \int \frac{d\alpha_j}{2\pi} \frac{\sqrt{\pi/\alpha_j}}{i^{1/4}\Gamma(1/4)}\right]$$

$$\exp i\left\{\sum_j \left[\alpha_j\beta_j + \eta_j\rho_j + \rho_j J(2\lambda m_R x_j)\right] + \frac{1}{2}\sum_{i,j} \rho_i\rho_j V_2(x_i - x_j) - \frac{1}{4}\sum_{i,j} \alpha_i\alpha_j V_0(x_i - x_j)\right\} , \tag{3.32}$$

where we have made a wave function renormalization as we discussed above.

This last expression clearly corresponds to a theory independent of λ provided we define

$$z_1 \equiv 2\lambda m_R \qquad \text{and} \qquad z_m \equiv \lambda^2 m^2 , \tag{3.33}$$

z_1 and z_m being parameters determined by normalization conditions.

Further, upon comparing (3.17), (3.18), and (3.19) in connection with (3.29), it becomes clear that we were justified in neglecting the latter exceptional cases for R above and writing (3.21). It appears therefore that the formal expression (3.21) does indeed correspond to a finite calculable representation of the large coupling limit of the Lagrangian (3.1). The result (3.32) gives this finite representation more explicitly.

C. Explicit Calculability

We are asserting that (3.32) is calculable, mainly on the basis of the calculability of the corresponding theory for small coupling, i.e., (AII.11) is known to be calculable. The representation (3.29), however, allows us to evaluate entirely unrenormalized quantities to see explicitly whether (3.32) is calculable. We consider as an example one of the basic vertices in the theory - the connected two point function:

$$- \langle 0|T(\varphi(y_1)\varphi(y_2))|0\rangle_c = \frac{\delta^2 iZ}{\delta J(y_1)\delta J(y_2)}\Big|_{J=0} = \frac{1}{\sum_{n=0}^{\infty} a_n t^n} \sum_{n=1}^{\infty} t^n \frac{\delta^2 a_n}{\delta J(y_1)\delta J(y_2)}\Big|_{J=0} \tag{3.34}$$

where we have defined

$$\begin{aligned} a_o &= 1 , \\ a_n &= \frac{1}{n!} \prod_{j=1}^{n} [\int d^4x_j\, m_R^4 \int_{-\infty}^{\infty} d\eta_j\, \eta_j^2 \frac{\int_{-\infty}^{\infty} d\rho_j}{2\pi} \frac{\int_{-\infty}^{\infty} d\beta_j}{\beta_j + i\epsilon} \frac{\int_{-\infty}^{\infty} d\alpha_j}{2\pi} \\ &\quad (\frac{1}{i^{1/4}\, \Gamma(1/4)\sqrt{\pi\alpha_j}})] \exp i[\sum_{j=1}^{n} (\alpha_j\beta_j + \rho_j\eta_j + 4\lambda^3\pi\rho_j J(2\lambda m_R x_j)) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{n} \rho_i\rho_j\, V_2(x_i - x_j) \\ &\quad - \frac{1}{4}\sum_{i,j=1}^{n} \alpha_i\alpha_j\, V_o(x_i - x_j)] \end{aligned} \tag{3.35}$$

for $n\geq 1$, and

$$t = -i/\sqrt{g} \quad . \tag{3.36}$$

In (3.34), the subscript c denotes the connected part. In (3.35), the $+i\epsilon$ prescription on β_j results from the requirement that the coefficient of $\varphi^2(x_j)$ in (3.12) have a small positive imaginary part; thus, here, $\epsilon \downarrow 0$.

Now, it is not supposed to matter how specifically the limit $J \to 0$ is taken in the functional derivative. That is to say, for any reasonably behaved functional $S \equiv \int d^4x\, F(J(x))$ where F is an ordinary function,

$$\frac{\delta S}{\delta J(y)}\Big|_{J=0} = \lim_{\tau \to 0} \int d^4x \left\{\frac{F(J(x) + \tau\delta(x-y)) - F(J(x))}{\tau}\right\}\Big|_{J=0}$$

$$= F'(J(y))\Big|_{J=0} \qquad (3.37)$$

appears to be independent of the manner in which $J \to 0$. Here, prime denotes differentiation with respect to argument. Hence, let us now restrict our attention to sources satisfying (3.31) in the neighborhood of $J = 0$. Then, as m_R was arbitrary, we identify it with m_p, the physical mass, without loss of physical content!

The lowest order contribution to $-i\bar{\Delta}_F\,(y_1 - y_2) \equiv -\langle 0|T(\varphi(y_1)\varphi(y_2))|0\rangle_c$ is then

$$-i\bar{\Delta}_F^{(I)}(y_1 - y_2) \equiv t\, \frac{\delta^2 a_1}{\delta J(y_1)\delta J(y_2)}\Big|_{J=0} = \left(\frac{-i}{\sqrt{g}}\right) m_p^4 \int_{-\infty}^{\infty} d\eta\, \eta^2 \int_{-\infty}^{\infty} \frac{d\rho}{2\pi} \int_{-\infty}^{\infty} \frac{d\beta}{\beta + i\epsilon} \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi}$$

$$\frac{1}{i^{1/4}\,\Gamma(1/4)\sqrt{\pi\alpha}} \exp i[\alpha\beta + \eta\rho + \tfrac{1}{2}\rho^2(2 + iz_m) - \alpha^2/4i]$$

$$\int d^4x \left[\left(i\rho\,\frac{\pi}{2m_p^3}\right)\delta(x - y_1)\left(i\rho\,\frac{\pi}{2m_p^3}\right)\delta(x - y_2)\right] . \qquad (3.38)$$

The integral over d^4x is straightforward. The integration over $d\beta$ is also straightforward, giving (Here, $\theta(s) = 1$ for $s > 0$ and $\theta(s) = 0$ for $s < 0$.)

$$\int_{-\infty}^{\infty} \frac{d\beta}{\beta + i\epsilon} \exp(i\alpha\beta) = \theta(-\alpha)\ (-2\pi i) \quad . \tag{3.39}$$

Thus, the subsequent integration over $d\alpha$ gives

$$\begin{aligned}
(-2\pi i) \int_{-\infty}^{0} \frac{d\alpha}{2\pi} \frac{\exp(-\alpha^2/4)}{\sqrt{\pi\alpha}} &= -\int_{0}^{\infty} \frac{d\alpha}{\sqrt{\pi\alpha}} \exp(-\alpha^2/4) \\
&= \frac{-1}{2\sqrt{\pi}} \int_0^\infty \frac{dr\, e^{-r/4}}{r^{3/4}} \\
&= -\frac{1}{\sqrt{2\pi}} \int_0^\infty dr\ r^{-3/4} \exp(-r) \\
&= -\frac{\Gamma(1/4)}{\sqrt{2\pi}} \quad .
\end{aligned} \tag{3.40}$$

Finally, the integrations over $d\rho$ and $d\eta$ give

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\eta\ \eta^2 \int_{-\infty}^{\infty} \frac{d\rho}{2\pi}\ \rho^2 \exp\{i[\eta\rho + \rho^2(1 + iz_m/2)]\} \\
&= \int_{-\infty}^{\infty} d\eta\ \left(-\frac{\partial^2}{\partial\gamma^2}\right) \int_{-\infty}^{\infty} \frac{d\rho}{2\pi}\ \rho^2 \exp\{i[\eta(\rho - \gamma) + \rho^2(1 + iz_m/2)]\}\Big|_{\gamma=0} \\
&= -\frac{\partial^2}{\partial\gamma^2} \int_{-\infty}^{\infty} d\rho\ \delta(\rho - \gamma)\rho^2 \exp[i\rho^2(1 + iz_m/2)]\Big|_{\gamma=0} \\
&= -2 \quad .
\end{aligned} \tag{3.41}$$

On introducing (3.40) and (3.41) into (3.38) we obtain

$$-i\bar{\Delta}_F^{(I)}(y_1 - y_2) \equiv t\, \frac{\delta^2 a_1}{\delta J(y_1)\delta J(y_2)}\Big|_{J=0} = \left(\frac{-i}{\sqrt{g}}\right) m_p^4 \left(\frac{1}{i^{1/4}\,\Gamma(1/4)}\right) \left(\frac{-\Gamma(1/4)}{\sqrt{2\pi}}\right) \left(\frac{\pi}{2m_p^3}\right)^2 (i)^2 (-2)$$

$$\times\ \delta(y_1 - y_2)$$

$$= \frac{i^{3/4}\pi^2 \delta(y_1 - y_2)}{2\sqrt{2\pi g}\ \ m_p^2} \quad . \qquad (3.42)$$

To compute the order 1/g contribution to $-i\bar{\Delta}_F(y_1 - y_2)$, we need the order t term in $\exp(-iZ)|_{J=0}$ times $-i\bar{\Delta}_F^{(I)}(y_1 - y_2)$, plus

$$t^2 \frac{\delta^2 a_2}{\delta J(y_1)\delta J(y_2)}\Big|_{J=0} \quad , \qquad (3.43)$$

as usual.

Turning first to $\exp(-iZ)|_{J=0}$ we have

$$\exp(-iZ)|_{J=0} = 1 - t a_1|_{J=0} + 0(1/g)$$

$$= 1 + \frac{i}{\sqrt{g}} \int d^4x\ m_p^4 \int_{-\infty}^{\infty} d\eta\ \eta^2 \int_{-\infty}^{\infty} \frac{d\rho}{2\pi} \int_{-\infty}^{\infty} \frac{d\beta}{\beta + i\epsilon} \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi}$$

$$\frac{1}{i^{1/4}\ \Gamma(1/4)\sqrt{\pi\alpha}} \exp\{i[\alpha\beta + \rho\eta + \tfrac{1}{2}\rho^2(2 + iz_m)$$

$$- \alpha^2/4i]\} + 0(1/g)$$

$$= 1 + \frac{i}{\sqrt{g}} \frac{m_p^4((2\pi)^4\delta_p^4(0))}{i^{1/4}\ \Gamma(1/4)} \left(\frac{-\Gamma(1/4)}{\sqrt{2\pi}}\right)\Big(-\frac{\partial^2}{\partial\gamma^2} \int_{-\infty}^{\infty} d\eta$$

$$\int_{-\infty}^{\infty} \frac{d\rho}{2\pi} \exp\{i[\eta(\rho - \gamma) + \tfrac{1}{2}\rho^2(2 + iz_m)]\}\Big|_{\gamma=0}\Big) + 0(1/g)$$

$$= 1 + \frac{i}{\sqrt{g}} m_p^4\left(\frac{(2\pi)^4\delta_p^4(0)}{i^{1/4}\ \Gamma(1/4)}\right) \left(\frac{-\Gamma(1/4)}{\sqrt{2\pi}}\right)$$

$$\times \Big(-\frac{\partial^2}{\partial\gamma^2} \exp[i\frac{\gamma^2}{2}(2 + iz_m)]\Big|_{\gamma=0}\Big)$$

$$= 1 - \frac{(2\pi)^4 \delta_p^4(0)\ m_p^4\ (2 + iz_m)}{\sqrt{2\pi g}\ i^{1/4}} + 0(1/g) \quad . \tag{3.44}$$

Here, $(2\pi)^4 \delta_p^4(0) \equiv VT$ is the total volume of space(V) - time(T) and (3.40) has been used. From (3.44) we have the contribution

$$-i\bar{\Delta}_F^{(IIa)}(y_1 - y_2) = - \frac{VT\ m_p^4\ (2 + iz_m)}{\sqrt{2\pi g}\ i^{1/4}} (-i\bar{\Delta}_F^{(I)}(y_1 - y_2)) = \frac{-i^{1/2}\pi m_p^2 (2 + iz_m)\delta(y_1 - y_2)VT}{4g} \tag{3.45}$$

to the order 1/g term in $-i\bar{\Delta}_F(y_1 - y_2)$.

The remaining order 1/g contribution to $-i\bar{\Delta}_F(y_1 - y_2)$ is explicitly given by

$$-i\bar{\Delta}_F^{(IIb)}(y_1 - y_2) \equiv t^2 \left.\frac{\delta^2 a_2}{\delta J(y_1)\delta J(y_2)}\right|_{J=0}$$

$$= \frac{1}{2}\left(\frac{-i}{\sqrt{g}}\right)^2 \int d^4x_1 \int d^4x_2\ m_p^8 \int_{-\infty}^{\infty} d\eta_1\ \eta_1^2$$

$$\frac{\int_{-\infty}^{\infty} d\rho_1}{2\pi} \int_{-\infty}^{\infty} d\eta_2\ \eta_2^2 \int_{-\infty}^{\infty} \frac{d\rho_2}{2\pi} \int_{-\infty}^{\infty} \frac{d\beta_1}{\beta_1 + i\epsilon} \int_{-\infty}^{\infty} \frac{d\alpha_1}{2\pi}$$

$$\int_{-\infty}^{\infty} \frac{d\beta_2}{\beta_2 + i\epsilon} \int_{-\infty}^{\infty} \frac{d\alpha_2}{2\pi} \left(\frac{1}{i^{1/2}\pi\sqrt{\alpha_1\alpha_2}\ \Gamma^2(1/4)}\right)$$

$$\exp\{i[\alpha_1\beta_1 + \alpha_2\beta_2 + \rho_1\eta_1 + \rho_2\eta_2$$

$$+ \rho_1\rho_2 V_2(x_1 - x_2) + \frac{1}{2}(\rho_1^2 + \rho_2^2)(2 + iz_m)$$

$$- \frac{1}{2}\alpha_1\alpha_2 V_o(x_1 - x_2) - (\alpha_1^2 + \alpha_2^2)/4i]\}$$

$$\times \; (i)^2 \{ (\frac{\pi^2}{4m_p^6}) \; (\rho_1 \delta(x_1 - y_1) + \rho_2 \delta(x_2 - y_1)) \times (\rho_1 \delta(x_1 - y_2) + \rho_2 \delta(x_2 - y_2)) \} \quad . \tag{3.46}$$

The integrations over $\{\eta_1, \eta_2\}$ and $\{\rho_1, \rho_2\}$ are effected in complete analogy with (3.41) and the $\eta - \rho$ integration in (3.44). We have

$$\int_{-\infty}^{\infty} d\eta_1 \eta_1^2 \int_{-\infty}^{\infty} \frac{d\rho_1}{2\pi} \int_{-\infty}^{\infty} d\eta_2 \, \eta_2^2 \int_{-\infty}^{\infty} \frac{d\rho_2}{2\pi} \, (\rho_1 \delta(x_1 - y_1) + \rho_2 \delta(x_2 - y_1))(\rho_1 \delta(x_1 - y_2) + \rho_2 \delta(x_2 - y_2))$$

$$\exp\{i[\rho_1 \eta_1 + \rho_2 \eta_2 + \rho_1 \rho_2 V_2(x_1 - x_2) + \tfrac{1}{2}(\rho_1^2 + \rho_2^2)(2 + iz_m)]\}$$

$$= \frac{\partial^4}{\partial\gamma_1^2 \partial\gamma_2^2} \, (\gamma_1 \delta(x_1 - y_1) + \gamma_2 \delta(x_2 - y_1))(\gamma_1 \delta(x_1 - y_2) + \gamma_2 \delta(x_2 - y_2)) \exp\{i[\gamma_1 \gamma_2 V_2(x_1 - x_2) + \tfrac{1}{2}(\gamma_1^2 + \gamma_2^2)(2 + iz_m)]\}\Big|_{\gamma_1, \gamma_2 = 0}$$

$$= 2 \; \delta(y_1 - y_2)[\delta(x_1 - y_1) + \delta(x_2 - y_1)] \; \{i(2 + iz_m)\} + [\delta(x_1 - y_1)\delta(x_2 - y_2) + \delta(x_1 - y_2)\delta(x_2 - y_1)](4i \; V_2(x_1 - x_2)) \; . \tag{3.47}$$

The required integrations over $\{\beta_1, \beta_2\}$ and $\{\alpha_1, \alpha_2\}$ are effected with the use of (3.39). We have

$$\int_{-\infty}^{\infty} \frac{d\beta_1}{\beta_1 + i\epsilon} \int_{-\infty}^{\infty} \frac{d\beta_2}{\beta_2 + i\epsilon} \int_{-\infty}^{\infty} \frac{d\alpha_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\alpha_2}{2\pi} \, (\frac{1}{\pi\sqrt{\alpha_1 \alpha_2}}) \exp\{i[\alpha_1 \beta_1 + \alpha_2 \beta_2 - \tfrac{1}{2} \alpha_1 \alpha_2 V_o(x_1 - x_2) - (\alpha_1^2 + \alpha_2^2)/4i]\}$$

$$= \int_{-\infty}^{\infty} \frac{d\alpha_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\alpha_2}{2\pi} \frac{\theta(-\alpha_1)\theta(-\alpha_2)(-2\pi i)^2}{\pi\sqrt{\alpha_1\alpha_2}} \exp\{i[-\tfrac{1}{2}\alpha_1\alpha_2 V_o(x_1 - x_2) - (\alpha_1^2 + \alpha_2^2)/4i]\}$$

$$= \int_o^{\infty} d\alpha_1 \int_o^{\infty} d\alpha_2 \frac{1}{\pi\sqrt{\alpha_1\alpha_2}} \exp\{i[-\tfrac{1}{2}\alpha_1\alpha_2 V_o(x_1 - x_2) - (\alpha_1^2 + \alpha_2^2)/4i]\}$$

$$= \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-)^n}{2^n n!} e^{-in(x_1 - x_2)^2 m_p^2} \int_o^{\infty} d\alpha_1 \int_o^{\infty} d\alpha_2\, \alpha_1^{n-1/2} \alpha_2^{n-1/2} e^{-(\alpha_1^2 + \alpha_2^2)/4}$$

$$= \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-)^n 2^{n-1}}{n!} \Gamma^2(\tfrac{n}{2} + 1/4)\, e^{-in(x_1 - x_2)^2 m_p^2} \quad . \tag{3.48}$$

On introducing (3.47) and (3.48) into (3.46) we find

$$-i\bar{\Delta}_F^{(IIb)}(y_1 - y_2) = \frac{1}{2}(-\frac{1}{g})(m_p^8)\frac{1}{i^{1/2}}\frac{1}{\Gamma^2(1/4)}\frac{\pi^2}{(4m_p^6)}$$

$$\times (-1)\int d^4x_1 \int d^4x_2 \{2i(2 + iz_m)\delta(y_1 - y_2)[\delta(x_1 - y_1) + \delta(x_2 - y_1)]$$

$$+ 4i\, V_2(x_1 - x_2)[\delta(x_1 - y_1)\delta(x_2 - y_2) + \delta(x_1 - y_2)\delta(x_2 - y_1)]\}$$

$$\times \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-)^n 2^{n-1}}{n!} \Gamma^2(\tfrac{n}{2} + 1/4) \exp[-in(x_1 - x_2)^2 m_p^2] \ . \tag{3.49}$$

Evidently, (3.49) requires

$$\int d^4x\, e^{-in(x-y)^2 m_p^2} = \sqrt{\frac{\pi}{inm_p^2}}\left(\frac{\pi}{-inm_p^2}\right)^{3/2} = \frac{i\pi^2}{n^2 m_p^4} \quad . \tag{3.50}$$

Using (3.50) we obtain

$$-i\bar{\Delta}_F^{(IIb)}(y_1 - y_2) = \frac{i^{1/2}\pi\, m_p^2}{2g\,\Gamma^2(1/4)}(2 + iz_m)\left\{\frac{VT\,\Gamma^2(1/4)}{2} + \frac{i\pi^2}{m_p^4}\sum_{n=1}^{\infty}\frac{(-)^n 2^{n-1}}{n^2 n!}\Gamma^2\left(\frac{n}{2}+1/4\right)\right\}$$

$$\times\,\delta(y_1 - y_2) + \frac{i^{-1/2}\,\pi m_p^2}{g}\{2i + (y_1 - y_2)^2 m_p^2 - z_m\}\sum_{n=0}^{\infty}\frac{(-)^n 2^{n-1}}{n!}$$

$$\times\left[\frac{\Gamma(\frac{n}{2} + 1/4)}{\Gamma(1/4)}\right]^2 \exp[-i(n+1)(y_1 - y_2)^2 m_p^2] \quad . \tag{3.51}$$

Hence, our result for the order $1/g$ contribution to $-i\bar{\Delta}_F(y_1 - y_2)$ is

$$-i\bar{\Delta}_F^{(II)}(y_1 - y_2) \equiv -i\bar{\Delta}_F^{(IIa)}(y_1 - y_2)\; -i\bar{\Delta}_F^{(IIb)}(y_1 - y_2)$$

$$= \frac{i^{3/2}\,\pi^3(2 + iz_m)}{2g\,m_p^2}\left(\sum_{n=1}^{\infty}\frac{(-)^n 2^{n-1}}{n^2 n!}\left[\frac{\Gamma(\frac{n}{2} + 1/4)}{\Gamma(1/4)}\right]^2\right)\delta(y_1 - y_2)$$

$$+\;\frac{i^{-1/2}\,\pi\, m_p^2}{g}\{2i + (y_1 - y_2)^2 m_p^2 - z_m\}\sum_{n=0}^{\infty}\frac{(-)^n 2^{n-1}}{n!}$$

$$\times\left[\frac{\Gamma(\frac{n}{2} + 1/4)}{\Gamma(1/4)}\right]^2 \exp[-i(n+1)(y_1 - y_2)^2 m_p^2] \,. \tag{3.52}$$

From (3.52) and (3.42) we have

$$\bar{\Delta}_F(y_1 - y_2) = \bar{\Delta}_F^{(I)}(y_1 - y_2) + \bar{\Delta}_F^{(II)}(y_1 - y_2) + 0(1/g^{3/2})$$

$$= \frac{i^{7/4}\,\pi^2\,\delta(y_1-y_2)}{2\sqrt{2\pi g}\,m_p^2} + \frac{i^{5/2}\,\pi^3\,(2+iz_m)}{2g\,m_p^2}\left(\sum_{n=1}^{\infty}\frac{(-)^n 2^{n-1}}{n^2 n!}\right.$$

$$\left.\times\left[\frac{\Gamma(\frac{n}{2}+1/4)}{\Gamma(1/4)}\right]^2\right)\delta(y_1-y_2)$$

$$+\frac{i^{1/2}\,\pi m_p^2}{g}\{2i+(y_1-y_2)^2 m_p^2 - z_m\}\sum_{n=0}^{\infty}\frac{(-)^n 2^{n-1}}{n!}\left[\frac{\Gamma(\frac{n}{2}+\frac{1}{4})}{\Gamma(1/4)}\right]^2$$

$$\exp[-i(n+1)(y_1-y_2)^2\,m_p^2] + 0(1/g^{3/2}) \quad . \qquad (3.53)$$

This expression (3.53) contains in it one unknown parameter, z_m. From this fact, it is apparent that $\langle 0|T(\varphi(y_1)\varphi(y_2))|0\rangle_c$ is calculable through order 1/g at least. As usual, the parameter z_m may be determined by the renormalized value of $\bar{\Delta}_F(p, -p)$ at some normalization point $p^2 = -\mu^2$, for example, where

$$\bar{\Delta}_F(p_1, p_2)(2\pi)^4\,\delta(p_1+p_2) \equiv \int d^4y_1 \int d^4y_2\;\bar{\Delta}_F(y_1-y_2)\exp\{i[p_1\cdot y_1 + p_2\cdot y_2]\} \qquad (3.54)$$

for 4-momenta p_1, p_2. This fact that only a mass renormalization parameter is necessary for the calculability of the large g limit of (3.1) to order 1/g is in complete agreement with the results of Wilson[7] for the strong coupling limit of the φ^6 interaction of the scalar field φ in two space and one time dimensions.

A more complete treatment of the connected Green's functions of (3.1) will appear elsewhere. Before concluding the present discussion of the two-point function, however, we would like to examine the large $|p^2|$ limit of

$\bar{\Delta}_F(p, -p)$ through order $1/g$. We have from (3.53) and (3.54) (using (3.50) and the derivative of (3.50) with respect to n)

$$\bar{\Delta}_F(p,-p) = \frac{\pi^2 i^{7/4}}{2\sqrt{2\pi g}\, m_p^2} - \frac{i^{1/2}\pi^3}{2gm_p^2}(2+iz_m)\sum_{n=1}^{\infty}\frac{(-)^n 2^{n-1}}{n^2 n!}\left[\frac{\Gamma(\frac{n}{2}+\frac{1}{4})}{\Gamma(1/4)}\right]^2$$

$$+ \frac{i^{1/2}\pi^3}{gm_p^2}\sum_{n=0}^{\infty}\frac{(-)^n 2^{n-1}}{n!}\left[\frac{\Gamma(\frac{n}{2}+1/4)}{\Gamma(1/4)}\right]^2\left(\frac{-2-iz_m}{(n+1)^2}+\frac{2}{(n+1)^3}\right.$$

$$\left.+\frac{ip^2}{4(n+1)^4 m_p^2}\right)\exp[ip^2/4(n+1)m_p^2] + O(1/g^{3/2}). \tag{3.55}$$

The p^2-dependent part of (3.55), as we shall show, is represented more conveniently by the use of

$$I_k \equiv \sum_{n=0}^{\infty}\frac{(-)^n 2^{n-1}}{n!}(\Gamma(\tfrac{n}{2}+1/4))^2\frac{1}{(n+1)^k}\exp i[p^2/4(n+1)m_p^2]$$

$$= \frac{i}{4}\oint_C dz\,\frac{2^z}{\Gamma(z+1)}(\Gamma(\tfrac{z}{2}+1/4))^2\frac{e^{i[p^2/4(z+1)m_p^2]}}{(z+1)^k\sin\pi z} \tag{3.56}$$

where k is a positive integer and C is the open curve encircling the non-negative integers in the complex z-plane, as shown in Fig. 1. (The arcs at ∞ and the line C' are not included in C.) Thus, we next study I_k in some detail.

First, we note that the identity

$$-z\Gamma(-z)\Gamma(z) = \frac{\pi}{\sin\pi z} \tag{3.57}$$

allows us to write

$$I_k = \frac{-i}{4\pi} \oint_C dz\, \Gamma(-z)\, 2^z \left(\Gamma\left(\frac{z}{2} + 1/4\right)\right)^2 \frac{e^{i[p^2/4(z+1)m_p^2]}}{(z+1)^k} . \tag{3.58}$$

But, the logarithm of $\Gamma(z)$ is

$$\ln \Gamma(z) = (z - 1/2)\ln z - z + \frac{1}{2}\ln(2\pi) + 2\int_o^\infty \frac{\arc\tan(t/z)}{e^{2\pi t} - 1}\, dt \tag{3.59}$$

where arc tan u is given by the equation

$$\arc\tan u = \int_o^u \frac{dt}{1 + t^2} \tag{3.60}$$

in which the path of integration is a straight line. Thus, we may use Cauchy's theorem to convert the integral over C to one over the line C$'$ in Fig. 1, giving

$$I_k = \frac{-i}{4\pi} \int_{-i\infty - \delta}^{i\infty - \delta} dz\, \Gamma(-z) 2^z \left(\Gamma\left(\frac{z}{2} + 1/4\right)\right)^2 \frac{e^{i[p^2/4(z+1)m_p^2]}}{(z+1)^k} \tag{3.61}$$

where $1 \gg \delta > 0$.(Hence, we have made a Sommerfeld-Watson transform.)

We will use the method of stationary phase to study (3.61). Toward this end, we observe that the imaginary part of the logarithm of the integrand in (3.61) is, for $z = iv - \delta$ with v real,

$$\begin{aligned}\mathrm{Phase}(v) = &- v\ln\sqrt{v^2 + \delta^2} + (\delta - 1/2)\tan^{-1}(-v/\delta) \\ &+ 2\,\mathrm{Im}\int_o^\infty \frac{dt}{(e^{2\pi t} - 1)} \int_o^{it/(v+i\delta)} \frac{du}{(1 + u^2)} + v\ln\sqrt{\frac{v^2}{4} + (1/4 - \delta/2)^2} \\ &-(\delta + 1/2)\tan^{-1}[v/(1/2 - \delta)] + 4\,\mathrm{Im}\int_o^\infty \frac{dt}{e^{2\pi t} - 1} \int_o^{-2it/(v+i\delta - i/2)} \frac{du}{1 + u^2}\end{aligned}$$

$$-k\ \tan^{-1}(v/(1-\delta)) + v\ \ln 2 + \frac{p^2(1-\delta)}{4m_p^2(v^2 + (1-\delta)^2)} . \tag{3.62}$$

Hence,

$$\frac{d\mathrm{Phase}(v)}{dv} = -\ln\sqrt{v^2+\delta^2} - \frac{v^2}{v^2+\delta^2} - \frac{(\delta-1/2)\delta}{v^2+\delta^2}$$

$$-2\,\mathrm{Im}\int_0^\infty \frac{dt\ it}{e^{2\pi t}-1}\ \frac{1}{((v+i\delta)^2 - t^2)} + \frac{1}{2}\ln(v^2 + (1/2-\delta)^2)$$

$$+ \frac{v^2}{v^2 + (1/2-\delta)^2} - \frac{(\delta+1/2)(1/2-\delta)}{v^2 + (1/2-\delta)^2}$$

$$+ 8\ \mathrm{Im}\int_0^\infty dt\ \frac{it}{(e^{2\pi t}-1)((v+i\delta-i/2)^2 - 4t^2)}$$

$$-\frac{k\ (1-\delta)}{v^2 + (1-\delta)^2} - \frac{p^2 v(1-\delta)}{2m_p^2(v^2 + (1-\delta)^2)^2} . \tag{3.63}$$

Evidently, if v is fixed, $d\mathrm{Phase}(v)/dv$ is nonzero for $|p^2| \to \infty$. Thus, any possible stationary point v_o for the phase in (3.61) must satisfy, for $|p^2| \to \infty$,

$$\left|\frac{p^2 v_o(1-\delta)}{2m_p^2(v_o^2 + (1-\delta)^2)^2}\right| < M < \infty \tag{3.64}$$

for some positive number M independent of p^2. Thus, for $|p^2| \to \infty$, we see by retaining the terms through order $1/v^3$ in (3.63) that

$$\left.\frac{d\mathrm{Phase}(v)}{dv}\right|_{v=v_o} = 0 \tag{3.65}$$

gives, approximately, for $\delta \sim (m_p^2/|p^2|)^{3/4}$ to be more specific ,

$$0 = -\ln|v_o| - 1 + \frac{6}{v_o^2}\left(\int_o^\infty dt \frac{t}{e^{2\pi t} - 1}\right) - \frac{1}{4v_o^2} - \frac{(k(1-\delta) - \delta/2)}{v_o^2}$$

$$+ \frac{1}{2} \ln v_o^2 + \frac{(1-4\delta)}{8v_o^2} + 1 - \frac{(1-4\delta)}{4v_o^2} - \frac{p^2}{2m_p^2 v_o^3} \quad . \tag{3.66}$$

(The region near $v_o \sim 0$, in which (3.64) can also be satisfied, can be shown not to give a solution to (3.65) by straightforward calculation so that only (3.66) is relevant to (3.65).) Hence, recalling that

$$4 \int_o^\infty dt \frac{t}{e^{2\pi t} - 1}$$

is the first Bernoullian number, which is 1/6, we have

$$(k + 1)\delta - (\frac{1}{8} + k) = p^2/2m_p^2 v_o$$

or

$$v_o \doteq -4p^2/m_p^2(1 + 8k) \quad . \tag{3.67}$$

Also, from (3.67),

$$\left.\frac{d^2\text{Phase}(v)}{dv^2}\right|_{v=v_o} \rightarrow (\frac{1}{4} + 2k)\frac{1}{v_o^3} + \frac{3p^2}{2m_p^2 v_o^4}$$

$$= (k + 1/8)\left(\frac{(1+8k)m_p^2}{4p^2}\right)^3 \tag{3.68}$$

and, from (3.62) ,

$$\text{Phase}(v_o) \underset{|p^2| \to \infty}{\rightarrow} -\frac{k\pi}{2}\,\varepsilon(-p^2) \tag{3.69}$$

with $\varepsilon(s) = \begin{cases} 1, & s > 0 \\ -1, & s < 0 \end{cases}$, to the accuracy of our approximations. Finally, the real part of the logarithm of the integrand in (3.61) is, for $v = v_o$,

$$-|v_o|\pi - (k+1)\ln|v_o| + \frac{1}{2}\ln 2 + \frac{3}{2}\ln(2\pi)$$

$$- (9+8k)/16 \qquad (3.70)$$

to the accuracy of our approximations. Thus, by the method of stationary phase,

$$I_k \simeq \frac{-i}{4\pi} \frac{e^{-\pi|v_o|}}{|v_o|^{k+1}} (2\pi)^{3/2}\sqrt{2}\, e^{-(9+8k)/16 - \frac{k\pi}{2}\epsilon(-p^2)i} \int_{-\infty}^{\infty} dv$$

$$\exp(-i(\frac{(1+8k)}{16v_o^3}(v-v_o)^2))$$

for $|p^2| \to \infty$. This gives, with $\arg i = \pi/2$, (3.71)

$$I_k \to -i\, \frac{(2\pi)^{3/2}\sqrt{2}\, e^{-(9+8k)/16 - \frac{k\pi}{2}\epsilon(-p^2)i - \pi|v_o|}}{4\pi|v_o|^{k+1}} \left(\frac{16\pi|v_o|^3}{i(1+8k)\epsilon(v_o)}\right)^{\frac{1}{2}}$$

$$= \frac{-i^{\frac{1}{2}}\ (4\pi)\ e^{-(9+8k)/16 - k\pi\epsilon(-p^2)i/2 - \pi|v_o|}}{\sqrt{1+8k}\ |v_o|^{k-\frac{1}{2}}\sqrt{\epsilon(-p^2)}} \qquad . \quad (3.72)$$

Hence, for $|p^2| \to \infty$, we have, from (3.55) and (3.72), with $\arg i = -3\pi/2 = \frac{1}{2}\arg -1$,

$$\bar{\Delta}_F(p,-p) \to \frac{\pi^2 i^{7/4}}{2\sqrt{2\pi g}\, m_p^2} - \frac{i^{1/2}\pi^3}{2gm_p^2}(2+iz_m) \sum_{n=1}^{\infty} \frac{(-)^n 2^{n-1}}{n^2 n!} \left[\frac{\Gamma(\frac{n}{2}+1/4)}{\Gamma(1/4)}\right]^2$$

$$+ \frac{i\pi^3(4\pi)}{gm_p^2\sqrt{33}} \frac{e^{-41/16 - \pi|v_o|}}{|v_o|^{7/2}\sqrt{\epsilon(-p^2)}} \frac{(ip^2/4m_p^2)}{(\Gamma(1/4))^2} \quad ,$$

(3.73)

where, here, $v_o = -4p^2/33m_p^2$; thus, for $|p^2| \to \infty$,

$$\bar{\Delta}_F(p,-p) \rightarrow \frac{\pi^2 i^{7/4}}{2\sqrt{2\pi g}\; m_p^2} - \frac{i^{1/2}\pi^3}{2gm_p^2}(2 + iz_m) \sum_{n=1}^{\infty} \frac{(-)^n 2^{n-1}}{n^2 n!} \left[\frac{\Gamma(\frac{n}{2}+\frac{1}{4})}{\Gamma(1/4)}\right]^2$$

$$+ \frac{35937\,\pi^4 i}{128g(\Gamma(1/4))^2} \left(\frac{m_p^3}{(p^2)^{5/2}}\right) e^{-41/16 - 4\pi|p^2|/33m_p^2} \quad . \tag{3.74}$$

As expected, the propagator approaches its asymptotic value essentially exponentially for large $|p^2|$. More elaboration of the meaning of this result will appear elsewhere.

IV. DISCUSSION

We have formulated the theory of strongly coupled renormalizable fields using scalar field theory as a prototype. To repeat, the more interesting applications will be taken up elsewhere.[1]

The restraint of renormalizability is clearly necessary in giving a meaningful formulation, for it assures us of only a finite number of parameters. We have no argument which would suggest that unrenormalizable theories possess a meaningful large coupling limit, although it would appear that in this latter limit formal expressions can be obtained for these theories just like the inverse power series expansion (3.21). We shall, therefore, not be concerned with such unrenormalizable theories in further reports, although, as we already admitted, we do not actually have a proof that such theories are without meaning in the respective limit.

We have given, for scalar field theory, an explicit demonstration of the renormalizability ("finiteness") of our formulation of the large coupling limit. Of course, this "finiteness" is intuitively clear (for all renormalizable theories) from the fact that the $g \rightarrow 0$ limits of the respective theories are "finite". In establishing this finiteness we have used an unconventional regularization procedure. In the Appendix II this unconventional procedure is compared with that of convention. The two would appear to be physically equivalent. This point will be discussed in more detail in the later works.

Our approach to the problem of strong coupling should be compared with that of Wentzel.[8] In this latter approach, an attempt is made to isolate the free and bound parts of the respective fields. On the other hand, in our approach, all aspects of the fields are treated on equal footing so that, for example, relativistic invariance is manifest throughout. Indeed, it is a special power of the path-space approach[2] that it permits all aspects of the theory, i.e., all aspects of the Lagrangian, to be treated on equal footing in a manifestly Lorentz invariant fashion.

We end by emphasizing that the methods in the text would appear to render accessible the large coupling limit of all renormalizable interactions.

APPENDIX I

A USEFUL EQUIVALENCE

In this short appendix, we shall establish the result (3.3b) of the text. It, like (3.3a), follows immediately from the work of Feynman.[2] For, from the definition of the path integral we have

$$\int \mathcal{D}\rho \mathcal{D}\pi \exp i \int d^4x \left[F(\rho) + \pi(\rho-\phi)\right] = \lim_{N\to\infty} \left[\prod_{j=1}^{N} \int_{-\infty}^{\infty} d\rho_j \int_{-\infty}^{\infty} d\pi_j\right] \exp(i \, \Delta X$$

$$\sum_j \left[F(\{\rho_{i_j}\}) + \pi_j(\rho_j-\phi_j)\right])$$

$$= \lim_{N\to\infty} \left[\prod_{j=1}^{N} \int_{-\infty}^{\infty} d\rho_j \, \frac{2\pi}{\Delta X} \, \delta(\rho_j-\phi_j)\right] \exp i \, \Delta X \sum_j F(\{\rho_{i_j}\})$$

$$= \lim_{N\to\infty} \left(\frac{2\pi}{\Delta X}\right)^N \exp i \, \Delta X \sum_j F(\{\phi_{i_j}\})$$

$$= C \exp i \int d^4x \, F(\phi) \qquad \text{(AI. 1)}$$

where the trivial infinite factor C could easily have been absorbed into the normalization of the functionals. This establishes (3.3b).

APPENDIX II

COMPARISON WITH CONVENTIONAL SMALL COUPLING THEORY

Here, we shall compare our method of regularizing the strong coupling limit with the conventional procedure for regularization. Since the conventional prescription has only been applied to small coupling, we shall effect such a comparison by applying our method of regularization to this same small coupling limit of (3.1), i.e., the theory

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) - g\phi^4 + J\phi \tag{AII.1}$$

with $g \rightarrow 0$.

Our generating functional is still

$$e^{iZ(J)} = \int \mathscr{D}\phi \exp i\int d^4x \left(\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) - g\phi^4 + J\phi\right) \quad . \tag{AII.2}$$

We find it convenient to write

$$\exp -i\int d^4x\, g\phi^4 = \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \prod_{j=1}^{n} \left[\int d^4x_j\, d\eta_j \eta_j^4 \frac{d\rho_j}{2\pi}\right] \exp i \sum_j \rho_j(\eta_j - \phi(x_j)) \tag{AII.3}$$

in view of the presumed smallness of g. Then

$$\begin{aligned} e^{iZ} &= \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \prod_{j=1}^{n} \left[\int d^4x_j d\eta_j \eta_j^4 \frac{d\rho_j}{2\pi}\right] \int \mathscr{D}\phi \exp\left\{ i\sum_j \rho_j\eta_j + i\int d^4x \left[\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi \right.\right. \\ &\qquad \left.\left. - m^2\phi^2) + J\phi - \sum_j \rho_j \delta(x-x_j)\phi\right]\right\} \\ &= \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \prod_{j=1}^{n} \left[\int d^4x_j d\eta_j \eta_j^4 \frac{d\rho_j}{2\pi}\right] \exp i\left\{\sum_j^n \rho_j\eta_j - \frac{1}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y) \right. \\ &\qquad \left. + \sum_j^n \rho_j \int J(x)\Delta_F(x-x_j) d^4x - \frac{1}{2}\sum_{i,j}^{n} \rho_i\rho_j\Delta_F(x_i-x_j)\right\} \end{aligned} \tag{AII.4}$$

where Δ_F is Feynman's solution of

$$-(\Box_x + m^2)\Delta_F = \delta(x-y) \qquad \text{(AII. 5)}$$

This last representation (AII. 4) of Z is the small coupling analogue of (3.21). It can easily be seen to correspond to the usual Feynman rules for the theory (AII. 1). It is, moreover, extremely convenient for applying our regularization procedure.

Since the function $\Delta_F(x_i - x_j)$ diverges badly for $x_i = x_j$, this small coupling limit of (AII. 1) also has singular ill-defined operator products at short distances, as is well known. Proceeding precisely as in Sec.III.B, we introduce the regularization

$$\Delta_F^\lambda(x-y) = \int d^4y' \Delta_F(x-y-y')\delta^\lambda(y') \qquad \text{(AII. 6)}$$

with $\delta^\lambda(x)$ given by (3.24a)

$$\delta^\lambda(x) \equiv \frac{e^{-ix^2/4\lambda^2}}{(2\lambda\sqrt{\pi})^4\sqrt{i}^2} \qquad \text{(AII. 7)}$$

In the limit $\lambda \downarrow 0$, we recover Δ_F

$$\Delta_F = \lim_{\lambda \downarrow 0} \Delta_F^\lambda \qquad \text{(AII. 8)}$$

Explicitly,

$$\Delta_F^\lambda = \int d^4y' \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y-y')}}{k^2-m^2+i\epsilon} \frac{e^{-iy'^2/4\lambda^2}}{(2\lambda\sqrt{\pi})^4\sqrt{i}^2} = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)+i\lambda^2k^2}}{k^2-m^2+i\epsilon} .$$

(AII. 9)

Now, in (AII. 4), we again scale

$$x_j \to \lambda m_R x_j$$

$$\eta_j \to \eta_j/\lambda m_R$$

$$\rho_j \rightarrow \rho_j \lambda m_R \quad , \tag{AII. 10}$$

giving

$$e^{iZ} = \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \prod_{j=1}^{n} \left[\int d^4x_j \, d\eta_j \eta_j^4 \frac{d\rho_j}{2\pi} \right] \exp i \left[\sum_j \rho_j \eta_j - \frac{1}{2} \int d^4x d^4y J(x) \Delta'_F(x-y) J(y) \right.$$

$$\left. + \int d^4x \sum_j \rho_j \, J(x) \Delta'_F(x-x_j) - \frac{1}{2} \sum_{i,j} \rho_i \rho_j \, \Delta'_F(x_i - x_j) \right] \tag{AII. 11}$$

where we have defined

$$\Delta'_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot (x-y) + ik^2/m_R^2}}{\left[k^2 - m_R^2 z_m + i\epsilon\right]} \quad , \tag{AII. 12}$$

$$z_m = \lim_{\lambda \to 0} \lambda^2 m^2$$

and have taken J so that

$$J(ax) = a^{-3} J(x) \tag{AII. 13}$$

in accordance with (3.31). Again, z_m is the mass renormalization parameter. The perturbation series (AII. 11) would appear to correspond physically to the usual series, except that now the "subtracted" propagator Δ'_F is exponentially damped at high energy. The precise relationship between our series (AII. 11) and that of convention will be taken up elsewhere.[1]

APPENDIX III

EVALUATION OF R

It may appear that there is a certain freedom associated with the phase of R - in view of the fractional powers appearing in it in (3.17). Thus, we should like to expose our choice of branch for these powers.

Specifically, for the evaluation of R we need, in the normal case, at order $(1/\sqrt{g})^n$,

$$R \equiv \prod_{j=1}^{n} \frac{\int_{-\infty}^{\infty} d\varphi_j \exp\{-i[g\Delta X\varphi_j^4 - \alpha_j\sqrt{g}\,\varphi_j^2]\}}{\int_{-\infty}^{\infty} d\varphi_j \exp\{-ig\Delta X\varphi_j^4\}} \quad . \qquad \text{(AIII.1)}$$

The numerator integrals are given by

$$N_j(\alpha_j) \equiv \int_{-\infty}^{\infty} d\varphi_j \exp\{-i[g\Delta X\varphi_j^4 - \alpha_j\sqrt{g}\,\varphi_j^2]\}$$

$$= \frac{1}{2}\left(\frac{\alpha_j}{-\Delta X\sqrt{g}}\right)^{\frac{1}{2}} e^{i\alpha_j^2/8\Delta X} K_{1/4}(i\alpha_j^2/(-)^2 8\Delta X) \qquad \text{(AIII.2)}$$

where $K_{1/4}$ is the modified Bessel function of the second kind of order 1/4. In defining the denominator of (AIII.1), we may simply write

$$\int_{-\infty}^{\infty} d\varphi_j \exp\{-ig\Delta X\varphi_j^4\} \equiv \lim_{\alpha_j \to 0} N_j(\alpha_j)$$

$$= \lim_{\alpha_j \to 0} \frac{1}{2^{7/4}}\left(\frac{\alpha_j}{-\Delta X\sqrt{g}}\right)^{\frac{1}{2}} \Gamma(1/4) \frac{((-)^2 8\Delta X)^{\frac{1}{4}}}{(i\alpha_j^2)^{1/4}}$$

$$= \frac{\Gamma(1/4)}{2i^{1/4}} \frac{1}{(\Delta Xg)^{1/4}} \qquad \text{(AIII.3)}$$

if $-\pi < \arg z \leq \pi$ for the complex variable z. Here, we have used the result that

$$K_{1/4}(z) \underset{z \to 0}{\to} \Gamma(1/4)\, 2^{-3/4} z^{-1/4} \quad \text{for } |\arg z| < \pi \quad . \qquad \text{(AIII.4)}$$

Indeed, by rotating the positive real φ_j axis to the axis arg $\varphi_j = -\pi/8$, we have

$$\int_{-\infty}^{\infty} d\varphi_j \exp\{-ig\Delta X\varphi_j^4\} = 2 \int_0^{\infty} d\rho\, e^{-i\pi/8} \exp\{-g\Delta X\rho^4\}$$

$$= \frac{\Gamma(1/4)}{2i^{1/4}\,(\Delta Xg)^{1/4}} \quad , \tag{AIII.5}$$

in agreement with (AIII.3).

Now, on taking the limit $\Delta X \downarrow 0$ in the numerator factor N_j, we have

$$N_j(\alpha_j) \underset{\Delta X \downarrow 0}{\rightarrow} \frac{1}{2}\left(\frac{-4\pi}{i\alpha_j\sqrt{g}}\right)^{\frac{1}{2}} = \left(\frac{-\pi}{i\alpha_j\sqrt{g}}\right)^{\frac{1}{2}} , \tag{AIII.6}$$

since

$$K_{1/4}(z) \underset{|z|\rightarrow\infty}{\rightarrow} e^{-z}\sqrt{\frac{\pi}{2z}} \text{ for } |\arg z| < \pi \quad . \tag{AIII.7}$$

Thus,

$$R \rightarrow \prod_{j=1}^{n} \frac{(-\pi/i\alpha_j\sqrt{g})^{\frac{1}{2}}}{(\Gamma(1/4)/2i^{1/4}(\Delta Xg)^{1/4})}$$

$$= \prod_{j=1}^{n} \frac{2i^{3/4}\sqrt{\pi/\alpha_j}\;(\Delta X)^{\frac{1}{4}}}{\Gamma(1/4)} \tag{AIII.8}$$

where $i^{3/4}$ is defined on the branch specified above: $-\pi < \arg i \leq \pi$.

From (AIII.8), we see that in equations (3.17) - (3.55) in the text, the branch of $z^{1/4}$ to be used in computing $i^{1/4}$ is given by

$$(2n-1)\pi < \arg z \leq (2n+1)\pi \tag{AIII.9}$$

such that

$$(e^{2n\pi i}i)^{-\frac{1}{4}} \equiv (e^{(2n+\frac{1}{2})\pi i})^{-\frac{1}{4}} = i^{3/4} \equiv e^{3\pi i/8} \quad . \tag{AIII.10}$$

This gives

$$-\frac{n}{2}-\frac{1}{8}=\frac{3}{8}$$

or

$$n=-1 \quad . \tag{AIII.11}$$

Hence,

$$i^{1/4} = e^{i\pi/8 \,-\, i\pi/2} = e^{-3\pi i/8} \tag{AIII.12}$$

in (3.17) - (3.55).

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5. Our use of the word "small" in the proof of the claim in Sec. II requires some comment. For, we do not simply mean that a term in $\mathcal{L}$ is small if its coupling coefficient is small. Rather, we intend that, after rescaling the theory, one separates out the small operators from those which are large and thereby expands the respective functional in a meaningful way. In the example treated in Sec. III, we expand scalar field theory, (3.4), in powers of $\pi^2/8\sigma\sqrt{g}$ and integrate out the usual free Lagrangian terms completely, since $-\int d^4x\,\pi^2/8\sigma\sqrt{g}$ would appear to be the small part of the action in (3.4). In this way, we have departed from the formal approach of Hori[9], who attempted to formally expand the free Lagrangian terms against the interaction terms, a procedure which would appear suspect. We thank Dr. U. Bar-Gadda for calling the work of Hori to our attention.

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FIG. 1.

Caption: The open curve C encircles the nonnegative integers and is otherwise infintesimally close to the real axis. The line C' is the line Rez = $-\delta$, where $1 \gg \delta > 0$.

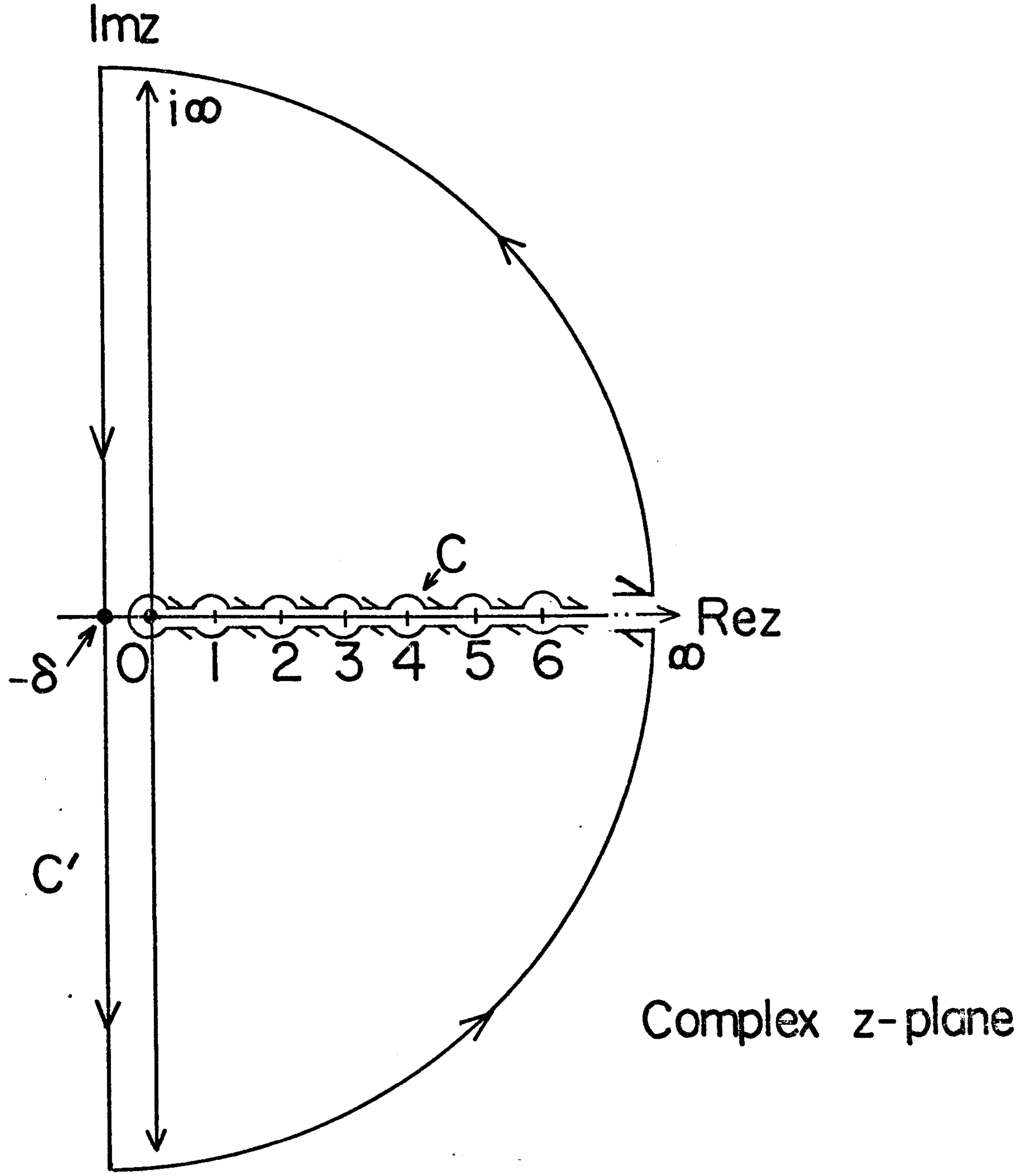

Fig. 1