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HIGHER ORDER MULTIPOLE MAGNET TOLERANCES*

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I. Introduction

Due to field impurities in the magnets in a storage ring or circular accelerator the values of the betatron frequencies for a given particle in a beam are dependent upon the energy and betatron amplitude of the particle as well as the values of the energy dispersion and betatron functions at the magnets. A method has been developed for finding the values of the betatron frequencies for any particle with given field impurities. This method has been used to study the quality of several preliminary designs of some of the quadrupole magnets in PEP by comparing the variations of the betatron frequencies over the maximum expected range of values of the particle energy and betatron amplitude.

The expressions for the values of betatron frequencies as functions of the various beam and machine parameters are derived in Section II. Some of the results for the evaluation of two types of the PEP magnets are presented in Section III. A discussion of these results is given in Section IV.

II. Betatron Frequency Shifts

In this analysis the contribution to the betatron frequency shift $\Delta\nu$ from the equilibrium orbit shift or the betatron oscillation in the vertical direction y is ignored. The values of $\Delta\nu_x$ and $\Delta\nu_y$ are functions of: x_e, the equilibrium orbit of a particle in the radial direction; x_β, the radial betatron oscillation amplitude about x_e; η and β, the values of the energy dispersion and betatron functions at the magnets. For a particle with energy $E = E_0 + \Delta E$, we define $\delta = \Delta E/E_0$ with E_0 the design energy of the machine.

Under these conditions the equations of motion for a single particle are:

$$\frac{d^2x}{ds^2} + K_x(s)x \approx -\frac{\Delta B_y(x,s)}{B\rho} + \frac{\delta}{\rho} \tag{1}$$

and

$$\frac{d^2y}{ds^2} + K_y(s)y \approx \frac{\frac{\partial}{\partial x}\Delta B_y(x,s)}{B\rho}\, y \tag{2}$$

where s is the longitudinal coordinate measured along the central orbit, $B\rho$ is the particle rigidity and ΔB_y is the field error in the magnets evaluated in the median plane.

The solution of Eq. (1) can be written as

$$x = x_e + x_\beta$$

with x_e satisfying the equation

$$\frac{d^2x_e}{ds^2} + K_x(s)x_e = -\frac{\Delta B_y(x_e,s)}{B\rho} + \frac{\delta}{\rho} \tag{3}$$

along with the periodic condition

$$x_e(s+L) = x_e(L)$$

where L is the length of one superperiod of the machine. In practice, the first term on the right hand side of Eq. (3) is small compared to the second term so that we can approximate x_e by $\eta\delta$.

Subtracting Eq. (3) from Eq. (1) we obtain the equation of motion for x_β:

$$\frac{d^2x_\beta}{ds^2} + K_x(s)x_\beta = -\frac{\Delta B_y(x_\beta+x_e,s) - \Delta B_y(x_e,s)}{B\rho} \tag{4}$$

By a suitable transformation Eq. (4) can be written in terms of the Courant-Synder variables[1] as

$$\frac{d^2}{d\theta^2}\xi + \nu_x^2\xi = F(\theta,\xi) \tag{5}$$

where

$$F(\theta,\xi) = -\frac{\nu_x^2\beta_x^{3/2}(\theta)}{B\rho}\left[\Delta B_y\left(x_e+\xi\sqrt{\beta_x(\theta)},\theta\right) - \Delta B_y(x_e,\theta)\right]$$

$$\xi = x_\beta\beta_x^{-1/2} \tag{6}$$

$$\theta = \int_0^s \frac{ds'}{\nu_x\beta_x(s')}$$

$$\nu_x = \frac{1}{2\pi}\int_0^C \frac{ds'}{\beta_x(s')}$$

and

C = machine circumference.

To find the value of $\Delta\nu_x$ we consider the amplitude and phase of ξ:

$$\xi = \sqrt{J}\cos\phi$$

and

$$\xi' = -\nu_x\sqrt{J}\sin\phi\ .$$

These quantities satisfy the differential equations:

$$\frac{dJ}{d\theta} = -\frac{2}{\nu_x}\sqrt{J}\sin\phi\ \ F\left(\theta,\sqrt{J}\cos\phi\right) \tag{7}$$

and

$$\frac{d\phi}{d\theta} = \nu_x - \frac{\cos\phi}{\nu_x\sqrt{J}}\,F\left(\theta,\sqrt{J}\cos\phi\right)\ . \tag{8}$$

Now if we ignore all resonance effects we expect that both J and $(\phi - \nu_x\theta)$ vary slowly in θ. Thus the shift in betatron frequency is given by

$$\Delta\nu_x = \left(\frac{d\phi}{d\theta}\right)_0 - \nu_x$$

$$\approx -\frac{1}{4\pi^2\nu_x\sqrt{J_0}}\int_0^{2\pi}d\theta\int_0^{2\pi}d\phi\cos\phi\ F\left(\theta,\sqrt{J_0}\cos\phi\right) \tag{9}$$

where the subscript 0 denotes the value of J and ϕ' averaged over ϕ and θ. The value of this shift of course will not be correct for the case where $\Delta\nu$ is so large as to place the frequency on resonance. We define the corresponding average peak betatron amplitude $\hat{x}_\beta = (\beta_x J_0)^{1/2}$.

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By a similar analysis the value of $\Delta\nu_y$ can be found from Eq. (2):

$$\Delta\nu_y \approx -\frac{\nu_y}{8\pi^2 B\rho}\int_0^{2\pi} d\theta\, \beta_y^2(\theta)\int_0^{2\pi} d\phi\, \frac{\partial}{\partial x}\Delta B_y\left(x_e+\hat{x}_\beta \cos\phi\right) \tag{10}$$

In practice it is often convenient to express the field errors in a power series in x. Furthermore if the η and β functions do not vary appreciably over the length of the magnets, we can approximate the field errors by

$$\Delta B_y(x, s) \approx \sum_n a_n\, x^n \ell_i\, \delta(s-s_i) \tag{11}$$

where s_i and ℓ_i are the location and length of the ith magnet. Under these assumptions the values of $\Delta\nu_x$ and $\Delta\nu_y$ caused by the field error in ith magnet are given by:

$$\Delta\nu_x \approx \frac{\ell_i\,\beta_{xi}}{2\pi B\rho}\sum_n a_n \sum_{k=1}^{n} (\delta\eta_i)^{n-k}\,(\hat{x}_\beta)^{k-1}\,\frac{n!}{k!\,(n-k)!}\,C_{k+1} \tag{12}$$

and

$$\Delta\nu_y \approx -\frac{\ell_i\,\beta_{yi}}{4\pi B\rho}\sum_n a_n \sum_{k=1}^{n} (\delta\eta_i)^{n-k}(\hat{x}_\beta)^{k-1}\,\frac{n!}{(n-k)!\,(k-1)!}\,C_{k-1} \tag{13}$$

where

$$C_k = \begin{cases} 0 & \text{if k odd} \\ \dfrac{(k-1)(k-3)\ldots 1}{k(k-2)\ldots 2} & \text{if k even} \end{cases} \tag{14}$$

III. PEP Magnet Study

The magnetic lattice for PEP has been designed to operate over a wide range of energy and configurations with $3.8\text{ m} \le \beta_x^* \le 7.0\text{ m}$, $.16\text{ m} \le \beta_y^* \le .2\text{ m}$ and $-2.2\text{ m} \le \eta^* \le 0\text{ m}$ where an asterisk indicates the values at the interaction point. The low-beta interaction region is obtained by using a pair of strong quadrupole doublets (Q3 and Q2) symmetrically located about the interaction point; Q3 is a vertical focussing magnet nearest to the interaction point and Q2 is a radially focussing magnet. The values of the betatron functions β_y and β_x are largest at Q3 and Q2, respectively. Typically $\beta_{y\,max}$ is about 600 meters and $\beta_{x\,max}$ is about 200 meters. Because of these properties, the effects of the field errors in these magnets must be carefully assessed.

Three designs A, B and C for the magnets Q3 and Q2 have been proposed by the magnet design group in LBL.[2] These designs were obtained by using a computer solution for different pole shapes and current distributions. Due to the symmetry of the magnet geometry used in the designs the field errors normalized to the ideal field value gx can be expressed in a power series as:

$$\left(\frac{B_y}{gx} - 1\right) = \sum_{n=1}^{N} A_n\left(\frac{x}{x_0}\right)^{4n} \quad \left(x \le x_0,\ y=0\right) \tag{15}$$

The values of A_n's depend upon the value of x_0 and N unless the circle of radius x_0 does not intersect the magnet poles. The values used are shown in Table I.

A plot of the normalized field errors is given in Fig. 1. For each case the values of $\Delta\nu_x$ and $\Delta\nu_y$ are computed[3] using Eqs. (12) and (13) over the region $0 \le |\delta/\sigma_e| \le 12$ and $0 \le \hat{x}_\beta/\sigma_x \le 12$, where σ_e and σ_x are the standard deviation values for gaussian distributions for the synchrontron and betatron motion, respectively. Some of the results for design B are shown in Figs. 2 to 5 for a typical configuration with $\beta_x^* = 3.8$ m, $\beta_y^* = .2$ m, $\eta^* = -.7$ m, $E_0 = 15$ GeV and

Table I

Values of A_n's for the power series expansions of the normalized field errors with $x_0 = 8.0$ cm and N = 6.

n	Design A	Design B	Design C
1	3.5×10^{-5}	7.9×10^{-5}	1.8×10^{-4}
2	5.8×10^{-5}	-9.3×10^{-4}	-1.8×10^{-3}
3	-4.3×10^{-4}	4.3×10^{-3}	6.5×10^{-3}
4	-4.1×10^{-3}	-6.4×10^{-3}	-7.4×10^{-3}
5	5.9×10^{-4}	-7.4×10^{-4}	-1.4×10^{-3}
6	1.7×10^{-3}	3.6×10^{-3}	4.6×10^{-3}

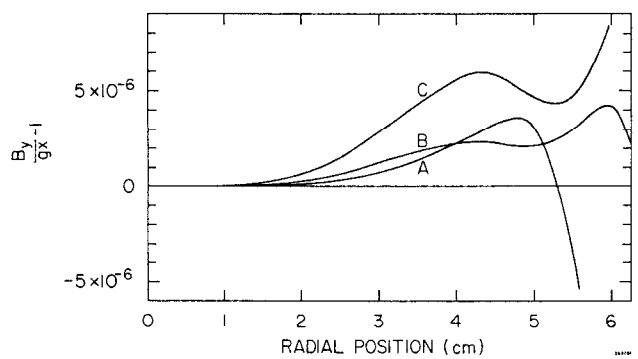

FIG. 1--Values of relative field errors for magnet design A, B and C.

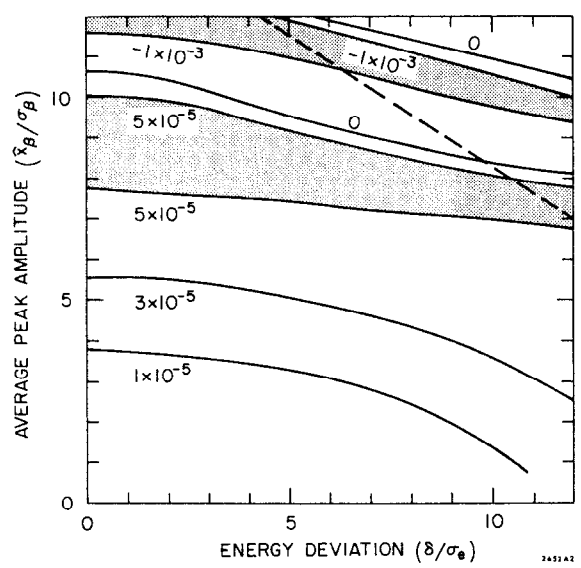

FIG. 2--A map of constant $\Delta\nu_x$ contours due to field errors in Q2 (design B) for a typical machine configuration.

$\nu_x = \nu_y = 18.75$. Figures 2 and 3 give a plot of the constant contour of $\Delta\nu_x$ for errors in Q2 and Q3, respectively. Figures 4 and 5 give a plot of the constant contour of $\Delta\nu_y$ for errors in Q2 and Q3, respectively. The aperture of the machine is indicated in these figures by a dashed line. Particles with values of δ and $\hat{x}_\beta$ above this line are lost in the machine. Similar results have been found for the other designs and for other configurations.

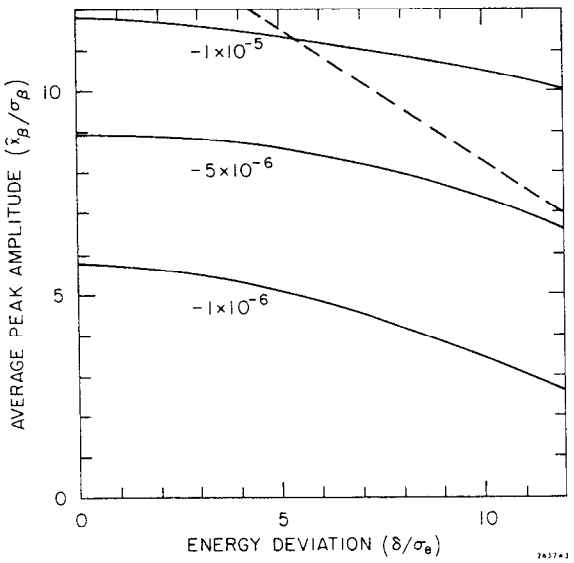

FIG. 3--A map of constant $\Delta\nu_x$ contours due to field errors in Q3 (design B) for a typical machine configuration.

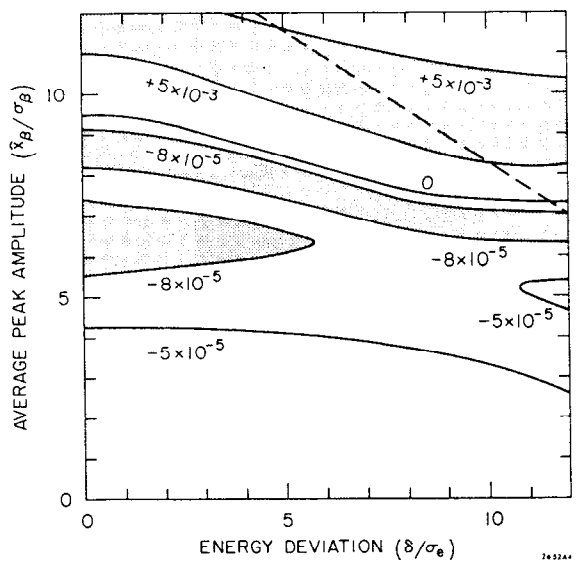

FIG. 4--A map of constant $\Delta\nu_y$ contours due to field errors in Q2 (design B) for a typical machine configuration.

A measure of the effects on the betatron frequency shifts may be indicated by the values of $\Delta\nu$ for particles on the dashed line. As an example, the range of values of $\Delta\nu$ for the configuration under consideration are shown in Table II for the three designs.

IV. Discussion

It is important to note that the values of $\Delta\nu$ obtained by this method is the sum of the $\Delta\nu$ values for all the multipole terms in the series expansion of the field errors. Comparison of the $\Delta\nu$ values of a given multipole term for the different designs does not give any indication of the relative quality of the designs. For example, from Table I it can be seen that most of the multipole coefficients for design B are greater in magnitude than those for design A. However, it can be seen from Table II that design B is superior to design A. This is resulted from the fact that there is more cancellation of the contributions from the different multipole terms in design B than in design A.

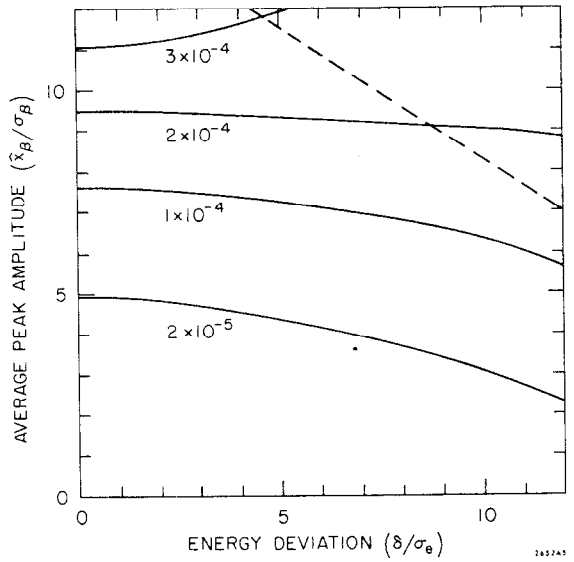

FIG. 5--A map of constant $\Delta\nu_y$ contours due to field errors in Q3 (design B) for a typical machine configuration.

Table II

Range of values of $\Delta\nu$ for particles whose maximum radial position is equal to the radial machine aperture.

	Design A		Design B		Design C	
Magnet	Q2	Q3	Q2	Q3	Q2	Q3
$\Delta\nu_x$	$-.6\times10^{-3}$ to -1.0×10^{-2}	$-.4\times10^{-5}$ to -1.0×10^{-5}	-1.0×10^{-3} to 6.0×10^{-5}	$-.7\times10^{-5}$ to -1.2×10^{-5}	$.2\times10^{-3}$ to 7.0×10^{-3}	-1.5×10^{-5} to -3.0×10^{-5}
$\Delta\nu_y$	$.5\times10^{-2}$ to 1.0×10^{-1}	$.9\times10^{-4}$ to 3×10^{-4}	$-.8\times10^{-4}$ to $.7\times10^{-2}$	1.4×10^{-4} to 3.2×10^{-4}	-4.0×10^{-4} to -1.0	3×10^{-4} to 8×10^{-4}

By comparing Figs. 3 and 4 it can be seen that for a given design the constant contour map of $\Delta\nu$ values can be different in character. For most cases the value of $\Delta\nu$ increases as the value of δ or $\hat{x}_\beta$ increases. However, for some cases the change in $\Delta\nu$ is not monotonic (see Fig. 4). For such cases maximum values of $\Delta\nu$ may not occur at the aperture limit and a detailed study of the $\Delta\nu$ contour is necessary.

In general, in order to evaluate a given magnet design first the aperture of the machine must be defined. The aperture limit may not appear as the same function of δ and $\hat{x}_\beta$ since the particles may be lost at different locations depending upon the machine configuration. The values of $\Delta\nu$ are calculated for all the desired machine configurations and the maximum values of $\Delta\nu$ within the aperture limit are noted for each configuration. If the values of $\Delta\nu_{max}$ are less than the tolerable value of $\Delta\nu$ then the design is acceptable. The tolerance for the value of $\Delta\nu_{max}$ is dependent upon the width of the machine resonances near the operating betatron frequencies. For a storage ring the aperture of the machine may be determined by the injection conditions, the allowed closed orbit deviations and the desired beam life time.

V. References

1. E. D. Courant and H. S. Synder, Ann. Phys. (N. Y.) 3, 1 (1958).
2. Robert Avery and Klaus Halback, Lawrence Berkeley Laboratory, private communication.
3. A. S. King, M. J. Lee and W. W. Lee, MAGIC—A Computer Code for Design Studies of Insertions and Storage Rings, Stanford Linear Accelerator Center report, to be published.