# A COVARIANT DYNAMICAL CALCULATION <br> OF THE NUCLEON-NUCLEON S-WAVES* 

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## ABSTRACT

We compute the binding energy of the ${ }^{3} \mathrm{~S}_{1}$ and ${ }^{1} \mathrm{~S}_{0}$ NN states (known to be bound by 2.22 and -0.07 MeV ) using a covariant singular core three-body model of the $N N \pi$ system with $r_{c}{ }^{N N}=0.7 \mathrm{fm}, 3_{f_{\infty}}=$ 1.8 and ${ }^{1} f_{\infty}=0.3$ as observed at high energy. For the $\pi N P_{11}$ input we use $\mathrm{r}_{\mathrm{c}}^{\pi \mathrm{N}}=0.18 \mathrm{fm}$, fitted (or $\hbar / \mathrm{Mc}=0.22 \mathrm{fm}$ postulated) and find 3.26 (2.59) for ${ }^{3} \mathrm{~S}_{1}$ and $1.41(0.73)$ for ${ }^{1} \mathrm{~S}_{0}$.
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[^0]If we assume that in first approximation nucleon-nucleon scattering can be treated as an $N N \pi$ system below pion production threshold, and that the short distance (high momentum) behavior of the NN and $\pi N$ subsystems can be unambiguously determined from experiment, the binding energy of the ${ }^{3} \mathrm{~S}_{1}$ ("deuteron," $\epsilon_{d}=2.2 \mathrm{MeV}$ ) and ${ }^{1} \mathrm{~S}_{0}$ ("singlet deuteron," $\epsilon_{0}=0.07 \mathrm{MeV}$ ) states near NN elastic scattering threshold can be predicted. The requisite relativistic three-body formalism has recently been developed by one of us, and successfully applied to the $3 \pi$ system to show that the $\rho$ generates the $\omega$ as the only low energy $I=0,1^{-} 3 \pi$ resonance, as well as to the relativistic $\pi$ d problem. ${ }^{2}$ In the NN system the fact that the $N$ can make a transition to a $P_{11} N \pi$ state with the pion then absorbed by the other nucleon gives a type of one-pion-exchange ladder, while the fact that the pion can scatter rather than being absorbed includes two-pion-exchange ladders in the multiple scattering series summed by the integral equation. To the extent that the physical $\pi \mathrm{N}$ amplitude we use reflects "crossed" and "uncrossed" two-pion diagrams and pion-pion scattering (including the "sigma"), we have included these effects without introducing any "renormalization" problems.

The obvious first approximation is to use the nucleon as an s-wave spectator of the $\pi N$ state which contains the nucleon pole ( $\mathrm{P}_{11}$ ), and the pion as a p-wave spectator of the appropriate NN s-wave $\left({ }^{1} S_{0}\right.$ to drive the ${ }^{3} S_{1}$ calculation and ${ }^{3} S_{1}$ to drive the ${ }^{1} \mathrm{~S}_{0}$ calculation). After antisymmetrization in the nucleon variables, our equation takes the form

$$
\begin{equation*}
x_{i}\left(q_{i}^{\prime}\right)=\bar{K}_{i 2}\left(q_{i}^{\prime}\right)+\sum_{j=1}^{2} \int_{0}^{Q_{j}} d q_{j} q_{j}^{2} K_{i j}\left(q_{i}^{\prime}, q_{j}\right) X_{j}\left(q_{j}\right) \tag{1}
\end{equation*}
$$

Here $X_{1}$ and $X_{2}$ represent series of pairwise rescatterings initiated by a $\pi N$-pair at the nucleon pole; $\mathrm{X}_{1}\left(\mathrm{X}_{2}\right)$ corresponds to a final $N N(\pi N)$ scattering. The
variable $q_{j}$ is the three-momentum of the spectator particle in the $c . m$. frame of the pair ( $j=1$ corresponds to a spectator pion, $j=2$ to a nucleon). The notation $\overline{\mathrm{K}}_{\mathrm{i} 2}$ represents the residue of $\mathrm{K}_{\mathrm{i} 2}$ at the nucleon pole $\left(q_{2}=q_{\mathrm{N}}\right)$; the NN amplitude is given by $t_{N N}=X_{2}\left(q_{N}\right)$. Because the two-body asymptotic wave functions start right at the singular cores, there is no region in which an extended "potential energy" forces the scatterings "off-shell", and all particles are always on mass-shell. This explains the one-variable character of the equation, even though the corresponding t-matrices are not separable.

The physical justification for using a singular core model ${ }^{1,2}$ to represent the high energy behavior of our two-body input is the well-known fact that all two-hadron channels can be well approximated by an absorbing disc of constant radius in the particle production region. Alternatively, we can interpret the boundary condition as approximating a rapid transition from a region where quark degrees of freedom are not much affected by the exterior dynamics to the region of free hadrons, or the stable point a quarter-wave outside an internal node in the wave function, which one of us suggested ${ }^{3}$ as a way to connect this model to Neudatchin's discussion of internal structure. Either interpretation allows any empirical result to be represented by an energy-dependent logarithmic derivative of the wave function at that radius $\lambda_{l}\left(k^{2}\right) \equiv\left[1+\ell+f_{l}\left(k^{2}\right)\right] / r_{c}$. The two-body amplitudes are then $t_{l}(\mathrm{k})=\mathrm{N}_{\ell}(\mathrm{k}) / \mathrm{D}_{\ell}(\mathrm{k})$ with

$$
\begin{align*}
N_{\ell}(k) & =\left(r_{c} \lambda_{\ell}(k)-\ell\right) j_{\ell}\left(r_{c} k\right)+r_{c} k j_{\ell+1}\left(r_{c} k\right) \\
D_{\ell}(k) & =i k\left[\left(r_{c} \lambda_{\ell}(k)-\ell\right) h_{\ell}\left(r_{c} k\right)+r_{c} k h_{\ell+1}\left(r_{c} k\right)\right] \tag{2}
\end{align*}
$$

The energy-dependent function $\lambda_{\ell}(\mathrm{k})$ is fitted to scattering data in the physical region $k^{2}>0$; at large $k$ it is taken to approach the constant value $\lambda_{\ell}^{\infty}$. Since $\lambda_{\ell}(k)$ must be meromorphic in $k^{2}$ in order for our formalism to produce unitary
three-body amplitudes, this fit permits analytic continuation of $N_{\ell}, D_{\ell}$ to $\mathrm{k}^{2}<0$. Below we use the notation $N_{\ell}^{\infty}, D_{\ell}^{\infty}$ to denote $N_{l}, D_{\ell}$ evaluated with $\lambda_{\ell} \rightarrow \lambda_{\ell}^{\infty}$. The dominant (singular) part of the kernel is

$$
\begin{gather*}
K_{i j}^{s}\left(q_{i}^{\prime}, q_{j}\right)=\Lambda_{i j} \frac{N_{i j}^{s}\left(q_{i}^{\prime}, q_{j}\right)}{D_{j}\left(k_{j}\right)} \cdot \frac{N_{j}\left(k_{j}\right)}{N_{j}^{\infty}\left(k_{j}\right)}, \\
N_{i j}^{S}\left(q_{i}^{\prime}, q_{j}\right)=-\frac{k_{j}}{\pi} \int_{-1}^{1} d z G_{i j}\left(z, \hat{K}_{i j} \cdot \hat{q}_{j}, \hat{Q}_{i j} \cdot \hat{q}_{j}\right) \frac{g_{i}\left(Q_{i} q_{i}^{\prime}, b_{i} Q_{i j}\right)}{q_{i}^{\prime 2}-Q_{i j}^{2}-i \epsilon} \frac{N_{i}^{\infty}\left(K_{i j}\right)}{Q_{i j}} \tag{3}
\end{gather*}
$$

with $\Lambda_{11}=0, \Lambda_{21}=\Lambda_{12}=-2 / \sqrt{3}$ and $\Lambda_{22}=-1 / 3$. Although the region where all three particles are close together contributes additional finite terms, which have been given explicitly elsewhere, ${ }^{1}$ all the significant dynamics comes from this structure.

In Eq. (3) $k_{j}$ is the $c . m$. momentum of the pair. $N_{j}$ denotes $N_{\ell(j)}$ where $\ell$ is the appropriate angular momentum for that pair. The three-vectors $K_{i j}$, $Q_{i j}$ are the values of $\underline{k}_{i}, \underline{q}_{i}$ in the (i)c.m. frame corresponding to $\underline{k}_{j}, \underline{q}_{j}$ in the (j) frame (only two such momenta are independent). The function $G_{i j}$ is the geometrical recoupling coefficient which would be unity if all the particles were in relative $s$-waves. As usual in such equations we have a Green's function denominator corresponding to free propagation, 4 and a factor related to the "off-shell" structure. In our case this is the product $g_{i} N_{i}$, where $g_{i}$ arises from the sharp cutoff at the pair radii $r_{c}^{i}$ (here simplified by using the slightly larger radial parameter $b_{i}$ ). Explicitly

$$
\begin{equation*}
g_{i}(a, b)=i b\left[b j_{\lambda}(a) h_{\lambda+1}(b)-a j_{\lambda+1}(a) h_{\lambda}(b)\right] \tag{4}
\end{equation*}
$$

Here $\lambda$ is the angular momentum of the spectator relative to the pair ( $\lambda=0$ if $\mathrm{i}=2$ ); note that $\mathrm{g}_{\mathrm{i}}(\mathrm{a}, \mathrm{a})=1$.

Although the concept of the analytic continuation of two-body amplitudes to obtain dynamical equations is a familiar one, our approach differs from field theory or dispersion theory in that we use the three-body equation to specify the requisite analytic continuation. The Lorentz frame we use for each pair is uniquely specified by requiring that the scattering pair remain in its own c.m. system when the spectator recedes to an infinite distance. This definition reduces to the usual one in the nonrelativistic limit, but introduces important kinematic effects in the covariant evaluation of the quantities ${\underset{-1}{i}}^{i}, \underline{K}_{i j}$, and $\underline{Q}_{i j}$, which enter the above equations. For three particles of mass $m_{\alpha}, m_{\beta}, m_{\gamma}$ treated as free outside the region excluded by the cores, and using a real spectator momentum $\mathrm{q} \geq 0$, the $c . \mathrm{m}$. energy for the $\beta \gamma$ pair is (with $\mathrm{m}=\Sigma_{\alpha} \mathrm{m}_{\alpha}$ )

$$
\begin{equation*}
\left(m_{\beta}^{2}+\mathrm{k}^{2}\right)^{\frac{1}{2}}+\left(\mathrm{m}_{\gamma}^{2}+\mathrm{k}^{2}\right)^{\frac{1}{2}}=\left[\mathrm{s}+\mathrm{m}^{2} \mathrm{q}^{2} /\left(\mathrm{m}_{\beta}+\mathrm{m}_{\gamma}\right)^{2}\right]^{\frac{1}{2}}-\left[\mathrm{m}_{\alpha}^{2}+\mathrm{m}^{2} \mathrm{q}^{2} /\left(\mathrm{m}_{\beta}+\mathrm{m}_{\gamma}\right)^{2}\right]^{\frac{1}{2}} \tag{5}
\end{equation*}
$$

with $s=P^{2}$ the invariant four-momentum squared of the three particle system. We see that the upper limit on this energy, and hence on the energy where we need the two-body input for our equation is achieved at $q^{2}=0$, while the lower limit implied by the fact that Eq. (5) can be satisfied only for $\mathrm{k}^{2} \geq$ - $\min \left(\mathrm{m}_{\beta}^{2}, \mathrm{~m}_{\gamma}^{2}\right)$ fixes an upper limit $\mathrm{q}=\mathrm{Q}_{\alpha}$ (infinite if $\mathrm{m}_{\beta}=\mathrm{m}_{\gamma}$ ). Since the $\mathrm{c} . \mathrm{m}$. energy of the $\beta \gamma$ pair is bounded by $\sqrt{s}-m_{\alpha}$, we see that any three-body treatment of the NN system requires two body input always a pion mass below the two-body output to be computed. In order to calculate NN scattering near elastic threshold $(\sqrt{\mathrm{S}} \sim 2 \mathrm{M})$, we require only NN input for $-\mathrm{M}^{2} \leq \mathrm{k}_{\mathrm{NN}}^{2} \leq-\mathrm{M} \mu(1-\mu / 4 \mathrm{M})$, and $\pi \mathrm{N}$ input in the narrow range, whose upper end is the position of the nucleon pole, $-\mu^{2} \leq \mathrm{k}_{\pi \mathrm{N}}^{2} \leq-\mu^{2}\left(1-\mu^{2} / 4 \mathrm{M}^{2}\right)$. These amplitudes are obtained from NN and $\pi \mathrm{N}$ scattering data analytically continued to the required region via $\lambda_{l}\left(\mathrm{k}^{2}\right)$, or
more conveniently $\mathrm{f}_{\ell}=\lambda_{\ell} \mathrm{r}_{\mathrm{c}}+\ell+1$. As is common in three-body equations, the left-hand cut structure enters only indirectly through the "off-shell" behavior, so our requirement that $\lambda_{l}$ be a meromorphic function of $k^{2}$ makes the extrapolation essentially unique.

For the ${ }^{3} S_{1}$ input parameters we use the ${ }^{3} S_{1}-{ }^{3} D_{1}$ coupled channel fit of Feshbach and Lomon ${ }^{5}$ with $\mathrm{r}_{\mathrm{c}}^{\mathrm{NN}}=0.70 \mathrm{fm},{ }^{3} \mathrm{f}_{\infty}=1.8$. We find that using the same core radius we can fit ${ }_{S_{0}}$ amplitudes up to 1 GeV with the simple parametrization ${ }^{1} \mathrm{f}(\mathrm{k})=0.30-0.271\left(1+1.16 \mathrm{k}^{2}+\mathrm{k}^{4}\right)^{-1}$ provided we take care to use a coupled channel formalism above pion production threshold and identify the eigenphase with the real part of the elastic scattering phase. Clearly only the core radius and $f_{\infty}$ are significant in the kinematic region needed for our calculation as specified above. By one iteration of the coupled system we can eliminate explicit reference to the $\mathrm{X}_{1}$ amplitude and isolate the usual OPE amplitude as the leading term in $t_{N N}$. Since the same term would occur if we were calculating higher partial waves, we can identify the coefficient as the constant $G_{n p \pi}^{2}$ measured in nucleon-nucleon scattering. This implies, for our simple model, that the residue at the nucleon pole in the $\dot{P}_{11}$ state is $G_{n p \pi}^{2}(1+1.5 \mu / M)^{-1}$. It is important to note that the integral in our iterated equation for $t_{N N}$ ends at $\mathrm{Q}_{2} \simeq \mathrm{~m}_{\pi}$ rathe than infinity. Therefore we cannot define an equivalent Lippmann-Schwinger equation or NN "potential" in any meaningful way. Of course we still have a meaningful nucleon-nucleon wave function in coordinate space, and via the coupled equation a wave function for the pion coordinate in this system as well. Since the form of the equation is the same for the ${ }^{1} \mathrm{~S}_{0}$ and the ${ }^{3} S_{1}$ amplitudes, and $r_{c}^{N N}$ is (empirically) the same for both, the splitting we find between these two states comes solely from the difference between ${ }^{1} f_{\infty}$ and
$3_{\mathrm{f}_{\infty}}$ in the input. We mirror the "tensor force" only through the difference between these two parameters empirically observed at high (i.e., $\mathrm{T}_{\mathrm{lab}}^{\mathrm{NN}} \geq 280 \mathrm{MeV}$ ) energy.

The input for the $P_{11}$ amplitude presents more of a problem, since the nucleon pole is only a pion mass below $\pi N$ threshold and, in contrast to the NN situation, we are most sensitive to data up to about a pion mass above threshold, where they are poorly known. We know the position of the pole and (as noted above) the residue at that pole in terms of $\mathrm{G}^{2}$, so the simplest fit has only $\mathrm{r}_{\mathrm{c}}^{\pi \mathrm{N}}$ as a parameter. Using the recent analysis of Carter, Bugg, and Carter ${ }^{6}{ }^{\prime}$ we obtain a good fit in the $\chi^{2}$ sense using all data up to 310 MeV , but for $\mathrm{G}^{2}=$ either 14.6 or 15.3 over half the $\chi^{2}$ comes from the 310 MeV point where the phase is starting to head toward the Roper resonance. For those who are bothered by our $G^{2}$ not being the same in $\pi N$ and NN scattering, we note that our approach will yield this only when we include antinucleons explicitly; empirically we note that Ball, Shaw, and Wong ${ }^{7}$ in fitting $P_{11}$ alone required a smaller value for $G^{2}$ than is usually observed either in NN scattering or forward $\pi N$ dispersion relations. So we present results for two values of $\mathrm{G}^{2}$, with and without the highest energy point. We also use a model with $r_{c}^{\pi N}$ fixed at the value $\hbar / \mathrm{Mc}=0.22 \mathrm{fm}$ and the value which gives about the right binding energy for the deuteron ( 0.234 fm ). More reliable results will have to await a better theoretical understanding of the energy dependence of the $P_{11}$ state, or an accurate value of the scattering length $a_{11}$, or preferably both.

The results of the calculation are given in Table I. We see that in spite of the uncertainties engendered by the uncertainty in the $P_{11}$ amplitude, the most significant features of the nucleon-nucleon S-waves - namely, two bound states close to zero in units of the pion mass and split by approximately 2 MeV - are
stably reproduced. Considering the simplicity of the model employed and its close connection to empirical results found in quite different experiments and hadronic phenomena, we find this close agreement with experiment truly remarkable.

In order to improve on our calculation we must include additional threeparticle states. The two states we think of next most importance to the ${ }^{3} \mathrm{~S}_{1}$ calculation are $\mathrm{P}_{13}$ coupled to an s -wave nucleon spectator and $\mathrm{P}_{11}$ coupled to a dwave spectator; since they are of approximately equal magnitude and of opposite sign, we expect the prediction for $\epsilon_{d}$ to change very little. The one state we would add to the ${ }^{1} S_{0}$ calculation is ${ }^{3} \mathrm{P}_{0}$ with an s-wave pion spectator; this will make a repulsive contribution, and could push the $\epsilon_{0}$ prediction up to being just virtual. Eq. (5) shows that we need only include elastic two-body amplitudes as input for output below the two-pion production threshold. If we go to higher energy or more particles the most important four-body channels will be those in which the system could separate into two "interacting" subsystems. This configuration is dominated by momenta such that at least one $\pi \mathrm{N}$ pair is near the nucleon pole. If it is precisely at the pole, that pair looks like a nucleon, and we are back to the problem already considered. Therefore we expect four-body corrections to be small.

Although we have not "derived" our input parameters from an elementary particle theory, it is suggestive that the NN radius has to be $\hbar / 2 \mathrm{~m}_{\pi} \mathrm{c}$, the usual estimate of where the NN channel gets lost among other hadronic degrees of freedom; the $N \pi$ radius is close to $\hbar / M_{N} c$, where the problem also becomes ultrarelativistic. In any case, our consistent covariant treatment of the NNT system using the singular core approach to the three-body problem allows us to understand why the nucleon-nucleon S-waves have bound states close to NN threshold, and to obtain a reasonable first approximation to them without any adjustment of the input parameters.

## REFERENCES

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## TABLE I

Dependence of the Results on $r_{c}^{\pi N}$
for Various Assumptions about the $P_{11} \pi N$ State
$\mathrm{r}_{\mathrm{c}}^{\pi \mathrm{N}}(\mathrm{fm}) \quad \mathrm{G}^{2} \quad \chi^{2} \quad \epsilon_{\mathrm{d}}(\mathrm{MeV}) \quad \epsilon_{0}(\mathrm{MeV})$
0.180
14.6
$8.8^{a}$
3.26
1.41
0.186
14.6
$4.2^{b}$
3. 14

1. 34
0.196
15.3
$9.0^{\mathrm{a}}$
2.96
1.10
0.198
15.3
$3.0^{\mathrm{b}}$
3.02
2. 17
0.220
15.3
2.59
0.73
0.234
15.3
3. 24
0.48
[^1]
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[^1]:    ${ }^{\text {b }} 9$ points from Ref. 6 (see text).

