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Theory of quantum metal to superconductor transitions in highly conducting systems

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Abstract

We derive the theory of the quantum (zero temperature) superconductor to metal transition in disordered materials when the resistance of the normal metal near criticality is small compared to the quantum of resistivity. This can occur most readily in situations in which "Anderson's theorem" does not apply. We explicitly study the transition in superconductor-metal composites, in an s-wave superconducting film in the presence of a magnetic field, and in a low temperature disordered d-wave superconductor. Near the point of the transition, the distribution of the superconducting order parameter is highly inhomogeneous. To describe this situation we employ a procedure which is similar to that introduced by Mott for description of the temperature dependence of the variable range hopping conduction. As the system approaches the point of the transition from the metal to the superconductor, the conductivity of the system diverges, and the Wiedemann-Franz law is violated. In the case of d-wave (or other exotic) superconductors we predict the existence of (at least) two sequential transitions as a function of increasing disorder: a d-wave to s-wave, and then an s-wave to metal transition.

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I. INTRODUCTION

The quantum (zero temperature) transition from a superconducting to a non-superconducting ground state is the poster child of quantum phase transitions. This transition is induced by changing external parameters at zero temperature T.

In this article we consider three representative problems in which a direct quantum phase transition occurs from a superconducting to a metallic phase in which $k_F l \gg 1$: the case of a composite of a superconducting and a non-superconducting metal in which the effective interaction between electrons changes sign as a function of position (Section II), the case when s-wave superconductivity is destroyed by an external magnetic field (Section III), and the case when d-wave (or other exotic) superconductivity is destroyed by quenched disorder (Section IV). Here k_F is the Fermi wave-vector, and l is the electron mean free path.

In three dimensions (3D), there is no question concerning the existence of both a superconducting and a metallic phase. In 2D, the existence of a metallic phase is problematic. However, for $k_F l \gg 1$, single particle localization occurs on such large length scales that its effects are mostly unobservable. Therefore, for most of this article, we will ignore the fundamental, but for our purposes purely academic question of whether or not a 2D interacting system of electrons can ever exhibit a metallic state in the asymptotic limit of zero temperature and infinite volume.

Before proceeding to discuss the findings of the present study, it is worth commenting briefly on the broader context. The central insight underlying the modern theory of critical phenomena is that, due to the divergent correlation length at criticality, the properties of the critical state are "universal." An important extension of this is the idea that, in systems with quenched disorder, the variations in local environments are self-averaging, so a near-critical system can be treated in terms of an effective, translationally invariant field theory [1, 2]. While this approach has had notable successes for the theory of classical phase transitions, it is more problematic in the case of quantum critical phenomena. This is most dramatically illustrated by the case[3, 4, 5] of the quantum critical point in the random transverse-field Ising model, where the physics of "rare events" results in the existence of a "quantum Griffith phase" in which, for a finite interval of parameters including the critical point, the susceptibility diverges as $T \to 0$. For somewhat analogous reasons, a generic feature that characterizes the transitions in all three cases mentioned above is that, at criticality, the spatial distribution of the superconducting order parameter is highly inhomogeneous. It is concentrated in "superconducting puddles" where, due to randomness, superconducting order is locally anomalously favorable, and the distance between "optimal" puddles is parametrically large. The transition occurs when the Josephson coupling between optimal puddles (which falls with a power of the separation) times the exponentially large local superconducting susceptibility on a puddle is strong enough to stabilize a macroscopically phase coherent state.

Near enough to the quantum phase transition and at low enough temperatures, where the correlation length is large compared to the distance between superconducting puddles, there is still, presumably, universal behavior described by appropriate critical exponents. However, a consequence of the large distance between optimal puddles is that this universal quantum critical regime is parametrically narrow. Conversely, there is an anomalously broad portion of the phase diagram in which the correlation length is comparable to or smaller than the distance between optimal puddles, but large compared to the size of an optimal puddle, where quantum fluctuations of the superconducting order-parameter dominate much of the physics. This broad quantum but not quantum critical regime is one of the characteristic signatures of the inhomogeneous nature of the critical state.

It is obvious that the presence of significant superconducting correlations in the "metallic" state near to the superconductor to metal critical point makes it highly anomalous: Its zero temperature conductivity diverges at the point of the transition, and can be much larger than the Drude value "near" criticality. In the same regime, the Hall conductivity decreases with respect to the Drude value and vanishes at the point of the transition. The Wiedemann-Franz law in such metals is also clearly violated.

One remarkable implication of the present analysis is that, in the case when d-wave superconductivity is destroyed by disorder, there are (at least) two quantum transitions: the first from a globally d-wave to a globally s-wave (although, possibly, still locally d-wave) state, and the second to the "normal" metal. Another outcome of this picture is peculiar temperature dependencies of the physical parameters of the near critical superconductor. Some of our findings are summarized in the schematic phase diagrams shown in Figs. 1-3.

To conclude this introduction, we would like to discuss the relation between our paper and those [6, 7, 8] in which it has been proposed that the quantum transition, especially in 2D, takes place between the superconducting and an insulating state. Several lines of reasoning led to the inference that near the $T = 0$ quantum critical point $k_F l \approx 1$, in which case localization (which we neglect) would necessarily be a serious issue:

1) Where Anderson's theorem applies at the level of mean-field theory, such as in the case in which s-wave superconductivity is destroyed by increasing disorder, the localization length must be comparable to or shorter than the coherence length in order for the disorder to have any substantial effect on T_c, at all [9].

2) It has been shown that in a system of superconducting grains linked by resistively shunted Josephson junctions, quantum fluctuations of the order parameter [6, 10, 11, 12] are strongly suppressed so long as $G^{(eff)} \gg 1$, independently of the strength of the Josephson coupling between puddles! Here $G^{(eff)}$ is a dimensionless shunt conductance measured in units e^2/h. An apparent implication of this result is that, so long as a small portion of a highly conducting system is superconducting, at low enough temperature the system will achieve global phase coherence so long as the dimensionless effective conductance, $G^{(eff)}$, is large compared to 1.

3) It was found in several theoretical studies [6, 7, 8] of 2D bosonic models of the transition that there is a universal value of $G^{(eff)} = G_c \sim \mathcal{O}(1)$ at the point of superconductor-metal transition. (These models assume the absence of gapless quasiparticle excitations, and therefore ignore dissipation of the sort that is represented by the shunt resistance in the previously discussed models.)

4) A large portion of the experimental realizations of such transitions involve two dimensional (2D) systems, such as films. Here, the transitions are often referred to as a "superconductor to insulator transition" [6] on the basis of the widely held theoretical belief that metallic phases are forbidden in 2D due to single particle localization [13, 14] – any non-superconducting phase is thus expected be insulating at zero-temperature.

Concerning point 1), in the present paper, for the most part, we consider problems in which Anderson's theorem does not apply, either due to the symmetry of the order parameter (*e.g.* d-wave) or the breaking of time reversal symmetry (*e.g.* by an applied magnetic field). Concerning point 3), we consider cases in which gapless quasiparticles are present near criticality, so the applicability of a bosonic model is questionable. Concerning point 4), as mentioned above, 2D localization is negligible, and hence a non-superconducting phase is "effectively metallic," whenever the parameter $k_F l$ is sufficiently large, since in this limit, the localization length is exponentially large, $\xi_{loc} \sim l \exp(\frac{\pi}{2} G_{2D})$. Here $G_{2D} \propto k_F^2 l d \gg 1$ is the dimensionless conductance per square measured in units of e^2/h of a 2D film of thickness d. (For a review, see [15].) Finally, concerning point 2), a large portion of the discussion in the present paper follows from the same considerations. The differences between the present results and those of the earlier studies spring from the fact that the effective model we develop from microscopics differs in a subtle manner from the phenomenological Ohmic heat bath considered in those earlier studies. This difference permits a transition to a phase incoherent state even under conditions in which $G^{(eff)} \gg 1$; however, a residual consequence of the same physics discovered in those earlier studies is that this transition occurs when the superconducting puddles are extremely dilute and so are weakly Josephson coupled to one another. This is one of the central results of the present study.

A. Effective action for the quantum superconductor-metal transition

In the present section, we develop the general features of the effective actions that govern the quantum fluctuations of the order parameter near criticality. Formally, such effective actions are obtained by integrating out the fermionic degrees of freedom, and all high energy collective modes, leaving us with a set of degrees of freedom, $\Delta_i = |\Delta_i| \exp[i\phi_i]$, identified as the phase and modulus of the order parameter on puddle i. In detail, the various terms are sensitive to the specific physical circumstances, but the overall structure of the effective action is the same in all cases studied in the present paper.

In the case of small puddles embedded in a normal metal, where the value of the order parameter is small, the Andreev reflection of quasiparticles from the metal-superconductor boundary is ineffective and can be taken into account in perturbation theory. Then, the imaginary time effective action that governs the superconducting fluctuations near criticality is of the form

$$\begin{aligned} S = \sum_j \Bigg\{ & \alpha_j \int d\tau \left[-\frac{(\gamma - \gamma_{jc})}{2} |\Delta_j|^2 + \frac{1}{4} \frac{|\Delta_j|^4}{\Delta_0^2} \right] \\ & + \frac{\beta_j}{2} \int d\tau d\tau' \frac{|\Delta_j(\tau) - \Delta_j(\tau')|^2}{(\tau - \tau')^2} \Bigg\} + \int d\tau H_J[\{\Delta\}] + \dots \\ H_J[\{\Delta\}] &= -(1/2) \sum_{i \neq j} [J_{ij} \Delta_i^\star \Delta_j + \text{c.c.}] \end{aligned} \tag{1}$$

where τ is imaginary time, j labels the randomly distributed superconducting puddles, γ is the parameter that tunes the phase transition (*e.g.* the magnetic field in units of the critical magnetic field), γ_{jc} is the critical value of γ in the j_{th} puddle , Δ_0 is the magnitude of the order parameter deep in the superconducting state , α_j and β_j depend on the local structure of the superconducting puddle (as discussed below), and J_{ij} is the Josephson coupling between nearby puddles. The $\ldots$ symbol represents high order terms in Δ that are negligible at the phase transition, including non-local quartic terms involving the order parameter on more than one puddle. (Some representative aspects of the derivation of Eq. 1 are sketched in the Appendix.)

The first and the third terms in Eq. 1 reflect the dynamics of the BCS Cooper instability [16, 17, 18], and hence

$$\alpha_i \sim \nu V_i, \quad \beta_i \sim \nu V_i/\Delta_0 \tag{2}$$

where V_i is the volume of the i^{th} puddle and ν is the density of states in the surrounding metal. (In two dimensional cases, naturally, V_i is the area of the puddle and $\nu = \nu_{3D}d$ is the two dimensional density of states.) To illustrate this, set $J_{ij} = 0$ and consider the dynamics of small amplitude fluctuations of the order parameter on an isolated puddle that is on the verge of becoming superconducting ($1 \gg (\gamma_{ic} - \gamma) > 0$) as described by the linearized version of Eq. 1:

$$\chi_i(\omega) \equiv \int dt e^{i\omega t} \langle \Delta_i^*(0)\Delta_i(t)\rangle = \frac{1}{2\pi(\beta_j|\omega| + 1/\tau_i)}; \qquad \frac{1}{\tau_i} = \frac{\alpha_i(\gamma - \gamma_{ic})}{2\pi}. \tag{3}$$

Comparing this with the usual calculation of gaussian fluctuations in the neighborhood of the superconducting transition leads to the expressions in Eq. 2, *i.e.* Eq. 3 simply describes the dynamical fluctuations which lead to the Cooper instability as $\gamma \to \gamma_{ic}$. Here brackets $\langle \; \rangle$ signifies the quantum expectation value.

In the opposite limit, when the puddles are large with big values of the order parameter, one can neglect quantum fluctuations of the modulus of the order parameter and write the effective action in terms of fluctuations of the phase only

$$S = \sum_j G_i^{(eff)} \int d\tau d\tau' \frac{\left|e^{i\phi_j(\tau)} - e^{i\phi_j(\tau')}\right|^2}{(\tau - \tau')^2} + \int d\tau H_J[\{\phi\}]$$
$$H_J[\{\phi\}] = -(1/2) \sum_{i \neq j} \tilde{J}_{ij} \cos(\phi_i - \phi_j) \tag{4}$$

The form of the effective action in Eq. 4 is familiar from many earlier studies of the quantum dynamics of a system of superconducting grains linked by resistively shunted Josephson junctions (See, for example, Refs. 6, 7, 10, 12, 19, 20, 21.) In particular, the dynamical term proportional to $G_i^{(eff)}$ in Eq. 4 has the familiar Caldeira-Leggett [22] form and describes the quantum dynamics of the order parameter of an isolated puddle. In this case the origin of the dynamical term in Eq. 4, is entirely different from that in Eq. 1: it reflects the interaction of the phase fluctuations of the superconducting order parameter with quantum fluctuations of the electromagnetic field. In this case $G_i^{(eff)}$ is the dimensionless effective conductance defined by injecting current into the i-th superconducting puddle embedded in the metallic host (ignoring the effect of other puddles), and measuring the voltage drop at infinity. There is a further subtlety in 2D, as discussed in Refs. [23, 24], associated with the fact that, $G_i^{(eff)}$, as so defined, vanishes logarithmically with the size of the system. To handle this problem properly, one needs to consider corrections to the dynamical term in the effective action, Eq. 4. When this is done, the result is equivalent to identifying $G_i^{(eff)} \sim \sqrt{G_{2D}}$.

We will show below that in different situations either electromagnetic fluctuations or the Cooper instability can make the dominant contribution to the quantum dynamics of the order parameter .

B. Josephson couplings between puddles

We will see that near criticality the typical inter-puddle distance is larger than their size. The other generically important aspect of the problem is the dependence of the Josephson couplings between puddles on their separation, $\mathbf{r}_i - \mathbf{r}_j$, which is long-ranged (power-law) in the limit that the temperature, $T \to 0$. Specifically, the coupling between small puddles in Eq. 1 is, up to logarithmic corrections which we will discuss later, of the form

$$J_{ij} \equiv J(\mathbf{r}_i, \mathbf{r}_j) \propto C_{ij} \frac{\nu V_i V_j}{|\mathbf{r}_i - \mathbf{r}_j|^D} \ \exp\left[-\frac{|\mathbf{r}_i - \mathbf{r}_j|}{L_T} \right], \tag{5}$$

where ν is the density of states in the normal metal, $L_T = \sqrt{D_{tr}/T}$ is the coherence length of normal metal which diverges as $T \to 0$, D_{tr} is the electron diffusion coefficient, and $C_{ij}(\mathbf{r}_i, \mathbf{r}_j)$ is, generally speaking, a complicated (random) dimensionless function of the coordinates. In the cases considered in Secs. II and IV, the value of C is determined by the average properties of the "normal" metal between puddles. In particular, in Sec. II, C_{ij} is mostly positive, and so can be approximated by its average value, $\overline{C_{ij}}$, while for the d-wave superconductor treated in Sec. IV, C_{ij} has a random sign, but at long distances, this sign is entirely determined by the character of the puddles, i and j, and is independent of the distance between puddles. By contrast, in the cases with a magnetic field considered in Sec. III , C_{ij} is determined by the random quantum interference between different paths through the normal metal. As a consequence, C_{ij} has a random phase.

In the limit of large puddles, the functional dependence of the Josephson coupling in Eq. 4,

$$\tilde{J}_{ij} \propto \tilde{C}_{ij} \frac{D_{tr}}{R^2} \frac{R^D}{|\mathbf{r}_i - \mathbf{r}_j|^D} \exp\left[-\frac{|\mathbf{r}_i - \mathbf{r}_j|}{L_T}\right] \tag{6}$$

on distance is the same as in Eq. 5. Here the random function $\tilde{C}_{ij}$ has properties similar to C_{ij}, and R is the typical size of the puddles.

C. Susceptibility of an individual puddle

The susceptibility of an individual puddle can be expressed in terms of the correlation function $\chi_i(\omega)$ (Eq. 3) as $\chi_i \equiv \chi_i(\omega = 0)$. Its value depends on the puddle size.

Let us start with the case when the puddle is large, so the dynamics of the order parameter is determined by the effective action given by Eq. 4 with $J_{ij} = 0$. The implications of this effective action are best appreciated by interpreting imaginary time as a fictive spatial dimension, making the single puddle problem equivalent to the classical inverse-square XY model [25, 26] at an effective temperature $T^{eff} = 1/G_i^{(eff)}$. The long-time correlation functions of the inverse X-Y model have been calculated in various ways, and are well understood. The characteristic decay time depends exponentially on $1/T^{eff}$, and the dynamic correlation function has a power-law fall-off [27],

$$\langle \Delta_i^\star(0)\Delta_i(\tau)\rangle \sim |\Delta_j|^2 \begin{cases} [\tau_j/\tau]^{x_j} & \text{for } \tau \ll \tau_j \\ [\tau_j/\tau]^2 & \text{for } \tau \gg \tau_j \end{cases} \tag{7}$$

where x_i is a non-universal exponent $x_i = T^{eff}/(2\pi)$, the relaxation time,

$$\tau_i \sim \exp[ZG^{(eff)}] \tag{8}$$

and $Z = 2\pi^2$. The susceptibility is thus

$$\chi_i \sim \Delta_0 \exp[ZG_i^{(eff)}]. \tag{9}$$

However, in 2D, due to the dimension specific subtlety [23, 24] discussed in Subsection IA, we must identify $G_i^{(eff)} \sim \sqrt{G_{2D}}$, so that

$$\chi_i \sim \Delta_0 \ \exp\left[\ Z' \sqrt{G_{2D}} \ \right] \tag{10}$$

where Z' is another number of order 1.

Let us now turn to the case when the modulus of the order parameter on a puddle is small and its dynamics are determined by Eq. 1. There is a complicated, and for our purposes not terribly important crossover that occurs for puddles which are right on the verge of a mean-field transition, $|\gamma - \gamma_{ic}| \ll 1$. Ignoring such puddles, there are two distinct asymptotic behaviors that are readily deduced:

$$\chi_i \sim \begin{cases} [\alpha_i(\gamma_{ic} - \gamma)]^{-1} & \text{for } (\gamma_{ic} - \gamma) > 0, \\ |\Delta_i| \ \exp[\ Z'' \beta_i |\Delta_i|^2 \ ] & \text{for } (\gamma_{ic} - \gamma) < 0, \end{cases} \tag{11}$$

where, in this expression, $|\Delta_i| = \Delta_0 \sqrt{\gamma - \gamma_{ic}}$ is the mean-field amplitude of the superconducting order on puddle i and Z'' is a renormalized relative of the same factor Z defined in Eq. 9, above. We now sketch the derivation of this result.

The result for a puddle that does not support a mean-field solution, *i.e.* for $\gamma_{ic} > \gamma$, is easily obtained by evaluating Eq. 3 at $\omega = 0$. In the opposite limit, $(\gamma - \gamma_{ic}) > 0$, there is a well developed mean field value of the order parameter on the puddle

$$\Delta_j = \Delta_0 \sqrt{\gamma - \gamma_{jc}} \exp[i\phi_j], \tag{12}$$

which has an arbitrary phase ϕ_j. To the extent that modulous fluctuations can be ignored, this problem is precisely equivalent to the large puddle problem, with the role of $G_i^{(eff)}$ played by $\beta_j \Delta_j^2$ on each grain. Although large amplitude modulus fluctuations are relatively costly, because the resulting expression for the susceptibility depends exponentially on $|\Delta_j|^2$, they cannot be neglected. However, since the modulus of the order parameter appears exponentially in the expression for χ_i, it is clear that the neglect of modulus fluctuations is not reasonable. However, they do not alter the asymptotic qualitatively, but rather result in a renormalization (reduction) of the factor Z'' in Eq. 11.

The most important feature of Eq. 11 is that the susceptibility increases exponentially as a function of $\gamma - \gamma_{ic}$ and of the volume, V_i, of the puddle, $\chi \sim \exp[Z'' \nu \Delta_0 V_i (\gamma - \gamma_{ic})]$.

D. Determination of the quantum critical point

We now outline the procedure for determination of the location of the quantum critical point under these circumstances.

Quantum fluctuations necessarily destroy the superconducting order in an isolated puddle. Thus, although the superconducting susceptibility of an individual puddle, χ_i, can, under some circumstances, be large, the transition to the globally phase coherent superconducting state is ultimately triggered by the Josephson coupling between puddles. Let us introduce a dimensionless coupling between two puddles, i and j,

$$X_{i,j} \equiv \chi_i J_{i,j} \chi_j J_{j,i}. \tag{13}$$

Two puddles fluctuate essentially independently of each other if $|X_{i,j}| \ll 1$, and they are phase locked to each other if $|X_{i,j}| \gg 1$. The transition to a globally phase coherent state occurs as a function of γ at the critical value, $\gamma = \gamma_c$, at which an infinite cluster of puddles is coupled together by links with $X_{i,j} \sim 1$. For an ordered array of puddles, the quantum superconductor-metal transition was discussed in this light in [16, 23, 24].

In disordered systems, the nature of the phase transitions described by the effective action in Eq. 1 depends on the distribution of the parameters, γ_{ic}, β_j, $G_i^{(eff)}$ and J_{ij}, and these in turn are somewhat different in the various cases we treat below.

However, what is common to the cases we will analyze is that, according to Eq.11 the susceptibilities of the puddles depend exponentially on the parameters of the action Eq.1. Thus in a generic situation in the neighborhood of the transition, the distribution of χ_i is extremely broad, and at criticality, rare puddles with exponentially large susceptibilities play a special role. In this case, the critical point can be identified by finding the set of "optimal puddles" which lie on the critical links of "the percolating cluster". This will be done in a way analogous to Mott's approach to the theory of the variable range hopping conductivity (See for example [28]).

Specifically, the optimal puddles are those in which γ_{ic} lies in an interval, $\gamma_{opt} - \Delta\gamma_{opt} < \gamma_{ic} < \gamma_{opt} + \Delta\gamma_{opt}$. Here both the optimal value, γ_{opt}, and the width of the interval, $\Delta\gamma_{opt}$, are determined by maximizing the quantity $X_{opt} = \chi_{opt}^2 J_{opt}^2$ with respect to these parameters, where χ_{opt} is the susceptibility of a puddle with $\gamma_{ic} = \gamma_{opt}$, and J_{opt} is the typical value of the Josephson coupling between two nearest-neighbor optimal puddles. Finally we find the critical value of $\gamma = \gamma_c$ from the requirement that, after maximizing, $max\{X_{opt}\} \approx 1$. In the following sections we consider several examples of this program.

II. A RANDOM MIXTURE OF METAL AND SUPERCONDUCTOR

As a first case, we consider a random set of s-wave superconducting grains is embedded in a normal metal host with no magnetic field or magnetic impurities. We identify the superconducting grains as regions in which the effective interaction between two electrons in the Cooper channel is attractive ($\lambda_S > 0$), while in the normal metal the interaction is repulsive ($\lambda_N < 0$). This system exhibits a metal-superconductor transition when the appropriate average value of $\lambda(\mathbf{r})$ changes sign, although the parameter $k_F l$ can still be arbitrarily large [16, 23, 24]. Some aspects of various closely related problems have been previously been analyzed using a variety of approaches[16, 23, 24, 29, 30, 31, 32]. However, we are able to obtain a much complete picture than has been obtained previously. Moreover, this problem serves as a useful warmup as it provides a simple explicit example of how the character of the optimal puddles and the nature of the quantum phase transition are determined from the present quantum percolation analysis.

To be concrete, we will assume that the diameters of the grains, R_i, are random quantities characterized by the Gaussian distribution

$$P(R_i) = \frac{N}{\sqrt{2\pi}\sigma_R \bar{R}} \exp\left[-\frac{(R_i - \bar{R})^2}{2\sigma_R^2 \bar{R}^2}\right] \tag{14}$$

where the average radius is $\bar{R}$, the dimensionless variance $\sigma_R \ll 1$, and the total concentration of grains is N. In the notation of the previous section, we can identify $\gamma_i - \gamma = R_i/R_c - 1$, where $R_c \sim \xi_0$ is the critical radius for the existence of a mean-field solution and ξ_0 is zero temperature coherence length in the superconductor.

Expressions for the susceptibilities of individual grains, in various limits, are given in Eqs. 9, 10, and 11.

The value of the Joshepson coupling between two superconducting grains embedded in a normal metal depends on whether the Andreev reflection on the N-S boundary is effective or not. When the puddles are larger than the coherence length determined by the magnitude of the order parameter in the puddle, one finds

$$\tilde{J}_{ij} \sim G_{eff} \frac{D_{tr}}{\bar{R}^2} \frac{\bar{V}}{|\mathbf{r}_i - \mathbf{r}_j|^D [1 + 2\lambda_N| \ln^2(|\mathbf{r}_i - \mathbf{r}_j|/\bar{R})|]} \exp\left[-\frac{|\mathbf{r}_i - \mathbf{r}_j|}{L_T}\right] \tag{15}$$

by solving the Usadel equation. (See for example [33]). Here $\bar{V}$ is the volume of a grain, $\bar{V} \sim \bar{R}^D$. In the opposite limit, when the value of the order parameter on the puddle is small and Andreev reflection is ineffective, the coupling can be computed from perturbation theory (See for example [16]:

$$J_{ij} \sim \frac{\nu \bar{V}^2}{|\mathbf{r}_i - \mathbf{r}_j|^D \left[1 + 2|\lambda_N| \ln^2(|\mathbf{r}_i - \mathbf{r}_j|/\bar{R})\right]} \exp\left[-\frac{|\mathbf{r}_i - \mathbf{r}_j|}{L_T}\right], \tag{16}$$

.

Due to the s-wave character of the superconducting order and the fact that the system is time-reversal invariant, and so long as certain effects of strong correlations in the metal [34] can be ignored, J_{ij} is always real and positive. Moreover, if $k_F l \gg 1$, the mesoscopic fluctuations of the magnitude of J_{ij} are small compared to the average, and can thus be neglected.

Our goal is to determine the critical concentration of grains N_c at which the superconductor-metal transition occurs. We can identify various regimes depending on values of the parameters σ_R, $(\bar{R}-R_c)/R_c$, G_{eff}, and the Ginzburg parameter $g \equiv \nu R_c^D \Delta_0$. ($g$ is roughly the number of electrons within energy window Δ_0 on an individual superconducting grain. Notice that, in the cases of interest here, where $R_c \sim \xi_0$, it is always the case that $g \gg 1$.) We analyze some representative cases:

A) Monodispersed superconducting grains: Let us start with the case where $\sigma_R \to 0$, so all puddles are essentially the same size $R_i \equiv \bar{R}$. At $T = 0$, it is simple to see that:

i) If $(R_c - \bar{R}) < 0$, individual puddles are not superconducting, even at mean-field level. In this case, which corresponds to the uniform "Cooper limit," the superconducting transition is, to first approximation, mean-field like and it occurs when the average interaction strength,

$$\bar{\lambda} \equiv N\bar{V}\lambda_N - (1 - N\bar{V})|\lambda_N| \tag{17}$$

changes sign (from attractive to repulsive). Thus, the critical concentration is $N_c\bar{V} \sim |\lambda_N|/[\lambda_S + |\lambda_N|]$. This estimate neglects the spatial variations in the local concentration of superconducting grains; even when N is, on average, smaller than this mean-field critical value, there occur regions in which the concentration of grains exceeds this critical value. These regions act as the superconducting puddles of a new level of analysis, which gives results similar to those discussed below in the case of larger σ_R.

ii) If $1 \gg (\bar{R} - R_c)/\bar{R} > 1/g$, there is a well-developed mean-field order, $\Delta = \sqrt{(\bar{R} - R_c)/R_c}\Delta_0$ on each grain, but the order parameter has a magnitude small compared to Δ_0. In this case, Eq. 1 governs the dynamics, and we can make use of the expressions Eqs. 11 and 16 for χ and J_{ij}, respectively. Since the grains are typically a distance of order $N^{-1/D}$ apart, the dimensionless coupling between two neighboring grains is

$$X \sim \left\{ g\bar{R}^D N \sqrt{(\bar{R}-R_c)/R_c} \exp[Z'' g(\bar{R}-R_c)/R_c] \right\}^2 . \tag{18}$$

This coupling is larger than 1 when N exceeds the critical density,

$$N_c \sim \frac{1}{g\bar{R}^D} \left[\frac{\bar{R}-R_c}{R_c} \right]^{1/2} \exp\left[-Z'' \; g \; \left(\frac{\bar{R}-R_c}{R_c} \right) \right] \quad \text{at} \quad T=0. \tag{19}$$

iii) If $(R_c - \bar{R})/\bar{R} \geq 1$, the grains act, more or less, like pieces of bulk superconductor. Here, the dynamics of the quantum fluctuations are governed by electric field fluctuations as in Eq. 4, and consequently the same analysis leads to

$$N_c \sim \begin{cases} \bar{R}^{-2} \exp(-Z'\sqrt{G_{2D}}) & \text{in} \quad D=2 \quad \text{at} \quad T=0 \\ \bar{R}^{-3} \exp(-ZG^{eff}) & \text{in} \quad D=3 \quad \text{at} \quad T=0. \end{cases} \tag{20}$$

Let us turn now to the temperature dependence of $N_c(T)$. Conventional arguments suggest that at low enough temperatures, and arbitrarily close to the zero temperature critical point, there is a universal quantum critical regime[35, 36], where, for example, $(N_c(T) - N_c) \sim N_c T^x$, where x signifies an appropriate universal critical exponent. This regime, if it exists, applies only so long as $L_T \gg N_c^{-1/D}$, and so is exponentially narrow. Beyond the quantum critical regime, there is a broad range of temperatures in which $N_c^{-1/D} \gtrsim L_T \gg \bar{R}$, where the fluctuations are highly quantum mechanical in the sense that one can still neglect the T-dependencies of $\chi(T)$. In this case there are two sources of the T-dependence of $N_c(T)$: a) the classical fluctuations which destroy the coherence between puddles when $J_{ij}|\Delta_i||\Delta_j| \approx T$, and b) the T-dependencies of $J_{ij}(T)$, given by Eq. 16. The relative importance of these two effects depends on the dimensionality of space D, and the value of the parameter g. The second mechanism dominates the T-dependence of $N_c(T)$ in the 2D case at arbitrary T, and in the 3D case in the wide interval of temperatures where $L_T/\bar{R} > g$. Then the criterion $X_{opt} \sim 1$ corresponds to a typical distance between puddles of order L_T, and hence

$$N_c(T) \sim \frac{1}{(L_T)^D}, \qquad \bar{R} \ll L_T < N_c(0)^{-1/D}, \tag{21}$$

(In this article we will ignore relatively small temperature interval $L_T > \bar{R}$ where in 3D the temperature dependence $N_C(T)$ is determined by the first mechanism.) Notice, subtleties, such as whether the superconducting state has long-range order (the 3D case) or only power-law order (the 2D case), do not affect the validity of this estimate.

B) Optimal radius grains: If the variance σ_R is not too small, the susceptibilities of individual grains χ_i have an exponentially broad distribution. As a result, at $T = 0$ the transition point occurs when $\bar{R} < R_c$ and is determined by relatively rare "optimal" puddles with anomalously large values of $(R_i - R_c)/R_c$, and consequently with exponentially large susceptibilities. However, as we shall see, if $\bar{R}$ is too much smaller than R_c, the optimal grains become so rare that, yet again, a new regime occurs in which the optimal puddle is formed in regions with an anomalously large concentration of sub-critical grains.

Let us focus on those grains with radii, R_i, within ΔR_{opt} of a still to be determined optimal radius, $R_{opt} > \bar{R}$. (It can be shown that the relevant range is $\Delta R_{opt} \sim \sigma_R \bar{R}$). We will ignore puddles which do not belong to this optimal set since puddles with much larger R_j are extraordinarily rare, and those with much smaller R_j have much smaller susceptibilities. Under the assumption (to which we will return below) that the optimal puddles are still small enough that Eq. 1 applies, the concentration of the optimal puddles is

$$N_{opt} \sim N\sigma_R \exp[-(R_{opt} - \bar{R})^2/2\sigma_R^2\bar{R}^2] \tag{22}$$

so, according to Eqs. 11,16 and 13

$$X_{opt} \sim [N_{opt} g\ ]^2 \exp\left[Z'' g \frac{(R_{opt} - R_c)}{\xi_0}\right]. \tag{23}$$

Maximizing Eq. 23 with respect to R_{opt} gives

$$R_{opt}^{(max)} = \bar{R} + Z'' g \bar{R}^2 \sigma_R^2/\xi_0 \tag{24}$$

and $max\{N_{opt}\} \sim N \exp\left[-(Z'' g \bar{R}\sigma_R)^2/2\xi_0^2\right] \ll N$ (*i.e.* most puddles are smaller than R_{opt} and hence play no direct role in the transition). Finally, using the criteria $X_{opt} \sim 1$, we find

$$N_c \sim \frac{1}{\bar{R}^{D-1}\sigma_R} \exp\left[\frac{Z'' g}{\xi_0}\left(R_c - \bar{R} - \frac{Z'' g \bar{R}^2 \sigma_R}{2\xi_0}\right)\right] \quad \text{at} \quad T = 0. \tag{25}$$

We note that the critical concentration in Eq. 25 can be extremely small as a consequence of the fact that rare larger than average puddles contribute significantly to the global phase coherence.

Let us now discuss the limits of applicability of Eq. 25. It is manifestly necessary that σ_R be small enough that $N_c \bar{R}^D \ll 1$, *i.e.* that

$$(R_c - \bar{R}) > Z'' g \bar{R}^2 \sigma_R^2/2\xi_0. \tag{26}$$

Note that that the same criterion leads to the inequality $\Delta R_{opt} \sim \bar{R}\sigma_R \ll (R_{opt} - \bar{R})$, which justifies the saddle point approximation.

The temperature dependence of N_c can be obtained in similar ways as in **A**, above. When $L_T \gg N_c^{1/D}$, the behavior is presumably governed by universal properties of the quantum critical point. However, for $N_c^{1/D} > L_T \gg \bar{R}$, the temperature dependence of N_c again derives from the temperature dependence of J_{ij}, *i.e.* that the Josephson coupling falls rapidly to zero at distances large compared to L_T. Here, the character of the optimal puddles is still determined by Eqs. 23 and 24, but the critical concentration is determined by the condition that the distance between optimal puddles is typically of order L_T, *i.e.*

$$N_c(T) \sim L_T^{-D} \exp\left[\left(Z'' n_c \bar{R}\sigma_R\right)^2 / 2\xi_0^2\right]. \tag{27}$$

The resulting phase diagram is shown schematically in Fig. 1.

C) Large puddles: The expressions in **B** were derived under the assumption that the optimal grains are sufficiently small that the mean-field order parameter has a magnitude small compared to Δ_0, and consequently that the susceptibility grows exponentially with the radius of the grain, according to Eqs. 2 and 11. However, if σ_R is sufficiently large, the optimal grains get to be large enough that $R_{opt} - R_c \geq A\xi_0$ (where A is of order 1), and consequently $|\Delta_i| \sim \Delta_0$. In this limit, the quantum dynamics of the order parameter is determined by quantum fluctuations of the electromagnetic field, and χ_i is determined by the normal state conductance of the metal, G_{eff}, as in Eqs. 9 and 10. We can estimate the conditions for this by extrapolating Eq. 24 to the point $R_{opt} - R_c = \xi_0$, from which we deduce that the optimal puddles are "large" when $\sigma_R^2 > (\xi_0/\bar{R})^2(1/Z''g)$.

In this limit, since χ grows with size of the grain only relatively weakly ($\log[\chi] \sim R^{D-2}$), while the density of grains of large size falls exponentially with their volume, the density of optimal puddles is simply the density of grains with radius larger than $R_0 \equiv R_c + A\xi_0$,

$$N_{large} = \int_{R>R_0} dRP(R) \sim N \exp\left[-\left(\bar{R} - R_c\right)^2/\bar{R}^2\sigma_R^2\right)\right]. \tag{28}$$

With N_{large} playing the role of N_{opt}, and with the expressions in Eq. 9 and 10 for the susceptibility, the same analysis can be applied as in part **B** to obtain N_c. For instance, in 3D,

$$N_c \sim R_c^{-3} \exp[-ZG^{eff} + (\bar{R} - R_c)^2/2\bar{R}^2\sigma_R^2]. \tag{29}$$

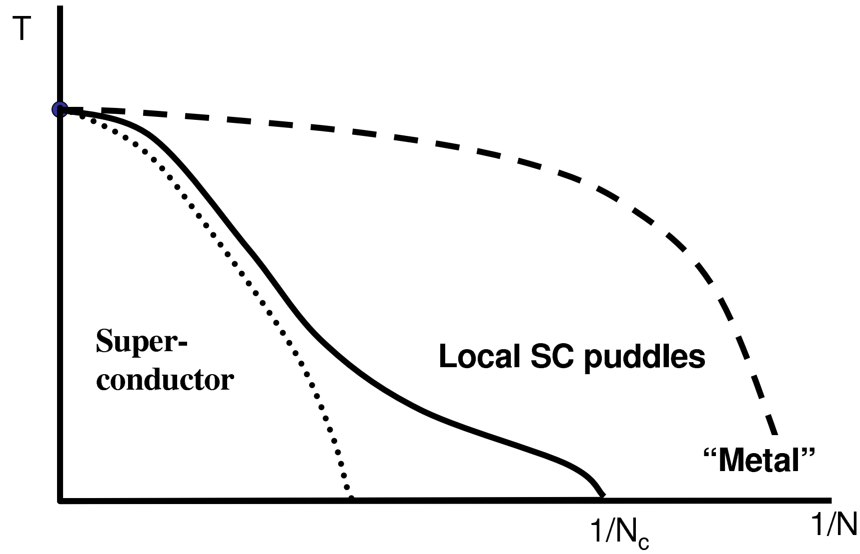

FIG. 1: Schematic phase diagram for the case considered in Section IIB, in which a concentration N of superconducting grains with a distribution of sizes is embedded in a metallic host. The solid line represents a phase transition. The dashed line represents a crossover where a "small fraction" of the sample first supports a mean-field solution - although near this line, global phase coherence is destroyed by quantum and thermal phase fluctuations, measurable manifestations of local superconductivity onset below this line. The dotted line represents the expected phase boundary in the Cooper limit, where mesoscopic fluctuations are ignored and a single uniform effective interaction between electrons is assumed as in Eq. 17.

We would like to stress that Eq. 29 holds only at exponentially small temperatures for which $L_T \gg N_c^{-1/D}$. In the opposite limit, puddles with $R_i > R_o$ are irrelevant, and the Cooper instability contribution represented by Eq.11 dominates the physics of the phase transition, even for relatively large values of σ_R.

III. S-WAVE SUPERCONDUCTOR IN A MAGNETIC FIELD

A. A magnetic field perpendicular to a superconducting film

A magnetic field, H, applied perpendicular to a metallic film ($D = 2$) couples primarily to the electron's orbital motion. In this case the Zeeman coupling of the magnetic field to the electron spin can be neglected. To the extent that mesoscopic fluctuations of the order parameter can be neglected, the problem of s-wave superconductivity in a magnetic field was solved by Abrikosov and Gorkov. In this approximation, at $H_{c2}^{(0)} = \Phi_o/\pi\xi_o^2$, the order parameter can be represented as a superposition of wave-functions in the first Landau level,

$$\phi(r) = \frac{1}{\sqrt{2\pi L_H d}} \exp(-r^2/L_H^2), \tag{30}$$

where $\Phi_o = hc/2e$ is the flux quantum, $L_H = \sqrt{\Phi_0/2\pi H_{c2}^{(0)}}$ is the magnetic length. (We consider the "dirty limit" $\xi_0 \gg l \gg k_F^{-1}$ where $\xi_0 = \sqrt{D_{tr}/\Delta_0}$). Roughly speaking, the same form of the wave-function applies even when mesoscopic fluctuations are taken into account. This simplifies the analysis in that it implies that, near the point of the transition, the puddles have a typical size, $L_j \approx L_H$. However, the critical magnetic field H_i varies randomly as a function of position, so that, in the notation of Section I, we can identify

$$\gamma_i - \gamma \equiv \frac{(H_i - H)}{H_{c2}^{(0)}}. \tag{31}$$

We will assume that the distribution of H_i is approximately gaussian

$$P(H_i) = \frac{1}{\sqrt{2\pi}\sigma_H \bar{H}_c} \exp[-\frac{(H_i - \bar{H}_c)^2}{2\sigma_H^2 \bar{H}_c^2}] \tag{32}$$

and is characterizing by the average $\bar{H}_c$ and a dimensionless variance σ_H. (This ignores the existence of long, but for present purposes irrelevant tails of the distribution produced by mesoscopic effects [37].) We assume that $\sigma_H \ll 1$, and thus that $H_{c2}^{(0)} \approx \bar{H}_c$. Generally, there are two contributions to the variance σ_H: one contribution comes from classical fluctuations in the strength of the local scattering potential and one from non-local quantum "interference" effects,

$$\sigma_H = \sigma^{(int)} + \sigma^{(cl)}. \tag{33}$$

The classical contribution is due to random fluctuations of the concentration of impurities,

$$\sigma_H^{(cl)}(L) \sim (\Lambda/L_{\bar{H}}) \tag{34}$$

where $\Lambda \ll L_{\bar{H}}$ is the correlation length of the disorder potential. The electron interference contribution is [38]:

$$\sigma^{(int)} \sim 1/G_{2D} \ll 1. \tag{35}$$

Note that, although the interference term is independent of puddle size, and hence is the larger term for big enough puddles, for large G_{2D} there is a parametrically wide range of puddle sizes for which the simple statistical variations in impurity concentrations dominates the variance of local critical fields.

The configuration dependent (mesoscopic) variations in the Josephson coupling, J_{ij} are more important here than in the previous example. One can see this by noticing that at large $|\mathbf{r}_i - \mathbf{r}_j|$ (up to possible logarithmic corrections) the averages

$$\begin{aligned} \overline{J_{ij}} &\sim \nu L_H^2 d \frac{L_H^2}{|\mathbf{r}_i - \mathbf{r}_j|^2} \exp\left[-\frac{|\mathbf{r}_i - \mathbf{r}_j|}{L_H} - \frac{|\mathbf{r}_i - \mathbf{r}_j|}{L_T}\right] \\ \overline{\tilde{J}_{ij}} &\sim G_{eff} \frac{D_{tr}}{|\mathbf{r}_i - \mathbf{r}_j|^2} \exp\left[-\frac{|\mathbf{r}_i - \mathbf{r}_j|}{L_H} - \frac{|\mathbf{r}_i - \mathbf{r}_j|}{L_T}\right] \end{aligned} \tag{36}$$

are much smaller than the variances

$$\begin{aligned} \left[\overline{|J_{ij}|^2}\right]^{1/2} &\sim \left(\frac{L_H^2}{v_F l}\right) \left(\frac{L_H}{|\mathbf{r}_i - \mathbf{r}_j|}\right)^2 \exp\left[-\frac{|\mathbf{r}_i - \mathbf{r}_j|}{L_T}\right]. \\ \left[\overline{|\tilde{J}_{ij}|^2}\right]^{1/2} &\sim \frac{D_{tr}}{|\mathbf{r}_i - \mathbf{r}_j|^2} \exp\left[-\frac{|\mathbf{r}_i - \mathbf{r}_j|}{L_T}\right]. \end{aligned} \tag{37}$$

where $\overline{O}$ indicates the average of O over realizations of the random scattering potential., and v_F is the Fermi velocity. As a consequence, at long distances the Josephson coupling in Eqs. 1 and 4 is of the form [38]

$$\begin{aligned} J_{ij} &\sim F_{ij} \nu L_H^2 d \frac{L_H^2}{|\mathbf{r}_i - \mathbf{r}_j|^2} \exp\left[-\frac{|\mathbf{r}_i - \mathbf{r}_j|}{L_T}\right] \\ \tilde{J}_{ij} &\sim \tilde{F}_{ij} \frac{D_{tr}}{|\mathbf{r}_i - \mathbf{r}_j|^2} \exp\left[-\frac{|\mathbf{r}_i - \mathbf{r}_j|}{L_T}\right] \end{aligned} \tag{38}$$

where $F_{ij} = F_{ij}(\mathbf{r}_i, \mathbf{r}_j)$, and $\tilde{F}_{ij} = \tilde{F}_{ij}(\mathbf{r}_i, \mathbf{r}_j)$ are dimensionless functions which varies randomly in phase, and $\overline{|\tilde{F}_{ij}|^2} \sim \overline{|F_{ij}|^2} \sim 1$.

To find the critical magnetic field H_c, we employ the same optimization procedure we used in the Section 2. We introduce an interval in the space of H_i which is centered at H_{opt} with width $\Delta H \sim \sigma_H \bar{H}_c$. We will see that for sufficiently large values of σ_H, the distance between the "optimal puddles" is large enough that the Joshepson coupling between puddles is dominated by the mesoscopic contribution given by Eq. 38.

Depending on the value of $G_{2D} = e^2 \nu D_{tr} d$, the quantum dynamics of the order parameter is determined by either the Cooper instability or the quantum fluctuations of the electromagnetic field.

If $(H_{opt} - H)/H_{c2}^{(0)} \ll 1$, and hence $\Delta_{opt} \ll \Delta_0$, the quantum dynamics of the order parameter is determined by the Cooper instability, so we can use Eqs. 2 and 5 to obtain

$$X_{opt} \sim a_H \exp\left[Z''g\left(\frac{H_{opt} - H}{\bar{H}_c}\right) - \frac{(H_{opt} - \bar{H}_c)^2}{\sigma_H^2 \bar{H}_c^2}\right] \quad \text{at} \quad T = 0. \tag{39}$$

Here $a_H \approx \Delta_0/E_F$, and $g = \nu\Delta_0\xi_0^2 d \approx G_{2D}$. As before, H_{opt} is the value which maximizes this expression, and the true critical field is then determined as the value of $H = H_c$ at which $X_{opt} = 1$:

$$H_{opt} = \bar{H}_c\left[1 + (ZG_{2D}/2)\sigma_H^2\right]; \quad H_c = \bar{H}_c\left[1 + (ZG_{2D}/4)\sigma_H^2\right]. \tag{40}$$

These expressions are self-consistent so long as

$$\frac{2}{\sqrt{ZG_{2D}}} \gg \sigma_H \gg \frac{2}{ZG_{2D}} \tag{41}$$

where the first inequality guarantees that $(H_c - \bar{H}_c)/\bar{H}_c \ll 1$ and the second that the optimal puddles are dilute. The expression for H_c in Eq. 40 has been obtained in Ref. [18] by a different method.

In any puddle for which $(H_i - H)/H_{c2}^{(0)} \gtrsim 1$, the superconducting order is "fully developed," $\Delta_i \approx \Delta_0$, so the dynamics of the order parameter is determined by the quantum fluctuations of the electromagnetic field. In this case, the susceptibility of the puddle depends only on G_{2D} as in Eq. 10, and is independent of H_i. Since the probability of finding such a puddle decreases with increasing H_i, the optimal puddles of this sort are those with $H_i \approx H_{opt} \approx H$. The corresponding dimensionless coupling between these puddles is thus

$$X_{opt} \sim \exp\left[2Z'\sqrt{G_{2D}} - \frac{(H_{opt} - \bar{H}_c)^2}{\sigma_H^2 \bar{H}_c^2}\right]. \tag{42}$$

These puddles are always dilute compared to their size so long as $[H - \bar{H}_c] > \sigma_H \bar{H}_c$. The critical value of H, determined by the condition $X_{opt} \sim 1$, is

$$H_c = \bar{H}_c(1 + \sqrt{2}\sigma_H G_{2D}^{1/4}) \quad \text{at} \quad T = 0. \tag{43}$$

The issue of whether the global superconducting properties are dominated by weakly superconducting or "fully developed" puddles is settled by determining which of the expressions for X_{opt} in Eqs. 39 and 42 give the largest value. In particular, the critical field is determined by the larger of the values given by Eqs. 40 and 43.

Since in both these cases, the puddles are dilute, the phase of the Josephson couplings between optimal puddles is random. Thus, the $T = 0$ ordered state is glassy, and is an example of a "gauge-glass." While this phase does not have long-range order, it supports a non-zero Edwards-Anderson order and is generally thought to have zero resistance.[41, 42]

In 2D, however, there is no ordered state at non-zero T, so there are only crossovers as a function of H and T. There is a characteristic field $H^\star(T)$, such that for $H < H^\star$, the coupling between optimal puddles is large (in magnitude) compared to the temperature; here, the resistivity is, presumably, due to some form of variable-range-hopping of vortices, and so decreases exponentially with decreasing T. Clearly, $H^\star(T) \to H_c$ as $T \to 0$. It is only weakly T dependent at low T, but in the temperature range such that L_T is smaller than the typical spacing between optimal puddles, but large compared to the puddle diameter, $H^\star$ is determined by the condition that the concentration of superconducting puddles must exceed L_T^{-2}; this leads to the unusual T dependence:

$$H^\star(T) \sim \bar{H}_c \left[1 + 2\sigma_H \ln^{1/2}\left(L_T/L_H\right)\right]. \tag{44}$$

The resulting schematic phase diagram is shown in Fig.2. Note that there are a series of additional crossovers that we have not addressed here, and which are not shown in the figure, which occur at fields greater than $H^\star$. These crossovers characterize various energy scales in the anomalous metallic phase proximate to the superconducting glass. More of the physics of the anomalous metal will be addressed in future studies.

B. The case of a parallel magnetic field

We now consider the opposite limit, in which the coupling of an applied magnetic field to the electron spin (Zeeman coupling) is significant, and the coupling to the orbital motion of the electrons can be neglected. To a good approximation, this can be realized in a thin film of an s-superconducting metal in which the superconductivity is destroyed by an in-plane magnetic field $H_\parallel$. The situation in this case critically depends on the value of the parameter $\Delta_0\tau_{so}$ (or, in other words, on the atomic weight of the metal) where $1/\tau_{so}$ is the spin-orbit scattering rate. In the case of relatively strong spin-orbit coupling, $\Delta_0\tau_{so} \ll 1$, on mean field level and in the absence of mesocopic fluctuations, the transition is second order. In this case, the effect of mesoscopic fluctuations on the character of the transition is qualitatively similar to the transition in the perpendicular magnetic field considered in the previous subsection. In the case $\Delta_0\tau_{so} \ll 1$, however, the situation is very different because on the mean field level the transition is first order. In this section, we will consider this case, and to simplify the discussion, we will consider it in the limit of zero spin-orbit coupling, $\Delta_0\tau_{so} \to \infty$. In this limit, the microscopic physics responsible for the emergence of a puddle state is quite different, and in particular the energy associated with the formation of puddles is larger than in the previous examples. Therefore the various manifestations of the physics, and especially the glassy character of the phase at intermediate fields, is more robust.

If mesoscopic fluctuations are ignored, the zero temperature transition is first order, with a discontinuous jump in the spin-magnetization density from $m = 0$ (in the superconducting state) to $m = \chi_{sp} H_\parallel$ in the metallic state, where $\chi_{sp} = \nu\hbar\mu_B^2$ is the normal-state spin (Pauli) susceptibility. The critical field is given by the well-known Chandrasekar-Clogston limit,

$$H_{c\parallel}^{(0)} = \Delta_0/\hbar\mu_B. \tag{45}$$

(In the disorder free case, there might appear a narrow regime of fields near $H_{c\parallel}^{(0)}$ in which a partially polarized Fulde-Ferrel-Larkin-Ovchinikov (FFLO) state occurs, but this is not relevant in the case $l \ll \xi_0$, considered here.)

In disordered systems the critical magnetic field $H_{c\parallel}(\mathbf{r})$ exhibits spatial fluctuations. In the absence of spin-orbit scattering the value of $H_{c\parallel}^{(0)}$ given by Eq. 45 is independent of l, which means that there is no classical contribution to the dimensionless variance $\sigma_\parallel$, which is, instead, determined entirely by mesoscopic interference effects. Thus, the dimensionless variance is [39]

$$\sigma_\parallel \equiv \frac{\left[\overline{\left(H_{c\parallel} - \bar{H}_{c\parallel}\right)^2}\right]^{1/2}}{\bar{H}_{c\parallel}} \approx \frac{1}{G_{2D}}. \tag{46}$$

and the correlation length of $H_{c\parallel}(\mathbf{r})$ is of order ξ_0.

According to general theorems [43], in 2D quenched disorder destroys first order transitions. Rather, near a putative first order transition, a domain structure occurs with a characteristic domain size L_{dom} which is exponentially large in the small disorder limit, $L_{dom} \sim \exp[Z''(1/\sigma_\parallel)^2]$. In some cases, the putative first order transition is simply smeared and replaced by a crossover which becomes increasingly sharp as the disorder becomes weaker. However, in the present case, since there is clearly a superconducting phase for small enough $H_\parallel$ and (as we shall confirm) a non-superconducting phase for large enough

$H_\parallel$, there must still be a sharp, continuous quantum phase transition at a shifted critical field, $H_{c\parallel}$.

Specifically, in the present case, the superconducting domains are regions with magnetization m near 0, and with the local magnitude of the order-parameter, $|\Delta_i| \approx \Delta_0$, while the metallic regions have $m \approx \chi_{sp} H_\parallel$ and miniscule magnitude of the superconducting order. The volume fraction of the two phases is a function of $H_\parallel$; it is roughly a 50-50 mixture when $H_\parallel \approx H_{c\parallel}^{(0)}$, and the superconducting fraction decreases monotonically with increasing $H_\parallel$. However, global phase coherence is not lost at $H_\parallel \approx H_{c\parallel}^{(0)}$, where on the mean field level the superconducting fraction first fails to percolate. Rather, as in the other problems we have examined, it occurs when the Josephson coupling between superconducting regions becomes sufficiently weak, which in turn occurs when the superconducting fraction is small and the superconducting regions far separated.

Because the superconducting regions have a characteristic size large compared to ξ_0, and the magnitude of the order parameter is large, the dynamics of phase fluctuations is determined by electric field fluctuations, and consequently (according to Eq. 10) $\chi_i \approx \Delta_0 \exp\left[Z' \sqrt{G_{2D}}\right]$.

To determine the distribution of Josephson couplings, we note that in an SNS junction, when the normal part of the junction is partially spin polarized, [40] the Josephson coupling oscillates in sign as a function the coordinates. Specifically, at $T = 0$,

$$
\begin{aligned}
\overline{\tilde{J}(\mathbf{r}, \mathbf{r}')} &\sim \frac{G_{2D} D_{tr}}{|\mathbf{r}-\mathbf{r}'|^2} \exp(-\frac{|\mathbf{r}-\mathbf{r}'|}{L_{H\parallel}}) \cos(\frac{|\mathbf{r}-\mathbf{r}'|}{L_{H\parallel}}), \\
\left[\overline{\left|\tilde{J}(\mathbf{r}, \mathbf{r}')\right|^2}\right]^{1/2} &\sim \frac{D_{tr}}{e|\mathbf{r}-\mathbf{r}'|^2} \\
\tilde{J}(\mathbf{r}, \mathbf{r}') &\sim F(\mathbf{r}, \mathbf{r}') \frac{G_{2D} D_{tr}}{|\mathbf{r}-\mathbf{r}'|^2} \cos(\frac{|\mathbf{r}-\mathbf{r}'|}{L_{H\parallel}}),
\end{aligned}
\tag{47}
$$

where $L_{H\parallel} = \sqrt{D_{tr}/\mu H_\parallel}$, and $F(\mathbf{r}, \mathbf{r}')$ is a sample specific function ($|F| \sim 1$) which has random variations both in modulus, and in sign.

The mesoscopic fluctuations of $\tilde{J}$ again dominate the average at distances large compared to $L_{H\parallel}$. Thus we can estimate the critical magnetic field $H_{c\parallel}$ at which the zero temperature phase transition to the metallic phase takes place. For $H_\parallel > \bar{H}_{c\parallel}$, the probability of finding a superconducting puddle is $\sim \exp\left[-\left(H_\parallel - \bar{H}_{c\parallel}\right)^2 / 2\sigma_\parallel^2 \bar{H}_{c\parallel}^2\right]$. As a result, following the same

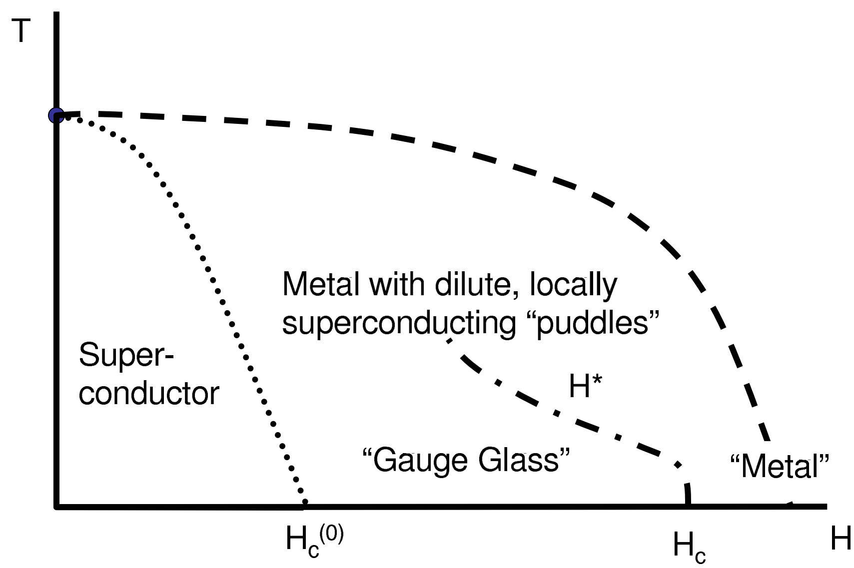

FIG. 2: Schematic phase diagram for the cases considered in Sections IIIA and IIIB in which superconductivity in a thin film of an s-wave superconductor is destroyed, respectively, by application of a perpendicular or a parallel magnetic field. The crossover scale, $H^\star$, (indicated by the dot-dashed line) is described in the text. The "gauge glass" refers to a zero temperature phase in which the resistance vanishes as $T \to 0$, but for which there is no finite temperature phase transition. The mean-field phase boundary has a continuous portion (indicated by the narrower dotted line) and a first order portion (indicated by the heavier line) separated by a tricritical point (indicated by the solid circle).

line of reasoning as in previous sections,

$$\frac{H_{c\|} - \bar{H}_{c\|}}{\bar{H}_{c\|}} \approx \sqrt{Z'}\sigma_{\|} G_{2D}^{1/4} \tag{48}$$

.

Notice that the average Josephson coupling in Eq. 47, itself, oscillates in sign, so even when the superconducting puddles are closely spaced, the Josephson couplings are random in sign. The resulting glassiness of the superconducting state is more robust than in the case of the perpendicular magnetic field because of the large size of the superconducting domains, the fact that the magnitude of the order parameter is fully developed within each domain, and due to the fact that the randomness in sign is not solely a long distance subtlety. In the absence of spin-orbit coupling, the magnetic field does not couple directly to any orbital degrees of freedom, and hence the glass phase can be precisely characterized as in an XY spin-glass, in which the ordered state supports spontaneously generated equilibrium orbital currents.

In the presence of spin-orbit coupling, a sharp definition of the glass phase based on the presence of orbital currents is not possible; even a normal disordered metal phase will support small orbital currents under these circumstances. As in the previous section, the superconducting coherent state near the critical field is some form of a gauge glass.

IV. DESTRUCTION OF D-WAVE SUPERCONDUCTIVITY BY DISORDER

We shall now consider the case in which, in the zero disorder limit, the superconducting state is a BCS state with d-wave symmetry due to a weak attractive interaction in the d-wave particle-particle channel.

In the d-wave case, it is necessary to explicitly treat the dependence of the superconducting order parameter on the relative coordinate. Specifically, in the absence of disorder and in a bulk sample, $\Delta(\mathbf{r},\mathbf{r}') = \Delta^{(d)}\kappa_d^{(0)}(\mathbf{r}-\mathbf{r}')$, where $\kappa_d^{(0)}(\mathbf{r})$ changes sign under rotation by $\pi/2$, and is a short-ranged function, with range, $b \ll \xi_0$, determined by the range of the effective attractive interaction.

Although the "d-wave" notation is inherited from spectroscopic notation for an "$l = 2$" irreducible representation of the rotation group, in a crystal, it refers to an appropriate irreducible representation of the point group. We will treat the case in which the point group has at least two distinct even parity one dimensional representations - a trivial one and a non-trivial one. For instance, in a tetragonal crystal, in addition to the trivial (s-wave) representation, there are three other even parity irreducible representations: a $\mathrm{d}_{x^2-y^2}$-wave, a d_{xy}-wave, and a g-wave (which transforms like $(x^2-y^2)xy$). We consider the case in which in the zero disorder limit, there is an effective attraction only for one of these representations, which we will call simply the "d-wave."

The most clear-cut manifestation of the d-wave nature of the ground state order parameter comes from "phase sensitive" measurements [44, 45] of the symmetry of the order parameter. Specifically, in a corner SQUID[45] of the sort described in Fig. 4, in which the external circle is a conventional s-wave superconducting wire, the ground state of the system will contain a half-flux quantum trapped in the SQUID for the case in which the sample is a d-wave superconductor, and no flux if it is an s-wave superconductor.

The fact that, in the absence of disorder, $\Delta(\mathbf{r},\mathbf{r}')$ changes sign under rotation makes the system much more sensitive to the strength of the disorder than a conventional s-wave superconductor. We will characterize the disorder strength by the electron mean free path l in the normal metal. What happens to the system in the presence of relatively strong disorder depends on the sign of the interaction in s-channel. If the interaction in the s-wave channel is attractive but much weaker than the attraction in the d-wave channel, then when the disorder is weak enough ($l > \xi_0$), the d-wave state dominates, but when the disorder strength increases enough to destroy the d-wave superconductivity ($l < \xi_0$), the system undergoes a phase transition to an s-wave state (See for example [46]). The s-wave state is destroyed only when $k_F l \approx 1$. However, in this article we consider the more interesting case in which the interaction in the s-channel is repulsive, so when disorder suppresses d-wave superconductivity, it drives the system to a normal state when the mean free path is still relatively large, $k_F l \gg 1$. This case may be relevant, for instance, to the destruction of superconductivity in the "overdoped" high temperature superconductors.

In the (conventional) approximation in which spacial fluctuations of the electron mean free path are neglected, d-wave superconductivity is destroyed when $l \sim l_0 = 1.78\xi_o$. Thus disordered d-wave superconductors are another example of a system which may have a quantum superconductor-metal transition in a situation in which the conductance is large. This case exhibits both similarities and differences with the cases we have already considered in Sections 2 and 3.

In the presence of disorder, a material has no particular spatial symmetry, and so the order parameter cannot be said precisely to have any particular symmetry at all. Nevertheless, in bulk samples, symmetry is restored upon configuration averaging. It is therefore legitimate to ask questions concerning the global symmetry of the order parameter. Hence, we can ask whether $\overline{\Delta(\mathbf{r},\mathbf{r}')}$, or $\overline{\mathcal{F}(\mathbf{r},\mathbf{r}')}$ have d-wave or s-wave symmetry. Here the overline stands for a configurational averaging, and $\mathcal{F}(\mathbf{r},\mathbf{r}') \equiv \mathcal{F}(\mathbf{r},\mathbf{r}',t=t')$ is the anomalous Green function which is connected to $\Delta(\mathbf{r},\mathbf{r}')$ by the interaction constant.

It is important to realize that it is possible (indeed, as we shall see, inevitable near criticality) to have a situation in which the local pairing is "d-wave like" and yet the global superconductivity has s-wave symmetry. In fact we will show that there are at least two quantum phase transition as disorder increases: the transition from d-wave to s-wave global symmetry, and subsequent transition from s-wave superconductor to the normal metal. The existence of the second (d-s) transition is the main difference with the cases considered in previous sections. (In fact, we consider it likely that rather than a sharp d to s transition, there is an intermediate glass phase in which time reversal symmetry is broken and s and d-wave ordering coexist. However, we have not fully explored this scenario.)

We propose several different definitions of the global symmetry of the order parameter: a) The best operational definition is provided by the result of a phase sensitive experiment, such as the corner SQUID experiment shown in Fig. 4. b) The quantity $\overline{\Delta(\mathbf{r}, \mathbf{r}')}$ can be characterized as having d-wave or s-wave symmetry. It can also provide a definition of a state with coexisting order if it has mixed symmetry. c) A specific diagnostic for a globally s-wave component of the order parameter can be defined in terms of the local component of the anomalous Green function $\mathcal{F}(\mathbf{r} = \mathbf{r}') \equiv \mathcal{F}^{(s)}(\mathbf{r})$. If we define $P_{\pm}$ to be the volume fraction of a sample where $F^{(s)}(\mathbf{r})$ has a positive or negative sign, respectively, then the system has an s-wave component if $(P^{+} - P^{-}) \neq 0$. These definitions are not equivalent under all circumstances. However, for the purposes of this article, we are not primarily interested in the most general definition of the global symmetry of the superconducting state. For the most part, we will deal with the interval of parameters in which all these definitions are approximately interchangeable.

1. *The d-wave to s-wave transition as a function of disorder*

This transition takes place in the region of concentrations where quantum fluctuations of the order parameter can be neglected, and therefore it can be understood on the mean filed level. As a warmup exercise, consider a cartoon picture of a system of superconducting puddles of a size large compared to ξ_0 and of a rectangular shape which are embedded in a noninteracting diffusive normal metal (See Fig. 5). The rectangles are identical, and they are oriented either vertically, or horizontally in a random fashion. The order parameter inside the rectangles has d-wave symmetry, and the orientation of the gap nodes is assumed to be pinned by the crystalline anisotropy.

In a d-wave superconductor, in addition to an overall phase of the order parameter, there is an arbitrary sign associated with the internal structure of the pair wavefunction. Specifically, we adopt a uniform phase convention such that when the phase of the order

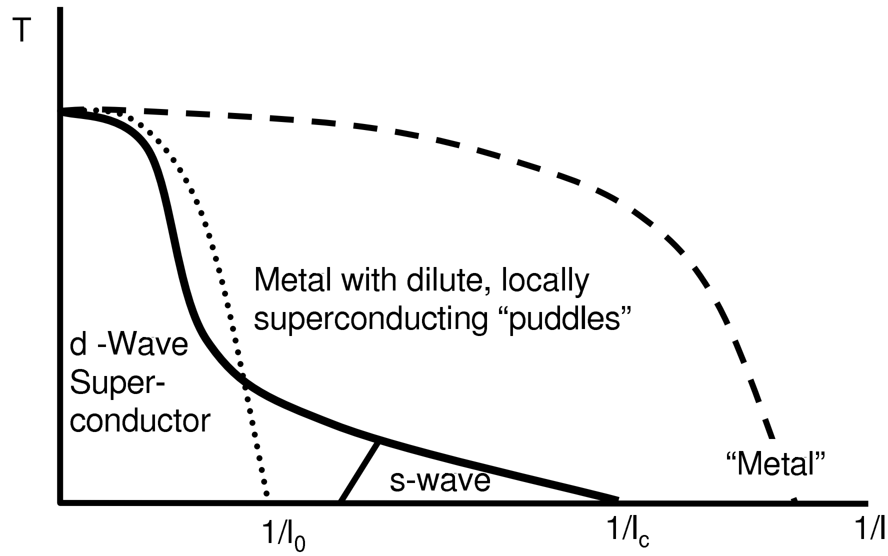

FIG. 3: Schematic phase diagram for the case considered in Section IV in which a BCS (weak-coupling) d-wave superconducting state (in 3D) is destroyed as a function of increasing disorder strength. The dotted line represents a transition between d-wave superconuctor and normal metal calculated by the conventional theory. In the presence of disorder, the labels "s-wave" and "d-wave" refer to the behavior of the system in a macroscopic phase sensitive measurement, as described in the text. (The negative slope of the boundary which divides globally d- and s-wave global superconductors shown in the figure can be justified only in the case when the electron interaction in the s-channel is attractive, so that the entropy of the s-wave superconductor is smaller than the entropy of d-wave. More generally, the slope of this boundary is to be determined.)

parameter $\phi_i = 0$, this implies $\Delta(\mathbf{r}, \mathbf{r}')$ in puddle i is real and has its positive lobes along the (appropriately defined) y axis and its negative lobes along the x axis.

It is obvious that at a high concentration of puddles, the order parameter in the ground state has global d-wave symmetry (See Fig. 5a.). However at small puddle concentrations, the situation is different. If the distances between puddles $|\mathbf{r}_i - \mathbf{r}_j| \gg R$ are much larger than the characteristic size of the puddles, R, the Josephson coupling between puddles inevitably favors globally s-wave superconductivity, even though the order parameter on each puddle looks locally d-wave like. In this case the mean field exchange energy of the system has a form

$$E_{Jos} = \sum_{i \neq j} \eta_i \eta_j \tilde{J}_{ij}^{(S)} \cos(\phi_i - \phi_j) \tag{49}$$

where $\eta_i = \pm 1$ are random numbers such that $\eta_i = 1$ for a rectangle oriented in the x-direction and $\eta_i = -1$ for a y-directed rectangle.

Eq. 49 represents the Mattis model which is well known in the theory of spin glasses [47]. The ground state of this model corresponds to

$$\cos(\phi_i) = -\eta_i. \tag{50}$$

Thus the distribution of $\exp(\phi_i)$ between puddles looks completely random, as shown in Fig. 5b. However the system is not a glass because it's ground state has a hidden symmetry, which in the present problem corresponds to to a global s-symmetry of the order parameter according to any of our proposed definitions!

A qualitative explanation of this fact is as follows: The inter-puddle Josephson coupling originates from the proximity effect in the normal metal, which is characterized by the anomalous Green function $\mathcal{F}(\mathbf{r}, \mathbf{r}')$. Due to lack of symmetry at the boundary of a puddle, an s-wave component $\mathcal{F}(\mathbf{r} = \mathbf{r}') = \mathcal{F}^{(s)}(\mathbf{r}) \neq 0$ of the anomalous Green function is generated in the neighboring metal. Specifically, at a normal metal-superconductor boundary, the sign of $\mathcal{F}^{(s)}(\mathbf{r})$ is determined by the sign of the d-wave order parameter in the $\mathbf{k}$-direction perpendicular to the boundary (See Fig. 5). (Thus the sign of $\mathcal{F}^{(s)}(\mathbf{r})$ changes along the boundary of a puddle.) On distances from the boundary larger than l, the anomalous Green function becomes isotropic. In other words, only the s-component $F^{(s)}(\mathbf{r})$ survives elastic scattering. It is this component which penetrates through the metal and carries the Josephson current between puddles. At distances larger than the size of the puddle (but smaller than $|\mathbf{r}_i - \mathbf{r}_j|$) the quantity $\mathcal{F}^s(\mathbf{r})$ has a definite sign which is determined by an integral around the surface, which sign gives us the value of η_i.

On intermediate distances, the situation is more complicated. Areas with different signs of $\mathcal{F}^{(s)}(\mathbf{r})$ mix in a random fashion. We argue that the most important aspects of this complex situation can be modelled by the following effective Josephson energy:

$$E_{Jos} = \sum_{i \neq j} [\eta_i \eta_j \tilde{J}_{ij}^{(s)} + \tilde{J}_{ij}^{(d)}] \cos(\phi_i - \phi_j) \tag{51}$$

where $\tilde{J}_{ij}^{(d)}$ characterizes the strength of the exchange interaction between the d-wave components of the order parameter. Typically at small $|\mathbf{r}_i - \mathbf{r}_j|$, $\tilde{J}_{ij}^{(d)} > \tilde{J}_{ij}^{(s)}$, but at large $|\mathbf{r}_i - \mathbf{r}_j|$ the

FIG. 4: A schematic picture of a phase sensitive "corner SQUID" experiment, introduced in Ref. [45]. If the square piece of superconductor has global d-wave superconductor symmetry, then there is a magnetic flux trapped in the ground state of the system. Pluses and minuses inside rosettes indicate the sign of $\Delta(\mathbf{k})$ as a function of the direction of $\mathbf{k}$

coupling strength $\tilde{J}_{ij}^{(s)}$ decays more slowly than $\tilde{J}_{ij}^{(d)}$. Thus between the asymptotic d-wave dominated regime where the puddles are dense, and the s-wave dominated regime where they are dilute, it is likely that there is an intermediate region in which the system will exhibit spin glass features and/or coexistence of d-wave and s-wave ordering. In this article, however, we will not further explore this fascinating but complex aspect of this problem.

While the above discussion was based on a cartoon model with regularly shaped puddles, we would like to stress that our conclusions do not rely on this. In particular, as it is qualitatively illustrated in Fig. 5c, Eq. 49 holds at arbitrary shape of the puddles provided that the typical distance between them is larger than their characteristic size.

To quantify our conclusions, we compute the Josephson coupling between a pair of far separated puddles in two extreme limits, large puddles and small puddles:

If the size of the puddles is large enough, the Josephson coupling has to be obtained from the solution of the Usadel equations (See for example [33]) for the configuration averaged

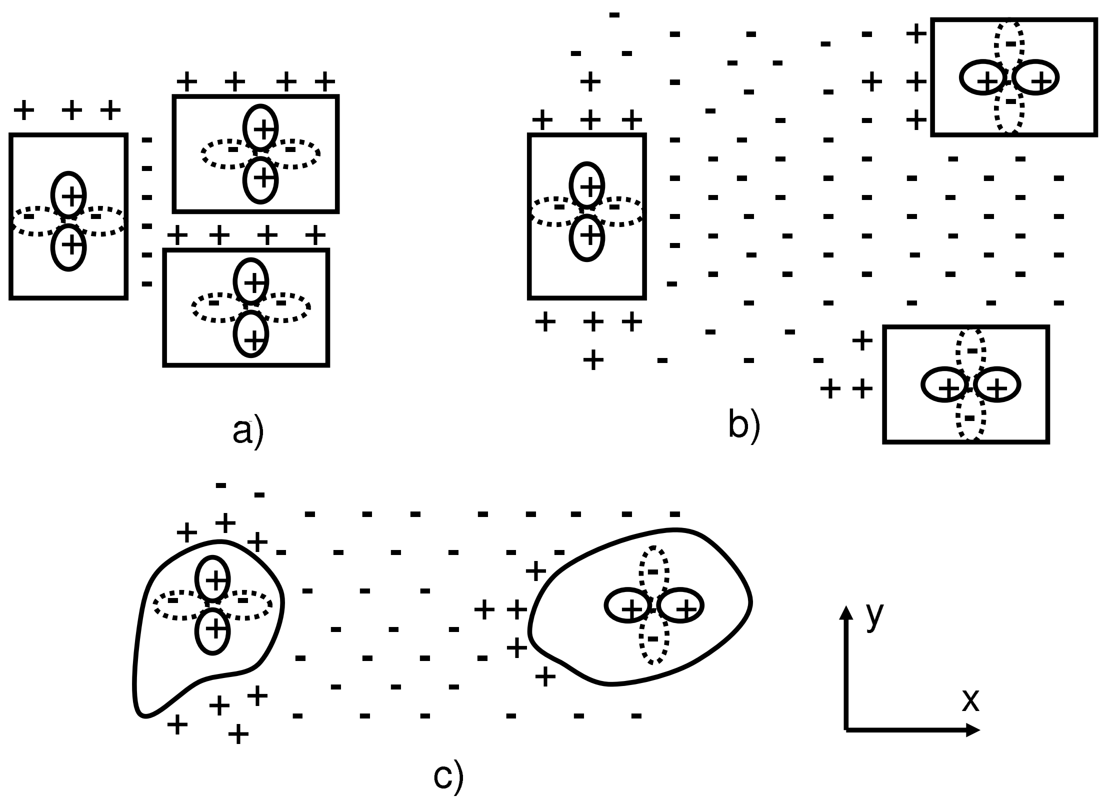

FIG. 5: A qualitative illustration of the global d-wave to s-wave transition. Solid lines represent boundaries of d-wave superconducting puddles embedded into a normal metal. Pluses and minuses indicate the areas were the s-wave component of the anomalous Green function $F^s(\mathbf{r}, \mathbf{r})$ is positive and negative respectively. a) The "cartoon" case in which the concentration of regular rectangular d-wave puddles is large, so the system has d-wave global symmetry. b) The case when the concentration of d-wave puddles is small so the system has s-wave global symmetry. c) The case in which the concentration of d-wave puddles is small and they have arbitrary shapes. Here, they are shown embedded in a normal metal, and the system has global s-wave symmetry.

anomalous Green function $\overline{\mathcal{F}_\epsilon^{(s)}(\mathbf{r})} \equiv -i \sin\theta(\epsilon, \mathbf{r})$ in the metal,

$$\frac{D_{tr}}{2} \partial_{\mathbf{r}}^2 \theta(\epsilon, \mathbf{r}) + i\epsilon \sin\theta(\epsilon, \mathbf{r}) = 0. \tag{52}$$

Here $\mathcal{F}_\epsilon^{(s)}(\mathbf{r})$ is the Fourier transform of $\mathcal{F}^{(s)}(\mathbf{r}, t-t')$, and the symbol $\overline{0}$ indicates averaging over the random scattering potential between the puddles at given shape of the puddles. The boundary conditions for Eq. 52 at the normal-superconductor surface have been derived in Ref. [48]. They are valid as long as the size of the puddles is large and the Andreev reflection on the puddles is effective.

Since the relevant energy for computing the Josephson coupling, $\epsilon \approx D_{tr}/|\mathbf{r}-\mathbf{r}_j|^2$, is much smaller than the value of the order parameter in the puddles, the boundary condition for $\theta(\mathbf{r},\epsilon)$ is independent of ϵ, and depends only on the angle between the unit vector parallel to the direction of a gap node, $\hat{\mathbf{n}}_\Delta$, and a unit vector, $\hat{\mathbf{n}}(\mathbf{r})$, normal to the boundary at point $\mathbf{r}$ at the surface, :

$$\theta_s(\epsilon,\mathbf{r}) = f[\alpha(\mathbf{r})], \quad \sin[\alpha(\mathbf{r})] \equiv \hat{\mathbf{n}}(\mathbf{r})\cdot\hat{\mathbf{n}}_\Delta. \tag{53}$$

Here $f(\alpha)$ is a smooth, approximately odd and periodic function, $f(\alpha) \approx -f(-\alpha)$, $f(\alpha) \approx f(\alpha+\pi)$, which grows from $f(\alpha) \approx 0$ at $\alpha = 0$, to $f(\alpha) \approx \pm\zeta$ for $\alpha = \pi/4$. Here $\zeta \sim 1$.

Solving Eq. 52 with the boundary conditions Eq. 53 and using the standard procedure of calculation of the Josephson energy we get

$$\eta_i = \text{sign}\left\{\int_i ds f(\alpha)\right\} \tag{54}$$

where the integral is taken over the surface of the i_{th} puddle. Moreover, the value of $\tilde{J}^{(s)}_{ij}$ in Eq. 49 turns out to be of the same order as in Eq. 4. It is this long range nature of the decay which ensures the existence of the phase in which the puddles separates by a large distance and the system has global s-wave phase coherence.

In the second case, the Andreev reflection is ineffective and the interactions between puddles can be computed in perturbation theory. Thus we can write an analog of Eq. 49.

$$E_{Jos} = -\sum_{i\neq j} \eta'_i\eta'_j[J^{(s)}_{ij}\Delta^\star_i\Delta_j + \text{c.c.}] \tag{55}$$

where, Δ_i is (up to a sign, $\eta'_j = \pm 1$) the average of the order parameter over puddle i,

$$\Delta_i \equiv \eta_i \int_{\text{puddle } i} \Delta(\mathbf{r},\mathbf{r}')\frac{d\mathbf{r}d\mathbf{r}'}{V_i^2}, \tag{56}$$

and η'_i is a random variable that we have introduced in Eq. 55, (and then cancelled in the definition of Δ_i). The strength of the Josephson interaction between s-component of the order parameter in puddles characterized by $J^{(s)}$ is (up to a numerical factor smaller than one) of order of Eq. 16. Again, we have neglected in Eq. 55 the interactions between the d-wave components of the order parameter, since they fall faster with separation between puddles. Notice that in the fine tuned case of a fully symmetric puddle, $\Delta_i = 0$ due to the d-wave symmetry.

The important point is that, in both the small and large puddle limits, Eqs. 55, and 49 yield the same qualitative picture: at large inter-puddle distances the Josephson coupling favors s-wave symmetry. It now remains to show that, near the point of quantum SMT, the distance between optimal puddles is indeed much larger than their size.

2. Globally s-wave superconductor to metal transition

The quantum transition between a globally s-wave superconducting state and the metal does not differ in a crucial way from the transition which has been considered in Section 2. For reasons that should by now be familiar, near the critical value of the disorder the spacial dependence of the order parameter can be visualized as defining a system of far separated superconducting puddles with anomalously large values of the order parameter separated by large areas of the normal metal. In particular, this results in a smaller value of l_c (larger critical magnitude of the disorder strength) for the destruction of superconductivity than l_{co} which is given by the conventional theory. The difference between the present problem and that treated in Section 2 is that, to identify a set of optimal puddles, one has to characterize them by two generally independent parameters: the size of the puddles and the value of the s-component of the order parameter associated with an optimal puddle.

As we have seen in previous sections, depending on whether the Andreev reflection from the superconductor-metal boundary is effective or not, two scenarios are possible. In the first case the amplitude of the Josephson energy is independent of the value of the order parameter in the puddles, and is given by Eq. 15. In the second case the amplitude of Josephson energy is given by Eq. 16. The susceptibility of an isolated puddle does not depend in any essential way on the symmetry of the superconducting order parameter, and as such, the analysis follows along identical lines as for the case of s-wave puddles, discussed in Sec. II. Thus, for small puddles, the susceptibility is determined by the Cooper instability Eq. 11, while puddles of radius R which is large compared to the coherence length, the susceptibility is determined, as in Eq. 9, by the effective conductance, $G^{eff} \sim R^{D-2}$, where R is the radius of the puddle.

Let us consider a situation where the mean free path $l(\mathbf{r})$ is a random quantity which exhibits classical spatial fluctuations with a Gaussian distribution

$$P(l) = \frac{1}{\sqrt{2\pi}\sigma_l \bar{l}} \exp[-\frac{(l-\bar{l})^2}{2\sigma_l^2 \bar{l}^2}] \tag{57}$$

characterized by the average $\bar{l}$, a dimensionless variance σ_l, and a correlation length, Λ.

To be concrete we consider the 3D case, and neglect the mesoscopic fluctuations of the mean free path of an interference nature. We also assume that the conductance of the metal is isotropic.

The size of the optimal puddles is readily seen to be of order the coherence length $R_{opt} \sim \xi_{opt} \sim \xi \sim \xi_0 \sqrt{l_0/(l_{opt}-l_0)}$, and therefore the susceptibility of the puddles is given by the large puddle result in Eq. 9. (The self-consistency of this assumption can be checked starting with the assumption that the small-puddle expression in Eq. 11 can be used, and then determining the optimal puddle size – this procedure leads to the inconsistent conclusion that the optimal puddles are arbitrarily large.)

The effective conductance that determines the susceptibility is the conductance of a region of characteristic linear dimension ξ_{opt}; this, in turn, depends on the local value of the mean-free path as $G_{opt} = G_{\xi_0}\sqrt{l_0/(l_{opt}-l_0)}$, where G_{ξ_o} is a conductance of a region of size ξ_0. The probability to find a puddle of this size is determined by the variance of the mean free path averaged over it's volume, which is of order $\sigma_l(\Lambda/\xi)^{3/2}$. Thus we have

$$X_{opt} \sim \exp\left[2ZG_{\xi_0}\frac{l_0^{1/2}}{(l_{opt}-l_0)^{1/2}} - \frac{(l_{opt}-\bar{l})^2}{\sigma_l^2(l_{opt}-l_0)^{3/2}l_0^{1/2}}\left(\frac{\xi_0}{\Lambda}\right)^3\right]. \tag{58}$$

As usual, l_{opt} is the value which maximizes Eq. 58, and the critical disorder, $\bar{l} = l_c$, is obtained by equating the result to unity. Although $l_c > l_0$, as long as σ_l is sufficiently small, $(l_0 - \bar{l}_c)/\bar{l} \ll 1$. In this limit, the result of this procedure can be expressed as

$$(l_0 - l_c)/l_0 \sim 2ZG_{\xi_0}\sigma_l^2\left(\frac{\Lambda}{\xi_0}\right)^3 \tag{59}$$

Near the critical point $\bar{l} = l_c$, the distance between the optimal puddles $R_{opt}\exp[G_{\xi_o}^{1/2}(\xi_0/\Lambda)^{3/2}/\sigma_l] \gg R_{opt}$ is exponentially large. The assumption that the distance between optimal puddles is large which we made in the previous subsection is, thus, justified for $\bar{l}$ near l_c.

The temperature dependence of the critical value of the mean-free path, $l_c(T)$, can be obtained from similar considerations to those used to determine $N_c(T)$ in Section 2.

The resulting phase diagram for a d-wave superconductor in the presence of quenched disorder is shown schematically in Fig. 3. At large disorder ($l < l_c$) the system is in the normal metal phase. At $l_0 > l_c$ the system in a state with a dominant s-wave component of the order parameter, and a d-wave component, whose sign is locally slaved to the s-wave component, in a way which varies randomly in space. At still larger values of $l > l_0$, there is a dominantly d-wave state, in which the s-component has a sign is locally slaved to the d-component and varies randomly in space.

We would like to note that both d-wave and s-wave superconducting phase are expected to exhibit glassy behavior associated with rare regions where the uniform phased order is strongly frustrated. We consider the existence of a glass phase which spontaneously breaks time reversal symmetry likely but not proven, so we have not shown it in the phase diagram in Fig. 3.

V. DISCUSSION

In this article we have focussed attention on several systems in which, at the point of the quantum superconductor-metal transition, the conductivity is still large compared to the quantum of conductance, and hence localization effects are unimportant. The key general aspects of this transition are: 1) The $T = 0$ transition occurs at a point at which the superconducting order parameter is small in most of the sample, other than in a dilute set of locally superconducting puddles. 2) The transition is triggered by the quantum fluctuations of the phase of the order parameter on these puddles, whose quantum dynamics can either be governed by the dynamics of the Cooper instability (when the optimal puddles are small) or the electric field fluctuations (when the optimal puddles are large). 3) While there is probably a small quantum critical regime at exceedingly low temperatures and very close to the quantum critical point where the physics is universal and can be described by a suitable quantum critical scaling theory, there is a much larger regime where quantum phase fluctuations dominate the physics, but the long-distance properties are more properly described by the thermally truncated percolation of phase coherence between puddles.

We have ignored the fundamental, but for our purposes purely academic question of electron localization in highly conducting samples. However, there remains the issue of the ultimate fate of the “metallic phase” in the true $T \to 0$ limit. In 3D this is, presumably, not an issue, but in 2D, where all single particle states are localized, this is an important point of principle. For $G_{2D} \gg 1$ the temperature below which interference effects are relevant is exponentially small, so the issue is not of practical relevance. Still even the point of principle is interesting, and, in our opinion, unresolved partially because calculations of weak localization corrections close to the point of the metal-superconductor transition, where the conductivity is much larger than the Drude value, remains a challenge.

3. Previous studies of the problem

There have, of course, been many theoretical studies of the quantum phase transition from the superconducting to the non-superconducting state in the presence of quenched disorder. Many of these studies concerned a scenario in which there is a direct superconductor to insulator transition. Under circumstances in which, near criticality, $k_F l \approx 1$, it is conceivable that there is a direct superconductor to insulator transition. Moreover, in common with the superconductor to metal transition considered by us, near such a superconductor to insulator transition the electron wave functions become strongly non-uniform, and exhibit fractal features [49]. Therefore it is not surprising that the order parameter at the point of the transition also non-uniform [50, 51]. Despite the similarity of this aspect of the two transitions, the superconductor to insulator transition differs from the superconductor to metal transition in significant ways, and is outside the scope of the present paper.

There have also been many previous studies of the superconductor to metal transition, starting with the mean-field studies of Abrikosov and Gorkov of the transition in a magnetic field. In this context, we would like to mention the paper Ref. [52] where a renormalization group approach was proposed for disordered 2D s-wave superconductors in the absence of magnetic field. The conclusion reached in this article bears some similarity to ours: The superconductivity is quenched under conditions such that $G_{2D} > 1$. In the framework of the analysis in Ref. 52, the reason is that the diagrams responsible for suppression of T_c by fluctuations of the phase of the order parameter are proportional to $\ln^3(L_T/l)$, while the weak localization corrections to the conductivity are only proportional to $\ln(k_F L_T)$. The theory presented in this work was also supported by a comparison between theory and the experiments of Ref. 53. The corrections of order $\ln^3(L_T/l)$ in Ref.[52] are of the similar physical origin as the power law in Eq. 7. However, there are significance differences in our analysis, the most important of which is that in the situations we have considered, the existence of a superconductor-metal transition is generically unconnected with interference effects governed by the parameter $k_F l$. Therefore the quantum superconductor to insulator transition and 2D localization can be treated separately from each other. As a result, the effects we have considered are much larger than those considered in Ref. 52.

Pioneering studies of the quantum d-wave superconductor (or more exotic superconductor) to metal transition were carried out in Refs. [17, 64], which employed standard diagramatic techniques and the replica trick to implement the disorder averaging. Our results differ significantly from those in Refs. [17, 64], principally due to the fact that these earlier studies did not account for existence and crucial role of rare suprconducting puddles near the transition.

A. Experiments on quantum superconductor-metal transitions

There have been a vast number of experiments on the destruction of superconductivity in thin-film systems - too many for us to comment on here. For example there are many experiments (for a review see, for example, Ref. [8]) in which s-superconductivity in films is destroyed by disorder in the absence of a magnetic field, and the transition takes place at $G_{eff} \sim 1$. Since our theory is developed in the case $G_{eff} \gg 1$, the present theory has no direct relevance to these experiments. We will only discuss experiments in which $G_{eff} \gg 1$, and even here, only a very small subset of them.

Before discussing the relation between our results and experiments we must address a question of terminology. An unambiguous distinction between metallic and insulating states can be made only in the limit $T \to 0$. The metallic state has finite resistance in this limit, while the insulating state has infinite resistance. (The superconducting state, of course, has zero resistance.) The complication concerns the way in which finite temperature data is extrapolated to the $T \to 0$ limit.

One relatively widely used criterion is to study the sign of the dimensionless quantity

$$R_T \equiv d \log[\rho]/d \log[T] \tag{60}$$

at the lowest accessible temperatures, and to identify the insulating state with $R_T < 0$, a superconducting state with $R_T > 0$, and a metallic state with $R_T \approx 0$. Clearly, in the extremes, this is a sensible criterion, since in order for the resistivity to either vanish or diverge in the $T \to 0$ limit, R_T must have the stated sign. The problem comes when R_T in the accessible range of temperatures is relatively small in magnitude. There are certainly well documented ways for a metal to exhibit $R_T < 0$ (for instance, in the Kondo effect), so an observation of $R_T < 0$ cannot be safely taken as evidence that ρ will diverge as $T \to 0$. Of course, it is also common for metals to have $R_T > 0$, so this observation, by itself, cannot be taken as a sure indication of a zero temperature superconducting state. It is always also important to pay attention to the absolute magnitude of the resistance in such discussions. Metals at low temperatures typically have resistances smaller than the quantum of resistance, while insulators, at low enough temperatures, always have resistances large compared to this value.

1. Transition in a perpendicular magnetic field in films with large G_{2D}

When highly conducting films of a superconducting metal, such as MoGe, are subjected to a perpendicular field, two asymptotic behaviors are expected, and observed as a function of H: For small enough H, the resistance tends [54] to arbitrarily small values at low T, which can be interpreted as a superconducting state. At large enough H, the resistance tends to a roughly temperature and H independent value, $\rho = \rho_D$, which can be interpreted as the "normal" state (Drude) value. (ρ_D is expected to be almost field independent so long as $r_c l \ll 1$ where $r_c(H)$ is the cyclotron radius.)

It has been observed in [54], and latter in [55], that at small T there is a large interval of magnetic fields where the resistance is independent of T, and can be up to four orders of magnitude smaller than ρ_D.

Although we have not calculated the conductance, and so cannot propose direct comparisons between theory and experiment, we believe that the significant enhancement of the conductance takes place in the interval of magnetic fields in which somewhat isolated puddles of the sample have a local value of $H_{c2} > H$. (It is not completely clear, to us, whether the observed behaviors should be interpreted as the finite T behavior of a system in the range of fields, $H_{c2}^{(0)} < H < H_c$, where the system forms a gauge glass phase in the $T \to 0$ limit, or whether it should be interpreted in terms of the anomalous metallic phase for $H \gtrsim H_c$, where even at $T = 0$ there is no global phase coherence.) In any case, to explain the broad range of H over which significant superconducting fluctuations occur, we must assume that $\sigma_H > \sqrt{G_{2D}}$ in Eq. 32.

In this context we would like to mention an interesting phenomenological observation made recently by Steiner *et al* [56]. They focussed attention on the critical value of the magnetic field, $H^\star$, at which $R_T(H)$ changes sign: $R_T(H) > 0$ for $H < H^\star$, $R_T(H) < 0$ for $H > H^\star$, and $R_T(H) \approx 0$ for $H = H^\star$. This is often identified as the point of a SIT, despite the fact that, on both side of the "transition," the T dependence of the resistance is sometimes sufficiently weak that an unbiased extrapolation to $T = 0$ would yield a finite result. Steiner *et al* found that the behaviors could be sorted into two classes. In some films, $\rho(H = H^\star) \approx h/4e^2$, and in these, a scaling collapse of the data suggestive of universal quantum critical phenomena can be achieved, and not only is $R_T(H) < 0$ for $H > H^\star$, but the resistance actually grows large enough at low T that it is suggestive of a truly insulating phase. In other films, $\rho(H = H^\star) \ll h/4e^2$, and in these, resistance appears to approach a finite "metallic" value for H on both sides of $H^\star$, and correspondingly any attempt to scale the data breaks down at low T. Moreover, in these low resistance films, the "critical" resistance, $\rho(H = H^\star)$, is manifestly non-universal. This analysis suggests that there are two possible limiting behaviors - the one in low resistance films, to which the present analysis is applicable, and that in the higher resistance films, which may, to some level of approximations, be exhibiting a superconductor to insulator transition.

However, even in the highly conducting films of Ref. [54], experimental indications of glassy behavior has not been reported for superconducting films in a perpendicular magnetic field, contrary to our expectations.

2. *Transition in a parallel magnetic field in films with large G_{2D}*

Most studies of superconducting films involve relatively heavy elements, such as Mo or even Pb, so that the spin-orbit scattering rate is substantial and $\Delta_0\tau_{so} > 1$. However, in Ref. 57, Wu and Adams studied aluminum films where $\Delta_0\tau_{so} \gg 1$. In these experiments, it has been observed that in the vicinity of $H_\parallel = H^{(0)}_{c\parallel}$, the time dependence of the resistance exhibits long time relaxations with characteristic times of order 10^3 sec. During this period of time, the resistance changes by orders of magnitude and exhibits avalanche-like jumps. This is the characteristic dynamics of a glassy system. We think that this behavior is compatible with our theory, as discussed in Section 3B. We do not know of any experiments reported to date on the quantum transition between this superconducting glass state and the normal metal – we believe such experiments could critically test the ideas presented here.

3. Transition in d-wave superconductors as a function of disorder

The cuprate high temperature superconductors are the best established example of a d-wave superconductor. Here, the critical temperature, T_c, is known to vary strongly as a function of the doped hole concentration, x, producing two quantum critical points at which T_c vanishes: a lower critical doping concentration, x_1, on the "underdoped" side, and an upper critical concentration, x_2, on the "overdoped" side of the phase diagram. On the underdoped side of the superconducting dome, the thermally accessible normal state above T_c is manifestly not a good Fermi liquid. Moreover, with increasing underdoping, these materials frequently appear to undergo a superconductor to insulator transition with a critical resistance that is typically large compared to $h/4e^2$ [58, 59] Thus, the present considerations may not be applicable. (However, in some instances of very high quality crystals of YBCO, the normal state revealed upon quenching superconductivity by underdoping can be somewhat metallic[60].)

It is still unclear to what extent a weak-coupling, Fermi liquid based approach is valid, even in the "overdoped" regime of these strongly correlated materials. If we assume that, despite the uncertainties inherent in the strong correlation physics of the cuprates, some of the more robust of our findings apply to the cuprates as $T_c \to 0$ with overdoping, there are a number of interesting predictions we can make, none of which (to the best of our knowledge) have so far been observed experimentally.

1) There should be a transition from a globally d-wave to a globally s-wave superconducting state at a doping concentration, x_{d-s}, which is less than the critical doping, x_2, at which T_c vanishes. While even for $x_{d-s} < x < x_2$, any local probe will see a d-wave-like gap structure, global phase sensitive measurements should record an s-wave state. (Some evidence of such a transition may already be present in the experiments of Ref. 61.)

2) For x near x_2, the superconducting state should consist of dilute puddles in which the pairing is strong, floating in an otherwise metallic sea. (Indirect evidence of such a situation in LSCO has been presented in Ref. 62.)

3) In the metallic state with $x > x_2$, the conductivity at low temperature should diverge as $x \to x_2$, the Hall resistance should vanish, and the Wiedemann-Franz law should be increasingly strongly violated, in the sense that the conductivity should be greater than anticipated on the basis of the thermal conductivity.

Finally we would like to mention that there are various other candidates for experimental studies of the above discussed effects. There is a growing consensus that there are multiple other examples of d-wave superconductors, including in the "115" family of heavy fermion superconductors and some organic superconductors. Moreover, Sr_2RuO_4 is known to be a p-wave superconductor. It has already been demonstrated [63] that superconductivity in these materials is suppressed by disorder when the parameter $k_F l$ of the system is still much larger than unity. Though the present theory was carried out specifically for the d-wave (*i.e.* spin singlet) case, we think that it is qualitatively applicable to the p-wave (*i.e.* spin triplet) case as well, at least in the presence of spin-orbit coupling.

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VI. APPENDIX: DERIVATION OF THE EFFECTIVE ACTION

We now sketch representative calculations required for the derivation of the effective actions presented in Eqs. 1, which describes the quantum dynamics of the local superconducting "puddles" in the regime in which the mean-field solution is highly inhomogeneous. To begin with, we consider the effective Euclidean action, $S[\Delta]$, to be a functional of the pair-field, Δ, obtained by performing a Hubbard-Stratonovich transformation on an underlying microscopic Hamiltonian, and then integrating out the electronic degrees of freedom. Approaching the transition from the non-superconducting side, we assume that the magnitude of Δ is everywhere small, so we can expand the action in powers of Δ

$$S = \int d\mathbf{r}d\mathbf{r}'dtdt'\Delta^\star(\mathbf{r},t)K(\mathbf{r},\mathbf{r}',t-t')\Delta(\mathbf{r}',t') + \dots$$

where $K(\mathbf{r},\mathbf{r}',t-t')$ is an appropriate imaginary time ordered four-fermion correlation function (which is dependent on the precise configuration of the quenched disorder) and ... represents higher order terms in powers of Δ. K and other response functions that enter the higher order terms in the effective action are evaluated in the normal state, *i.e.* they reflect the physics of disordered metals, not the superconducting state.

The time Fourier transform of K generically has the structure,

$$\tilde{K}(\mathbf{r}, \mathbf{r}'; \omega) = K_0(\mathbf{r}, \mathbf{r}') + |\omega| K_1(\mathbf{r}, \mathbf{r}') + \dots \tag{61}$$

where ... means higher order terms in powers of ω. The presence of the non-analytic $|\omega|$ dependence is generic in a metal, and reflects the fact that in real-time, superconducting fluctuations have an exponential dependence on time; they decrease exponentially if the normal metal state is stable, and increase exponentially if the normal metal is unstable.

In disordered systems $\tilde{K}(\mathbf{r}, \mathbf{r}'; \omega)$ is a random function of the coordinates. Consequently near the point of the quantum phase transition the distribution of the order parameter can be visualized as a sequence of superconducting puddles separated on by a large distance. More precisely, at the saddle point level, a superconducting state occurs whenever $\hat{K}_0$ (by which we mean the integral operator corresponding to K_0) has at least one negative eigenvalue, $\hat{K}_0 \Phi_\alpha = \varepsilon_\alpha \Phi_\alpha$. In other words, if $\mathrm{Min}[\varepsilon_\alpha]$ is the smallest eigenvalue of K_0, then the superconducting state occurs when $\mathrm{Min}[\varepsilon_\alpha] = 0$. Generically, the smallest eigenvalues (*i.e.* states deep in the "Lifshitz tails") are associated with wave-functions that are spatially localized in regions of the system that are anomalously favorable for superconductivity. However, the nature of these localized solutions (*i.e.* the spatial extent of the localized state), and the distribution of eigenvalues in the tails of the distribution depend on the circumstances, as we discuss in Sections II - IV of the paper.

The full saddle-point value of Δ_{sp}, obtained by minimizing $S[\Delta]$, can be expanded in terms of

$$\Delta_{sp}(\mathbf{r}) = \sum_\alpha \mathbf{\Delta}_\alpha \mathbf{\Phi}_\alpha(\mathbf{r}) \approx \sum_{\varepsilon_\alpha < \mathbf{0}} \mathbf{\Delta}_\alpha \mathbf{\Phi}_\alpha(\mathbf{r}) \tag{62}$$

In this expansion, Δ_α can be approximately interpreted as the superconducting amplitude on puddle α. One trouble with this, however, is that, like Wannier functions in a crystal, the wave-functions Φ_α are not quite as localized as they should be, because they have small admixtures of the wave-function from neighboring puddles which are necessitated by the orthogonality condition, $\int d\mathbf{r} \Phi^\star_\alpha(\mathbf{r}) \mathbf{\Phi}_{\alpha'}(\mathbf{r}) = \delta_{\alpha,\alpha'}$.

We thus obtain Eq. 1 when we substitute the approximate expression in Eq. 62 into Eq.

61. The coefficient β_i reflects the long-time dynamics of the order parameter,

$$\beta_i = \int d\mathbf{r} d\mathbf{r}' \Phi_i^\star(\mathbf{r}) K_1(\mathbf{r}, \mathbf{r}') \Phi_i(\mathbf{r}'). \tag{63}$$

The strength of the Josephson couplings between i-th and j-th puddles J_{ij} is given by the expression,

$$J_{ij} = \int d\mathbf{r} d\mathbf{r}' \Phi_i^\star(\mathbf{r}) K_0(\mathbf{r}, \mathbf{r}') \Phi_j(\mathbf{r}') \tag{64}$$

By dimensional analysis (as well as explicit calculation) it is clear that α_i and β_i are given by Eq. 2, while J_{ij} is given by Eq. 5.

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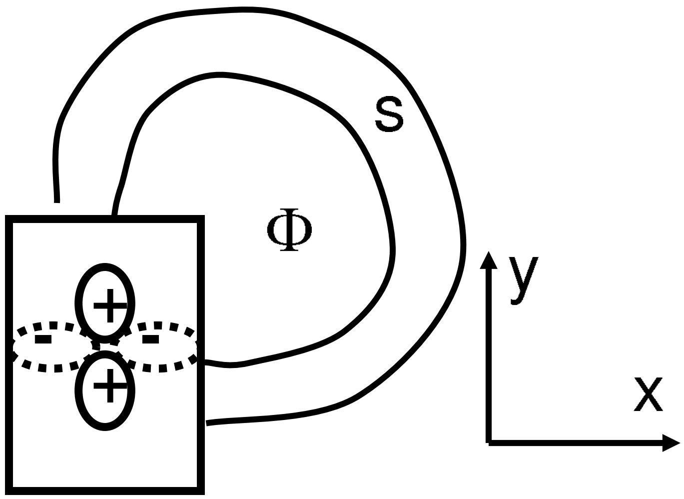

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