All-order results for infrared and collinear singularities in massless gauge theories

Lance J. Dixon[*]

SLAC National Accelerator Laboratory, Stanford University
E-mail: lance@slac.stanford.edu

Einan Gardi

School of Physics, The University of Edinburgh
E-mail: einan.gardi@gmail.com

Lorenzo Magnea[†‡]

CERN, PH Department, TH Unit;
Dipartimento di Fisica Teorica, Università di Torino, and INFN Sezione di Torino
E-mail: magnea@to.infn.it

We review recent results concerning the all-order structure of infrared and collinear divergences in massless gauge theory amplitudes. While the exponentiation of these divergences for non-abelian gauge theories has been understood for a long time, in the past couple of years we have begun to unravel the all-order structure of the anomalous dimensions that build up the perturbative exponent. In the large-N_c limit, all infrared and collinear divergences are determined by just three functions; one of them, the cusp anomalous dimension, plays a key role also for non-planar contributions. Indeed, all infrared and collinear divergences of massless gauge theory amplitudes with any number of hard partons may be captured by a surprisingly simple expression constructed as a sum over color dipoles. Potential corrections to this expression, correlating four or more hard partons at three loops or beyond, are tightly constrained and are currently under study.

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1. Introduction

The long-distance behavior of gauge theory amplitudes and cross sections has been the subject of theoretical studies for nearly three quarters of a century. After such a long time, one might imagine that further progress, if any, should be slow and incremental: on the contrary, the past few years have witnessed several new developments, some of them surprising. This has been due in part to the phenomenological requirements set by the beginning of LHC operation, which places unprecedented pressure on theorists to come up with precise and reliable predictions for very complex QCD processes. On the other hand, 'pure theory' has also played a role: for example, insights have come from the study of the maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory; also, part of the general motivation for these studies remains the fact that long-distance singularities provide a gateway from perturbative calculations to the non-perturbative content of the theory.

Much of the knowledge accumulated in earlier work on the structure of singularities for gauge theory *S*-matrix elements, which forms the basis for more recent developments, can be summarized in a single formula[1], expressing the factorization of fixed-angle amplitudes into separate functions, responsible for infrared poles, collinear poles, and finite remainders. Choosing a basis in the space of available color structures for a given *n*-particle amplitude, by picking suitable color tensors $c_L^{a_1,\ldots,a_n}$, the amplitude $M^{a_1,\ldots,a_n}$ can be expressed in terms of its components M_L. They obey

$$
M_L \left(p_i/\mu, \alpha_s(\mu^2), \varepsilon \right) = \mathscr{S}_{LK} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \varepsilon \right) H_K \left(\frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \varepsilon \right)
$$

$$
\times \prod_{i=1}^n \left[J_i \left(\frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \varepsilon \right) \middle/ \mathscr{J}_i \left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \varepsilon \right) \right], \quad (1.1)
$$

where p_i, $i = 1, \ldots, n$, are the external hard momenta, which are assumed to form invariants $p_i \cdot p_j$ of a common parametric size Q^2. In Eq. (1.1), the functions J_i collect all collinear singularities associated with virtual gluons emitted in direction of parton *i*. In order to factorize these singularities, it is necessary to introduce 'factorization vectors' n_i^μ, $n_i^2 \neq 0$, which play a threefold role: first, they ensure gauge invariance of the operator matrix element defining J_i, which includes an infinite Wilson line in the direction n_i; these Wilson lines act as absorbers, replacing the other hard partons in the amplitude, and collecting the gluons emitted by parton *i* without generating extra singularities; finally, n_i can be physically interpreted as a vector separating gluons which are collinear to p_i (those whose momenta *k* satisfy $k \cdot p_i < k \cdot n_i$) from (soft) gluons emitted at large angles; in this sense n_i can be properly interpreted as a factorization vector. Note that the jet functions J_i are color singlets: infrared (soft) singularities, on the other hand, are not color diagonal, and therefore they are organized in a matrix, $\mathscr{S}_{LK}$, which is purely eikonal. It is defined in terms of a product of Wilson lines characterized by the directions and color representations of the hard partons. Being purely eikonal, $\mathscr{S}_{LK}$ can only depend on the velocities β_i of the hard partons, defined by dividing out the common hard scale *Q*, taking $p_i \propto Q\beta_i$. With these definitions, gluons that are both soft and collinear to one of the hard partons have been counted twice; it is however simple to subtract the double counting: one just needs to divide each jet J_i by its own eikonal approximation, denoted by $\mathscr{J}_i$ in Eq. (1.1). The vector of hard functions H_K, finally, collects all finite remainders, and

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[1]For reviews of the methods leading to Eq. (1.1) see, for example, [1].

each component is finite as $\varepsilon \to 0$. Note that the matrix element M has been normalized to be dimensionless: the functional dependences of the various factors in Eq. (1.1) reflect this fact, and will be further discussed below. In the following, we will briefly outline the consequences of the factorization presented in Eq. (1.1), beginning with the simpler case of amplitudes at large N_c.

2. Form factors and large-N_c amplitudes

The simplest instance of Eq. (1.1) is given by parton form factors, which describe the scattering of a parton by an electroweak current[2]. This is a color-singlet process, so that the soft matrix $\mathscr{S}$ is just a single function. One may then derive an evolution equation by imposing that the form factor be independent of the factorization vectors n_i. The solution to this equation is especially simple and transparent in dimensional regularization [2]: within this framework, one may take advantage of the fact that the d-dimensional running coupling vanishes as a power of the scale for $d > 4$, in order to impose as a boundary condition that radiative corrections should vanish at $Q^2 = 0$. One finds that the form factor exponentiates as [3]

$$\Gamma\left(Q^2, \varepsilon\right) = \exp\left\{\frac{1}{2} \int_0^{-Q^2} \frac{d\xi^2}{\xi^2} \left[G\left(\overline{\alpha}\left(\xi^2, \varepsilon\right), \varepsilon\right) - \frac{1}{2} \gamma_K\left(\overline{\alpha}\left(\xi^2, \varepsilon\right)\right) \log\left(\frac{-Q^2}{\xi^2}\right) \right] \right\}. \quad (2.1)$$

The first remarkable fact about Eq. (2.1) is that all singularities are generated by integrating over the scale of the d-dimensional coupling $\overline{\alpha}$: the functions G and γ_K are finite as $\varepsilon \to 0$ and universal. Specifically, $\gamma_K(\alpha_s)$ is the cusp anomalous dimension [4], governing ultraviolet singularities for correlators of pairs of light-like Wilson lines originating at a cusp, and responsible in this case for double (infrared-collinear) poles; it depends only on the color representation of the hard parton. The function G generates single poles, as well as finite contributions; it can be expressed in terms of operator matrix elements involving Wilson lines as well as elementary fields [3], so it also depends on the the spin of the hard parton. Perhaps more interestingly, it can be decomposed as

$$G(\alpha_s, \varepsilon) = 2B_\delta(\alpha_s) + G_{\text{eik}}(\alpha_s) + \beta(\alpha_s) \partial E_H(\alpha_s, \varepsilon) / \partial \alpha_s. \quad (2.2)$$

In Eq. (2.2), $B_\delta(\alpha_s)$ is the virtual part of the diagonal Altarelli-Parisi splitting function, while $G_{\text{eik}}(\alpha_s)$ is a subleading anomalous dimension for Wilson line correlators, associated with the eikonal approximation to the form factor; the last term generates finite contributions only, and it vanishes in a conformal theory. We learn that the only non-eikonal contributions to the singular behavior of the form factor are contained in the collinear function $B_\delta(\alpha_s)$. Notably, in the conformal case, Eq. (2.2) holds also at strong coupling, in the large N_c limit [5].

The second remarkable fact about Eq. (2.1) is that it generates all singularities not only for form factors, but for all large-N_c multiparton fixed-angle amplitudes as well [6, 7]. In the planar limit, gluons are confined to propagate inside 'wedges' bounded by neighboring hard partons: the full amplitude becomes then a product of (square roots of) form factors, up to finite corrections.

Finally, the fact that long-distance singularities are completely encoded in the running coupling has important consequences for conformal theories, such as $\mathscr{N} = 4$ Super-Yang-Mills (SYM). In dimensional regularization, the coupling for these theories runs simply according to $\overline{\alpha}(\mu^2, \varepsilon) =$

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[2]We thank George Sterman, who co-authored Ref. [3], where the results of this section were derived.

$\left(\mu^2/\mu_0^2\right)^{-\varepsilon}\overline{\alpha}(\mu_0^2,\varepsilon)$, so that all integrations in Eq. (2.1) are trivially performed. This fact was exploited in [7] to study and test the all-order behavior of amplitudes in $\mathcal{N}=4$ SYM. By the same token, one derives [3] a strikingly simple relation tying the analytic continuation of the form factor to the cusp anomalous dimension. One finds that in any conformal gauge theory

$$\left|\frac{\Gamma(Q^2)}{\Gamma(-Q^2)}\right|^2 = \exp\left[\frac{\pi^2}{4}\gamma_K\left(\alpha_s(Q^2)\right)\right] \ . \tag{2.3}$$

Eq. (2.3) ties together two quantities that are finite (and actually independent of Q^2) in $d=4$, and are non-perturbatively defined. It may thus be argued to be an exact result for a class of non-trivial four dimensional gauge theories, which may at some point become testable at strong coupling.

3. Beyond the large-N_c limit

In order to go beyond the large-N_c limit, one must revisit the functional dependences in Eq. (1.1). Functions defined in terms of Wilson lines with velocities n_i (with $n_i^2 \neq 0$) depend homogeneously on the velocity vectors, reflecting the classical invariance of their operator definitions under rescalings $n_i^\mu \to \kappa_i n_i^\mu$. This is not the case for functions of the light-like Wilson lines with velocities β_i. The reason can be traced to the fact that these functions acquire new collinear divergences, which make the classical invariance under $\beta_i^\mu \to \kappa_i \beta_i^\mu$ anomalous at the quantum level. The anomaly is precisely expressed by the cusp anomalous dimension, which governs the superposition of soft and collinear singularities. This fact has remarkable consequences on the singularity structure. Indeed, one may construct a reduced soft matrix, where all soft-collinear double poles are cancelled, so that the anomaly in rescaling invariance is absent. It is defined by [6, 8]

$$\overline{\mathscr{S}}_{LK}\left(\rho_{ij},\alpha_s(\mu^2),\varepsilon\right) = \frac{\mathscr{S}_{LK}\left(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\varepsilon\right)}{\displaystyle\prod_{i=1}^{n}\mathscr{J}_i\left(\frac{(\beta_i\cdot n_i)^2}{n_i^2},\alpha_s(\mu^2),\varepsilon\right)} \ . \tag{3.1}$$

While the functions $\mathscr{S}_{LK}$ and $\mathscr{J}_i$ separately have double poles, and can thus depend on variables like $\beta_i\cdot\beta_j$, or $x_i \equiv (\beta_i\cdot n_i)^2/n_i^2$, the reduced matrix $\overline{\mathscr{S}}_{LK}$ has only single poles, and can only depend on rescaling-invariant combinations of β_i and n_i, which can only be constructed out of the variables $\rho_{ij} \equiv (\beta_i\cdot\beta_j)^2/x_i x_j$. Since all functions entering Eq. (3.1) are multiplicatively renormalizable, this recombination must be reflected in their respective anomalous dimensions, which must obey

$$\Gamma_{IJ}^{\overline{\mathscr{S}}}\left(\rho_{ij},\alpha_s(\mu^2)\right) = \Gamma_{IJ}^{\mathscr{S}}\left(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\varepsilon\right) - \delta_{IJ}\sum_{k=1}^{n}\gamma_{\mathscr{J}_k}\left(x_k,\alpha_s(\mu^2),\varepsilon\right) \ . \tag{3.2}$$

We see from Eq. (3.2) that singular terms in the matrix $\Gamma^{\mathscr{S}}$ must be canceled by those in the eikonal jet anomalous dimensions $\gamma_{\mathscr{J}}$, and must therefore be diagonal. Furthermore, finite diagonal contributions must conspire to reconstruct a dependence on ρ_{ij}, combining $\beta_i\cdot\beta_j$ with x_i and x_j. Finally, off-diagonal terms in $\Gamma^{\mathscr{S}}$ must be finite, and must by themselves depend only on rescaling-invariant combinations of β_i's. Such combinations exist only for amplitudes with at least four-particles, and must be built out of conformal cross-ratios of the form $\rho_{ijkl} \equiv (\beta_i\cdot\beta_j)(\beta_k\cdot\beta_l)/(\beta_i\cdot\beta_k)(\beta_j\cdot\beta_l)$. These powerful constraints can be summarized in a single set of equations [8], linking

the matrix $\Gamma^{\overline{\mathscr{S}}}$ to the cusp anomaly, and correlating kinematic and color degrees of freedom for any number of partons and at finite N_c. They are given by

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{\mathscr{S}}}_{MN} (\rho_{ij}, \alpha_s) = \frac{1}{4} \gamma^{(i)}_K (\alpha_s) \, \delta_{MN} \qquad \forall i, \tag{3.3}$$

where we made explicit the fact that the cusp anomalous dimension depends on the color representation of the selected parton i. An analogous equation was independently derived, using the methods of soft-collinear effective field theory, in [9].

One may now make the further assumption that the cusp anomalous dimension depend on the representation only through an overall factor of the quadratic Casimir operator, *i.e.* $\gamma^{(i)}_K (\alpha_s) = C_i \widehat{\gamma}_K (\alpha_s)$, with $\widehat{\gamma}_K (\alpha_s)$ a universal function. This assumption ('Casimir scaling') is true up to three loops, and arguments were given in [9] indicating that it should remain valid at four loops. Adopting the basis-independent notation of color generators, and thus writing $C_i = \mathrm{T}_i \cdot \mathrm{T}_i$, one may construct an explicit solution of Eq. (3.3). Indeed, the sum-over-dipoles formula

$$\Gamma^{\overline{\mathscr{S}}}_{\mathrm{dip}} (\rho_{ij}, \alpha_s) = -\frac{1}{8} \widehat{\gamma}_K (\alpha_s) \sum_{j \neq i} \ln(\rho_{ij}) \, \mathrm{T}_i \cdot \mathrm{T}_j + \frac{1}{2} \widehat{\delta}_{\overline{\mathscr{S}}}(\alpha_s) \sum_i \mathrm{T}_i \cdot \mathrm{T}_i \tag{3.4}$$

satisfies Eq. (3.3), as is easily shown using color conservation, $\sum_i \mathrm{T}_i = 0$. If desired, the residual anomalous dimension $\widehat{\delta}_{\overline{\mathscr{S}}}(\alpha_s)$ can be recombined with the color singlet contributions arising from jet functions in Eq. (1.1): a sum-over-dipoles formula then organizes all infrared and collinear divergences of fixed-angle amplitudes, for an arbitrary number of partons [9, 10].

4. Beyond the dipole formula?

Eq. (3.4) is remarkable, as it implies that color correlations induced by soft gluons are drastically simpler than expected on the basis of a diagrammatic analysis. One must ask what corrections, if any, might arise, that would be compatible with the constraint equation (3.3). Clearly, one possibility is a breakdown of Casimir scaling, *i.e.* the presence of higher-rank Casimir operators in the cusp anomalous dimension at sufficiently high order. The only other possibility allowed by Eq. (3.3) is the addition of a solution to the associated homogeneous equation. One may write in full generality

$$\Gamma^{\overline{\mathscr{S}}} (\rho_{ij}, \alpha_s) = \Gamma^{\overline{\mathscr{S}}}_{\mathrm{dip}} (\rho_{ij}, \alpha_s) + \Delta^{\overline{\mathscr{S}}} (\rho_{ij}, \alpha_s) , \tag{4.1}$$

where the correction term $\Delta^{\overline{\mathscr{S}}}$ must satisfy

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Delta^{\overline{\mathscr{S}}} (\rho_{ij}, \alpha_s) = 0 \qquad \Leftrightarrow \qquad \Delta^{\overline{\mathscr{S}}} = \Delta^{\overline{\mathscr{S}}} \left(\rho_{ijkl}, \alpha_s \right) . \tag{4.2}$$

In words, corrections to the dipole formula must be functions of the conformal cross rations ρ_{ijkl}: thus, by eikonal exponentiation, they must arise from gluon webs connecting at least four hard partons, and therefore they can first appear at three loops. This analysis explains the results of Ref. [11]: indeed, the dipole formula correctly reproduces all existing finite-order results.

One may proceed to ask whether further constraints are available, going beyond Eq. (3.3), that might force the correction term $\Delta^{\overline{\mathscr{S}}}$ to vanish, or determine its color structure, and its functional dependence on ρ_{ijkl}'s. This analysis, started in Ref. [9], was pursued in Ref. [12].

A general constraint on $\Delta^{\overline{\mathscr{S}}}$ is dictated by its behavior in the limit where two or more of the hard partons become collinear. In this limit Eq. (1.1) breaks down, however it is expected that the new singularities that arise should be captured by a splitting matrix depending only on the degrees of freedom of the partons becoming collinear. As shown in Refs. [9, 12], this essentially forces $\Delta^{\overline{\mathscr{S}}}$ to have trivial collinear limits. One may furthermore impose Bose symmetry (since $\Delta^{\overline{\mathscr{S}}}$ arises diagrammatically from webs of gluons), and require that, at any given order, the functions comprising $\Delta^{\overline{\mathscr{S}}}$ should satisfy a transcendentality bound (at g loops, one needs $\tau_{\mathrm{max}} = 2g - 1$). Remarkably, the set of functions satisfying all the constraints is quite small, though not empty. As an example, if one considers functions built out of products of logarithms of ρ_{ijkl}'s (which would naturally occur in Feynman diagram calculations), then Bose symmetry and collinear consistency single out a unique class of functions. Defining $L_{ijkl} \equiv \log \rho_{ijkl}$, the quadrupole component of $\Delta^{\overline{\mathscr{S}}}$, which is the basic building block for higher-point corrections as well, must be of the form

$$\Delta_4(\rho_{ijkl}) = \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \left[f_{ade} f_{cb}{}^e L_{1234}^{h_1} \left(L_{1423}^{h_2} L_{1342}^{h_3} - (-1)^{h_1 + h_2 + h_3} L_{1342}^{h_2} L_{1423}^{h_3} \right) + \text{cycl.} \right], \quad (4.3)$$

where h_i are positive integers, and only cyclic permutations of the $(2, 3, 4)$ labels must be added. Including the transcendentality bound, at three loops precisely one function in this class survives [12]: it is given by Eq. (4.3) with $h_1 = 1$ and $h_2 = h_3 = 2$. If one considers more general classes of functions, including for example polylogarithms, at least two further consistent examples can be found.

The question whether the dipole formula receives corrections involving multigluon correlations at high perturbative orders remains thus open. It is clear however that such corrections, if any, are constrained in a much stronger way than might have been expected, and one may concretely hope to bring them under control in the not too distant future.

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