

On soft singularities at three loops and beyond

Lance J. Dixon

*SLAC National Accelerator Laboratory, Stanford University,
Stanford, CA 94309, USA*

Einan Gardi

*School of Physics, The University of Edinburgh
Edinburgh EH9 3JZ, Scotland, UK*

Lorenzo Magnea

*Dipartimento di Fisica Teorica, Università di Torino, and
INFN, Sezione di Torino, Via P. Giuria 1, I-10125 Torino, Italy*

ABSTRACT: We report on further progress in understanding soft singularities of massless gauge theory scattering amplitudes. Recently, a set of equations was derived based on Sudakov factorization, constraining the soft anomalous dimension matrix of multi-leg scattering amplitudes to any loop order, and relating it to the cusp anomalous dimension. The minimal solution to these equations was shown to be a sum over color dipoles. Here we explore potential contributions to the soft anomalous dimension that go beyond the sum-over-dipoles formula. Such contributions are constrained by factorization and invariance under rescaling of parton momenta to be functions of conformally invariant cross ratios. Therefore, they must correlate the color and kinematic degrees of freedom of at least four hard partons, corresponding to gluon webs that connect four eikonal lines, which first appear at three loops. We analyze potential contributions, combining all available constraints, including Bose symmetry, the expected degree of transcendentality, and the singularity structure in the limit where two hard partons become collinear. We find that if the kinematic dependence is solely through products of logarithms of cross ratios, then at three loops there is a unique function that is consistent with all available constraints. If polylogarithms are allowed to appear as well, then at least two additional structures are consistent with the available constraints.

KEYWORDS: perturbative QCD, resummation, soft singularities.

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1. Introduction

Understanding the structure of gauge theory scattering amplitudes is important from both the fundamental field-theoretic perspective, and the pragmatic one of collider phenomenology. Infrared singularities, in particular, open a window into the all-order structure of perturbation theory and the relation between the weak and strong coupling limits; at the same time, they provide the key to resummation of large logarithms in a variety of phenomenological applications.

The study of infrared singularities in QCD amplitudes, which has a three-decade-long history [1–40], recently received a major boost [41, 42]. The factorization properties of soft and collinear modes, also referred to as Sudakov factorization, were combined with the symmetry of soft-gluon interactions under rescaling of hard parton momenta, and were shown to constrain the structure of singularities of any massless gauge theory amplitude, to any loop order, and for a general number of colors N_c . A remarkably simple structure emerges as the simplest solution to these constraints. All non-collinear soft singularities

are generated by an anomalous dimension matrix in color space [9, 10, 14, 15, 21, 28, 30, 34, 36]. In the simplest solution, this matrix takes the form of a sum over color dipoles, corresponding to pairwise interactions among hard partons. This interaction is governed by a single function of the strong coupling, the cusp anomalous dimension, $\gamma_K(\alpha_s)$. The simplicity of this result is remarkable, especially given the complexity of multi-leg amplitude computations beyond tree level. The color dipole structure of soft singularities appears naturally at the one-loop order [24, 28–30], where the interaction is genuinely of the form of a single gluon exchange between any two hard partons. The validity of this structure at two loops was not obvious *a priori*; it was discovered through the explicit computation of the anomalous dimension matrix [37, 38].

This remarkable simplicity is peculiar to the case of massless gauge theories: recent work [43–49] has shown that the two-loop matrix, when at least two colored legs are massive, is not proportional to the one-loop matrix, except in particular kinematic regions. In general, in the massive case, there are new $\mathcal{O}(m^4/s^2)$ contributions correlating color and momentum degrees of freedom of at least three partons, starting at two loops.

Given that all existing massless results are consistent with the sum-over-dipoles formula, it is tempting to conjecture that it gives the full answer [41, 42, 50–52]. As emphasized in ref. [41], however, constraints based on Sudakov factorization and momentum rescaling alone are not sufficient to determine uniquely the form of the soft anomalous dimension. A logical possibility exists that further contributions will show up at the multi-loop level, which directly correlate the kinematic and color degrees of freedom of more than two hard partons. It is very interesting to establish whether these corrections exist, and, if they do not, to gain a complete understanding of the underlying reason. Beyond the significance of the soft singularities themselves, a complete understanding of their structure may shed light on the structure of the finite parts of scattering amplitudes.

Ref. [41] showed that precisely two classes of contributions may appear as corrections to the sum-over-dipoles formula. The first class stems from the fact that the sum-over-dipoles formula provides a solution to the factorization-based constraints only if the cusp anomalous dimension, $\gamma_K^{(i)}(\alpha_s)$, associated with a hard parton in representation i of the gauge group, obeys $\gamma_K^{(i)}(\alpha_s) = C_i \hat{\gamma}_K(\alpha_s)$, where $\hat{\gamma}_K$ is universal and C_i is the quadratic Casimir of representation i , $C_i = C_A$ or C_F for gluons and quarks, respectively. This property is referred to as ‘Casimir scaling’ henceforth. Casimir scaling holds through three loops [53, 54]; an interesting open question is whether it holds at four loops [55]. In the strong coupling limit, it is known to break down for $\mathcal{N} = 4$ super-Yang-Mills theory in the large- N_c limit [56], at least when γ_K is computed for Wilson lines in a special class of representations of the gauge group.

The second class of corrections, the one on which we focus here, can occur even if the cusp anomalous dimension obeys Casimir scaling. In this case, the sum-over-dipoles formula solves a set of inhomogeneous linear differential equations, which follow from the constraints of Sudakov factorization and momentum rescalings. However, we can contemplate adding solutions to the homogeneous differential equations, which are provided by arbitrary functions of conformally (and rescaling) invariant cross ratios built from the mo-

menta of four hard partons [41]. Thus any additional terms must correlate directly the momenta, and colors, of four legs. Due to the non-Abelian exponentiation theorem [8,11,13] such contributions must originate in webs that connect four hard partons, which first appear at three loops. From this perspective then, the absence of new correlations at two loops [37,38], or in three-loop diagrams involving matter fields [57], is not surprising, and it does not provide substantial new evidence in favor of the minimal, sum-over-dipoles solution. The first genuine test is from the matter-independent terms at three loops. At this order, purely gluonic webs may connect four hard partons, possibly inducing new types of soft singularities that correlate the color and kinematic variables of the four partons.

The most recent step in addressing this issue was taken in Ref. [42], in which an additional strong constraint on the singularity structure of the amplitude was established, based on the properties of amplitudes as two partons become collinear. Recall that the primary object under consideration is the fixed-angle scattering amplitude, in which all ratios of kinematic invariants are taken to be of order unity. This fixed-angle limit is violated upon considering the special kinematic situation where two of the hard partons become collinear. An additional class of singularities, characterized by the vanishing invariant mass of the two partons, arises in this limit. The splitting amplitude is defined to capture this class of singularities. It relates an n -parton amplitude with two collinear partons to an $(n-1)$ -parton amplitude, where one of the legs carries the total momentum and color charge of the two collinear partons. The basic, universal property of the splitting amplitude is that it depends only on the momentum and color degrees of freedom of the collinear partons, and not on the rest of the process.

Splitting amplitudes have been explicitly computed, or extracted from known scattering amplitudes, at one [58–61] and two [62,63] loops. A derivation of splitting-amplitude universality to all loop orders, based on unitarity, has been given in the large- N_c limit [64]. The light-cone-gauge method for computing two-loop splitting amplitudes [62], in which only the two collinear legs and one off-shell parton appear, strongly suggests that the same all-orders universality extends to arbitrary color configurations, not just planar ones.

Based on splitting-amplitude universality, Ref. [42] established additional constraints on the singularity structure of amplitudes. Using these constraints in conjunction with the Sudakov factorization constraints discussed above, that paper excluded any possible three-loop corrections depending linearly on logarithms of cross ratios¹.

In the present paper we re-examine the structure of soft singularities at three loops. We put together all available constraints, starting with the Sudakov factorization constraints and Bose symmetry, and including the properties of the splitting amplitude and the expected degree of transcendentality of the functions involved². We make only modest assumptions on the kinematic dependence, and consider all possible products of logarithms,

¹More recently, in a revised version of Ref. [42], the collinear constraint was applied to a more general class of functions, leading to the conclusion that functions of conformal cross ratios cannot altogether be ruled out.

²Transcendentality here refers to the assignment of an additive integer τ for each type of factor in a given term arising in an amplitude or an anomalous dimension: $\tau = 0$ for rational functions, $\tau = 1$ for factors of π or single logarithms, $\ln x$; $\tau = n$ for factors of $\zeta(n)$, $\ln^n x$ or $\text{Li}_n(x)$, *etc.* [65].

and eventually also polylogarithms. We find that potential contributions beyond the sum-over-dipoles formula are still possible at three loops, but their functional form is severely constrained.

The paper is organized as follows. We begin with three short sections in which we review the main relevant results of Refs. [41, 42]. In Sec. 2 we briefly summarize the Sudakov factorization of the amplitude and the constraints imposed on the soft anomalous dimension matrix by rescaling invariance of Wilson lines. In Sec. 3 we present the sum-over-dipoles formula, the simplest possible solution to these constraints. In Sec. 4 we review the splitting amplitude constraint. The main part of our study is Sec. 5, in which we put together all available constraints and analyze the possible color and kinematic structures that may appear beyond the sum-over-dipoles formula. Most of the discussion is general, and applies to any loop order, but specific analysis is devoted to potential three-loop corrections. At the end of the section we make a few comments concerning four-loop corrections. Our discussion throughout Sec. 5 focuses on amplitudes involving four colored partons, plus any number of color-singlet particles. The generalization to the multi-parton case is presented in Sec. 6. Our conclusions are summarized in Sec. 7, while an appendix discusses the special case of four-parton scattering at three loops.

2. Sudakov factorization and its consequences

We summarize here the infrared and collinear factorization properties of fixed-angle scattering amplitudes $\mathcal{M}(p_i/\mu, \alpha_s(\mu^2), \epsilon)$ involving n massless partons, plus any number of color-singlet particles, evaluated in dimensional regularization with $D = 4 - 2\epsilon$. We refer the reader to Ref. [41] for technical details and operator definitions of the various functions involved. Multi-parton fixed-angle amplitudes can be expressed in terms of their color components \mathcal{M}_L in a chosen basis in the vector space of available color structures for the scattering process at hand. All infrared and collinear singularities of \mathcal{M}_L can be factorized [10, 18, 26, 32, 38, 39, 41] into jet functions J_i , one for each external leg i , multiplied by a (reduced) soft matrix $\overline{\mathcal{S}}_{LK}$,

$$\mathcal{M}_L(p_i/\mu, \alpha_s(\mu^2), \epsilon) = \overline{\mathcal{S}}_{LK}(\rho_{ij}, \alpha_s(\mu^2), \epsilon) H_K \left(\frac{2p_i \cdot p_j}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon \right) \times \prod_{i=1}^n J_i \left(\frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon \right), \quad (2.1)$$

leaving behind a vector of hard functions H_K , which are finite as $\epsilon \rightarrow 0$. A sum over K is implied. The hard momenta³ p_i defining the amplitude \mathcal{M} are assumed to be light-like, $p_i^2 = 0$, while the n_i are auxiliary vectors used to define the jets in a gauge-invariant way, and they are not light-like, $n_i^2 \neq 0$. The reduced soft matrix $\overline{\mathcal{S}}_{LK}$ can be computed from the expectation value of a product of eikonal lines, or Wilson lines, oriented along the hard parton momenta, dividing the result by n eikonal jet functions \mathcal{J}_i , which remove

³In our convention momentum conservation reads $q + \sum_{i=1}^n p_i = 0$, where q is the recoil momentum carried by colorless particles.

collinear divergences and leave only singularities from soft, wide-angle virtual gluons. It is convenient to express the color structure of the soft matrix $\overline{\mathcal{S}}$ in a basis-independent way, in terms of operators \mathbf{T}_i^a , $a = 1, 2, \dots, N_c^2 - 1$, representing the generators of $\text{SU}(N_c)$ acting on the color of parton i ($i = 1, 2, \dots, n$) [24].

The partonic (quark or gluon) jet function solves two evolution equations simultaneously, one in the factorization scale μ and another in the kinematic variable $(2p_i \cdot n_i)^2/n_i^2$ (see *e.g.* ref. [41]). The latter equation generalizes the evolution of the renormalization-group invariant form factor [22]. The resulting solution to these equations can be written as [52]

$$J_i \left(\frac{(2p_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon \right) = H_{J_i} \left(1, \alpha_s \left(\frac{(2p_i \cdot n_i)^2}{n_i^2} \right), \epsilon \right) \exp \left\{ -\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \gamma_{J_i}(\alpha_s(\lambda^2), \epsilon) \right. \\ \left. + \frac{\mathbf{T}_i \cdot \mathbf{T}_i}{2} \int_0^{(2p_i \cdot n_i)^2/n_i^2} \frac{d\lambda^2}{\lambda^2} \left[\frac{1}{4} \widehat{\gamma}_K(\alpha_s(\lambda^2), \epsilon) \ln \left(\frac{n_i^2 \lambda^2}{(2p_i \cdot n_i)^2} \right) + \frac{1}{2} \widehat{\delta}_{\overline{\mathcal{S}}}(\alpha_s(\lambda^2), \epsilon) \right] \right\}, \quad (2.2)$$

where H_{J_i} is a finite coefficient function, and all singularities are generated by the exponent. The solution depends on just three anomalous dimensions, which are functions of the D -dimensional coupling alone: γ_{J_i} is the anomalous dimension of the quark or gluon field defining the jet, while $\widehat{\gamma}_K = 2\alpha_s/\pi + \dots$ and $\widehat{\delta}_{\overline{\mathcal{S}}} = \alpha_s/\pi + \dots$ are, respectively, the cusp anomalous dimension and an additional eikonal anomalous dimension defined in Sec. 4.1 of Ref. [41]. In eq. (2.2) we have already assumed that the latter two quantities admit Casimir scaling, and we have factored out the quadratic Casimir operator $C_i \equiv \mathbf{T}_i \cdot \mathbf{T}_i$.

Our main interest here is the reduced soft matrix $\overline{\mathcal{S}}$, which takes into account non-collinear soft radiation. It is defined entirely in terms of vacuum correlators of operators composed of semi-infinite Wilson lines (see *e.g.* Ref. [41]), and depends on the kinematic variables

$$\rho_{ij} \equiv \frac{(-\beta_i \cdot \beta_j)^2}{\frac{2(\beta_i \cdot n_i)^2}{n_i^2} \frac{2(\beta_j \cdot n_j)^2}{n_j^2}} = \frac{|\beta_i \cdot \beta_j|^2 e^{-2i\pi\lambda_{ij}}}{\frac{2(\beta_i \cdot n_i)^2}{n_i^2} \frac{2(\beta_j \cdot n_j)^2}{n_j^2}}, \quad (2.3)$$

which are invariant with respect to rescaling of all the Wilson line velocities β_i . The β_i are related to the external momenta by $p_i^\mu = (Q/\sqrt{2})\beta_i^\mu$, where Q is a hard scale whose precise value will not be important here. The phases λ_{ij} are defined by $\beta_i \cdot \beta_j = -|\beta_i \cdot \beta_j|e^{-i\pi\lambda_{ij}}$, where $\lambda_{ij} = 1$ if i and j are both initial-state partons, or both final-state partons, and $\lambda_{ij} = 0$ otherwise. Note that the sign of the phase in $e^{-i\pi\lambda_{ij}}$ is determined by the $+i\epsilon$ prescription for the Feynman propagator.

The reduced soft matrix obeys the renormalization group equation

$$\mu \frac{d}{d\mu} \overline{\mathcal{S}}_{IK}(\rho_{ij}, \alpha_s, \epsilon) = - \sum_J \Gamma_{IJ}^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s) \overline{\mathcal{S}}_{JK}(\rho_{ij}, \alpha_s, \epsilon). \quad (2.4)$$

The soft anomalous dimension matrix $\Gamma_{IJ}^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s)$, in turn, obeys the equation [41]

$$\sum_{j \neq i} \frac{\partial}{\partial \ln \rho_{ij}} \Gamma_{IJ}^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s) = \frac{1}{4} \gamma_K^{(i)}(\alpha_s) \delta_{IJ}, \quad \forall i, I, J, \quad (2.5)$$

found by considering a rescaling of the eikonal velocity β_i . The simplest solution of this equation is the sum-over-dipoles formula [41],

$$\Gamma_{\text{dip}}^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s) = -\frac{1}{8} \widehat{\gamma}_K(\alpha_s) \sum_{i=1}^n \sum_{j \neq i} \ln \rho_{ij} \mathbf{T}_i \cdot \mathbf{T}_j + \frac{1}{2} \widehat{\delta}_{\overline{\mathcal{S}}}(\alpha_s) \sum_{i=1}^n \mathbf{T}_i \cdot \mathbf{T}_i. \quad (2.6)$$

In this expression the dependence on the scale μ appears exclusively through the argument of the D -dimensional coupling in $\widehat{\gamma}_K$ and $\widehat{\delta}_{\overline{\mathcal{S}}}$. Therefore eq. (2.4) is easily integrated to give the corresponding formula for the reduced soft matrix $\overline{\mathcal{S}}$,

$$\overline{\mathcal{S}}_{\text{dip}}(\rho_{ij}, \alpha_s, \epsilon) = \exp \left\{ -\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \left[\frac{1}{2} \widehat{\delta}_{\overline{\mathcal{S}}}(\alpha_s(\lambda^2, \epsilon)) \sum_{i=1}^n \mathbf{T}_i \cdot \mathbf{T}_i - \frac{1}{8} \widehat{\gamma}_K(\alpha_s(\lambda^2, \epsilon)) \sum_{i=1}^n \sum_{j \neq i} \ln \rho_{ij} \mathbf{T}_i \cdot \mathbf{T}_j \right] \right\}. \quad (2.7)$$

Equation (2.6) satisfies the constraints (2.5) if and only if the cusp anomalous dimension admits Casimir scaling, namely $\gamma_K^{(i)}(\alpha_s) = C_i \widehat{\gamma}_K(\alpha_s)$, with $\widehat{\gamma}_K$ independent of the color representation of parton i . In this paper we shall assume that this is the case, postponing to future work the analysis of how higher-order Casimir contributions to γ_K would affect the soft anomalous dimension matrix (the starting point for such an analysis is eq. (5.5) of Ref. [41]).

Even under the assumption of Casimir scaling for $\gamma_K^{(i)}$, eq. (2.7) may not be the full result for $\overline{\mathcal{S}}$, because $\Gamma^{\overline{\mathcal{S}}}$ may receive additional corrections $\Delta^{\overline{\mathcal{S}}}$ going beyond the sum-over-dipoles ansatz. In this case the full anomalous dimension can be written as a sum,

$$\Gamma^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s) = \Gamma_{\text{dip}}^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s) + \Delta^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s). \quad (2.8)$$

Here $\Delta^{\overline{\mathcal{S}}}$ is a matrix in color space, which is constrained to satisfy the homogeneous differential equation

$$\sum_{j \neq i} \frac{\partial}{\partial \ln \rho_{ij}} \Delta^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s) = 0 \quad \forall i. \quad (2.9)$$

This equation is solved by any function of conformally invariant cross ratios of the form

$$\rho_{ijkl} \equiv \frac{\beta_i \cdot \beta_j \beta_k \cdot \beta_l}{\beta_i \cdot \beta_k \beta_j \cdot \beta_l}, \quad (2.10)$$

which are related to the kinematic variables ρ_{ij} in eq. (2.3), and to the momenta p_i , by

$$\rho_{ijkl} = \left(\frac{\rho_{ij} \rho_{kl}}{\rho_{ik} \rho_{jl}} \right)^{1/2} = \frac{p_i \cdot p_j p_k \cdot p_l}{p_i \cdot p_k p_j \cdot p_l} = \left| \frac{p_i \cdot p_j p_k \cdot p_l}{p_i \cdot p_k p_j \cdot p_l} \right| e^{-i\pi(\lambda_{ij} + \lambda_{kl} - \lambda_{ik} - \lambda_{jl})}. \quad (2.11)$$

Each leg that appears in ρ_{ijkl} does so once in the numerator and once in the denominator, thus cancelling in the combination of derivatives in eq. (2.9). Hence we define

$$\Delta^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s) = \Delta(\rho_{ijkl}, \alpha_s). \quad (2.12)$$

Any additional correction Δ must introduce new correlations between at least four partons into the reduced soft function. Such additional corrections are known not to appear at two loops [37, 38], as expected from the fact that two-loop webs can correlate at most three hard partons. By the same token, they cannot show up in matter-dependent diagrams at three loops, as verified explicitly in Ref. [57]. On the other hand, they might be generated at three loops by purely gluonic diagrams, such as the one shown in fig. 1.

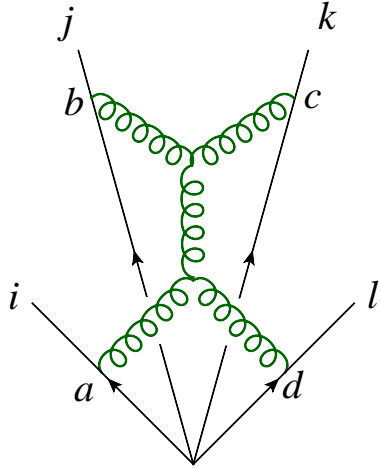


Figure 1: A purely gluonic diagram connecting the four hard partons labeled i, j, k, l , which may contribute to the soft anomalous dimension matrix at three loops. It correlates the colors of the four partons via the operator $\mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d f^{ade} f^{cbe}$.

The main purpose of the present paper is to examine all available constraints on the soft anomalous dimension matrix, and check whether they are sufficient to rule out a non-vanishing Δ at three loops. We will show that, despite the powerful constraints available, corrections to the sum-over-dipoles formula may indeed appear at this order. We will see, however, that by making some reasonable assumptions about the admissible kinematic behavior we will be able to reduce the space of solutions for Δ at three loops to a few very specific functions.

3. Minimal ansatz for the singularities of the amplitude

The factorization formula (2.1) has the attractive property that each of the singular factors is defined in a gauge-invariant way. It requires the introduction of the auxiliary vectors n_i , which have been very useful [41] in revealing the properties of the soft anomalous dimension. At the end of the day, however, the singularities of the amplitude \mathcal{M} cannot depend on these auxiliary vectors, but only on the kinematic invariants built out of external parton momenta. Indeed, as discussed below, the cancellation of the dependence of the singular terms on the vectors n_i can be explicitly performed, and one can write the factorization of

the amplitude in a more compact form:

$$\mathcal{M} \left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = Z \left(\frac{p_i}{\mu_f}, \alpha_s(\mu_f^2), \epsilon \right) \mathcal{H} \left(\frac{p_i}{\mu}, \frac{\mu_f}{\mu}, \alpha_s(\mu^2), \epsilon \right), \quad (3.1)$$

as used in Refs. [42,51]. Here the (matrix-valued) Z factor absorbs all the infrared (soft and collinear) singularities, while the hard function \mathcal{H} is finite as $\epsilon \rightarrow 0$. We distinguish between two scales, the renormalization scale μ , which is present in the renormalized amplitude \mathcal{M} on the left-hand side of eq. (3.1), and μ_f , a factorization scale that is introduced through the Z factor. The function \mathcal{H} (a vector in color space) plays the role of H in the factorization formula (2.1), but differs from it by being independent of the auxiliary vectors n_i .

Sudakov factorization implies that the Z matrix is renormalized multiplicatively. We can then define the anomalous dimension matrix Γ , corresponding to Z , by

$$\frac{d}{d \ln \mu_f} Z \left(\frac{p_i}{\mu_f}, \alpha_s(\mu_f^2), \epsilon \right) = -\Gamma \left(\frac{p_i}{\mu_f}, \alpha_s(\mu_f^2) \right) Z \left(\frac{p_i}{\mu_f}, \alpha_s(\mu_f^2), \epsilon \right). \quad (3.2)$$

Note that the matrix Γ is finite, but it can depend implicitly on ϵ when evaluated as a function of the D -dimensional running coupling; it will then generate the infrared poles of Z , as usual, through integration over the scale.

The sum-over-dipoles ansatz for $\Gamma^{\overline{\mathcal{S}}}$, eq. (2.6), implies an analogous formula for Γ . In order to see it, one may use the factorization formula (2.1), substitute in eqs. (2.2) and (2.7), use color conservation, $\sum_{j \neq i} \mathbf{T}_j = -\mathbf{T}_i$, and apply the identity

$$\underbrace{\ln \left(\frac{(2p_i \cdot n_i)^2}{n_i^2} \right)}_{J_i} + \underbrace{\ln \left(\frac{(2p_j \cdot n_j)^2}{n_j^2} \right)}_{J_j} + \underbrace{\ln \left(\frac{(|\beta_i \cdot \beta_j| e^{-i\pi\lambda_{ij}})^2}{\frac{2(\beta_i \cdot n_i)^2}{n_i^2} \frac{2(\beta_j \cdot n_j)^2}{n_j^2}} \right)}_{\overline{\mathcal{S}}} = 2 \ln(2 |p_i \cdot p_j| e^{-i\pi\lambda_{ij}}). \quad (3.3)$$

Note also that the poles associated with $\widehat{\delta}_{\overline{\mathcal{S}}}(\alpha_s)$ cancel out between the soft and jet functions. In this way, one arrives at the sum-over-dipoles ansatz for Γ ,

$$\Gamma_{\text{dip}} \left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2) \right) = -\frac{1}{4} \widehat{\gamma}_K(\alpha_s(\lambda^2)) \sum_{i=1}^n \sum_{j \neq i} \ln \left(\frac{2 |p_i \cdot p_j| e^{-i\pi\lambda_{ij}}}{\lambda^2} \right) \mathbf{T}_i \cdot \mathbf{T}_j + \sum_{i=1}^n \gamma_{J_i}(\alpha_s(\lambda^2)). \quad (3.4)$$

The Z matrix which solves eq. (3.2) can be written in terms of the sum-over-dipoles ansatz (3.4) as an exponential, in a form similar to eq. (2.7). However, $\overline{\mathcal{S}}_{\text{dip}}$ has only simple poles in ϵ in the exponent, while the integration of Γ_{dip} over the scale λ of the D -dimensional running coupling will generate double (soft-collinear) poles within Z , inherited from the jet functions in eq. (2.2), because of the explicit dependence of Γ_{dip} on the logarithm of the scale λ .

If a non-trivial correction Δ appears in the reduced soft function (2.8), then the full anomalous dimension is

$$\Gamma\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2)\right) = \Gamma_{\text{dip}}\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2)\right) + \Delta(\rho_{ijkl}, \alpha_s(\lambda^2)). \quad (3.5)$$

In terms of this function the solution of eq. (3.2) takes the form

$$Z\left(\frac{p_i}{\mu_f}, \alpha_s(\mu_f^2), \epsilon\right) = \text{P exp} \left\{ -\frac{1}{2} \int_0^{\mu_f^2} \frac{d\lambda^2}{\lambda^2} \Gamma\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2, \epsilon)\right) \right\}, \quad (3.6)$$

where P stands for path-ordering: the order of the color matrices after expanding the exponential coincides with the ordering in the scale λ . We emphasize that path ordering is only necessary in eq. (3.6) if $\Delta \neq 0$ and $[\Delta, \Gamma_{\text{dip}}] \neq 0$. Indeed, the ansatz (3.4) has the property that the scale-dependence associated with non-trivial color operators appears through an overall factor, $\widehat{\gamma}_K(\alpha_s(\lambda^2))$, so that color matrices Γ corresponding to different scales are proportional to each other, and obviously commute. This is no longer true for a generic $\Delta \neq 0$, starting at a certain loop order l . In this case eq. (3.5) would generically be a sum of two non-commuting matrices, each of them having its own dependence on the coupling and thus on the scale λ . Considering two scales λ_1 and λ_2 , we would then have $[\Gamma(\lambda_1), \Gamma(\lambda_2)] \neq 0$, and the order of the matrices in the expansion of eq. (3.6) would be dictated by the ordering of the scales. It should be noted, though, that the first loop order in Z that would be affected is order $l + 1$, because Γ starts at one loop, so that

$$[\Gamma(\lambda_1), \Gamma(\lambda_2)] \sim [\Gamma^{(1)}(\lambda_1), \Delta^{(l)}(\lambda_2)] = \mathcal{O}(\alpha_s^{l+1}). \quad (3.7)$$

The issue of ordering can thus be safely neglected at three loops, the first order at which a non-vanishing Δ can arise.

4. The splitting-amplitude constraint

Let us now consider the limit where two of the hard partons in the amplitude become collinear. Following Ref. [42], we shall see that this limit provides an additional constraint on the structure of Δ . However, the way we use this constraint in the next section will be different, and will go beyond what was done in Ref. [42].

The Sudakov factorization described by eq. (2.1), and subsequently the singularity structure encoded in Z in eq. (3.1), apply to scattering amplitudes at fixed angles. All the invariants $p_i \cdot p_j$ are taken to be of the same order, much larger than the confinement scale Λ^2 . The limit in which two of the hard partons are taken collinear, *e.g.* $p_1 \cdot p_2 \rightarrow 0$, is a singular limit, which we are now about to explore. In this limit, $p_1 \rightarrow zP$ and $p_2 \rightarrow (1-z)P$, where the longitudinal momentum fraction z obeys $0 < z < 1$ (for time-like splitting). We will see, following Ref. [42], that there is a relation between the singularities that are associated with the splitting — the replacement of one parton by two collinear partons — and the singularities encoded in Z in eq. (3.1).

It is useful for our derivation to make a clear distinction between the two scales μ_f and μ introduced in eq. (3.1). Let us first define the splitting amplitude, which relates the

dimensionally-regularized amplitude for the scattering of $n - 1$ partons to the one for n partons, two of which are taken collinear. We may write

$$\mathcal{M}_n(p_1, p_2, p_j; \mu, \epsilon) \xrightarrow{1\|2} \mathbf{Sp}(p_1, p_2; \mu, \epsilon) \mathcal{M}_{n-1}(P, p_j; \mu, \epsilon) . \quad (4.1)$$

Here the two hard partons that become collinear are denoted by p_1 and p_2 , and all the other momenta by p_j , with $j = 3, 4, \dots, n$. We have slightly modified our notation for simplicity: the number of colored partons involved in the scattering is indicated explicitly; the dependence of each factor on the running coupling is understood; finally, the matrix elements have dimensionful arguments (while in fact they depend on dimensionless ratios, as indicated in the previous sections). The splitting described by eq. (4.1) preserves the total momentum $p_1 + p_2 = P$ and the total color charge $\mathbf{T}_1 + \mathbf{T}_2 = \mathbf{T}$. We assume eq. (4.1) to be valid in the collinear limit, up to corrections that must be finite as $P^2 = 2p_1 \cdot p_2 \rightarrow 0$.

The splitting matrix \mathbf{Sp} encodes all singular contribution to the amplitude \mathcal{M}_n arising from the limit $P^2 \rightarrow 0$, and, crucially, it must depend only on the quantum numbers of the splitting partons. The matrix element \mathcal{M}_{n-1} , in contrast, is evaluated at $P^2 = 0$, and therefore it obeys Sudakov factorization, eq. (3.1), as applied to an $(n-1)$ -parton amplitude. The operator \mathbf{Sp} is designed to relate color matrices defined in the n -parton color space to those defined in the $(n - 1)$ -parton space: it multiplies on its left the former and on its right the latter. Thus, the initial definition of \mathbf{Sp} is not diagonal. Upon substituting $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2$, however, one can use the n -parton color space only. In this space \mathbf{Sp} is diagonal; all of its dependence on \mathbf{T}_1 and \mathbf{T}_2 can be expressed in terms of the quadratic Casimirs, using $2\mathbf{T}_1 \cdot \mathbf{T}_2 = \mathbf{T}^2 - \mathbf{T}_1^2 - \mathbf{T}_2^2$.

Because the fixed-angle factorization theorem in eq. (2.1) breaks down in the collinear limit, $p_1 \cdot p_2 \rightarrow 0$, we expect that some of the singularities captured by the splitting matrix \mathbf{Sp} will arise from the hard functions \mathcal{H} . Specifically, if the Z factor in eq. (3.1) is defined in a minimal scheme, \mathcal{H} will contain all terms in \mathcal{M}_n with logarithmic singularities in $p_1 \cdot p_2$ associated with non-negative powers of ϵ . We then define $\mathbf{Sp}_{\mathcal{H}}$, in analogy with eq. (4.1), by the collinear behavior of the hard functions,

$$\mathcal{H}_n(p_1, p_2, p_j; \mu, \mu_f, \epsilon) \xrightarrow{1\|2} \mathbf{Sp}_{\mathcal{H}}(p_1, p_2; \mu, \mu_f, \epsilon) \mathcal{H}_{n-1}(P, p_j; \mu, \mu_f, \epsilon) , \quad (4.2)$$

where all factors are finite as $\epsilon \rightarrow 0$. As was the case for eq. (4.1), eq. (4.2) is valid up to corrections that remain finite in the limit $P^2 \rightarrow 0$. Singularities in that limit are all contained in the splitting matrix $\mathbf{Sp}_{\mathcal{H}}$, while the function \mathcal{H}_{n-1} is evaluated at $P^2 = 0$.

Next, recall the definition of the Z factors in eq. (3.1) for both the n - and $(n-1)$ -parton amplitudes. In the present notation, they read

$$\mathcal{M}_n(p_1, p_2, p_j; \mu, \epsilon) = Z_n(p_1, p_2, p_j; \mu_f, \epsilon) \mathcal{H}_n(p_1, p_2, p_j; \mu, \mu_f, \epsilon) , \quad (4.3)$$

$$\mathcal{M}_{n-1}(P, p_j; \mu, \epsilon) = Z_{n-1}(P, p_j; \mu_f, \epsilon) \mathcal{H}_{n-1}(P, p_j; \mu, \mu_f, \epsilon) . \quad (4.4)$$

Substituting eq. (4.4) into eq. (4.1) yields

$$\mathcal{M}_n(p_1, p_2, p_j; \mu, \epsilon) \xrightarrow{1\|2} \mathbf{Sp}(p_1, p_2; \mu, \epsilon) Z_{n-1}(P, p_j; \mu_f, \epsilon) \mathcal{H}_{n-1}(P, p_j; \mu, \mu_f, \epsilon) . \quad (4.5)$$

On the other hand, substituting eq. (4.2) into eq. (4.3) we get

$$\mathcal{M}_n(p_1, p_2, p_j; \mu, \epsilon) \xrightarrow{1||^2} Z_n(p_1, p_2, p_j; \mu_f, \epsilon) \mathbf{Sp}_{\mathcal{H}}(p_1, p_2; \mu, \mu_f, \epsilon) \mathcal{H}_{n-1}(P, p_j; \mu, \mu_f, \epsilon). \quad (4.6)$$

Comparing these two equations we immediately deduce the relation between the full splitting matrix \mathbf{Sp} , which is infrared divergent, and its infrared-finite counterpart $\mathbf{Sp}_{\mathcal{H}}$,

$$\mathbf{Sp}_{\mathcal{H}}(p_1, p_2; \mu, \mu_f, \epsilon) = Z_n^{-1}(p_1, p_2, p_j; \mu_f, \epsilon) \mathbf{Sp}(p_1, p_2; \mu, \epsilon) Z_{n-1}(P, p_j; \mu_f, \epsilon), \quad (4.7)$$

where Z_n is understood to be evaluated in the collinear limit. This equation (*cf.* eq. (55) in Ref. [42]) is a non-trivial constraint on both Z and the splitting amplitude \mathbf{Sp} , given that the left-hand side must be finite as $\epsilon \rightarrow 0$, and that the splitting amplitude depends only on the momenta and color variables of the splitting partons — not on other hard partons involved in the scattering process.

To formulate these constraints, we take a logarithmic derivative of eq. (4.7), using the definition of Γ_n and Γ_{n-1} according to eq. (3.2). It is straightforward to show that

$$\frac{d}{d \ln \mu_f} \mathbf{Sp}_{\mathcal{H}} = Z_n^{-1} \Gamma_n Z_n \mathbf{Sp}_{\mathcal{H}} - \mathbf{Sp}_{\mathcal{H}} Z_{n-1}^{-1} \Gamma_{n-1} Z_{n-1}, \quad (4.8)$$

where, as above, $(n-1)$ -parton matrices are evaluated in collinear kinematics ($P^2 = 0$), and corrections are finite in the collinear limit.

The splitting matrix \mathbf{Sp} commutes with the Γ and Z matrices, as an immediate consequence of the fact that it can only depend on the color degrees of freedom of the partons involved in the splitting, *i.e.* \mathbf{T}_1 , \mathbf{T}_2 and $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2$, and it is therefore color diagonal. In contrast, Γ_n may well fail to commute with Z_n , if Δ in eq. (3.5) does not vanish. According to eq. (3.7), however, this violation can only affect terms of one order higher than Δ itself. Therefore, up to the first order at which Δ does not vanish, we can further simplify eq. (4.10) to obtain

$$\frac{d}{d \ln \mu_f} \mathbf{Sp}_{\mathcal{H}}(p_1, p_2; \mu, \mu_f, \epsilon) = \Gamma_{\mathbf{Sp}}(p_1, p_2; \mu_f) \mathbf{Sp}_{\mathcal{H}}(p_1, p_2; \mu, \mu_f, \epsilon), \quad (4.9)$$

where we defined

$$\Gamma_{\mathbf{Sp}}(p_1, p_2; \mu_f) \equiv \Gamma_n(p_1, p_2, p_j; \mu_f) - \Gamma_{n-1}(P, p_j; \mu_f). \quad (4.10)$$

Note that we have adapted eq. (4.10) to our current notation with dimensionful arguments; as before, the matrices involved acquire implicit ϵ dependence when evaluated as functions of the D -dimensional coupling. Note also that the matrix Γ_{n-1} operates on the n -parton color space, through the identification $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2$.

We may now solve eq. (4.9) for the μ_f dependence of $\mathbf{Sp}_{\mathcal{H}}$, with the result

$$\mathbf{Sp}_{\mathcal{H}}(p_1, p_2; \mu, \mu_f, \epsilon) = \mathbf{Sp}_{\mathcal{H}}^{(0)}(p_1, p_2; \mu, \epsilon) \exp \left[\frac{1}{2} \int_{\mu^2}^{\mu_f^2} \frac{d\lambda^2}{\lambda^2} \Gamma_{\mathbf{Sp}}(p_1, p_2; \lambda) \right]. \quad (4.11)$$

The initial condition for evolution

$$\mathbf{Sp}_{\mathcal{H}}^{(0)}(p_1, p_2; \mu, \epsilon) = \mathbf{Sp}_{\mathcal{H}}(p_1, p_2; \mu, \mu_f = \mu, \epsilon) \quad (4.12)$$

will, in general, still be singular as $p_1 \cdot p_2 \rightarrow 0$, although it is finite as $\epsilon \rightarrow 0$. We may, in any case, use eq. (4.11) by matching the μ -dependence in eq. (4.7), which yields an expression for the full splitting function \mathbf{Sp} . We find

$$\mathbf{Sp}(p_1, p_2; \mu, \epsilon) = \mathbf{Sp}_{\mathcal{H}}^{(0)}(p_1, p_2; \mu, \epsilon) \exp \left[-\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \Gamma_{\mathbf{Sp}}(p_1, p_2; \lambda) \right]. \quad (4.13)$$

While collinear singularities accompanied by non-negative powers of ϵ are still present in the initial condition, all poles in ϵ in the full splitting matrix arise from the integration over the scale of the D -dimensional running coupling in the exponent of eq. (4.13).

The restricted kinematic dependence of $\Gamma_{\mathbf{Sp}}$, which generates the poles in the splitting function \mathbf{Sp} , is sufficient to provide nontrivial constraints on the matrix Δ , as we will now see. Indeed, substituting eq. (3.5) into eq. (4.10) we obtain

$$\Gamma_{\mathbf{Sp}}(p_1, p_2; \lambda) = \Gamma_{\mathbf{Sp}, \text{dip}}(p_1, p_2; \lambda) + \Delta_n(\rho_{ijkl}; \lambda) - \Delta_{n-1}(\rho_{ijkl}; \lambda), \quad (4.14)$$

where

$$\begin{aligned} \Gamma_{\mathbf{Sp}, \text{dip}}(p_1, p_2; \lambda) = & -\frac{1}{2} \widehat{\gamma}_K(\alpha_s(\lambda^2)) \left[\ln \left(\frac{2|p_1 \cdot p_2| e^{-i\pi\lambda_{12}}}{\lambda^2} \right) \mathbf{T}_1 \cdot \mathbf{T}_2 \right. \\ & \left. - \mathbf{T}_1 \cdot (\mathbf{T}_1 + \mathbf{T}_2) \ln z - \mathbf{T}_2 \cdot (\mathbf{T}_1 + \mathbf{T}_2) \ln(1-z) \right] \\ & + \gamma_{J_1}(\alpha_s(\lambda^2)) + \gamma_{J_2}(\alpha_s(\lambda^2)) - \gamma_{J_P}(\alpha_s(\lambda^2)). \end{aligned} \quad (4.15)$$

Equation (4.15) is the result of substituting the sum-over-dipoles ansatz (3.4) for Γ_n and Γ_{n-1} . The terms in eq. (4.14) going beyond eq. (4.15) depend on conformally invariant cross ratios in the n -parton and $(n-1)$ -parton amplitudes, respectively. Their difference should conspire to depend only on the kinematic variables p_1 and p_2 and on the color variables \mathbf{T}_1 and \mathbf{T}_2 . In this way eq. (4.14) provides a non-trivial constraint on the structure of Δ , which we will implement in Sec. 5.4.

5. Constraining corrections to the sum-over-dipoles formula

5.1 Functions of conformally invariant cross ratios

Our task here is to analyze potential contributions of the form $\Delta(\rho_{ijkl}, \alpha_s)$ to the soft singularities of any n -leg amplitude. Our starting point is the fact that these contributions must be written as functions of conformally invariant cross ratios of the form (2.10). Because we are dealing with renormalizable theories in four dimensions, we do not expect Z to contain power-law dependence on the kinematic variables; instead the dependence should be “slow”, *i.e.* logarithmic in the arguments, through variables of the form

$$L_{ijkl} \equiv \ln \rho_{ijkl} = \ln \left(\frac{p_i \cdot p_j \ p_k \cdot p_l}{p_i \cdot p_k \ p_j \cdot p_l} \right). \quad (5.1)$$

Eventually, at high enough order, dependence on ρ_{ijkl} through polylogarithms and harmonic polylogarithms might arise. We will not assume here that Δ is linear in the variables L_{ijkl} . We will allow logarithms of different cross ratios to appear in a product, raised to various powers, and this will be a key to finding solutions consistent with the collinear limits. Subsequently, we will examine how further solutions may arise if polylogarithmic dependence is allowed.

A further motivation to consider a general logarithmic dependence through the variables in eq. (5.1) is provided by the collinear limits, which can take certain cross ratios ρ_{ijkl} to 0, 1, or ∞ , corresponding to physical limits where logarithmic divergences in Δ will be possible. Other values of the cross ratios, on the other hand, should not cause (unphysical) singularities in $\Delta(\rho_{ijkl})$. This fact limits the acceptable functional forms. For example, in specifying a logarithmic functional dependence to be through eq. (5.1), we explicitly exclude the form $\ln(c + \rho_{ijkl})$ for general⁴ constant c . Such a shift in the argument of the logarithm would generate unphysical singularities at $\rho_{ijkl} = -c$, and would also lead to complicated symmetry properties under parton exchange, which would make it difficult to accommodate Bose symmetry. We will thus focus our initial analysis on kinematic dependence through the variables L_{ijkl} . Although it seems less natural, polylogarithmic dependence on ρ_{ijkl} cannot be altogether ruled out, and will be considered in the context of the three-loop analysis in Sec. 5.5.

The fact that the variables (5.1) involve the momenta of four partons, points to their origin in webs that connect (at least) four of the hard partons in the process, exemplified by fig. 1. The appearance of such terms in the exponent, as a correction to the sum-over-dipoles formula, implies, through the non-Abelian exponentiation theorem, that they cannot be reduced to sums of independent webs connecting just two or three partons, neither diagrammatically nor algebraically. Indeed, for amplitudes composed of just three partons the sum-over-dipoles formula is exact [41]. Similarly, because two-loop webs can connect at most three different partons, conformally invariant cross ratios cannot be formed. Consequently, at two loops there are no corrections to the sum-over-dipoles formula, independently of the number of legs. Thus, the first non-trivial corrections can appear at three loops, and if they appear, they are directly related to webs that connect four partons. For the remainder of this section, therefore, we will focus on corrections to the sum-over-dipoles formula that arise from webs connecting precisely four partons, although other partons or colorless particles can be present in the full amplitude. Our conclusions are fully general at three loops, as discussed in Sec. 6, because at that order no web can connect more than four partons.

We begin by observing that, independently of the loop order at which four-parton corrections appear, their color factor must involve at least one color generator corresponding to each of the four partons involved. For example, the simplest structure a term in Δ can have in color space is

$$\Delta_4(\rho_{ijkl}) = h^{abcd} \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d \Delta_4^{\text{kin}}(\rho_{ijkl}), \quad (5.2)$$

⁴In Sec. 5.6 we will briefly consider the possibility of including a dependence of the form $\ln(1 - \rho)$.

where h^{abcd} is some color tensor built out of structure constants corresponding to the internal vertices in the web that connects the four partons (i, j, k, l) to each other. Note that h^{abcd} may receive contributions from several different webs at a given order, and furthermore, for a given h^{abcd} , the kinematic coefficient $\Delta_4^{\text{kin}}(\rho_{ijkl})$ can receive corrections from higher-order webs. In what follows, we will not display the dependence on the coupling of the kinematic factors, because it does not affect our arguments. As we will see, symmetry arguments will, in general, force us to consider sums of terms of the form (5.2), with different color tensors h^{abcd} associated with different kinematic factors.

More generally, at sufficiently high orders, there can be other types of contributions in which each Wilson line in the soft function is attached to more than one gluon, and hence to more than one index in a color tensor. Such corrections will be sums of terms of the form

$$\Delta_4(\rho_{ijkl}) = \Delta_4^{\text{kin}}(\rho_{ijkl}) h^{a_1, \dots, a_{m_1}, b_1, \dots, b_{m_2}, c_1, \dots, c_{m_3}, d_1, \dots, d_{m_4}} (\mathbf{T}_i^{a_1} \mathbf{T}_i^{a_2} \dots \mathbf{T}_i^{a_{m_1}} \mathbf{T}_j^{b_1} \mathbf{T}_j^{b_2} \dots \mathbf{T}_j^{b_{m_2}} \mathbf{T}_k^{c_1} \mathbf{T}_k^{c_2} \dots \mathbf{T}_k^{c_{m_3}} \mathbf{T}_l^{d_1} \mathbf{T}_l^{d_2} \dots \mathbf{T}_l^{d_{m_4}})_+, \quad (5.3)$$

where the $(\)_+$ indicates symmetrization with respect to all the indices corresponding to a given parton. Note that generators carrying indices of different partons commute, while the antisymmetric components have been excluded from eq. (5.3), because they reduce, via the commutation relation $[\mathbf{T}_i^a, \mathbf{T}_i^b] = if^{abc}\mathbf{T}_i^c$, to shorter strings⁵. In the following subsections, we will focus on (combinations of) color structures of the form (5.2), and we will not consider further the more general case of eq. (5.3), which, in any case, can only arise at or beyond four loops.

5.2 Bose symmetry

The Wilson lines defining the reduced soft matrix are effectively scalars, as the spin-dependent parts have been stripped off and absorbed in the jet functions. Consequently, the matrices Γ and Δ should admit Bose symmetry and be invariant under the exchange of any pair of hard partons. Because Δ depends on color and kinematic variables, this symmetry implies correlation between color and kinematics. In particular, considering a term of the form (5.2), the symmetry properties of h^{abcd} under permutations of the indices a, b, c and d must be mirrored in the symmetry properties of the kinematic factor $\Delta_4^{\text{kin}}(\rho_{ijkl})$ under permutations of the corresponding momenta p_i, p_j, p_k and p_l . The requirement of Bose symmetry will lead us to express Δ as a sum of terms, each having color and kinematic factors with a definite symmetry under some (or all) permutations.

Because we are considering corrections arising from four-parton webs, we need to analyze the symmetry properties under particle exchanges of the ratios ρ_{ijkl} that can be constructed with four partons. There are 24 different cross ratios of this type, corresponding to the number of elements of the permutation group acting on four objects, S_4 . However,

⁵One can make a stronger statement for Wilson lines in the fundamental representation of the gauge group. In that case the symmetric combination in eq. (5.3) can also be further reduced, using the identity $\{\mathbf{t}_a, \mathbf{t}_b\} = \frac{1}{N_c} \delta_{ab} + d_{abc} \mathbf{t}_c$, so that the generic correction in eq. (5.3) turns into a combination of terms of the form (5.2). We are not aware of generalizations of this possibility to arbitrary representations.

a $Z_2 \times Z_2$ subgroup of S_4 leaves each ρ_{ijkl} (and hence each L_{ijkl}) invariant. Indeed, one readily verifies that

$$\rho_{ijkl} = \rho_{jilk} = \rho_{klij} = \rho_{lkji}. \quad (5.4)$$

The subgroup $Z_2 \times Z_2$ is an invariant subgroup of S_4 . Thus, we may use it to fix one of the indices, say i , in ρ_{ijkl} . This leaves six cross ratios, transforming under the permutation group of three objects, $S_3 \simeq S_4/(Z_2 \times Z_2)$.

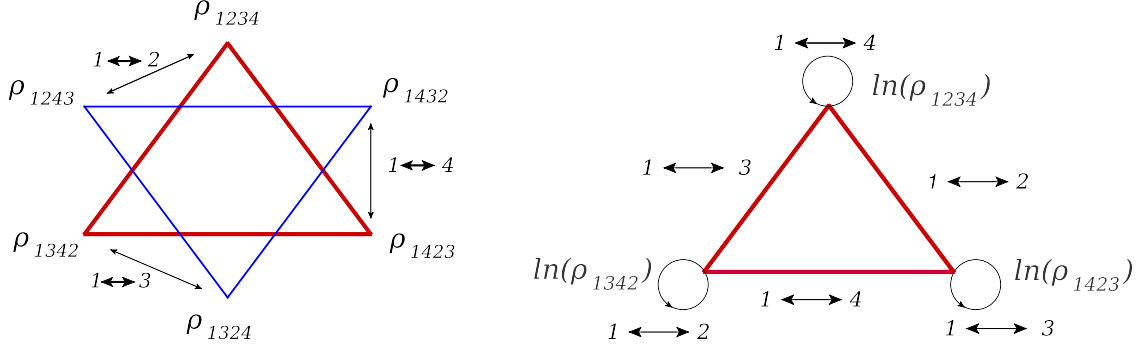


Figure 2: Symmetry properties of conformally invariant cross ratios. Each of the two triangles in the left-hand figure connects three cross ratios. The cross ratios associated with the two triangles are related by inversion, and one moves between the two triangles with odd permutations of momentum labels. The right-hand figure shows the resulting antisymmetry of the logarithms of the three conformally invariant cross ratios under permutations. The three variables L_{1234} , L_{1423} and L_{1342} transform into one another under permutations, up to an overall minus sign. For example, under the permutation $1 \leftrightarrow 2$, we have $L_{1234} \rightarrow -L_{1423}$, $L_{1423} \rightarrow -L_{1234}$, and $L_{1342} \rightarrow -L_{1342}$.

The permutation properties of the remaining six cross ratios are displayed graphically in fig. 2, where we made the identifications $\{i, j, k, l\} \rightarrow \{1, 2, 3, 4\}$ for simplicity. The analysis can be further simplified by noting that odd permutations in S_4 merely invert ρ_{ijkl} , so that, for example,

$$\rho_{ijkl} = \frac{1}{\rho_{ikjl}} \quad \longrightarrow \quad L_{ijkl} = -L_{ikjl}. \quad (5.5)$$

This inversion corresponds to moving across to the diametrically opposite point in the left-hand plot in fig. 2. We conclude that there are only three different cross ratios (corresponding to the cyclic permutations of $\{j, k, l\}$ associated with $S_3/Z_2 \simeq Z_3$), namely ρ_{ijkl} , ρ_{iljk} and ρ_{iklj} . They correspond to triangles in fig. 2. Finally, the logarithms of the three cross ratios are linearly dependent, summing to zero:

$$L_{ijkl} + L_{iljk} + L_{iklj} = 0. \quad (5.6)$$

These symmetry properties lead us to consider for Δ_4^{kin} in eq. (5.2) the general form

$$\Delta_4^{\text{kin}}(\rho_{ijkl}) = (L_{1234})^{h_1} (L_{1423})^{h_2} (L_{1342})^{h_3}, \quad (5.7)$$

where we have adopted the labeling of hard partons by $\{1, 2, 3, 4\}$ as in fig. 2. Here the h_i are non-negative integers, and eq. (5.6) has not yet been taken into account. Our general

strategy will be to construct linear combinations of the monomials in eq. (5.7) designed to match the symmetries of the available color tensors, h^{abcd} in eq. (5.2). Such combinations can be constructed for general h_i . As we shall see, however, transcendentality constraints restrict the integers h_i to be small at low loop orders. In the three-loop case, this will suffice to eliminate all solutions to the constraints, except for a single function.

We begin by noting that the antisymmetry of L_{1234} under the permutation $1 \leftrightarrow 4$ (or under $2 \leftrightarrow 3$, see fig. 2) is mirrored by the antisymmetry of the color factor $h^{abcd} \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d$ if the tensor $h^{abcd} = f^{ade} f^{cbe}$, where f^{ade} are the usual, fully antisymmetric $SU(N_c)$ structure constants. The same is obviously true for any odd power of L_{1234} , while in the case of even powers, an appropriate type of color tensor is $h^{abcd} = d^{ade} d^{cbe}$, where d^{ade} are the fully symmetric $SU(N_c)$ tensors. Fig. 2, however, shows that under other permutations, the different cross ratios transform into one another. Therefore, if we are to write a function with a definite symmetry under all permutations, it must be a function of all three variables. Specifically, in order for a term of the form (5.7) to have, by itself, a definite symmetry under all permutations, the powers h_1 , h_2 and h_3 , must all be equal. Alternatively, one can consider a linear combination of several terms of the form (5.7), yielding together a function of the kinematic variables with definite symmetry. In this respect it is useful to keep in mind that the sum of the three logarithms (all with a single power) is identically zero, by eq. (5.6).

Let us now construct the different structures that realize Bose symmetry, by considering linear combinations of terms of the form of eq. (5.2), with Δ_4^{kin} given by eq. (5.7). We consider first three examples, where the logarithms L_{ijkl} are raised to a single power h . As we will see, none of these examples will satisfy all the constraints; they are useful however for illustrating the available structures.

a) We first consider simply setting $h_1 = h_2 = h_3$ in eq. (5.7), obtaining

$$\Delta_4(\rho_{ijkl}) = h^{abcd} \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \left[L_{1234} L_{1423} L_{1342} \right]^h. \quad (5.8)$$

For odd h the color tensor h^{abcd} must be completely antisymmetric in the four indices, while for even h it must be completely symmetric. We anticipate that odd h is ruled out, because completely antisymmetric four-index invariant tensors do not exist for simple Lie groups [66]. Furthermore, while symmetric tensors do exist, eq. (5.8) is ruled out at three loops, because from fig. 1 it is clear that only $h^{abcd} = f^{ade} f^{cbe}$ (or permutations thereof) can arise in Feynman diagrams at this order.

b) Our second example is

$$\Delta_4(\rho_{ijkl}) = \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \left[f^{ade} f^{cbe} L_{1234}^h + f^{cae} f^{dbe} L_{1423}^h + f^{bae} f^{cde} L_{1342}^h \right], \quad (5.9)$$

where h must be odd. Alternatively, each f^{ade} may be replaced by the fully symmetric $SU(N_c)$ tensor d^{ade} , and then h must be even. In eq. (5.9) each term has a definite symmetry only with respect to certain permutations, but the three terms transform into one another in such a way that their sum admits full Bose symmetry. We will see shortly that the structure in eq. (5.9) does not satisfy the collinear constraints.

c) Finally, one may consider the case where two of the three logarithms appear together in a product, raised to some power h ,

$$\Delta_4(\rho_{ijkl}) = \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \left[d^{abe} d^{cde} (L_{1234} L_{1423})^h + d^{dae} d^{cbe} (L_{1423} L_{1342})^h + d^{cae} d^{bde} (L_{1342} L_{1234})^h \right]. \quad (5.10)$$

Once again, we observe that these color tensors cannot arise in three-loop webs. Furthermore, as we will see, eq. (5.10), at any loop order, fails to satisfy the collinear constraints.

We are led to consider more general structures, using eqs. (5.2) and (5.7) with arbitrary integers h_i . As announced, we will satisfy Bose symmetry by constructing polynomial kinematical factors mimicking the symmetry of the available color tensors. One may write for example

$$\begin{aligned} \Delta_4(\rho_{ijkl}) &= \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \\ &\times \left[f^{ade} f^{cbe} L_{1234}^{h_1} \left(L_{1423}^{h_2} L_{1342}^{h_3} - (-1)^{h_1+h_2+h_3} L_{1342}^{h_2} L_{1423}^{h_3} \right) \right. \\ &+ f^{cae} f^{dbe} L_{1423}^{h_1} \left(L_{1234}^{h_2} L_{1342}^{h_3} - (-1)^{h_1+h_2+h_3} L_{1342}^{h_2} L_{1234}^{h_3} \right) \\ &\left. + f^{bae} f^{cde} L_{1342}^{h_1} \left(L_{1423}^{h_2} L_{1234}^{h_3} - (-1)^{h_1+h_2+h_3} L_{1234}^{h_2} L_{1423}^{h_3} \right) \right], \end{aligned} \quad (5.11)$$

where h_1 , h_2 and h_3 can be any non-negative integers. The first line is invariant, for example, under the permutation $1 \leftrightarrow 4$ (when applied to both kinematics and color), the second line is invariant under $1 \leftrightarrow 3$, and the third is invariant under $1 \leftrightarrow 2$. The other exchange symmetries are realized by the transformation of two lines into one another. For example, under $1 \leftrightarrow 4$ the second line transforms into the third and vice versa. In eq. (5.11) the color and kinematic factors in each line are separately antisymmetric under the corresponding permutation. Note that eq. (5.9) corresponds to the special case where h_1 in eq. (5.11) is odd, while $h_2 = h_3 = 0$.

One can also construct an alternative Bose symmetrization using the symmetric combination,

$$\begin{aligned} \Delta_4(\rho_{ijkl}) &= \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \\ &\times \left[d^{ade} d^{cbe} L_{1234}^{h_1} \left(L_{1423}^{h_2} L_{1342}^{h_3} + (-1)^{h_1+h_2+h_3} L_{1342}^{h_2} L_{1423}^{h_3} \right) \right. \\ &+ d^{cae} d^{dbe} L_{1423}^{h_1} \left(L_{1234}^{h_2} L_{1342}^{h_3} + (-1)^{h_1+h_2+h_3} L_{1342}^{h_2} L_{1234}^{h_3} \right) \\ &\left. + d^{bae} d^{cde} L_{1342}^{h_1} \left(L_{1423}^{h_2} L_{1234}^{h_3} + (-1)^{h_1+h_2+h_3} L_{1234}^{h_2} L_{1423}^{h_3} \right) \right], \end{aligned} \quad (5.12)$$

Note that eq. (5.10) is reproduced by setting $h_1 = 0$ and $h_2 = h_3 = h$ in eq. (5.12).

Eqs. (5.11) and (5.12) both yield non-trivial functions for both even and odd powers h_i , with the following exceptions: For even h_1 , eq. (5.11) becomes identically zero if $h_2 = h_3$; similarly, for odd h_1 eq. (5.12) becomes identically zero if $h_2 = h_3$. It is interesting to note that eq. (5.8) with odd h cannot be obtained as a special case of eq. (5.11). Indeed, by

choosing $h_1 = h_2 = h_3$ one obtains the correct kinematic dependence, but then the color structure factors out and vanishes by the Jacobi identity,

$$h^{abcd} = f^{ade} f^{cbe} + f^{cae} f^{dbe} + f^{bae} f^{cde} = 0. \quad (5.13)$$

In contrast, for even h , eq. (5.8) can be obtained as a special case of eq. (5.12), setting

$$h^{abcd} = d^{ade} d^{cbe} + d^{cae} d^{dbe} + d^{bae} d^{cde}, \quad (5.14)$$

which is totally symmetric, as required. This is expected from the general properties of symmetric and antisymmetric invariant tensors for simple Lie algebras [66].

At any fixed number of loops l , the total power of the logarithms in eqs. (5.11) and (5.12), $h_{\text{tot}} \equiv h_1 + h_2 + h_3$, will play an important role. Indeed, h_{tot} is the degree of transcendentality of the function Δ_4 , as defined in the Introduction, and it is bounded from above by the maximal allowed transcendentality of the anomalous dimension matrix at l loops, as described in Sec. 5.3. We expect then that at l loops there will be a finite number of sets of integers h_i satisfying the available constraints. The most general solution for the correction term Δ_4 will then be given by a linear combination of symmetric and antisymmetric polynomials such as those given in eqs. (5.11) and (5.12), with all allowed choices of h_i .

Such combinations include also contributions related to higher-order Casimir operators. Indeed, summing over permutations of $\{h_1, h_2, h_3\}$ in the symmetric version of Δ_4 , eq. (5.12), one finds a completely symmetric kinematic factor, multiplying a color tensor which is directly related to the quartic Casimir operator (with a suitable choice of basis in the space of symmetric tensors over the Lie algebra [66]),

$$\begin{aligned} \Delta_4(\rho_{ijkl}) &= \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \times \left[d^{ade} d^{cbe} + d^{cae} d^{dbe} + d^{bae} d^{cde} \right] \\ &\times \left[L_{1234}^{h_1} L_{1423}^{h_2} L_{1342}^{h_3} + L_{1423}^{h_1} L_{1342}^{h_2} L_{1234}^{h_3} + L_{1342}^{h_1} L_{1234}^{h_2} L_{1423}^{h_3} \right. \\ &\left. + (-1)^{h_1+h_2+h_3} \left(L_{1234}^{h_1} L_{1342}^{h_2} L_{1423}^{h_3} + L_{1423}^{h_1} L_{1234}^{h_2} L_{1342}^{h_3} + L_{1342}^{h_1} L_{1423}^{h_2} L_{1234}^{h_3} \right) \right]. \end{aligned} \quad (5.15)$$

For even $h_{\text{tot}} \equiv h_1 + h_2 + h_3$ this function is always non-trivial, while for odd h_{tot} it is only non-trivial if all three powers h_i are different. We note once again that, due to the Jacobi identity (5.13), eq. (5.15) does not have an analog involving the antisymmetric structure constants.

5.3 Maximal transcendentality

Our next observation is that, at a given loop order, the total power of the logarithms, h_{tot} , cannot be arbitrarily high. It is well known (although not proven mathematically) that the maximal transcendentality τ_{max} of the coefficient of the $1/\epsilon^k$ pole in an l -loop amplitude (including $k = 0$) is $\tau_{\text{max}} = 2l - k$. Because Γ , $\Gamma_{\mathbf{Sp}}$ and Δ are associated with the $1/\epsilon$ single pole, their maximal transcendentality is $\tau_{\text{max}} = 2l - 1$. For $\mathcal{N} = 4$ super-Yang-Mills theory, in every known instance the terms arising in this way are purely of this maximal transcendentality; there are no terms of lower transcendentality. This property is relevant

also for non-supersymmetric massless gauge theories, particularly at three loops. Indeed, in any massless gauge theory the purely-gluonic web diagrams that we need to consider at three loops are the same as those arising in $\mathcal{N} = 4$ super-Yang-Mills theory. We conclude that at three loops Δ should have transcendentality $\tau = 5$ [57], while for $l > 3$ some relevant webs may depend on the matter content of the theory, so that Δ is only constrained to have a transcendentality at most equal to $2l - 1$.

It should be emphasized that some transcendentality could be attributed to constant prefactors. For example, the sum-over-dipoles formula (3.4) for Γ_{dip} attains transcendentality $\tau = 2l - 1$ as the sum of $\tau = 2l - 2$ from the (constant) cusp anomalous dimension γ_K (associated with a $1/\epsilon^2$ double pole in the amplitude) and $\tau = 1$ from the single logarithm.

Because the functions Δ_4^{kin} are defined up to possible numerical prefactors, which may carry transcendentality, terms of the form (5.7), (5.11) or (5.12) must obey

$$h_{\text{tot}} = h_1 + h_2 + h_3 \leq 2l - 1. \quad (5.16)$$

We note furthermore that constants of transcendentality $\tau = 1$, *i.e.* single factors of π , do not arise in Feynman diagram calculations, except for imaginary parts associated with unitarity phases. We conclude that whenever the maximal transcendentality argument applies to Γ , the special case in which our functions Δ_4 have $\tau = 2l - 2$ is not generally allowed.

The sum of the powers of all the logarithms in the product must then be no more than $h_{\text{tot}} = 5$ at three loops, or $h_{\text{tot}} = 7$ at four loops, and so on. In the special cases considered above, at three loops, the constraint is: $3h \leq 5$, *i.e.* $h \leq 1$ in eq. (5.8), $h \leq 5$ in eq. (5.9), and $2h \leq 5$, *i.e.* $h \leq 2$ in eq. (5.10). Clearly, at low orders, transcendentality imposes strict limitations on the admissible functional forms. We will take advantage of these limitations at three loops in Sec. 5.5.

5.4 Collinear limits

Equipped with the knowledge of how Bose symmetry and other requirements may be satisfied, let us return to the splitting amplitude constraint, namely the requirement that the difference between the two Δ terms in eq. (4.14) must conspire to depend only on the color and kinematic variables of the two partons that become collinear.

We begin by analyzing the case of an amplitude with precisely four colored partons, possibly accompanied by other colorless particles (we postpone the generalization to an arbitrary number of partons to Sec. 6). The collinear constraint simplifies for $n = 4$ because for three partons there are no contributions beyond the sum-over-dipoles formula, so that $\Delta_{n-1} = \Delta_3 = 0$ [41].⁶ In eq. (4.14) we therefore have to consider Δ_4 on its own, and require that when, say, p_1 and p_2 become collinear Δ_4 does not involve the kinematic or color variables of other hard particles in the process. Because in this limit there remains no non-singular Lorentz-invariant kinematic variable upon which Δ_4 can depend, it essentially

⁶For $n = 4$ we should add a colorless particle carrying off momentum; otherwise the three-parton kinematics are ill-defined (for real momenta), and the limit $p_1 \cdot p_2 \rightarrow 0$ is not really a collinear limit but a forward or backward scattering limit.

means that Δ_4 must become trivial in this limit, although it does not imply, of course, that Δ_4 vanishes away from the limit. In the following we shall see how this can be realized.

To this end let us first carefully examine the limit under consideration. We work with strictly massless hard partons, $p_i^2 = 0$ for all i . In a fixed-angle scattering amplitude we usually consider $2p_i \cdot p_j = Q^2 \beta_i \cdot \beta_j$ where Q^2 is taken large, keeping $\beta_i \cdot \beta_j = \mathcal{O}(1)$ for any i and j . Now we relax the fixed-angle limit for the pair of hard partons p_1 and p_2 . Defining $P \equiv p_1 + p_2$ as in Sec. 4, we consider the limit $2p_1 \cdot p_2 / Q^2 = P^2 / Q^2 \rightarrow 0$. The other Lorentz invariants all remain large; in particular for any $j \neq 1, 2$ we still have $2p_1 \cdot p_j = Q^2 \beta_1 \cdot \beta_j$ and $2p_2 \cdot p_j = Q^2 \beta_2 \cdot \beta_j$ where $\beta_1 \cdot \beta_j$ and $\beta_2 \cdot \beta_j$ are of $\mathcal{O}(1)$. In order to control the way in which the limit is approached, it is useful to define

$$p_1 = zP + k, \quad p_2 = (1 - z)P - k, \quad (5.17)$$

so that z measures the longitudinal momentum fraction carried by p_1 in P , namely

$$z = \frac{p_1^+}{P^+} = \frac{p_1^+}{p_1^+ + p_2^+}, \quad (5.18)$$

where we assume, for simplicity, that the “+” light-cone direction⁷ is defined by p_1 , so that $p_1 = (p_1^+, 0^-, 0_\perp)$. In eq. (5.18) both the numerator and denominator are of order Q , so z is of $\mathcal{O}(1)$ and remains fixed in the limit $P^2 / Q^2 \rightarrow 0$. In eq. (5.17) k is a small residual momentum, making it possible for P to be off the light-cone while p_1 and p_2 remain strictly light-like. Using the mass-shell conditions $p_1^2 = p_2^2 = 0$ one easily finds

$$k^2 = -z(1 - z)P^2, \quad k \cdot P = \frac{1}{2}(1 - 2z)P^2, \quad (5.19)$$

so that the components of k are

$$k = \left(0^+, -\frac{P^2}{2P^+}, -\sqrt{z(1 - z)P^2} \right). \quad (5.20)$$

Note that in the collinear limit k^- / Q scales as P^2 / Q^2 , while k_\perp / Q scales as $\sqrt{P^2 / Q^2}$.

We can now examine the behavior of the logarithms of the three cross ratios entering Δ_4^{kin} in eq. (5.7), in the limit $P^2 \rightarrow 0$. Clearly, L_{1234} and L_{1423} , which contain the vanishing invariant $p_1 \cdot p_2$ either in the numerator or in the denominator, will be singular in this limit. Similarly, it is easy to see that L_{1342} must vanish, because $\rho_{1342} \rightarrow 1$. More precisely, the

⁷One can then further specify the frame by choosing the “−” direction along the momentum of one of the other hard partons, say p_3 .

collinear behavior may be expressed using the parametrization of eq. (5.17), with the result

$$\begin{aligned}
L_{1234} &= \ln \left(\frac{p_1 \cdot p_2 p_3 \cdot p_4}{p_1 \cdot p_3 p_2 \cdot p_4} \right) \\
&\simeq \underbrace{\ln \left(\frac{P^2 p_3 \cdot p_4}{2z(1-z) P \cdot p_3 P \cdot p_4} \right)}_{\mathcal{O}(\ln(P^2/Q^2))} - \underbrace{\frac{k \cdot p_3}{z P \cdot p_3}}_{\mathcal{O}(\sqrt{P^2/Q^2})} + \underbrace{\frac{k \cdot p_4}{(1-z) P \cdot p_4}}_{\mathcal{O}(\sqrt{P^2/Q^2})} \rightarrow \infty, \quad (5.21)
\end{aligned}$$

$$\begin{aligned}
L_{1423} &= \ln \left(\frac{p_1 \cdot p_4 p_2 \cdot p_3}{p_1 \cdot p_2 p_4 \cdot p_3} \right) \\
&\simeq \underbrace{\ln \left(\frac{2z(1-z) P \cdot p_4 P \cdot p_3}{P^2 p_4 \cdot p_3} \right)}_{\mathcal{O}(\ln(P^2/Q^2))} - \underbrace{\frac{k \cdot p_3}{(1-z) P \cdot p_3}}_{\mathcal{O}(\sqrt{P^2/Q^2})} + \underbrace{\frac{k \cdot p_4}{z P \cdot p_4}}_{\mathcal{O}(\sqrt{P^2/Q^2})} \rightarrow -\infty, \quad (5.22)
\end{aligned}$$

$$L_{1342} = \ln \left(\frac{p_1 \cdot p_3 p_4 \cdot p_2}{p_1 \cdot p_4 p_3 \cdot p_2} \right) = \frac{1}{z(1-z)} \left(\frac{k \cdot p_3}{P \cdot p_3} - \frac{k \cdot p_4}{P \cdot p_4} \right) = \mathcal{O} \left(\sqrt{P^2/Q^2} \right) \rightarrow 0, \quad (5.23)$$

where we expanded in the small momentum k . As expected, two of the cross-ratio logarithms diverge logarithmically with P^2/Q^2 , with opposite signs, while the third cross-ratio logarithm *vanishes linearly with* $\sqrt{P^2/Q^2}$. We emphasize that this vanishing is independent of whether the momenta p_i are incoming or outgoing, except, of course, that the two collinear partons p_1 and p_2 must either be both incoming or both outgoing. Indeed, according to (2.11), ρ_{1342} carries no phase when p_1 and p_2 are collinear:

$$\rho_{1342} = \left| \frac{p_1 \cdot p_3 p_4 \cdot p_2}{p_1 \cdot p_4 p_3 \cdot p_2} \right| e^{-i\pi(\lambda_{13} + \lambda_{42} - \lambda_{14} - \lambda_{32})} \rightarrow 1, \quad (5.24)$$

since $\lambda_{13} = \lambda_{32}$ and $\lambda_{42} = \lambda_{14}$.

Let us now examine a generic term with a kinematic dependence of the form (5.7) in this limit. Substituting eqs. (5.21) through (5.23) into eq. (5.7) we see that, if h_3 (the power of L_{1342}) is greater than or equal to 1, then the result for Δ_4^{kin} in the collinear limit is zero. This vanishing is not affected by the powers of the other logarithms, because they diverge only logarithmically as $P^2/Q^2 \rightarrow 0$, while L_{1342} vanishes as a power law in the same limit. In contrast, if $h_3 = 0$, and h_1 or h_2 is greater than zero, then the kinematic function Δ_4^{kin} in eq. (5.7) diverges when p_1 and p_2 become collinear, due to the behavior of L_{1234} and L_{1423} in eqs. (5.21) and (5.22). The first term in each of these equations introduces explicit dependence on the non-collinear parton momenta p_3 and p_4 into Δ_n in eq. (4.14), which would violate collinear universality. We conclude that consistency with the limit where p_1 and p_2 become collinear requires $h_3 \geq 1$.

Obviously we can consider, in a similar way, the limits where other pairs of partons become collinear, leading to the conclusion that all three logarithms, L_{1234} , L_{1423} and L_{1342} must appear raised to the first or higher power. Collinear limits thus constrain the powers of the logarithms by imposing

$$h_i \geq 1, \quad \forall i. \quad (5.25)$$

This result puts a lower bound on the transcendentality of Δ_4^{kin} , namely

$$h_{\text{tot}} = h_1 + h_2 + h_3 \geq 3. \quad (5.26)$$

5.5 Three-loop analysis

We have seen that corrections to the sum-over-dipoles formula involving four-parton correlations are severely constrained. We can now examine specific structures that may arise at a given loop order l , beginning with the first nontrivial possibility, $l = 3$. Because we consider webs that are attached to four hard partons, at three loops they can only attach once to each eikonal line, as in fig. 1, giving the color factor in eq. (5.2), where h^{abcd} must be constructed out of the structure constants f^{ade} . The only possibility is terms of the form $f^{ade} f^{bce}$ — the same form we obtained in the previous section starting from the symmetry properties of the kinematic factors depending on L_{ijkl} . In contrast, the symmetric tensor d^{ade} cannot arise in three-loop webs.

Taking into account the splitting amplitude constraint (5.25) on the one hand, and the maximal transcendentality constraint (5.16) on the other, there are just a few possibilities for the various powers h_i . These are summarized in Table 1.

The lowest allowed transcendentality for Δ_4^{kin} is $\tau = 3$, corresponding to $h_1 = h_2 = h_3 = 1$. This brings us to eq. (5.8), in which we would have to construct a completely antisymmetric tensor h^{abcd} out of the structure constants f^{ade} . Such a tensor, however, does not exist. Indeed, starting with the general expression (5.11), which is written in terms of the structure constants, and substituting $h_1 = h_2 = h_3 = 1$, we immediately see that the color structure factorizes, and vanishes by the Jacobi identity (5.13). The possibility $h_1 = h_2 = h_3 = 1$ is thus excluded by Bose symmetry.

Next, we may consider transcendentality $\tau = 4$. Ultimately, we exclude functions with this degree of transcendentality at three loops, because we are dealing with purely gluonic webs, which are the same as in $\mathcal{N} = 4$ super Yang-Mills theory. We expect then that the anomalous dimension matrix will have a uniform degree of transcendentality $\tau = 5$, and there are no constants with $\tau = 1$ that might multiply functions with $h_{\text{tot}} = 4$ to achieve the desired result, as discussed in Sec. 5.3. However, it is instructive to note that symmetry alone does not rule out this possibility. Indeed, having excluded eq. (5.12), involving the symmetric tensor d^{ade} , we may consider eq. (5.11) with $h_1 + h_2 + h_3 = 4$. Bose symmetry and the splitting amplitude constraint in eq. (5.25) leave just two potential structures, one with $h_1 = 2$ and $h_2 = h_3 = 1$, and a second one with $h_1 = h_2 = 1$ and $h_3 = 2$ ($h_1 = h_3 = 1$ and $h_2 = 2$ yields the latter structure again). The former vanishes identically, while the latter could provide a viable candidate,

$$\begin{aligned} \Delta_4^{(112)}(\rho_{ijkl}) &= \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \\ &\times \left[f^{ade} f^{cbe} L_{1234} \left(L_{1423} L_{1342}^2 - L_{1342} L_{1423}^2 \right) \right. \\ &\quad + f^{cae} f^{dbe} L_{1423} \left(L_{1234} L_{1342}^2 - L_{1342} L_{1234}^2 \right) \\ &\quad \left. + f^{bae} f^{cde} L_{1342} \left(L_{1423} L_{1234}^2 - L_{1234} L_{1423}^2 \right) \right], \end{aligned} \quad (5.27)$$

h_1	h_2	h_3	h_{tot}	comment
1	1	1	3	vanishes identically by Jacobi identity (5.13)
2	1	1	4	kinematic factor vanishes identically
1	1	2	4	allowed by symmetry, excluded by transcendentality
1	2	2	5	viable possibility, eq. (5.28)
3	1	1	5	viable possibility, eq. (5.29)
2	1	2	5	viable possibility, eq. (5.31)
1	1	3	5	viable possibility, eq. (5.32)

} all coincide using eq. (5.13)

Table 1: Different possible assignments of the powers h_i in eq. (5.11) at three loops. We only consider $h_i \geq 1$ because of the splitting amplitude constraint (5.25) and $h_{\text{tot}} \leq 5$ because of the bound on transcendentality, eq. (5.16). We also omit the combinations that can be obtained by interchanging the values of h_2 and h_3 ; this interchange yields the same function, up to a possible overall minus sign.

We rule out eq. (5.27) based only on its degree of transcendentality.

We consider next the highest attainable transcendentality at three loops, $\tau = 5$. Equation (5.11) yields four different structures, summarized in Table 1. The first structure we consider has $h_1 = 1$ and $h_2 = h_3 = 2$. It is given by

$$\begin{aligned} \Delta_4^{(122)}(\rho_{ijkl}) &= \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \left[f^{ade} f^{cbe} L_{1234} (L_{1423} L_{1342})^2 \right. \\ &\quad \left. + f^{cae} f^{dbe} L_{1423} (L_{1234} L_{1342})^2 + f^{bae} f^{cde} L_{1342} (L_{1423} L_{1234})^2 \right]. \end{aligned} \quad (5.28)$$

The second structure has $h_1 = 3$ and $h_2 = h_3 = 1$, yielding

$$\begin{aligned} \Delta_4^{(311)}(\rho_{ijkl}) &= \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \left[f^{ade} f^{cbe} (L_{1234})^3 L_{1423} L_{1342} \right. \\ &\quad \left. + f^{cae} f^{dbe} (L_{1423})^3 L_{1234} L_{1342} + f^{bae} f^{cde} (L_{1342})^3 L_{1423} L_{1234} \right]. \end{aligned} \quad (5.29)$$

We now observe that the two functions (5.28) and (5.29) are, in fact, one and the same. To show this, we form their difference, and use relation (5.6) to substitute $L_{1234} = -L_{1423} - L_{1342}$. We obtain

$$\begin{aligned} \Delta_4^{(122)} - \Delta_4^{(311)} &= \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d L_{1234} L_{1423} L_{1342} \\ &\quad \times \left[f^{ade} f^{cbe} (L_{1423} L_{1342} - L_{1234}^2) + f^{cae} f^{dbe} (L_{1234} L_{1342} - L_{1423}^2) \right. \\ &\quad \left. + f^{bae} f^{cde} (L_{1423} L_{1234} - L_{1342}^2) \right] \\ &= -\mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d L_{1234} L_{1423} L_{1342} \left[f^{ade} f^{cbe} + f^{cae} f^{dbe} + f^{bae} f^{cde} \right] \\ &\quad \times (L_{1342}^2 + L_{1342} L_{1423} + L_{1423}^2) = 0, \end{aligned} \quad (5.30)$$

vanishing by the Jacobi identity (5.13).

The last two structures in Table 1 are given by

$$\begin{aligned} \Delta_4^{(212)}(\rho_{ijkl}) = & \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \left[f^{ade} f^{cbe} L_{1234}^2 \left(L_{1423} L_{1342}^2 + L_{1423}^2 L_{1342} \right) \right. \\ & + f^{cae} f^{dbe} L_{1423}^2 \left(L_{1234} L_{1342}^2 + L_{1234}^2 L_{1342} \right) \\ & \left. + f^{bae} f^{cde} L_{1342}^2 \left(L_{1423} L_{1234}^2 + L_{1423}^2 L_{1234} \right) \right], \end{aligned} \quad (5.31)$$

and

$$\begin{aligned} \Delta_4^{(113)}(\rho_{ijkl}) = & \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \left[f^{ade} f^{cbe} L_{1234} \left(L_{1423} L_{1342}^3 + L_{1423}^3 L_{1342} \right) \right. \\ & + f^{cae} f^{dbe} L_{1423} \left(L_{1234} L_{1342}^3 + L_{1234}^3 L_{1342} \right) \\ & \left. + f^{bae} f^{cde} L_{1342} \left(L_{1423} L_{1234}^3 + L_{1423}^3 L_{1234} \right) \right]. \end{aligned} \quad (5.32)$$

One easily verifies that they are both proportional to $\Delta_4^{(122)} = \Delta_4^{(311)}$. Consider first eq. (5.31). In each line we can factor out the logarithms and use eq. (5.6) to obtain a monomial. For example, the first line may be written as:

$$\begin{aligned} L_{1234}^2 \left(L_{1423} L_{1342}^2 + L_{1423}^2 L_{1342} \right) &= L_{1234}^2 L_{1423} L_{1342} (L_{1423} + L_{1342}) \\ &= -L_{1234}^3 L_{1423} L_{1342}, \end{aligned} \quad (5.33)$$

where we recognise that this function coincides with eq. (5.29) above. Consider next eq. (5.32), where, for example, the first line yields

$$\begin{aligned} L_{1234} \left(L_{1423} L_{1342}^3 + L_{1423}^3 L_{1342} \right) &= L_{1234} L_{1423} L_{1342} \left(L_{1342}^2 + L_{1423}^2 \right) \\ &= L_{1234} L_{1423} L_{1342} \left((L_{1342} + L_{1423})^2 - 2L_{1342} L_{1423} \right) \\ &= L_{1234} L_{1423} L_{1342} \left(L_{1234}^2 - 2L_{1342} L_{1423} \right), \end{aligned} \quad (5.34)$$

which is a linear combination of eqs. (5.28) and (5.29), rather than a new structure.

We conclude that there is precisely one function, $\Delta_4^{(122)} = \Delta_4^{(311)}$, that can be constructed out of arbitrary powers of logarithms and is consistent with all available constraints at three loops. We emphasize that this function is built with color and kinematic factors that one expects to find in the actual web diagram computations, and it is quite possible that it indeed appears. Because this structure saturates the transcendentality bound, its coefficient is necessarily a rational number.

Note that color conservation has not been imposed here, but it is implicitly assumed that for a four-parton amplitude

$$\mathbf{T}_1^a + \mathbf{T}_2^a + \mathbf{T}_3^a + \mathbf{T}_4^a = 0. \quad (5.35)$$

Importantly, upon using this relation, the structure (5.28) (or, equivalently, (5.29)) remains non-trivial.

5.6 Three-loop functions involving polylogarithms

Additional functions can be constructed upon removing the requirement that the kinematic dependence be of the form (5.7), where only powers of logarithms are allowed. Three key features of the function $\ln \rho$ were essential in the examples above: it vanishes like a power at $\rho = 1$, it has a definite symmetry under $\rho \rightarrow 1/\rho$, and it only diverges logarithmically as $\rho \rightarrow 0$ and $\rho \rightarrow \infty$. These properties can be mimicked by a larger class of functions. In particular, allowing dilogarithms one can easily construct a function of transcendentality $\tau = 4$, which is consistent with Bose symmetry and collinear constraints. It is given by

$$\begin{aligned} \Delta_4^{(211, \text{Li}_2)}(\rho_{ijkl}) &= \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \quad (5.36) \\ &\left[f^{ade} f^{cbe} \left(\text{Li}_2(1 - \rho_{1234}) - \text{Li}_2(1 - 1/\rho_{1234}) \right) \ln \rho_{1423} \ln \rho_{1342} \right. \\ &\quad + f^{cae} f^{dbe} \left(\text{Li}_2(1 - \rho_{1423}) - \text{Li}_2(1 - 1/\rho_{1423}) \right) \ln \rho_{1234} \ln \rho_{1342} \\ &\quad \left. + f^{bae} f^{cde} \left(\text{Li}_2(1 - \rho_{1342}) - \text{Li}_2(1 - 1/\rho_{1342}) \right) \ln \rho_{1423} \ln \rho_{1234} \right]. \end{aligned}$$

The key point here is that the function $\text{Li}_2(1 - \rho_{1234}) - \text{Li}_2(1 - 1/\rho_{1234})$ is odd under $\rho_{1234} \rightarrow 1/\rho_{1234}$, which allows it to be paired with the antisymmetric structure constants f^{ade} . It is also easy to verify that the collinear constraints are satisfied.

We note that it is also possible to construct a potentially relevant function containing logarithms with a more complicated kinematic dependence. Indeed, the structure

$$\begin{aligned} \Delta_4^{(211, \text{mod})}(\rho_{ijkl}) &= \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \quad (5.37) \\ &\left[f^{ade} f^{cbe} \ln \rho_{1234} \ln \left(\frac{\rho_{1234}}{(1 - \rho_{1234})^2} \right) \ln \rho_{1423} \ln \rho_{1342} \right. \\ &\quad + f^{cae} f^{dbe} \ln \rho_{1423} \ln \left(\frac{\rho_{1423}}{(1 - \rho_{1423})^2} \right) \ln \rho_{1234} \ln \rho_{1342} \\ &\quad \left. + f^{bae} f^{cde} \ln \rho_{1342} \ln \left(\frac{\rho_{1342}}{(1 - \rho_{1342})^2} \right) \ln \rho_{1423} \ln \rho_{1234} \right] \end{aligned}$$

fulfills the symmetry requirements discussed above, because $\ln(\rho_{1234}/(1 - \rho_{1234})^2)$ is even under $\rho_{1234} \rightarrow 1/\rho_{1234}$. Thanks to the extra power of the cross-ratio logarithm, eq. (5.37) also vanishes in all collinear limits, as required. Logarithms with argument $1 - \rho_{ijkl}$ cannot be directly rejected on the basis of the fact that they induce unphysical singularities, because $\rho_{ijkl} \rightarrow 1$ corresponds to a physical collinear limit⁸. We conclude that eqs. (5.36) and (5.37) would be viable based on symmetry and collinear requirements alone. However, we can exclude them on the basis of transcendentality: as discussed in Sec. 5.3, a function with $h_{\text{tot}} = 4$ cannot arise at three loops, because it cannot be upgraded to maximal transcendentality $\tau = 5$ by constant prefactors.

⁸Note that the analogue structure containing $\ln(\rho_{1234}/(1 + \rho_{1234})^2)$ can be excluded because the limit $\rho_{ijkl} \rightarrow -1$ should not be singular. Indeed, by construction the variables ρ_{ijkl} always contain an even number of negative momentum invariants, so their real part is always positive, (although unitarity phases may add up and bring their logarithm to the second Riemann sheet).

At transcendentality $\tau = 5$, there are at least two further viable structures that involve polylogarithms, in which second and third powers of logarithms are replaced, respectively, by appropriate combinations Li_2 and Li_3 . The first structure can be obtained starting from eq. (5.28), and using the same combination of dilogarithms that was employed in eq. (5.36). One finds

$$\begin{aligned} \Delta_4^{(122, \text{Li}_2)}(\rho_{ijkl}) &= \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \tag{5.38} \\ &\times \left[f^{ade} f^{cbe} \ln \rho_{1234} \left(\text{Li}_2(1 - \rho_{1342}) - \text{Li}_2(1 - 1/\rho_{1342}) \right) \left(\text{Li}_2(1 - \rho_{1423}) - \text{Li}_2(1 - 1/\rho_{1423}) \right) \right. \\ &+ f^{cae} f^{dbe} \ln \rho_{1423} \left(\text{Li}_2(1 - \rho_{1234}) - \text{Li}_2(1 - 1/\rho_{1234}) \right) \left(\text{Li}_2(1 - \rho_{1342}) - \text{Li}_2(1 - 1/\rho_{1342}) \right) \\ &\left. + f^{bae} f^{cde} \ln \rho_{1342} \left(\text{Li}_2(1 - \rho_{1234}) - \text{Li}_2(1 - 1/\rho_{1234}) \right) \left(\text{Li}_2(1 - \rho_{1423}) - \text{Li}_2(1 - 1/\rho_{1423}) \right) \right]. \end{aligned}$$

Here it was essential to replace both \ln^2 terms in order to keep the symmetry properties in place. Starting instead from eq. (5.29), there is one possible polylogarithmic replacement, which, however, requires introducing trilogarithms, because using Li_2 times a logarithm would turn the odd function into an even one, which is excluded. One may write instead

$$\begin{aligned} \Delta_4^{(311, \text{Li}_3)}(\rho_{ijkl}) &= \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \tag{5.39} \\ &\times \left[f^{ade} f^{cbe} \left(\text{Li}_3(1 - \rho_{1234}) - \text{Li}_3(1 - 1/\rho_{1234}) \right) L_{1423} L_{1342} \right. \\ &+ f^{cae} f^{dbe} \left(\text{Li}_3(1 - \rho_{1423}) - \text{Li}_3(1 - 1/\rho_{1423}) \right) L_{1234} L_{1342} \\ &\left. + f^{bae} f^{cde} \left(\text{Li}_3(1 - \rho_{1342}) - \text{Li}_3(1 - 1/\rho_{1342}) \right) L_{1423} L_{1234} \right]. \end{aligned}$$

Neither eq. (5.38) nor eq. (5.39) can be excluded at present, as they satisfy all available constraints. We can, however, exclude similar constructions with higher-order polylogarithms. For example, Li_4 has transcendentality $\tau = 4$, so it could be accompanied by at most one logarithm; this product would not satisfy all collinear constraints. We do not claim to be exhaustive in our investigation of polylogarithmic functions; additional possibilities may arise allowing arguments of the polylogarithms that have a different functional dependence on the cross ratios.

5.7 Four-loop analysis

Let us briefly turn our attention to contributions that may arise beyond three loops. At the four-loop level several new possibilities open up. First, there are potential quartic Casimirs in γ_K . Corresponding corrections to the soft anomalous dimension would satisfy inhomogeneous differential equations, eq. (5.5) of Ref. [41]. Beyond that, new types of corrections may appear even if γ_K admits Casimir scaling. First, considering the logarithmic expressions of eq. (5.7), purely gluonic webs might give rise to functions of transcendentality up to $h_{\text{tot}} = h_1 + h_2 + h_3 = 7$. At this level, there are four potential functions: (a) $h_1 = 5$ and $h_2 = h_3 = 1$; (b) $h_1 = 4$, $h_2 = 2$, $h_3 = 1$; (c) $h_1 = 1$ and $h_2 = h_3 = 3$; (d) $h_1 = 3$ and $h_2 = h_3 = 2$. Of course, as in the three-loop case, also polylogarithmic structures may

appear, and functions with $h_{\text{tot}} \leq 5$ might be present (of the type already discussed at three loops), multiplied by transcendental constants with $\tau \geq 2$.

It is interesting to focus in particular on color structures that are related to quartic Casimir operators, which can appear at four loops not only in γ_K but also in four-parton correlations. Indeed, a structure allowed by Bose symmetry and collinear constraints is given by eq. (5.8), where the group theory factor h^{abcd} is generated by a pure-gluon box diagram attached to four different hard partons, giving rise to a trace of four adjoint matrices. It is of the form

$$\Delta_4^{C_{4,A}}(\rho_{ijkl}) = \text{Tr} \left[F^a F^b F^c F^d \right] \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \left[\ln \rho_{1234} \ln \rho_{1423} \ln \rho_{1342} \right]^h, \quad (5.40)$$

where F^a are the $\text{SU}(N_c)$ generators in the adjoint representation, $(F^a)_{bc} = -if_{bc}^a$. This expression may be relevant *a priori* for both odd and even h , projecting respectively on the totally antisymmetric or symmetric parts of $\text{Tr} [F^a F^b F^c F^d]$. As noted above, however, a totally antisymmetric tensor cannot be constructed with four adjoint indices, so we are left with the completely symmetric possibility, which indeed corresponds to the quartic Casimir operator. The transcendentality constraint comes into play here: the only even integer h that can give transcendentality $\tau \leq 7$ is $h = 2$. For $\mathcal{N} = 4$ super-Yang-Mills theory, eq. (5.40) with $h = 2$ can be excluded at four loops, because there is no numerical constant with $\tau = 1$ that could bring the transcendentality of eq. (5.40) from 6 up to 7.

On the other hand, in theories with a lower number of supersymmetries, and at four loops, there are potentially both pure-gluon and matter-loop contributions of lower transcendentality, because only the specific loop content of $\mathcal{N} = 4$ super-Yang-Mills theory is expected to be purely of maximal transcendentality. Thus eq. (5.40) may be allowed for $h = 2$ for generic adjoint-loop contributions (for example a gluon box in QCD), and analogously

$$\Delta_4^{C_{4,F}}(\rho_{ijkl}) = \text{Tr} \left[t^a t^b t^c t^d \right] \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \left[\ln \rho_{1234} \ln \rho_{1423} \ln \rho_{1342} \right]^h, \quad (5.41)$$

for loops of matter in the fundamental representation (with generators t^a), *e.g.* from quark box diagrams. As before, the other power allowed by transcendentality, $h = 1$, is excluded by symmetry, because there is no projection of $\text{Tr} [t^a t^b t^c t^d]$ that is totally antisymmetric under permutations.

While eq. (5.40) is excluded by transcendentality for $\mathcal{N} = 4$ super-Yang-Mills theory, another construction involving the quartic Casimir is allowed: eq. (5.15) for $h_1 = 2$, $h_2 = h_3 = 1$ can be used, after multiplying it by the transcendentality $\tau = 3$ constant $\zeta(3)$. Finally, as already mentioned, there are a number of other purely logarithmic structures with partial symmetry in each term, as represented by eqs. (5.11) and (5.12), that may appear at four loops.

6. Generalization to n -leg amplitudes

The above analysis focused on the case of four partons, because this is the first case where cross ratios can be formed, and thus Δ may appear. However, rescaling-invariant ratios

involving more than four momenta can always be split into products of cross ratios involving just four momenta. Therefore it is straightforward to generalize the results we obtained to any n -parton process at three loops. Indeed, contributions to Δ_n are simply constructed as a sum over all possible sets of four partons,

$$\Delta_n = \sum_{i,j,k,l} \Delta_4(\rho_{ijkl}), \quad (6.1)$$

just as the sum-over-dipoles formula (2.7) is written as a sum over all possible pairs of legs. The indices in the sum in eq. (6.1) are of course all unequal. Assuming a purely logarithmic structure, at three loops the function Δ_4 in eq. (6.1) is given by $\Delta_4^{(122)}$ in eq. (5.28) (or, equivalently, $\Delta_4^{(311)}$ in eq. (5.29)). Of course the overall prefactor to Δ_4 could still be zero; its value remains to be determined by an explicit computation. The total number of terms in the sum increases rapidly with the number of legs: for n partons, there are $\binom{n}{4}$ different terms.

Now we wish to show that this generalization is a consistent one. To do so, we shall verify that the difference between Δ_n and Δ_{n-1} in eq. (4.14), for the splitting amplitude anomalous dimension, does not introduce any dependence on the kinematics or color of any partons other than the collinear pair. The verification is non-trivial for $n \geq 5$ because Δ_{n-1} is no longer zero.

Consider the general n -leg amplitude, in which the two legs p_1 and p_2 become collinear. The terms entering eq. (4.14) include:

- A sum over $\binom{n-2}{4}$ different terms in Δ_n that do not involve any of the legs that become collinear. They depend on the cross ratios ρ_{ijkl} where none of the indices is 1 or 2. However, exactly the same terms appear in Δ_{n-1} , so they cancel in eq. (4.14).
- A sum over $\binom{n-2}{2}$ different terms in Δ_n depending on the variables ρ_{12ij} (and permutations), where $i, j \neq 1, 2$. These variables involve the two legs that become collinear. According to eq. (6.1), each of these terms is Δ_4 , namely it is given by a sum of terms that admit the constraint (5.25). Therefore each of them is guaranteed to vanish in the collinear limit, and we can discard them from eq. (4.14). The same argument applies to any Δ_4 that is consistent at the four-parton level, such as the polylogarithmic constructions (5.38) and (5.39).
- Finally, Δ_n brings a sum over $2 \times \binom{n-2}{3}$ terms involving just one leg among the two that become collinear. These terms depend on ρ_{1jkl} or ρ_{2jkl} , where $j, k, l \neq 1, 2$. In contrast, Δ_{n-1} brings just one set of such terms, because the (12) leg, $P = p_1 + p_2$, is now counted once. Recalling, however, that this leg carries the color charge

$$\mathbf{T}^a = \mathbf{T}_1^a + \mathbf{T}_2^a, \quad (6.2)$$

it becomes clear that any term of this sort having a color factor of the form (5.2)

would cancel out in the difference. Indeed

$$\begin{aligned}
& \Delta_n(\rho_{1jkl}, \rho_{2jkl}) - \Delta_{n-1}(\rho_{(12)jkl}) \\
&= \sum_{j,k,l} h^{abcd} \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d \left[\underbrace{\mathbf{T}_1^a \Delta^{\text{kin}}(\rho_{1jkl}) + \mathbf{T}_2^a \Delta^{\text{kin}}(\rho_{2jkl})}_{\text{from } \Delta_n} - \underbrace{\mathbf{T}^a \Delta^{\text{kin}}(\rho_{(12)jkl})}_{\text{from } \Delta_{n-1}} \right] \quad (6.3) \\
&= 0.
\end{aligned}$$

To show that this combination vanishes we used eq. (6.2) and the fact that the kinematic factor Δ^{kin} in all three terms is identical because of rescaling invariance, that is, it depends only on the directions of the partons, which coincide in the collinear limit.

We conclude that eq. (6.1) is consistent with the limit as any two of the n legs become collinear.

A similar analysis also suggests that eq. (6.1) is consistent with the triple collinear limit in which p_1 , p_2 and p_3 all become parallel. We briefly sketch the analysis. We assume that there is a universal factorization in this limit, in which the analog of \mathbf{Sp} again only depends on the triple-collinear variables: $P^2 \equiv (p_1 + p_2 + p_3)^2$, which vanishes in the limit; $2p_1 \cdot p_2 / P^2$ and $2p_2 \cdot p_3 / P^2$; and the two independent longitudinal momentum fractions for the p_i , namely z_1 and z_2 (and $z_3 = 1 - z_1 - z_2$) — see *e.g.* ref. [67] for a discussion at one loop. In the triple-collinear limit there are the following types of contributions:

- $\binom{n-3}{4}$ terms in Δ_n that do not involve any of the collinear legs. They cancel in the analog of eq. (4.14) between Δ_n and Δ_{n-2} , exactly as in the double-collinear case.
- $3 \times \binom{n-3}{3}$ terms containing cross ratios of the form ρ_{1jkl} , or similar terms with 1 replaced by 2 or 3. These contributions cancel exactly as in eq. (6.3), except that there are three terms and the color conservation equation is $\mathbf{T}^a = \mathbf{T}_1^a + \mathbf{T}_2^a + \mathbf{T}_3^a$.
- $3 \times \binom{n-3}{2}$ terms containing cross ratios of the form ρ_{12kl} , or similar terms with $\{1, 2, 3\}$ permuted. These terms cancel for the same reason as the ρ_{12ij} terms in the double-collinear analysis, namely one of the logarithms is guaranteed to vanish.
- $(n-3)$ terms containing cross ratios of the form ρ_{123l} . This case is non-trivial, because no logarithm vanishes (no cross ratio goes to 1). However, it is easy to verify that in the limit, each of the cross ratios that appears depends only on the triple-collinear kinematic variables, and in a way that is independent of p_l . Therefore the color identity $\sum_{l \neq 1,2,3} \mathbf{T}_l^a = -\mathbf{T}^a$ can be used to express the color, as well as the kinematic dependence, of the limit of Δ_n solely in terms of the collinear variables, as required by universality.

Thus all four contributions are consistent with a universal triple-collinear limit. However, because the last type of contribution is non-vanishing in the limit, in contrast to the double-collinear case, the existence of a non-trivial Δ_n would imply a new type of contribution to the $1/\epsilon$ pole in the triple-collinear splitting function, beyond that implied by the sum-over-dipoles formula.

We conclude that eq. (6.1) provides a straightforward and consistent generalization of the structures found in the four-parton case to n partons. At the three-loop level, if four-parton correlations arise, they contribute to the anomalous dimension matrix through a sum over color ‘quadrupoles’ of the form (6.1). At higher loops, of course, structures that directly correlate the colors and momenta of more than four partons may also arise.

7. Conclusions

Building upon the factorization properties of massless scattering amplitudes in the soft and collinear limits, recent work [41,42] determined the principal structure of soft singularities in multi-leg amplitudes. It is now established that the cusp anomalous dimension γ_K controls all pairwise interactions amongst the hard partons, to all loops, and for general N_c . The corresponding contribution to the soft anomalous dimension takes the elegant form of a sum over color dipoles, directly correlating color and kinematic degrees of freedom. This recent work also led to strong constraints on any additional singularities that may arise, thus opening a range of interesting questions.

In the present paper we studied multiple constraints on the form of potential soft singularities that couple directly four hard partons, which may arise at three loops and beyond. We focused on potential corrections to the sum-over-dipoles formula that do not require the presence of higher Casimir contributions to the cusp anomalous dimension γ_K . The basic property of these functions is that they satisfy the homogeneous set of differential equations (2.9), and therefore they can be written in terms of conformally invariant cross ratios [41].

Our main conclusion is that indeed, potential structures of this kind may arise starting at three loops. Their functional dependence on both color and kinematic variables is, however, severely constrained by

- Bose symmetry;
- Sudakov factorization and momentum-rescaling symmetry, dictating that corrections must be functions of conformally invariant cross ratios;
- collinear limits, in which the (expected) universal properties of the splitting amplitude force corrections to vanish (for $n = 4$ partons) or be smooth (for $n > 4$ partons) in these limits;
- transcendentality, a bound on which is expected to be saturated at three loops, based on the properties of $\mathcal{N} = 4$ super-Yang-Mills theory.

In the three-loop case, assuming purely logarithmic dependence on the cross ratios, these constraints combine to exclude all but one specific structure. The three-loop result for Δ_n can therefore be written in terms of the expression $\Delta_4^{(122)}$ in eq. (5.28), up to an overall numerical coefficient. Because this structure has the maximal possible transcendentality, $\tau = 5$, its coefficient is a rational number. For all we know now, however, this coefficient may vanish. It remains for future work to decide whether this contribution is present or

not. Considering also polylogarithmic functions of conformally invariant cross ratios in Δ_4 , we find that at three loops at least two additional acceptable functional forms arise, eqs. (5.38) and (5.39).

The range of admissible functions at four loops is even larger. A particularly interesting feature at this order is the possible appearance of contributions proportional to quartic Casimir operators, not only in the cusp anomalous dimension, but in four-parton correlations as well.

Explicit computations at three and four loops will probably be necessary to take the next steps toward a complete understanding of soft singularities in massless gauge theories.

Acknowledgments

We thank Thomas Binoth, David Kosower and George Sterman for stimulating discussions. L. D. and L. M. thank CERN for hospitality while this work was completed. This research was supported by the US Department of Energy under contract DE-AC02-76SF00515, by MIUR (Italy) under contract 2006020509_004, and by the European Community's Marie-Curie Research Training Network 'Tools and Precision Calculations for Physics Discoveries at Colliders' ('HEPTOOLS'), under contract MRTN-CT-2006-035505.

A. The four-parton amplitude with no momentum recoil

Here we investigate briefly the simplifications of the potential forms for Δ_4 at three loops that result from using momentum-conservation relations special to the four-parton amplitude, $p_1 + p_2 + p_3 + p_4 = 0$. Thus we exclude here the presence in the amplitude of other colorless particles that might carry recoil momentum. As before, all momenta are light-like, $p_i^2 = 0$, so the momentum invariants are now related by

$$p_1 \cdot p_2 + p_2 \cdot p_3 + p_1 \cdot p_3 = 0,$$

as well as $p_3 \cdot p_4 = p_1 \cdot p_2$, *etc.* Using this relation, all three cross ratios entering Δ_4 can be expressed in terms of a single dimensionless ratio,

$$r \equiv \frac{p_2 \cdot p_3}{p_1 \cdot p_2} = -1 - \frac{p_1 \cdot p_3}{p_1 \cdot p_2}. \quad (\text{A.1})$$

Substituting into eq. (2.11), we have

$$\begin{aligned} L_{1234} &= \ln \left(\left| \frac{1}{1+r} \right|^2 e^{-i\pi(\lambda_{12} + \lambda_{34} - \lambda_{13} - \lambda_{24})} \right), \\ L_{1423} &= \ln \left(|r|^2 e^{-i\pi(\lambda_{14} + \lambda_{23} - \lambda_{12} - \lambda_{34})} \right), \\ L_{1342} &= \ln \left(\left| \frac{1+r}{r} \right|^2 e^{-i\pi(\lambda_{13} + \lambda_{24} - \lambda_{14} - \lambda_{23})} \right), \end{aligned} \quad (\text{A.2})$$

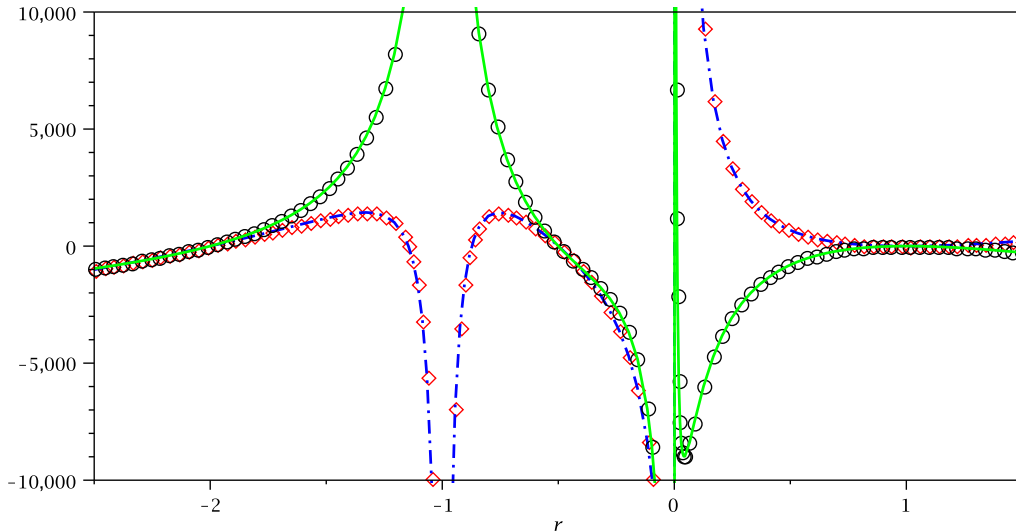


Figure 3: The logarithmic functions Δ_4 with transcendentality $\tau = 5$ introduced in eqs. (5.28) and (5.29), respectively, in the case of a four-parton amplitude with no recoil. In each case we separately display the real (dot-dash) and imaginary (solid line) parts as a function of the ratio r , defined in eq. (A.1), multiplying the color coefficient $f^{ade} f^{cbe}$ after the Jacobi identity has been taken into account, as in eq. (A.3). In each case symbols represent eq. (5.28) while the lines stand for eq. (5.29). The plot demonstrates that the two are identical once the Jacobi identity is taken into account.

where we used the variables λ_{ij} , defined below eq. (2.3), to keep track of the unitarity phases.

Let us now examine the behavior of the three expressions for Δ_4 that we found admissible, as functions of r . As mentioned in Sec. 5.4, in the four-parton case with no momentum recoil, $p_i \cdot p_j$ is not a collinear limit, but a forward or backward scattering limit. There are three channels to consider:

- a) p_1 and p_2 incoming, p_3 and p_4 outgoing. The physical region is $-1 < r < 0$.
- b) p_1 and p_3 incoming, p_2 and p_4 outgoing. The physical region is $0 < r < \infty$.
- c) p_1 and p_4 incoming, p_2 and p_3 outgoing. The physical region is $-\infty < r < -1$.

The two endpoints of each physical interval are the forward and backward scattering limits. Using eq. (A.2), we can read off the phases associated with each of the logarithms of the cross ratios in these three physical regions. The results are summarized in Table 2.

We have seen that, due to the Jacobi identity, the three terms in Δ_4 corresponding to antisymmetrization of any pair of color indices are related to each other, leaving just two independent terms. For example, in eq. (5.28) we can substitute $f^{bae} f^{cde} = -f^{ade} f^{cbe} -$

	a) $-1 < r < 0$	b) $0 < r < \infty$	c) $-\infty < r < -1$
$L_{1234}/2$	$-\ln 1+r - i\pi$	$-\ln 1+r + i\pi$	$-\ln 1+r $
$L_{1423}/2$	$\ln r + i\pi$	$\ln r $	$\ln r - i\pi$
$L_{1342}/2$	$\ln (1+r)/r $	$\ln (1+r)/r - i\pi$	$\ln (1+r)/r + i\pi$

Table 2: Analytic continuation of the three cross ratios into the three physical regions in a $2 \rightarrow 2$ scattering amplitude.

$f^{cae} f^{dbe}$, getting

$$\Delta_4^{(122)}(\rho_{ijkl}) = \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d L_{1234} L_{1423} L_{1342} \left\{ f^{ade} f^{cbe} L_{1423} [L_{1342} - L_{1234}] + f^{cae} f^{dbe} L_{1234} [L_{1342} - L_{1423}] \right\}. \quad (\text{A.3})$$

The resulting dependence on r is shown in Figure 3, which displays the coefficient of $f^{ade} f^{cbe}$, after using the Jacobi identity, as in eq. (A.3). The plot shows that the result is the same when starting with either eq. (5.28) or eq. (5.29), as proven in (5.30). In a similar way one can use the Jacobi identity for the other admissible functions given in eqs. (5.38) and (5.39). Of course, each of them yields a different function of r .

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